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Existence of global attractor for the Trojan Y Chromosome model

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Abstract. This paper is concerned with the long time behavior of solution for the equation derived by the Trojan Y Chromosome (TYC) model with spatial spread. Based on the regularity estimates for the semigroups and the classical existence theorem of global attractors, we prove that this equations possesses a global attractor in $H^k(\Omega)^4$ $(k \geq 0)$ space.
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1 Introduction

An exotic species is a species which resides outside its native habitat, when it causes some sort of measurable damage, it is often referred to as an invasive species. In recent history, the economic process of globalization has accelerated the pace at which the exotic species are introduced into the new enviromnents. Once it is established, these species can be extremely difficult to manage and almost impossible to deracinate (see [6, 11]). The effect of these invaders is thus devastating (see [3]). The present approaches for controlling exotic fish species are limited to general chemical control methods applied to small water bodies and/or small isolated populations that not only kill the exotic species but also the native fish in addition to the target fish (see [8]).

In 2006, a strategy for eradication of invasive fish in which a Trojan fish is added to the population was reported by Gutierrez and Teem [4]. This strategy is relevant to species a menable to sex reversal and with an XY sex-determination system. In this strategy males are the heterogametic sex (carrying one X chromosome and one Y chromosome, XY), females are the homogametic sex (carrying two X chromosomes, XX). The eradication strategy requires adding a sex-reversed "Trojan" female individual bearing two Y chromosomes, that is, feminized supermales (r), at a constant rate μ, to a target population of an invasive species, containing normal females and males denoted as f and m, respectively. Matings involving the introduced r generate a disproportionate number of males over time. The higher incidence of males decrease the female to male ratio. Finally, the number of f decline to zero, causing local extinction. This is the Trojan Y Chromosome (TYC) strategy.

Recently, Gutierrez et al. [5] considered the spatial spread in aquatic settings for the Trojan Y Chromosome (TYC) model, which resulting in a PDE model. In [9], Parshad and Gutierrez demonstrated the existence of a unique

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weak solution to the infinite dimensional TYC system. Furthermore, they obtained improved estimates on the upper bounds for the Hausdorff and fractal dimensions of the global attractor of the TYC system, via the use of weighted Sobolev spaces. These results confirmed that the TYC eradication strategy is a sound theoretical method of eradication of invasive species in a spatial setting. Parshad and Gutierrez [10] also give the the existence of a global attractor for the TYC system, which is H^2 regular, attracting orbits uniformly in the $L^2(\Omega)$ metric. They also showed that this attractor supports a state, in which the female population is driven to zero, then resulting in local extinction.

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, then the model with spatial spread is given by

$$\frac{\partial f}{\partial t} = D\Delta f + \frac{1}{2} fm\beta L - \delta f, \quad f|_{\partial\Omega} = 0, \quad f(\cdot, 0) = f_0, \tag{1.1}$$

$$\frac{\partial m}{\partial t} = D\Delta m + (\frac{1}{2} fm + \frac{1}{2} rm + fs)\beta L - \delta m, \quad m|_{\partial\Omega} = 0, \quad m(\cdot, 0) = m_0, \tag{1.2}$$

$$\frac{\partial s}{\partial t} = D\Delta s + (\frac{1}{2} rm + rs)\beta L - \delta s, \quad s|_{\partial\Omega} = 0, \quad s(\cdot, 0) = s_0, \tag{1.3}$$

$$\frac{\partial r}{\partial t} = D\Delta r + \mu - \delta r, \quad r|_{\partial\Omega} = 0, \quad r(\cdot, 0) = r_0. \tag{1.4}$$

Also

$$L = 1 - (\frac{f + m + r + s}{K}), \tag{1.5}$$

where K is the carrying capacity of the ecosystem, D is a diffusivity coefficient, β is a birt coefficient (i. e. what proportion of encounters between males and females result in progeny), and δ is a death coefficient (i. e. what proportion of the population is dying a any given moment). Assume that initial data in $L^2(\Omega)$, define the phase space for the model as follows

$$H = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega),$$

$$Y = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega),$$

There are many studies on the existence of global attractors for diffusion equations. For the classical results we refer the reader to [2, 14, 17]. Recently, based on the iteration technique for regularity estimates, combining with the classical existence theorem of global attractors, Song et al [12, 13] considered the global attractor for some parabolic equations, such as Cahn-Hilliard equation, Swift-Hohenberg equation and so on, in H^k $(0 \leq k \leq \infty)$ space. Zhao and Liu [15] studied the global attractor for a fourth order parabolic equation modeling epitaxial thin-film growth in H^k $(0 \leq k < 5)$ space. However, since the difficulty arise from the components' interaction, there's none consider the H^k-global attractor for the diffusion systems.

In this paper, we are interested in the existence of global attractors for the diffusion system (1.1)-(1.4). Based on R. D. Parshad, J. B. Gutierrez's article [9] and T. Ma, S. Wang's recent work [7], we shall prove that the equations (1.1)-(1.4) possesses a global attractor in $H^k(\Omega)^4$ $(0 \le k < \infty)$ space.

The outline of this paper is as follows: In the next section, we give preliminary considerations, we also give the main result on the existence of global attractor for the problem (1.1)-(1.4); In section 3, the main result is proved; Finally in Section 4, conclusions are obtained.

2 Preliminaries

Assume X and X_1 are two Banach spaces, $X_1 \subset X$ a compact and dense inclusion. Consider the following equation defined on X,

$$\begin{cases} U_t = AU + GU, \\ U(0) = U_0, \end{cases} \tag{2.1}$$

where U is an unknown function, $A : X_1 \to X$ a linear operator and $G : X_1 \to X$ a nonlinear operator. Then the solution of (2.1) can be expressed as

$$U(t, U_0) = S(t)U_0,$$

where $S(t) : X \to X$ $(t \ge 0)$ is a semigroup generated by (2.1).

We used to assume that the linear operator $A : X_1 \to X$ in (2.1) is a sectorial operator, which generates an analytic semigroup e^{tA}, and A induces the fractional power operators $\mathscr{L}^\alpha$ and fractional order spaces X_α as follows,

$$\mathscr{L}^\alpha = (-A)^\alpha : X_\alpha \to X, \ \alpha \in R, \tag{2.2}$$

where $X_\alpha = D(\mathscr{L}^\alpha)$ is the domain of $\mathscr{L}^\alpha$. By the semigroup theory of linear operators, $X_\beta \subset X_\alpha$ is a compact inclusion for any $\beta > \alpha$. If you want to know more about the space H_α, I recommend you read [7].

Now, we introduce a lemma on the existence of global attractor which can be found in [7, 12, 13].

Lemma 2.1 *Assume that $U(t, U_0) = S(t)U_0$ $(U_0 \in X, t \ge 0)$ is a solution of (2.1) and $S(t)$ the semigroup generated by (2.1). Assume further that X_α is the fractional order space generated by A and*

(B1) For some $\alpha \ge 0$ there is a bounded set $B \subset X_\alpha$, which means for any $U_0 \in X_\alpha$, there exists $t_{U_0} > 0$ such that

$$U(t, U_0) \in B, \quad \forall t > t_{U_0};$$

(B2) There is a $\beta > \alpha$, for any bounded set $E \subset X_\beta$, $\exists T > 0$ and $C > 0$ such that

$$\|U(t, U_0)\|_{X_\beta} \le C, \quad \forall t > T, \ U_0 \in E.$$

Then (2.1) has a global attractor $\mathscr{A} \subset X_\alpha$ which attracts any bounded set of X_α in the $X_\alpha-$norm.

We also have the following lemma which can be found in [7, 12, 13].

Lemma 2.2 *Assume that $A : X_1 \to X_\alpha$ is a sectorial operator which generates an analytic semigroup $T(t) = e^{tA}$. If all eigenvalues λ of A satisfy $Re\lambda < -\lambda_0$ for some real number $\lambda_0 > 0$, then for $\mathscr{L}^\alpha (\mathscr{L} = -A)$ we have*
(C1) $T(t) : X \to X_\alpha$ is bounded for all $\alpha \in R^1$ and $t > 0$;
(C2) $T(t)\mathscr{L}^\alpha x = \mathscr{L}^\alpha T(t)x, \forall x \in X_\alpha$;
(C3) For each $t > 0$, $\mathscr{L}^\alpha T(t) : X \to X$ is bounded, and

$$\|\mathscr{L}^\alpha T(t)\| \leq C_\alpha t^{-\alpha} e^{-\delta t};$$

where some $\delta > 0$ and $C_\alpha > 0$ is a constant depending only on α;
(C4) The $X_\alpha-$norm can be defined by $\|x\|_{X_\alpha} = \|\mathscr{L}^\alpha x\|_X$.

For the problem (1.1)-(1.4), we introduce the spaces as follows

$$\begin{cases} \mathcal{H} = H = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega), \\ \mathcal{H}_{\frac{1}{2}} = Y = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega), \\ \mathcal{H}_1 = \big(H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)\big) \bigcap \mathcal{H}_{\frac{1}{2}}, \end{cases} \tag{2.3}$$

Let $u = (f, m, s, r)$, where (f, m, s, r) represents a column vector. Define the operators L and G_i $(i = 1, 2, 3, 4)$ by

$$\begin{cases} Lf = D\Delta f, \ Lm = D\Delta m, \ Ls = D\Delta s, \ Lr = D\Delta r, \\ G_1 u = g_1(f, m, s, r) = \frac{1}{2} m\beta L - \delta f, \\ G_2 u = g_2(f, m, s, r) = (\frac{1}{2} fm + \frac{1}{2} rm + fs)\beta L - \delta m, \\ G_3 u = g_3(f, m, s, r) = (\frac{1}{2} rm + rs)\beta L - \delta s, \\ G_4 u = g_4(f, m, s, r) = \mu - \delta r, \end{cases} \tag{2.4}$$

where $g_i(f, m, s, r)$ $(i = 1, 2, 3, 4)$ are nonlinear functions. Obviously, the linear operator $L : H^2(\Omega) \to L^2(\Omega)$ given by (2.4) is a sectorial operator.
Define

$$A = \begin{pmatrix} D\Delta & 0 & 0 & 0 \\ 0 & D\Delta & 0 & 0 \\ 0 & 0 & D\Delta & 0 \\ 0 & 0 & 0 & D\Delta \end{pmatrix} = \begin{pmatrix} L & & & \\ & L & & \\ & & L & \\ & & & L \end{pmatrix} : \mathcal{H}_1 \to \mathcal{H},$$

and

$$Gu = \begin{pmatrix} G_1 u \\ G_2 u \\ G_3 u \\ G_4 u \end{pmatrix} = \begin{pmatrix} g_1(f, m, s, r) \\ g_2(f, m, s, r) \\ g_3(f, m, s, r) \\ g_4(f, m, s, r) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} m\beta L - \delta f \\ (\frac{1}{2} fm + \frac{1}{2} rm + fs)\beta L - \delta m \\ (\frac{1}{2} rm + rs)\beta L - \delta s \\ \mu - \delta r \end{pmatrix},$$

then the initial boundary value problem (1.1)-(1.4) is formulated into the following problem:

$$\frac{du}{dt} = Au + Gu, \quad t > 0, \tag{2.5}$$

where $u = (f, m, s, r)$ for any initial data $u(0) = u_0 = (f_0, m_0, s_0, r_0)$. We define $\|u\|_{\mathcal{H}_\alpha} = (\|f\|^2_{\mathrm{H}_\alpha} + \|m\|^2_{\mathrm{H}_\alpha} + \|s\|^2_{\mathrm{H}_\alpha} + \|r\|^2_{\mathrm{H}_\alpha})^{\frac{1}{2}}$. Clearly, in (2.5), A is a linear operator and G a nonlinear operator.

Compared with (2.1), it is easy to see that $X = \mathcal{H}$, $X_1 = \mathcal{H}_1$, $A : \mathcal{H}_1 \to \mathcal{H}$ is a linear sectorial operator and G a nonlinear operator in (2.5). We can define the fractional order spaces $\mathscr{L}^\alpha$ as (2.2), where $\mathcal{H}_\alpha = D(\mathscr{L}^\alpha) = \mathrm{H}_\alpha \times \mathrm{H}_\alpha \times \mathrm{H}_\alpha \times \mathrm{H}_\alpha = D((-L)^\alpha) \times D((-L)^\alpha) \times D((-L)^\alpha) \times D((-L)^\alpha)$ is the domain of $\mathscr{L}^\alpha$.

We summarize the following results in [9, 10].

Proposition 2.1 *Consider the Trojan Y Chromosome model, (1.1)-(1.4). The solution (f, m, r, s) of the system are bounded as follows:*

$$\|f\|_{L^\infty} \leq K, \quad \|m\|_{L^\infty} \leq K, \quad \|s\|_{L^\infty} \leq K, \quad \|r\|_{L^\infty} \leq K, \tag{2.6}$$

where $K > 0$ is a is the carrying capacity of the ecosystem which can be seen as a constant.

Proposition 2.2 *Consider the Trojan Y Chromosome model, (1.1)-(1.4). There exists a (H, H) global attractor $\mathscr{A}$ for this system which is compact and invariant in H and attracts all the bounded subsets of H in the H metric.*

Now, we give the main result, which provides the existence of global attractors of the equations (1.1)-(1.4) in any kth space $H^k(\Omega)^4$.

Theorem 2.1 *Consider the Trojan Y Chromosome model, (1.1)-(1.4). For any $\alpha \geq 0$, $u_0 = (f_0, m_0, s_0, r_0) \in \mathcal{H}_\alpha$, the semigroup $S(t)$ associated with problem (1.1)-(1.4) possesses a global attractor $\mathscr{A}$ in $\mathcal{H}_\alpha$ space and $\mathscr{A}$ attractors any bounded set of $\mathcal{H}_\alpha$ in the $\mathcal{H}_\alpha$-norm.*

3 Proof of Theorem 2.1

We are now in a position to state and prove the main theorem in this paper, which provides the existence of a global attractor of the equations (1.1)-(1.4) in spaces $\mathcal{H}_\alpha$ of any αth differentiable function.

For any $(f_0, m_0, s_0, r_0) \in \mathcal{H}$, the solution (f, m, s, r) of the problem (1.1)-(1.4) can be written as

$$\begin{aligned} f(t, f_0) &= e^{t\mathbf{L}} f_0 + \int_0^t e^{(t-\tau)\mathbf{L}} G_1 u d\tau \\ &= e^{t\mathbf{L}} f_0 + \int_0^t e^{(t-\tau)\mathbf{L}} g_1(f, m, s, r) d\tau, \end{aligned} \tag{3.1}$$

$$\begin{aligned} m(t,m_0) &= e^{t\mathbf{L}}m_0 + \int_0^t e^{(t-\tau)\mathbf{L}}G_2 u d\tau \\ &= e^{t\mathbf{L}}m_0 + \int_0^t e^{(t-\tau)\mathbf{L}}g_2(f,m,s,r)d\tau, \end{aligned} \tag{3.2}$$

$$\begin{aligned} s(t,s_0) &= e^{t\mathbf{L}}s_0 + \int_0^t e^{(t-\tau)\mathbf{L}}G_3 u d\tau \\ &= e^{t\mathbf{L}}s_0 + \int_0^t e^{(t-\tau)\mathbf{L}}g_3(f,m,s,r)d\tau, \end{aligned} \tag{3.3}$$

$$\begin{aligned} r(t,r_0) &= e^{t\mathbf{L}}r_0 + \int_0^t e^{(t-\tau)\mathbf{L}}G_4 u d\tau \\ &= e^{t\mathbf{L}}r_0 + \int_0^t e^{(t-\tau)\mathbf{L}}g_4(f,m,s,r)d\tau, \end{aligned} \tag{3.4}$$

By Lemma 2.1, in order to prove Theorem 2.1, we first prove the following lemma.

Lemma 3.1 *If (f,m,s,r) is a solution to the Trojan Y Chromosome model, (1.1)-(1.4), then, for any $\alpha \geq 0$, $u_0 \in \mathcal{H}_\alpha$, the semigroup $S(t)$ associated with problem (1.1)-(1.4) is uniformly compact in $\mathcal{H}_\alpha$.*

Proof. It suffices to prove that for any bounded set $E \subset \mathcal{H}_\alpha$ with initial value $u_0 = (f_0, m_0, s_0, r_0) \in E \subset \mathcal{H}_\alpha$, there exists $C > 0$ such that

$$\|u(t,u_0)\|_{\mathcal{H}_\alpha} \leq C, \quad \forall t \geq 0,\ \alpha \geq 0. \tag{3.5}$$

Obviously, if we get

$$\|f(t,f_0)\|^2_{\mathrm{H}_\alpha} + \|m(t,m_0)\|^2_{\mathrm{H}_\alpha} + \|s(t,s_0)\|^2_{\mathrm{H}_\alpha} + \|r(t,r_0)\|^2_{\mathrm{H}_\alpha} \leq C, \quad \forall t \geq 0,\ \alpha \geq 0,$$

then (3.5) is obtained immediately.

For $\alpha = 0$, this follows from Proposition 2.2, i.e. for any bounded set $E \subset \mathcal{H}$ with initial value $(f_0, m_0, s_0, r_0) \in E \subset \mathcal{H}$, there exists a constant $C > 0$ such that

$$\|f(t,f_0)\|^2_{\mathrm{H}} + \|m(t,m_0)\|^2_{\mathrm{H}} + \|s(t,s_0)\|^2_{\mathrm{H}} + \|r(t,r_0)\|^2_{\mathrm{H}} \leq C, \quad \forall t \geq 0, \tag{3.6}$$

Hence

$$\|u(t,u_0)\|_{\mathcal{H}} \leq C, \quad \forall t \geq 0.$$

So, we only need to show (3.5) for any $\alpha > 0$. There are three steps for us to prove it.

Step 1. We prove that for any bounded set $E \subset \mathcal{H}_\alpha$ $(0 < \alpha < 1)$, there exists a positive constant C such that $\forall t \geq 0,\ 0 < \alpha < 1$,

$$\|f(t,f_0)\|^2_{\mathrm{H}_\alpha} + \|m(t,m_0)\|^2_{\mathrm{H}_\alpha} + \|s(t,s_0)\|^2_{\mathrm{H}_\alpha} + \|r(t,r_0)\|^2_{\mathrm{H}_\alpha} \leq C. \tag{3.7}$$

It follows from Proposition 2.1 that

$$\|f\|_{L^\infty} \le K, \quad \|m\|_{L^\infty} \le K, \quad \|s\|_{L^\infty} \le K, \quad \|r\|_{L^\infty} \le K,$$

where K is a positive constant. Hence

$$\begin{aligned}
& \|g_1(f,m,s,r)\|_{\mathrm{H}}^2 = \int_\Omega (\frac{1}{2} fm\beta L - \delta f)^2 dx \\
= \;& \int_\Omega (\frac{\beta}{2} fm - \frac{\beta}{2K} f^2 m - \frac{\beta}{2K} fm^2 - \frac{\beta}{2K} mrf - \frac{\beta}{2K} msf - \delta f)^2 dx \\
\le \;& C \int_\Omega (f^2 m^2 + \frac{1}{K^2} f^4 m^2 + \frac{1}{K^2} f^2 m^4 + \frac{1}{K^2} f^2 m^2 r^2 \\
& + \frac{1}{K^2} f^2 m^2 s^2 + f^2) dx \\
\le \;& C \int_\Omega (\sup_{x\in\Omega} m^2 \cdot f^2 + \frac{1}{K^2} \sup_{x\in\Omega} f^4 \cdot m^2 + \frac{1}{K^2} \sup_{x\in\Omega} m^4 \cdot f^2 \\
& + \frac{1}{K^2} \sup_{x\in\Omega} m^2 f^2 \cdot r^2 + \frac{1}{K^2} \sup_{x\in\Omega} m^2 f^2 \cdot s^2 + f^2) dx \\
\le \;& C \int_\Omega (f^2 + m^2 + r^2 + s^2) dx \\
= \;& C(\|f\|^2 + \|m\|^2 + \|s\|^2 + \|r\|^2) \le C.
\end{aligned} \tag{3.8}$$

Based on Proposition 2.1, simple calculations show that

$$\|g_2(f,m,s,r)\|_{\mathrm{H}}^2 \le C(\|f\|^2 + \|m\|^2 + \|s\|^2 + \|r\|^2) \le C, \tag{3.9}$$

$$\|g_3(f,m,s,r)\|_{\mathrm{H}}^2 \le C(\|f\|^2 + \|m\|^2 + \|s\|^2 + \|r\|^2) \le C, \tag{3.10}$$

$$\|g_4(f,m,s,r)\|_{\mathrm{H}}^2 \le C(\|f\|^2 + \|m\|^2 + \|s\|^2 + \|r\|^2) \le C, \tag{3.11}$$

By (3.1), (3.6) and (3.8), using the properties of Lemma 2.2, we obtain

$$\begin{aligned}
\|f(t,f_0)\|_{\mathrm{H}_\alpha} &= \|e^{t\mathbf{L}} f_0 + \int_0^t e^{(t-\tau)\mathbf{L}} g_1(f,m,s,r) d\tau\|_{\mathrm{H}_\alpha} \\
&\le \|e^{t\mathbf{L}} f_0\|_{\mathrm{H}_\alpha} + \|\int_0^t e^{(t-\tau)\mathbf{L}} g_1(f,m,s,r) d\tau\|_{\mathrm{H}_\alpha} \\
&\le C\|f_0\|_{\mathrm{H}_\alpha} + \int_0^t \|(-\mathbf{L})^\alpha e^{(t-\tau)\mathbf{L}}\| \cdot \|g_1(f,m,s,r)\|_{\mathrm{H}} d\tau \\
&\le C\|f_0\|_{\mathrm{H}_\alpha} + C\int_0^t \tau^{-\alpha} e^{-\delta\tau} d\tau \\
&\le C, \quad \forall t \ge 0, \ (f_0, m_0, s_0, r_0) \in E,
\end{aligned} \tag{3.12}$$

where $0 < \alpha < 1$. By (3.2), (3.3), (3.4), (3.6), (3.9), (3.10) and (3.11), simple calculations shows that

$$\|m(t,m_0)\|_{\mathrm{H}_\alpha} \le C, \quad \forall t \ge 0, \ (f_0, m_0, s_0, r_0) \in E, \tag{3.13}$$

$$\|s(t,s_0)\|_{\mathrm{H}_\alpha} \le C, \quad \forall t \ge 0,\ (f_0,m_0,s_0,r_0) \in E, \tag{3.14}$$

$$\|r(t,r_0)\|_{\mathrm{H}_\alpha} \le C, \quad \forall t \ge 0,\ (f_0,m_0,s_0,r_0) \in E, \tag{3.15}$$

where $0 < \alpha < 1$. By (3.12), (3.13), (3.14) and (3.15), we obtain (3.7) immediately.

Step 2. We prove that for any bounded set $E \subset \mathcal{H}_\alpha$ ($\frac{1}{2} < \alpha < \frac{3}{2}$), there exists a positive constant C such that $\forall t \ge 0$, $\frac{1}{2} < \alpha < \frac{3}{2}$,

$$\|f(t,f_0)\|^2_{\mathrm{H}_\alpha} + \|m(t,m_0)\|^2_{\mathrm{H}_\alpha} + \|s(t,s_0)\|^2_{\mathrm{H}_\alpha} + \|r(t,r_0)\|^2_{\mathrm{H}_\alpha} \le C. \tag{3.16}$$

By Proposition 2.1 and the following embedding theorems of fractional order spaces

$$\mathrm{H}_\alpha \hookrightarrow H^1(\Omega), \quad \forall \alpha > \frac{1}{2}, \tag{3.17}$$

we obtain

$$\begin{aligned}
& \|g_1(f,m,s,r)\|_{\mathrm{H}_{\frac{1}{2}}} \\
= & \int_\Omega |\nabla g_1(f,m,s,r)|^2 dx = \int_\Omega |\nabla(\frac{1}{2} fm\beta L - \delta f)|^2 dx \\
= & \int_\Omega (\frac{\beta}{2} f\nabla m + \frac{\beta}{2} m\nabla f - \frac{\beta}{2K}(f+m+r+s)(f\nabla m + m\nabla f) \\
& -\frac{\beta}{2K} fm(\nabla f + \nabla m + \nabla s + \nabla r) - \delta\nabla f)^2 dx \\
\le & C\int_\Omega (f^2|\nabla m|^2 + m^2|\nabla f|^2 + \frac{1}{K^2} f^4|\nabla m|^2 + \frac{1}{K^2} f^2m^2|\nabla m|^2 \\
& + \frac{1}{K^2} f^2r^2|\nabla m|^2 + \frac{1}{K^2} f^2s^2|\nabla m|^2 + \frac{1}{K^2} f^2m^2|\nabla f|^2 + \frac{1}{K^2} m^4|\nabla f|^2 \\
& + \frac{1}{K^2} m^2r^2|\nabla f|^2 + \frac{1}{K^2} m^2s^2|\nabla f|^2 + \frac{1}{K^2} f^2m^2|\nabla s|^2 \\
& + \frac{1}{K^2} f^2m^2|\nabla r|^2 + |\nabla f|^2) dx \\
\le & C\int_\Omega (\sup_{x\in\Omega} f^2 \cdot |\nabla m|^2 + \sup_{x\in\Omega} m^2 \cdot |\nabla f|^2 + \frac{1}{K^2} \sup_{x\in\Omega} f^4 \cdot |\nabla m|^2 \\
& + \frac{1}{K^2} \sup_{x\in\Omega} f^2m^2 \cdot |\nabla m|^2 + \frac{1}{K^2} \sup_{x\in\Omega} f^2r^2 \cdot |\nabla m|^2 + \frac{1}{K^2} \sup_{x\in\Omega} f^2s^2 \cdot |\nabla m|^2 \\
& + \frac{1}{K^2} \sup_{x\in\Omega} f^2m^2 \cdot |\nabla f|^2 + \frac{1}{K^2} \sup_{x\in\Omega} m^4 \cdot |\nabla f|^2 + \frac{1}{K^2} \sup_{x\in\Omega} m^2r^2 \cdot |\nabla f|^2 \\
& + \frac{1}{K^2} \sup_{x\in\Omega} m^2s^2 \cdot |\nabla f|^2 + \frac{1}{K^2} \sup_{x\in\Omega} f^2m^2 \cdot |\nabla s|^2 \\
& + \frac{1}{K^2} \sup_{x\in\Omega} f^2m^2 \cdot |\nabla r|^2 + |\nabla f|^2) dx \\
\le & C\int_\Omega (|\nabla f|^2 + |\nabla m|^2 + |\nabla s|^2 + |\nabla r|^2) dx \\
\le & C(\|f\|^2_{\mathrm{H}_\alpha} + \|m\|^2_{\mathrm{H}_\alpha} + \|s\|^2_{\mathrm{H}_\alpha} + \|r\|^2_{\mathrm{H}_\alpha}) \le C.
\end{aligned} \tag{3.18}$$

Based on Proposition 2.1 and (3.17), simple calculations show that

$$\|g_2(f,m,s,r)\|^2_{\mathrm{H}_{\frac{1}{2}}} \le C(\|f\|^2_{\mathrm{H}_\alpha} + \|m\|^2_{\mathrm{H}_\alpha} + \|s\|^2_{\mathrm{H}_\alpha} + \|r\|^2_{\mathrm{H}_\alpha}) \le C, \tag{3.19}$$

$$\|g_3(f,m,s,r)\|^2_{\mathrm{H}_{\frac{1}{2}}} \le C(\|f\|^2_{\mathrm{H}_\alpha} + \|m\|^2_{\mathrm{H}_\alpha} + \|s\|^2_{\mathrm{H}_\alpha} + \|r\|^2_{\mathrm{H}_\alpha}) \le C, \tag{3.20}$$

$$\|g_4(f,m,s,r)\|^2_{\mathrm{H}_{\frac{1}{2}}} \le C(\|f\|^2_{\mathrm{H}_\alpha} + \|m\|^2_{\mathrm{H}_\alpha} + \|s\|^2_{\mathrm{H}_\alpha} + \|r\|^2_{\mathrm{H}_\alpha}) \le C, \tag{3.21}$$

By (3.1), (3.7) and (3.18), using the properties of Lemma 2.2, we obtain

$$\begin{aligned}
\|f(t,f_0)\|_{\mathrm{H}_\alpha} &= \|e^{t\mathbf{L}}f_0 + \int_0^t e^{(t-\tau)\mathbf{L}} g_1(f,m,s,r)d\tau\|_{\mathrm{H}_\alpha} \\
&\le \|e^{t\mathbf{L}}f_0\|_{\mathrm{H}_\alpha} + \|\int_0^t e^{(t-\tau)\mathbf{L}} g_1(f,m,s,r)d\tau\|_{\mathrm{H}_\alpha} \\
&\le C\|f_0\|_{\mathrm{H}_\alpha} + \int_0^t \|(-\mathbf{L})^{\alpha-\frac{1}{2}} e^{(t-\tau)\mathbf{L}}\| \cdot \|g_1(f,m,s,r)\|_{\mathrm{H}_{\frac{1}{2}}} d\tau \\
&\le C\|f_0\|_{\mathrm{H}_\alpha} + C\int_0^t \tau^{-\beta} e^{-\delta\tau} d\tau \\
&\le C, \quad \forall t \ge 0,\ (f_0,m_0,s_0,r_0) \in E,
\end{aligned} \tag{3.22}$$

where $\beta = \alpha - \frac{1}{2}$ $(0 < \beta < 1)$. By (3.2), (3.3), (3.4), (3.7), (3.19), (3.20) and (3.21), simple calculations shows that

$$\|m(t,m_0)\|_{\mathrm{H}_\alpha} \le C, \quad \forall t \ge 0,\ (f_0,m_0,s_0,r_0) \in E, \tag{3.23}$$

$$\|s(t,s_0)\|_{\mathrm{H}_\alpha} \le C, \quad \forall t \ge 0,\ (f_0,m_0,s_0,r_0) \in E, \tag{3.24}$$

$$\|r(t,r_0)\|_{\mathrm{H}_\alpha} \le C, \quad \forall t \ge 0,\ (f_0,m_0,s_0,r_0) \in E, \tag{3.25}$$

where $\frac{1}{2} < \alpha < \frac{3}{2}$. By (3.22), (3.23), (3.24) and (3.25, we obtain (3.16) immediately.

Step 3. We prove that for any bounded set $E \subset \mathcal{H}_\alpha$ $(1 < \alpha < 2)$, there exists a positive constant C such that $\forall t \ge 0,\ 1 < \alpha < 2$,

$$\|f(t,f_0)\|^2_{\mathrm{H}_\alpha} + \|m(t,m_0)\|^2_{\mathrm{H}_\alpha} + \|s(t,s_0)\|^2_{\mathrm{H}_\alpha} + \|r(t,r_0)\|^2_{\mathrm{H}_\alpha} \le C. \tag{3.26}$$

By Proposition 2.1 and the following embedding theorems of fractional order spaces

$$\mathrm{H}_\alpha \hookrightarrow H^2(\Omega), \quad \mathrm{H}_\alpha \hookrightarrow W^{1,4}(\Omega), \quad \forall \alpha > 1, \tag{3.27}$$

we obtain

$$
\begin{aligned}
&\|g_1(f,m,s,r)\|_{\mathrm{H}_1} = \int_\Omega |\Delta g_1(f,m,s,r)|^2 dx = \int_\Omega |\Delta(\frac{1}{2} fm\beta L - \delta f)|^2 dx \\
= \quad & \int_\Omega \bigg(\frac{\beta}{2} f\Delta m + \frac{\beta}{2} m\Delta f + \beta \nabla f \nabla m \\
& - \frac{\beta}{2K}(\nabla f + \nabla m + \nabla s + \nabla r)(f\nabla m + m\nabla f) \\
& - \frac{\beta}{2K}(f+m+s+r)(f\Delta m + m\Delta f + \nabla f\nabla m) \\
& - \frac{\beta}{2K} fm(\Delta f + \Delta m + \Delta s + \Delta r) - \frac{\beta}{2K} f\nabla m(\nabla f + \nabla m + \nabla s + \nabla r) \\
& - \frac{\beta}{2K} m\nabla f(\nabla f + \nabla m + \nabla r + \nabla s) - \delta\Delta f \bigg)^2 dx \\
\leq \quad & C \int_\Omega \bigg(f^2|\Delta m|^2 + m^2|\Delta f|^2 + |\nabla f\nabla m|^2 + \frac{1}{K^2}|f\nabla f\nabla m|^2 + \frac{1}{K^2} f^2|\nabla m|^4 \\
& + \frac{1}{K^2}|f\nabla m\nabla s|^2 + \frac{1}{K^2}|f\nabla m\nabla r|^2 + \frac{1}{K^2} m^2|\nabla f|^4 + \frac{1}{K^2}|m\nabla f\nabla m|^2 \\
& + \frac{1}{K^2}|m\nabla f\nabla s|^2 + \frac{1}{K^2}|m\nabla f\nabla r|^2 + \frac{1}{K^2}|(f+m+s+r)f\Delta m|^2 \\
& + \frac{1}{K^2}|(f+m+s+r)m\Delta f|^2 + \frac{1}{K^2}|(f+m+s+r)\nabla f\nabla m|^2 \\
& + \frac{1}{K^2}|fm\Delta f|^2 + \frac{1}{K^2}|fm\Delta m|^2 + \frac{1}{K^2}|fm\Delta r|^2 + \frac{1}{K^2}|fm\Delta s|^2 \\
& + \frac{1}{K^2}|f\nabla f\nabla m|^2 + \frac{1}{K^2} f^2|\nabla m|^4 + \frac{1}{K^2}|f\nabla m\nabla r|^2 + \frac{1}{K^2}|f\nabla m\nabla s|^2 \\
& + \frac{1}{K^2} m^2|\nabla f|^4 + \frac{1}{K^2}|m\nabla f\nabla m|^2 + \frac{1}{K^2}|m\nabla f\nabla s|^2 \\
& + \frac{1}{K^2}|m\nabla f\nabla r|^2 + |\Delta f|^2 \bigg) dx \\
\leq \quad & C \int_\Omega \bigg(\sup_{x\in\Omega} f^2 \cdot |\Delta m|^2 + \sup_{x\in\Omega} m^2 \cdot |\Delta f|^2 + |\nabla f|^4 + |\nabla m|^4 \\
& + \frac{1}{K^2} \sup_{x\in\Omega} f^2 \cdot (|\nabla f|^4 + |\nabla m|^4) + \frac{1}{K^2} \sup_{x\in\Omega} f^2 \cdot |\nabla m|^4 \\
& + \frac{1}{K^2} \sup_{x\in\Omega} f^2 \cdot (|\nabla m|^4 + |\nabla s|^4) + \frac{1}{K^2} \sup_{x\in\Omega} f^2 \cdot (|\nabla m|^4 + |\nabla r|^4) \\
& + \frac{1}{K^2} \sup_{x\in\Omega} m^2 \cdot |\nabla f|^4 + \frac{1}{K^2} \sup_{x\in\Omega} m^2 \cdot (|\nabla f|^4 + |\nabla m|^4) \\
& + \frac{1}{K^2} \sup_{x\in\Omega} m^2 \cdot (|\nabla f|^4 + |\nabla s|^4) + \frac{1}{K^2} \sup_{x\in\Omega} |(f+m+s+r)f|^2 \cdot |\Delta m|^2 \\
& + \frac{1}{K^2} \sup_{x\in\Omega} m^2 \cdot (|\nabla f|^4 + |\nabla r|^4) + \frac{1}{K^2} \sup_{x\in\Omega} |(f+m+s+r)m|^2 \cdot |\Delta f|^2
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{K^2}\sup_{x\in\Omega}(f+m+s+r)^2(|\nabla f|^4+|\nabla m|^4)+\frac{1}{K^2}\sup_{x\in\Omega}f^2m^2\cdot|\Delta f|^2 \\
& +\frac{1}{K^2}\sup_{x\in\Omega}f^2m^2\cdot|\Delta m|^2+\frac{1}{K^2}\sup_{x\in\Omega}f^2m^2\cdot|\Delta r|^2+\frac{1}{K^2}\sup_{x\in\Omega}f^2m^2\cdot|\Delta s|^2 \\
& +\frac{1}{K^2}\sup_{x\in\Omega}f^2\cdot(|\nabla f|^4+|\nabla m|^4)+\frac{1}{K^2}\sup_{x\in\Omega}f^2\cdot|\nabla m|^4 \\
& +\frac{1}{K^2}\sup_{x\in\Omega}f^2\cdot(|\nabla m|^4+|\nabla r|^4)+\frac{1}{K^2}\sup_{x\in\Omega}f^2\cdot(|\nabla m|^4+|\nabla s|^4) \\
& +\frac{1}{K^2}\sup_{x\in\Omega}m^2\cdot|\nabla f|^4+\frac{1}{K^2}\sup_{x\in\Omega}m^2\cdot(|\nabla f|^4+|\nabla m|^4) \\
& +\frac{1}{K^2}\sup_{x\in\Omega}m^2\cdot(|\nabla f|^4+|\nabla s|^4)+\frac{1}{K^2}\sup_{x\in\Omega}m^2\cdot(|\nabla f|^4+|\nabla r|^4)+|\Delta f|^2\Bigg)\,dx \\
\leq\ & C\int_\Omega(|\Delta f|^2+|\Delta m|^2+|\Delta s|^2+|\Delta r|^2+|\nabla f|^4 \\
& +|\nabla m|^4+|\nabla s|^4+|\nabla r|^4)dx \\
\leq\ & C(\|f\|^2_{\mathrm{H}_\alpha}+\|m\|^2_{\mathrm{H}_\alpha}+\|s\|^2_{\mathrm{H}_\alpha}+\|r\|^2_{\mathrm{H}_\alpha}+\|f\|^4_{\mathrm{H}_\alpha} \\
& +\|m\|^4_{\mathrm{H}_\alpha}+\|s\|^4_{\mathrm{H}_\alpha}+\|r\|^4_{\mathrm{H}_\alpha})\leq C. \qquad (3.28)
\end{aligned}
$$

Based on Proposition 2.1 and (3.27), simple calculations show that

$$
\begin{aligned}
\|g_2(f,m,s,r)\|^2_{\mathrm{H}_1} \leq\ & C(\|f\|^2_{\mathrm{H}_\alpha}+\|m\|^2_{\mathrm{H}_\alpha}+\|s\|^2_{\mathrm{H}_\alpha}+\|r\|^2_{\mathrm{H}_\alpha}+\|f\|^4_{\mathrm{H}_\alpha} \\
& +\|m\|^4_{\mathrm{H}_\alpha}+\|s\|^4_{\mathrm{H}_\alpha}+\|r\|^4_{\mathrm{H}_\alpha})\leq C, \qquad (3.29)
\end{aligned}
$$

$$
\begin{aligned}
\|g_3(f,m,s,r)\|^2_{\mathrm{H}_1} \leq\ & C(\|f\|^2_{\mathrm{H}_\alpha}+\|m\|^2_{\mathrm{H}_\alpha}+\|s\|^2_{\mathrm{H}_\alpha}+\|r\|^2_{\mathrm{H}_\alpha}+\|f\|^4_{\mathrm{H}_\alpha} \\
& +\|m\|^4_{\mathrm{H}_\alpha}+\|s\|^4_{\mathrm{H}_\alpha}+\|r\|^4_{\mathrm{H}_\alpha})\leq C, \qquad (3.30)
\end{aligned}
$$

$$
\begin{aligned}
\|g_4(f,m,s,r)\|^2_{\mathrm{H}_1} \leq\ & C(\|f\|^2_{\mathrm{H}_\alpha}+\|m\|^2_{\mathrm{H}_\alpha}+\|s\|^2_{\mathrm{H}_\alpha}+\|r\|^2_{\mathrm{H}_\alpha}+\|f\|^4_{\mathrm{H}_\alpha} \\
& +\|m\|^4_{\mathrm{H}_\alpha}+\|s\|^4_{\mathrm{H}_\alpha}+\|r\|^4_{\mathrm{H}_\alpha})\leq C, \qquad (3.31)
\end{aligned}
$$

By (3.1), (3.16) and (3.28), using the properties of Lemma 2.2, we obtain

$$
\begin{aligned}
\|f(t,f_0)\|_{\mathrm{H}_\alpha} =\ & \|e^{t\mathbf{L}}f_0+\int_0^t e^{(t-\tau)\mathbf{L}}g_1(f,m,s,r)d\tau\|_{\mathrm{H}_\alpha} \\
\leq\ & \|e^{t\mathbf{L}}f_0\|_{\mathrm{H}_\alpha}+\|\int_0^t e^{(t-\tau)\mathbf{L}}g_1(f,m,s,r)d\tau\|_{\mathrm{H}_\alpha} \\
\leq\ & C\|f_0\|_{\mathrm{H}_\alpha}+\int_0^t\|(-\mathbf{L})^{\alpha-1}e^{(t-\tau)\mathbf{L}}\|\cdot\|g_1(f,m,s,r)\|_{\mathrm{H}_1}d\tau \\
\leq\ & C\|f_0\|_{\mathrm{H}_\alpha}+C\int_0^t\tau^{-\beta}e^{-\delta\tau}d\tau \\
\leq\ & C,\quad \forall t\geq 0,\ (f_0,m_0,s_0,r_0)\in E, \qquad (3.32)
\end{aligned}
$$

where $\beta = \alpha - 1$ $(0 < \beta < 1)$. By (3.2), (3.3), (3.4), (3.16), (3.29), (3.30) and (3.31), simple calculations shows that

$$\|m(t, m_0)\|_{\mathrm{H}_\alpha} \leq C, \quad \forall t \geq 0, \ (f_0, m_0, s_0, r_0) \in E, \tag{3.33}$$

$$\|s(t, s_0)\|_{\mathrm{H}_\alpha} \leq C, \quad \forall t \geq 0, \ (f_0, m_0, s_0, r_0) \in E, \tag{3.34}$$

$$\|r(t, r_0)\|_{\mathrm{H}_\alpha} \leq C, \quad \forall t \geq 0, \ (f_0, m_0, s_0, r_0) \in E, \tag{3.35}$$

where $1 < \alpha < 2$. By (3.32), (3.33), (3.34) and (3.35), we obtain (3.26) immediately.

In the same fashion as in the proof of (3.26), by iteration we can prove that for any bounded set $E \subset \mathcal{H}_\alpha$, there exists a positive constant C such that

$$\|f(t, f_0)\|^2_{\mathrm{H}_\alpha} + \|m(t, m_0)\|^2_{\mathrm{H}_\alpha} + \|s(t, s_0)\|^2_{\mathrm{H}_\alpha} + \|r(t, r_0)\|^2_{\mathrm{H}_\alpha} \leq C, \quad \forall t \geq 0, \ \alpha \geq 0.$$

Therefore

$$\|u(t, u_0)\|_{\mathcal{H}_\alpha} \leq C, \quad \forall t \geq 0, \ \alpha \geq 0.$$

That is, for all $\alpha \geq 0$, the solution $u = (f, m, s, r)$ of (1.1)-(1.4) is uniformly bounded in $\mathcal{H}_\alpha$.

Hence, Lemma 3.1 is proved. Now, we give Lemma 3.2.

Lemma 3.2 *If (f, m, s, r) is a solution to the Trojan Y Chromosome model, (1.1)-(1.4), then, for any $\alpha \geq 0$, $u_0 \in \mathcal{H}_\alpha$, the problem (1.1)-(1.4) has a bounded absorbing set in $\mathcal{H}_\alpha$.*

Proof. It suffices to prove that for any bounded set $E \subset \mathcal{H}_\alpha$ $(\alpha \geq 0)$ with initial value $(f_0, m_0, s_0, r_0) \in E$, there exists $T > 0$ and a constant $C > 0$ independent of (f_0, m_0, s_0, r_0), such that

$$\|u(t, u_0)\|_{\mathcal{H}_\alpha} \leq C, \quad \forall t \geq T. \tag{3.36}$$

Obviously, if we have

$$\|f(t, f_0)\|^2_{\mathrm{H}_\alpha} + \|m(t, m_0)\|^2_{\mathrm{H}_\alpha} + \|s(t, s_0)\|^2_{\mathrm{H}_\alpha} + \|r(t, r_0)\|^2_{\mathrm{H}_\alpha} \leq C, \quad \forall t \geq T,$$

then (3.36) is obtained immediately.

For $\alpha = 0$, this follows from Proposition 2.2. So we shall prove (3.36) for any $\alpha > 0$. We prove the lemma in the following steps:

Step 1. we prove that for any $0 < \alpha < 1$, the problem (1.1)-(1.4) has a bounded absorbing set in $\mathcal{H}_\alpha$.

It then follows from (3.1)-(3.4) that

$$f(t, f_0) = e^{(t-T)\mathbf{L}} f(T, f_0) + \int_T^t e^{(t-\tau)\mathbf{L}} g_1(f, m, s, r) d\tau, \tag{3.37}$$

$$m(t, m_0) = e^{(t-T)\mathbf{L}} m(T, m_0) + \int_T^t e^{(t-\tau)\mathbf{L}} g_2(f, m, s, r) d\tau, \tag{3.38}$$

$$s(t,s_0) \quad = \quad e^{(t-T)\mathbf{L}}s(T,s_0)+\int_T^t e^{(t-\tau)\mathbf{L}}g_3(f,m,s,r)d\tau, \tag{3.39}$$

$$r(t,r_0) \quad = \quad e^{(t-T)\mathbf{L}}r(T,r_0)+\int_T^t e^{(t-\tau)\mathbf{L}}g_4(f,m,s,r)d\tau. \tag{3.40}$$

Assume B is the bounded absorbing set of the problem (1.1)-(1.4) and satisfy $B \subset \mathcal{H}$, we also assume $T_0 > 0$ the time such that $\forall t > T_0$, $(f_0, m_0, s_0, r_0) \in E \subset \mathcal{H}_\alpha$,

$$(f(t,f_0), m(t,m_0), s(t,s_0), r(t,r_0)) \in B, \quad \alpha > 0. \tag{3.41}$$

It is easy to check that

$$\|e^{t\mathbf{L}}\| \leq Ce^{-d\lambda_1 t},$$

here, $\lambda_1 > 0$ is the first eigenvalue of the equation

$$\begin{cases} -\Delta\Sigma = \lambda\Sigma, \\ \Sigma|_{\partial\Omega} = 0, \end{cases} \tag{3.42}$$

where $\Sigma = f, m, s, r$.

Then, for any given $T > 0$ and $(f_0, m_0, s_0, r_0) \in (E_1, E_2, E_3, E_4) \subset \mathcal{H}_\alpha$ $(\alpha > 0)$, we deduce that

$$\lim_{t\to\infty}\|e^{(t-T)\mathbf{L}}f(T,f_0)\|_{\mathrm{H}_\alpha} = 0. \quad \lim_{t\to\infty}\|e^{(t-T)\mathbf{L}}m(T,m_0)\|_{\mathrm{H}_\alpha} = 0.$$
$$\lim_{t\to\infty}\|e^{(t-T)\mathbf{L}}s(T,s_0)\|_{\mathrm{H}_\alpha} = 0. \quad \lim_{t\to\infty}\|e^{(t-T)\mathbf{L}}r(T,r_0)\|_{\mathrm{H}_\alpha} = 0.$$

Then, by (3.37) and (3.41), we obtain

$$\begin{aligned} & \|f(t,f_0)\|_{\mathrm{H}_\alpha} \\ \leq \quad & \|e^{(t-T_0)\mathbf{L}}f(T_0,f_0)\|_{\mathrm{H}_\alpha} + \int_{T_0}^t \|(-\mathbf{L})^\alpha e^{(t-\tau)\mathbf{L}}\| \cdot \|g_1(f,m,s,r)\|_{\mathrm{H}} d\tau \\ \leq \quad & \|e^{(t-T_0)\mathbf{L}}f(T_0,f_0)\|_{\mathrm{H}_\alpha} + C\int_{T_0}^t \|(-\mathbf{L})^\alpha e^{(t-\tau)\mathbf{L}}\| d\tau \\ \leq \quad & \|e^{(t-T_0)\mathbf{L}}f(T_0,f_0)\|_{\mathrm{H}_\alpha} + C\int_0^{T-T_0} \tau^{-\alpha}e^{-\delta\tau} d\tau \\ \leq \quad & \|e^{(t-T_0)\mathbf{L}}f(T_0,f_0)\|_{\mathrm{H}_\alpha} + C_1, \end{aligned} \tag{3.43}$$

where C_1 is a positive constant independent of u_0. Using the same method, we can also obtain

$$\|m(t,m_0)\|_{\mathrm{H}_\alpha} \leq \|e^{(t-T_0)\mathbf{L}}m(T_0,m_0)\|_{\mathrm{H}_\alpha} + C_2, \tag{3.44}$$

$$\|s(t,s_0)\|_{\mathrm{H}_\alpha} \leq \|e^{(t-T_0)\mathbf{L}}s(T_0,s_0)\|_{\mathrm{H}_\alpha} + C_3, \tag{3.45}$$

$$\|r(t,r_0)\|_{\mathrm{H}_\alpha} \leq \|e^{(t-T_0)\mathbf{L}} r(T_0,r_0)\|_{\mathrm{H}_\alpha} + C_4, \tag{3.46}$$

where $C_2,\ C_3,\ C_4$ are positive constants independent of u_0.

Then, by (3.43)-(3.46), we obtain (3.36) holds for all $0 < \alpha < 1$.

Step 2. We prove that for any $\frac{1}{2} < \alpha < \frac{3}{2}$, the problem (1.1)-(1.4) has a bounded absorbing set in $\mathcal{H}_\alpha$.

By (3.37) and (3.18), we obtain

$$\begin{aligned}
& \|f(t,f_0)\|_{\mathrm{H}_\alpha} \\
\leq\ & \|e^{(t-T_0)\mathbf{L}} f(T_0,f_0)\|_{\mathrm{H}_\alpha} + \int_{T_0}^{t} \|(-\mathbf{L})^{\alpha-\frac{1}{2}} e^{(t-\tau)\mathbf{L}}\| \cdot \|g_1(f,m,s,r)\|_{\mathrm{H}_{\frac{1}{2}}} d\tau \\
\leq\ & \|e^{(t-T_0)\mathbf{L}} f(T_0,f_0)\|_{\mathrm{H}_\alpha} + C\int_{T_0}^{t} \|(-\mathbf{L})^{\alpha-\frac{1}{2}} e^{(t-\tau)\mathbf{L}}\| d\tau \\
\leq\ & \|e^{(t-T_0)\mathbf{L}} f(T_0,f_0)\|_{\mathrm{H}_\alpha} + C\int_{T_0}^{t} \tau^{-(\alpha-\frac{1}{2})} e^{-\delta\tau} d\tau \\
\leq\ & \|e^{(t-T_0)\mathbf{L}} f(T_0,f_0)\|_{\mathrm{H}_\alpha} + C_5,
\end{aligned} \tag{3.47}$$

where C_5 is a positive constant independent of u_0. Using the same method, we can also obtain

$$\|m(t,m_0)\|_{\mathrm{H}_\alpha} \leq \|e^{(t-T_0)\mathbf{L}} m(T_0,m_0)\|_{\mathrm{H}_\alpha} + C_6, \tag{3.48}$$

$$\|s(t,s_0)\|_{\mathrm{H}_\alpha} \leq \|e^{(t-T_0)\mathbf{L}} s(T_0,s_0)\|_{\mathrm{H}_\alpha} + C_7, \tag{3.49}$$

$$\|r(t,r_0)\|_{\mathrm{H}_\alpha} \leq \|e^{(t-T_0)\mathbf{L}} r(T_0,r_0)\|_{\mathrm{H}_\alpha} + C_8, \tag{3.50}$$

where $C_6,\ C_7,\ C_8$ are positive constants independent of u_0.

Then, by (3.47)-(3.50), we obtain (3.36) holds for all $\frac{1}{2} < \alpha < \frac{3}{2}$.

By iteration, we can prove that for any $\alpha > 0$, (3.36) holds. Therefore, the problem (1.1) has a bounded absorbing set in $\mathcal{H}_\alpha$.

Then, Lemma 3.2 is proved.

Now, we give the proof the the main result.

Proof of Theorem 2.1. By Lemma 2.1, Lemma 3.1, Lemma 3.2, we immediately conclude that the proof of Theorem 2.1 is completed.

4 Conclusions

In this paper, we have shown the existence of global attractor for the Trojan Y Chromosome (TYC) model. It is well known that a necessary condition for the existence of a global attractor is the presence of a bounded absorbing sets in the phase space, whose existence implies that indeed the population of invasive species under consideration will be confined to bounded regions after a long time. The results on the existence of global attractor have an analytical complexity slightly above what biologists normally encounter, then potentially making the analysis more difficult to interpret for a non-mathematician. Since we provide a biological interpretation of these results, we believe that our approach is more satisfying than multiple numerical simulations because with computed solutions there is always the question of whether all interesting states of the system have been detected.

In [10], Parshad and Gutierrez considered the existence and finite dimensionality of global attractor for TYC model in $H^2(\Omega)^4$ space. Here, we introduce a generalized space $\mathcal{H}_\alpha$ ($\alpha \geq 0$), which is a fractional dimension space. Using the iteration technique for regularity estimates and Sobolev's embedding theorem, we extend the result on the existence of global attractor of TYC model to the generalized space $\mathcal{H}_\alpha$. Clearly, this results is the extend of [10], which is more generalized than [10]. We believe that this result will provide a big step forward in posing an effective strategy for eradication/containment of invasive aquatic species. We also believe that this result will help biologists to carry out the strategy in a realistic scenario, thus protecting the environment, aiding ailing fishing industries, and reducing other industry expenditures.

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