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Existence of positive solutions of elliptic equations with Hardy term

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Abstract. This paper is devoted to studying the existence of positive solutions of the problem:

$$\begin{cases} -\Delta u = \frac{u^p}{|x|^a} + h(x,u,\nabla u), & \text{in}\Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{$*$}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is an open bounded smooth domain with boundary $\partial\Omega$, and $1 < p < \frac{N-a}{N-2}$, $0 < a < 2$. Under suitable conditions of $h(x,u,\nabla u)$, we get *a priori* estimates for the positive solutions of problem ($*$). By making use of these estimates and topological degree theory, we further obtain some existence results for the positive solutions of problem ($*$) when $1 < p < \frac{N-a}{N-2}$.

Keywords: *a priori* estimates, Hardy term, positive solutions.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be an open bounded smooth domain with boundary $\partial\Omega$. We consider the following elliptic problem with Hardy term:

$$\begin{cases} -\Delta u = \frac{u^p}{|x|^a} + h(x,u,\nabla u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $0 < a < 2, 1 < p < \frac{N-a}{N-2}$. We mainly focus on the existence of solutions for problem (1.1). It is worth pointing out that problem (1.1) occurs in various branches of mathematical physics and biological models. Theoretically, when $a = 0$, there is no Hardy term in problem (1.1). As is known to all, that the processing without a Hardy term is much simpler than the processing with a Hardy term. When $h(x,u,\nabla u) = h(x,u)$, which means, no gradient terms appear in problem (1.1), in this case, problem (1.1) is reduced to the following problem:

$$\begin{cases} -\Delta u = u^p + h(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

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problem of this type was raised as an important issue in the survey paper [14]. For the existence of solutions to problem of this type, it was studied by many authors with different methods and techniques: upper and lower solution method, mountain pass theorem, *a priori* estimates, fixed points theorem and so on. We recall the papers [2,7,13,15,24] and the references therein. In [7], Figueiredo, Gossez and Ubilla concerned with the existence of solutions based on weak upper and lower solution method. Besides, Ambrosetti and Rabinowitz proposed the mountain pass theorem in [2], and proved the existence of nontrivial solutions. In [13], the author concerned the existence and regularity of solutions based on *a priori* estimates and blow up method by imposing suitable conditions on the coefficients and $h(x,u)$. It is worth mentioning that when $h(x,u)=0$, problem (1.2) becomes:

$$\begin{cases} -\Delta u = u^p, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

There are a lot of work related to this subject. For the existence of solutions, we refer to the pioneering work of [5,9,10,22]. It is well known that the Sobolev exponent $2^* = \frac{2N}{N-2}$ serves as the dividing number for existence and non-existence of solutions to (1.3); please see [5] and [9]. It is pointed out that the proof in [5] is based on Pohozaev identity and moving planes method. While in [10], the proof is based on a scaling argument reminiscent to that used in the theory of Minimal Surfaces to get *a priori* bounds.

As for the problem (1.1) containing gradients term, variational methods can not directly be applied for the problem generally. Thus, some other methods are proposed, we refer to [1,8,19,25] and the references therein. Specifically, for the following problem:

$$\begin{cases} -\Delta u = h(x,u,\nabla u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

The authors in [8] obtained the existence of positive solution through an iterative method based on mountain-pass techniques. In [1], the existence of solutions are obtained for this problem with convection term by using the Galerkin methods. It should be noted in particular that, the method Gidas and Spruck proposed in [10] is also applicable to the case with gradients term.

Recently, great attention has been focused on the study of the existence and non-existence solutions of the Hardy–Hénon equation:

$$-\Delta u = |x|^a u^p, \quad \text{in } \Omega. \tag{1.5}$$

Traditionally, the equation (1.5) is called Hardy (Hénon, or Lane–Emden) equation for $a < 0 (a > 0,\ a = 0)$. It is shown in [11] that for $a < -2, 1 < p < \frac{N+a}{N-2}$, equation (1.5) has no positive solutions in $\mathbb{R}^N$. Besides, in [21], Reichel and Zou proved that equation (1.5) do not admit any classical solutions in $\mathbb{R}^N$ if $1 < p < \frac{N+2+2a}{N-2}$ and $a > -2$. The non-existence results of Reichel and Zou was revisited by Phan and Souplet in [18], and a new proof of non-existence of bounded solutions in the case $N = 3$ is provided by using the technique introduced in [23]. For the Dirichlet boundary value problem of (1.5), Ni obtained the existence of multiplicity bounded positive solutions by using the upper and lower solution and approximation methods in [16]. Particularly, in [27], Zhu studied the following Hénon equation with perturbation terms in the unit ball B of $\mathbb{R}^N (N > 4)$:

$$\begin{cases} -\Delta u = |x|^\alpha |u|^{p-2} u + h(x), & \text{in } B, \\ u = 0, & \text{on } \partial B. \end{cases} \tag{1.6}$$

By applying the perturbation method in the unit sphere, the author obtained an infinite number of mutually different solutions to problem (1.6). The difficulty with this problem lies in the existence of the Hardy term, so we need to overcome the difficulties brought about by the Hardy term. It is worth pointing out that, in [27], the technique for handling $|x|^a$ is to impose special symmetry restrictions on u; the authors in [18] deal with the Hardy term $|x|^a$ by a change of variables and a doubling-rescaling argument. These methods provide us with good ideas for dealing with Hardy term.

In this paper, we focus on the existence of positive solutions for the problem (1.1) with Hardy term. Through the well-known Liouville-type theorem (see [10,18]), a change of variable and doubling-rescaling argument (see [20]). We firstly get the decay estimates of the solutions, then we derive *a priori* bounds for positive solutions of problem (1.1). Motivated by the works above, we can show the existence of solutions combined with the topological degree theory under some assumptions.

Firstly, we propose the definition of weak solution.

Definition 1.1. We say that $u \in H_0^1(\Omega)$ is a weak solution of problem (1.1) if

$$\int_\Omega \nabla u \cdot \nabla \varphi dx = \int_\Omega \frac{u^p}{|x|^a}\varphi dx + \int_\Omega h(x,u,\nabla u)\varphi dx, \qquad \forall \varphi \in C_0^\infty(\Omega).$$

Throughout this paper, we always denote by $\|.\|_q$ the norm of $L^q(\Omega)$ for any $q \geq 1$, which means $\|u\|_q = \|u\|_{L^q(\Omega)} = (\int_\Omega |u|^q dx)^{\frac{1}{q}}$, $1 \leq q < \infty$, and $\|u\|_\infty = \|u\|_{L^\infty(\Omega)} = \sup_\Omega |u|$, $q = \infty$.

Next, we introduce the assumptions required for this paper, to this end, we first introduce the following eigenvalue problem:

$$\begin{cases} -\Delta\varphi = \lambda\varphi, & \text{in } \Omega, \\ \varphi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

we denote by $\lambda_1(\Omega)$ the first eigenvalue of problem (1.1). Then we give the following hypotheses on $h(x,u,\nabla u)$:

(H_1) For $m > 0$, $h(x,m,\zeta)$ is Hölder continuous and $h(x,m,\zeta) \geq 0$.

(H_2) If $1 < p < \frac{N-a}{N-2}$, we assume that there exists a positive constant λ_0 such that $\lim_{m\to\infty} \frac{h(x,m,\zeta)}{m^p} = 0$, $\lim_{m\to 0} \frac{h(x,m,\zeta)}{m} = \lambda_0$, and $|h(x,m,\zeta)| \leq C(1+m^p+\zeta^b)$ for $m > 0$, $1 < b < \frac{2p}{p+1} < p < \frac{N-a}{N-2}$, appropriate constant $C > 0$.

Remark 1.2. If $1 < p < \frac{N-a}{N-2}$, then $h(x,u,\nabla u) = \lambda_0 u + \frac{|u|^{b-1}u|\nabla u|^2}{1+|\nabla u|^2}$ satisfies (H_1) and (H_2).

Now, we are turning to state the main results.

Theorem 1.3. *Let $N \geq 3, 0 < a < 2$ and $1 < p < \frac{N+2}{N-2}$. There exists a constant $\bar{C} = \bar{C}(N,p,a) > 0$ such that the following hold:*

(1) Any nonnegative solution of problem (1.1) in $\Omega = \{x \in \mathbb{R}^N; 0 < |x| < \rho\} (\rho > 0)$ satisfies that:

$$|u(x)| \leq \bar{C}|x|^{-\frac{2-a}{p-1}} \quad and \quad |\nabla u(x)| \leq \bar{C}|x|^{-\frac{p+1-a}{p-1}}, \qquad 0 < |x| < \frac{\rho}{2}.$$

(2) Any nonnegative solution of problem (1.1) in $\Omega = \{x \in \mathbb{R}^N; |x| > \rho\}$ $(\rho > 0)$ satisfies that:

$$|u(x)| \leq \bar{C}|x|^{-\frac{2-a}{p-1}} \quad and \quad |\nabla u(x)| \leq \bar{C}|x|^{-\frac{p+1-a}{p-1}}, \qquad |x| > 2\rho.$$

The proof of Theorem 1.3 is based on a change of variable and a generalization of a doubling-rescaling arguments; see [18].

Theorem 1.4. *Assume that* $1 < p < \frac{N-a}{N-2}$, $0 < a < 2$, *and that* (H_1) *and* (H_2) *hold with* $\lambda_0 < \lambda_1(\Omega)$. *Then there exist two universal positive constants* $\widetilde{C}$ *and* $\widehat{C}$ *such that for any positive solution* $u \in C^2(\Omega\backslash\{0\}) \cap C(\overline{\Omega})$ *of problem* (1.1), *there holds* $\widetilde{C} \leq \|u\|_{C(\Omega)} \leq \widehat{C}$.

The proof of Theorem 1.4 is based on the well-known blow up technique introduced by Gidas and Spruck (see [10]) and adopted by Phan (see [18]).

Therefore, according to Theorem 1.4 and the Leray–Schauder degree theory, we can get the existence of positive solutions to problem (1.1).

Theorem 1.5. *Assume that* $1 < p < \frac{N-a}{N-2}$, $0 < a < 2$, $\frac{u^p}{|x|^a} + h(x, u, \nabla u) \in L^k$, *where* $k < \min\{\frac{N}{a}, \frac{N(p-1)}{(p+1-a)b}\}$, (H_1) *and* (H_2) *hold, then problem* (1.1) *has at least one solution.*

The rest of paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we concern the decay estimates of solutions. Section 4 is devoted to the proof of Theorem 1.4, in which we establish *a priori* estimates to problem (1.1) by blow up technique. In Section 5, we prove the existence of solutions for problem (1.1) by topology degree theory and give the proof of Theorem 1.5.

2 Preliminaries

In this section, we will give some lemmas which will be used to prove the main results.

Lemma 2.1. *Let* $u(x)$ *be a nonnegative* C^2 *solution of the following equation:*

$$-\Delta u = u^p, \qquad x \in \mathbb{R}^N,$$

where $N > 2, 1 < p < \frac{N+2}{N-2}$. *Then* $u(x) \equiv 0$.

Lemma 2.2. *Let* $\mathbb{R}^N_+$ *be the half space* $\{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_N > 0\}$. *Suppose that* $u(x)$ *is a nonnegative* $C^2(\mathbb{R}^N_+) \bigcap C^0(\{x \in \mathbb{R}^N : x_N \geq 0\})$ *solution of the following problem:*

$$\begin{cases} -\Delta u = u^p, & x \in \mathbb{R}^N_+, \\ u = 0, & x_N = 0, \end{cases}$$

where $1 < p < \frac{N+2}{N-2}$. *Then* $u(x) \equiv 0$.

Remark 2.3. Lemma 2.1 and Lemma 2.2 follow directly from [10, Theorem 1.2, Theorem 1.3].

Lemma 2.4 ([18])**.** *Let* $N \geq 2$, $a > -2$, $p > 1$. *If* $p < min\{p_s, p_s(a)\}$ *or* $p \leq \frac{N+a}{N-2}$, $p_s = \frac{N+2}{N-2}$, $p_s(a) = \frac{N+2+2a}{N-2}$. *Then the following equation:*

$$-\Delta u = |x|^a u^p$$

has no positive solution in $\mathbb{R}^N$.

Lemma 2.5 (Hardy's inequality [4])**.** *Assume* $N \geq 3$ *and* $r > 0$. *Suppose that* $u \in H^1(B(0, r))$. *Then* $\frac{u}{|x|} \in L^2(B(0, r))$, *with the estimate*

$$\int_{B(0,r)} \frac{u^2}{|x|^2} dx \leq \int_{B(0,r)} \left(|Du|^2 + \frac{u^2}{r^2}\right) dx.$$

Lemma 2.6 ([20]). *Let (X,d) be a complete metric space, $\emptyset \neq D \subset \Sigma \subset X$ with Σ closed. Furthermore, assume that $M : D \to (0,\infty)$ is bounded on compact subsets of D, and fix a real $K > 0$. If $y \in D$ is such that*

$$M(y)\,\mathrm{dist}(y,\Gamma) > 2k, \qquad \Gamma = \Sigma \backslash D,$$

then there exists $x \in D$ such that

$$M(x)\,\mathrm{dist}(x,\Gamma) > 2k, \qquad M(x) \geq M(y),$$

and

$$M(z) \leq 2M(x) \quad \text{for all} \quad z \in D \cap \overline{B\big(x,kM^{-1}(x)\big)}.$$

Lemma 2.7 (Leray–Schauder [3]). *Assume that X is a real Banach space, Ω is a bounded, open subset of X and $\Phi : [a,b] \times \overline{\Omega} \to X$ is given by $\Phi(\lambda,u) = u - T(\lambda,u)$ with T a compact map. Define*

$$T_\lambda(u) = T(\lambda,u), \quad u \in X,$$

$$\Phi_\lambda = I - T_\lambda, \quad \lambda \in [a,b],$$

$$\Sigma = \{(\lambda,u) \in [a,b] \times \overline{\Omega} : \Phi(\lambda,u) = 0\},$$

and note $\Sigma_\lambda = \{u \in \overline{\Omega} : (\lambda,u) \in \Sigma\}$. We also suppose that,

$$\Phi(\lambda,u) = u - T(\lambda,u) \neq 0, \qquad \forall (\lambda,u) \in [a,b] \times \partial\Omega.$$

If $\deg(\Phi_a,\Omega,0) \neq 0$, then we have,

(1) $\Phi(\lambda,u) = u - T(\lambda,u) = 0$ has a solution $u \in X$ in Ω for every $a \leq \lambda \leq b$.

(2) Furthermore, there exists a compact connected set $\mathcal{C} \subset \Sigma$ such that

$$\mathcal{C} \cap (\{a\} \times \Sigma_a) \neq \emptyset, \mathcal{C} \cap (\{b\} \times \Sigma_b) \neq \emptyset.$$

3 Decay estimates

In this section, we concern the decay estimates of solutions to the problem (1.1). We need the following lemma, which is an extensive of [20, Theorem 6.1] and [18, Lemma 2.1].

Lemma 3.1. *Let $N \geq 3, 1 < p < \frac{N+2}{N-2}, \alpha \in (0,1]$. Assume in addition that $c(x) \in C^\alpha(\overline{B_1})$ satisfies that,*

$$\|c(x)\|_{C^\alpha(\overline{B_1})} \leq C_1 \quad \text{and} \quad c(x) \geq C_2, \quad x \in \overline{B_1}, \tag{3.1}$$

for some $C_1, C_2 > 0$, where $B_1 = \{x \in \mathbb{R}^N; |x| < 1\}$. Then there exists a constant C, depending only on α, C_1, C_2, p, N such that, for any nonnegative classical solution u of

$$-\Delta u = \frac{u^p}{c(x)} + h(x,u,\nabla u), \quad x \in \overline{B_1}, \tag{3.2}$$

u satisfies that,

$$|u(x)|^{\frac{p-1}{2}} + |\nabla u(x)|^{\frac{p-1}{p+1}} \leq C\big(1 + \mathrm{dist}^{-1}(x,\partial B_1)\big), \qquad x \in B_1.$$

Proof. Arguing by contradiction. Denote $N_k = |u_k|^{\frac{p-1}{2}} + |\nabla u_k|^{\frac{p-1}{p+1}}$. We suppose that there exist a sequence of c_k, u_k, y_k verifying that (3.1), (3.2), and $N_k(y_k) > 2k\big(1 + \operatorname{dist}^{-1}(y_k, \partial B_1)\big) > 2k \operatorname{dist}^{-1}(y_k, \partial B_1)$. By Lemma 2.6, there exists x_k such that

$$N_k(x_k) \geq N_k(y_k), \qquad N_k(x_k) > 2k \operatorname{dist}^{-1}(x_k, \partial B_1).$$

and

$$N_k(z) \leq 2N_k(x_k), \quad \text{for all } z \text{ satisfying } |z - x_k| \leq kN_k^{-1}(x_k).$$

Consequently, we have that,

$$\lambda_k = N_k^{-1}(x_k) \to 0, \qquad k \to +\infty, \tag{3.3}$$

due to $N_k(x_k) \geq N_k(y_k) > 2k$. Next we let

$$v_k(y) = \lambda_k^{\frac{2}{p-1}} u_k(x_k + \lambda_k y), \qquad \widetilde{c_k}(y) = c_k(x_k + \lambda_k y),$$

noting that $|v_k(0)|^{\frac{p-1}{2}} + |\nabla v_k(0)|^{\frac{p-1}{p+1}} = 1$,

$$|v_k(y)|^{\frac{p-1}{2}} + |\nabla v_k(y)|^{\frac{p-1}{p+1}} \leq 2, \qquad |y| \leq k, \tag{3.4}$$

and

$$\begin{aligned}
-\Delta v_k &= -\lambda_k^{\frac{2p}{p-1}} \Delta u_k(x_k + \lambda_k y) \\
&= \lambda_k^{\frac{2p}{p-1}} \left(\frac{u_k^p(x_k + \lambda_k y)}{c_k(x_k + \lambda_k y)} + h\big(x_k + \lambda_k y, u_k(x_k + \lambda_k y), \nabla u_k(x_k + \lambda_k y)\big) \right) \\
&= \frac{v_k^p}{c_k(x_k + \lambda_k y)} + \lambda_k^{\frac{2p}{p-1}} h(x_k + \lambda_k y, \lambda_k^{-\frac{2}{p-1}} v_k, \lambda_k^{-\frac{p+1}{p-1}} \nabla v_k) \\
&= \frac{v_k^p}{\widetilde{c_k}(y)} + \lambda_k^{\frac{2p}{p-1}} h(x_k + \lambda_k y, \lambda_k^{-\frac{2}{p-1}} v_k, \lambda_k^{-\frac{p+1}{p-1}} \nabla v_k).
\end{aligned} \tag{3.5}$$

So we see that v_k satisfies the following equation:

$$-\Delta v_k = \frac{v_k^p}{\widetilde{c_k}(y)} + \lambda_k^{\frac{2p}{p-1}} h\Big(x_k + \lambda_k y, \lambda_k^{-\frac{2}{p-1}} v_k, \lambda_k^{-\frac{p+1}{p-1}} \nabla v_k\Big) \tag{3.6}$$

where $|y| \leq k$. According to the condition (H_2) on $h(x, u, \nabla u)$, it implies that,

$$\lambda_k^{\frac{2p}{p-1}} h(x_k + \lambda_k y, \lambda_k^{-\frac{2}{p-1}} v_k, \lambda_k^{-\frac{p+1}{p-1}} \nabla v_k) \leq C, \quad |y| \leq k,$$

for k large enough, we deduce that there exist a subsequence of v_k converges in $C^1_{loc}(\mathbb{R}^N)$ to a function $v(y) > 0$. Fix $y \in \mathbb{R}^N$ and denote $\mu_k = \lambda_k^{-\frac{2}{p-1}} v_k(y)$, $\xi_k = v_k^{-\frac{p+1}{2}} \nabla v_k(y)$, we may write that,

$$\lambda_k^{\frac{2p}{p-1}} h(x_k + \lambda_k y, \lambda_k^{-\frac{2}{p-1}} v_k, \lambda_k^{-\frac{p+1}{p-1}} \nabla v_k) = v_k^p \mu_k^{-p} h(x_k + \lambda_k y, \mu_k, \mu_k^{\frac{p+1}{2}} \xi_k).$$

Obviously, $\mu_k \to \infty$ as $k \to +\infty$ and ξ_k is bounded. Besides, if $\{x_k\}$ is bounded, condition (H_2) implies that,

$$v_k^p(y) \mu_k^{-p} h(x_k + \lambda_k y, \mu_k, \mu_k^{\frac{p+1}{2}} \xi_k) \to 0, \qquad k \to +\infty. \tag{3.7}$$

On the other hand, due to (3.1), we have $C_2 \le \widetilde{c_k} \le C_1$, and for each $R > 0$ and $k > k_0(R)$ large enough, the following holds:

$$|\widetilde{c_k}(y) - \widetilde{c_k}(z)| \le C_1|\lambda_k(y-z)|^\alpha \le C_1|y-z|^\alpha, \qquad |y|, |z| \le R. \tag{3.8}$$

Therefore, by the Arzelà–Ascoli theorem; see [26], there exists $\widetilde{c}$ in $C(\mathbb{R}^N)$, with $\widetilde{c} \ge C_2$ such that, after extracting a subsequence, $\widetilde{c_k} \to \widetilde{c}$ in $C_{loc}(\mathbb{R}^N)$. Now for each $R > 0$ and $1 < q < \infty$, by (3.4), (3.6) and interior elliptic L^q estimates, the sequence v_k is uniformly bounded in $W^{2,q}(B_R)$. Using standard embeddings and interior elliptic Schauder estimates, after extracting a subsequence, we may assume that $v_k \to v$ in $C^2_{loc}(\mathbb{R}^N)$. Moreover, (3.3) and (3.8) imply that $|\widetilde{c_k}(y) - \widetilde{c_k}(z)| \to 0$ as $k \to +\infty$, so that the function $\widetilde{c}$ is actually a constant $C > 0$. Therefore, we have that,

$$\frac{v_k^p}{\widetilde{c_k}(y)} \to Cv^p, \qquad k \to +\infty. \tag{3.9}$$

According to (3.7) and (3.9), it follows that $v > 0$ is a classical solution of

$$-\Delta v = Cv^p, \qquad y \in \mathbb{R}^N$$

and satisfying $|v(0)|^{\frac{p-1}{2}} + |\nabla v(0)|^{\frac{p-1}{p+1}} = 1$, this contradicts the Liouville-type theorem. □

By Lemma 3.1, we are ready to prove the decay estimates of solutions to problem (1.1) as follows.

Proof of Theorem 1.3. Assume either $\Omega = \{x \in \mathbb{R}^N; 0 < |x| < \rho\}$ and $0 < |x_0| < \frac{\rho}{2}$, or $\Omega = \{x \in \mathbb{R}^N; |x| > \rho\}$ and $|x_0| > 2\rho$. We denote $R_0 = \frac{1}{2}|x_0|$, and observe that, for all $y \in B_1$, $\frac{|x_0|}{2} < |x_0 + R_0 y| < \frac{3|x_0|}{2}$, so that $x_0 + R_0 y \in \Omega$ in either case. Let us thus define that,

$$U(y) = R_0^{\frac{2-a}{p-1}} u(x_0 + R_0 y).$$

Therefore,

$$\begin{aligned}
-\Delta U(y) &= -R_0^{\frac{2p-a}{p-1}} \Delta u(x_0 + R_0 y) \\
&= R_0^{\frac{2p-a}{p-1}} \left(\frac{u^p(x_0 + R_0 y)}{|x_0 + R_0 y|^a} + h\big(x_0 + R_0 y, u(x_0 + R_0 y), \nabla u(x_0 + R_0 y)\big) \right) \\
&= \frac{U^p(y)}{|\frac{x_0}{R_0} + y|^a} + R_0^{\frac{2p-a}{p-1}} h\Big(x_0 + R_0 y, R_0^{-\frac{2-a}{p-1}} U(y), R_0^{-\frac{p+1-a}{p-1}} \nabla U(y)\Big) \\
&= \frac{U^p(y)}{c(y)} + R_0^{\frac{2p-a}{p-1}} h\Big(x_0 + R_0 y, R_0^{-\frac{2-a}{p-1}} U(y), R_0^{-\frac{p+1-a}{p-1}} \nabla U(y)\Big),
\end{aligned} \tag{3.10}$$

where $c(y) = |\frac{x_0}{R_0} + y|^a$. Then U is a solution of

$$-\Delta U(y) = \frac{U^p(y)}{c(y)} + R_0^{\frac{2p-a}{p-1}} h\Big(x_0 + R_0 y, R_0^{-\frac{2-a}{p-1}} U(y), R_0^{-\frac{p+1-a}{p-1}} \nabla U(y)\Big), \qquad y \in B_1.$$

Notice that $|y + \frac{x_0}{R_0}| \in [1, 3]$ for all $y \in \overline{B_1}$ and $\|c(y)\|_{C^1(\overline{B_1})} \le C(a)$ according to Lemma 3.1, where $C(a)$ is a constant depending on a. Applying Lemma 3.1 again, we have that $|U(0)| + |\nabla U(0)| \le C$. Hence,

$$|u(x_0)| \le \bar{C} R_0^{-\frac{2-a}{p-1}}, \qquad |\nabla u(x_0)| \le \bar{C} R_0^{-\frac{p+1-a}{p-1}},$$

which yields the desired conclusion. □

4 *A priori* estimates

We will show *a priori* bounds for the positive solutions to problem (1.1) in this section. Owing to the well-known Liouville-type results (Lemma 2.1, Lemma 2.2 and Lemma 2.4), we can get *a priori* estimates. Now, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. To get the lower bound, we argue by contradiction. Assume that $\|u\|_{C(\Omega)} < \widetilde{C}$ holds for any $\widetilde{C} > 0$. Therefore, there exists a sequence solution $\{u_k\}$ of problem (1.1) such that

$$M_k = \sup_{x\in\Omega} u_k(x) \to 0, \quad \text{as } k \to +\infty.$$

Multiplying the first equation of problem (1.1) by u_k, and integrating the result over Ω, by Hölder inequality, Young's inequality with ε and Hardy's inequality, then we have that

$$\begin{aligned}
\int_\Omega |\nabla u_k|^2 dx &= \int_\Omega \frac{u_k^{p+1}}{|x|^a} + u_k h(x, u_k, \nabla u_k) dx \\
&= \int_\Omega \frac{u_k^a}{|x|^a} \cdot u_k^{p+1-a} dx + \int_\Omega u_k h(x, u_k, \nabla u_k) dx \\
&\le \Big(\int_\Omega \frac{u_k^2}{|x|^2} dx \Big)^{\frac{a}{2}} \Big(\int_\Omega u_k^{\frac{2(p+1-a)}{2-a}} dx \Big)^{\frac{2-a}{2}} + C \int_\Omega u_k (1 + u_k^p + |\nabla u_k|^b) dx \\
&\le \varepsilon \int_\Omega \frac{u_k^2}{|x|^2} dx + C \Big(\int_\Omega u_k^{\frac{2(p+1-a)}{2-a}} dx + \int_\Omega u_k + u_k^{p+1} + u_k |\nabla u_k|^b \Big) dx \\
&\le C \Bigg[\varepsilon \int_\Omega (|\nabla u_k|^2 + u_k^2) dx + \int_\Omega u_k^{\frac{2(p+1-a)}{2-a}} dx + \int_\Omega u_k dx \\
&\quad + \int_\Omega u_k^{p+1} dx + \Big(\int_\Omega |\nabla u_k|^2 dx \Big)^{\frac{b}{2}} \Big(\int_\Omega u_k^{\frac{2}{2-b}} dx \Big)^{\frac{2-b}{2}} \Bigg] \\
&\le C \Bigg[\varepsilon \int_\Omega |\nabla u_k|^2 dx + \varepsilon \int_\Omega u_k^2 dx + \int_\Omega u_k^{\frac{2(p+1-a)}{2-a}} dx + \int_\Omega u_k dx \\
&\quad + \int_\Omega u_k^{p+1} dx + \varepsilon \int_\Omega |\nabla u_k|^2 dx + \int_\Omega u_k^{\frac{2}{2-b}} dx \Bigg].
\end{aligned} \tag{4.1}$$

Hence,

$$\begin{aligned}
&(1 - \varepsilon C) \int_\Omega |\nabla u_k|^2 dx \\
&\quad \le C \Big(\varepsilon \int_\Omega |u_k|^2 dx + \int_\Omega u_k^{\frac{2(p+1-a)}{2-a}} dx + \int_\Omega u_k dx + \int_\Omega u_k^{p+1} dx + \int_\Omega u_k^{\frac{2}{2-b}} dx \Big).
\end{aligned} \tag{4.2}$$

Let $\varepsilon \to 0$, then we have that

$$\begin{aligned}
\|\nabla u_k\|_2^2 &\le C \Big(\|u_k\|_{\frac{2(p+1-a)}{2-a}}^{\frac{2(p+1-a)}{2-a}} + \|u_k\|_1 + \|u_k\|_{p+1}^{p+1} + \|u_k\|_{\frac{2}{2-b}}^{\frac{2}{2-b}} \Big) \\
&\le C \|u_k\|_{C(\Omega)} \\
&= C M_k \to 0, \quad \text{as } k \to +\infty.
\end{aligned} \tag{4.3}$$

Further, let $u_k(x) = M_k U_k(x)$, obviously, $U_k(x)$ satisfies that,

$$\begin{cases} -\Delta U_k = \frac{M_k^{p-1} U_k^p}{|x|^a} + M_k^{-1} h(x, M_k U_k, M_k \nabla U_k), & \text{in } \Omega, \\ U_k > 0, & \text{in } \Omega, \\ U_k = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Owing to

$$\begin{aligned} \int_\Omega |\nabla U_k|^2 dx &= \int_\Omega \left(M_k^{p-1} \frac{U_k^{p+1}}{|x|^a} + U_k \frac{h(x, M_k U_k, M_k \nabla U_k)}{M_k} \right) dx \\ &= \int_\Omega M_k^{p-1} \frac{U_k^a}{|x|^a} U_k^{p+1-a} dx + \int_\Omega U_k \frac{h(x, M_k U_k, M_k \nabla U_k)}{M_k} dx \\ &\leq M_k^{p-1} \Big(\int_\Omega \frac{U_k^2}{|x|^2} dx \Big)^{\frac{a}{2}} \Big(\int_\Omega U_k^{\frac{2(p+1-a)}{2-a}} dx \Big)^{\frac{2-a}{2}} + \int_\Omega U_k^2 \frac{h(x, M_k U_k, M_k \nabla U_k)}{M_k U_k} dx \\ &\leq \Big(M_k^{\frac{p-1}{a}} \int_\Omega \frac{U_k^2}{|x|^2} dx \Big)^{\frac{a}{2}} \Big(M_k^{\frac{p-1}{2-a}} \int_\Omega U_k^{\frac{2(p+1-a)}{2-a}} dx \Big)^{\frac{2-a}{2}} + \int_\Omega U_k^2 \frac{h(x, M_k U_k, M_k \nabla U_k)}{M_k U_k} dx \\ &\leq \frac{a}{2} C M_k^{\frac{p-1}{a}} \int_\Omega (U_k^2 + |\nabla U_k|^2) dx + \frac{2-a}{2} M_k^{\frac{p-1}{2-a}} \int_\Omega U_k^{\frac{2(p+1-a)}{2-a}} dx \\ &\quad + \int_\Omega U_k^2 \frac{h(x, M_k U_k, M_k \nabla U_k)}{M_k U_k} dx. \end{aligned} \tag{4.5}$$

By (H_2) and the standard elliptic estimates; see [12], we can easily see that, the subsequence in U_k converges to a positive function v in $C^2(\Omega)$. Moreover, v satisfies

$$\begin{cases} -\Delta v = \lambda_0 v, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

On the other hand, problem (4.6) has no positive solution due to $\lambda_0 < \lambda_1(\Omega)$. This reaches a contradiction. Consequently, there exists a universal constant $\widetilde{C} > 0$ such that for any positive solution u of problem (1.1), we have that,

$$\|u\|_{C(\Omega)} \geq \widetilde{C}. \tag{4.7}$$

To get the upper bound, we also proceed by contradiction. Assume that $\|u\|_{C(\Omega)} > \widehat{C}$ holds. Therefore, there exists a sequence of solutions u_k and a sequence of points $P_k \in \Omega$ such that

$$M_k = \sup_{x \in \Omega} u_k(x) = u_k(P_k) \to \infty, \quad \text{as } k \to +\infty.$$

We may assume that $P_k \to P \in \overline{\Omega}$ as $k \to +\infty$, and we divide the proof into the following two cases:

Case 1. $P \in \Omega \backslash \{0\}$ or $P \in \partial\Omega$. In this case, we rescale the solution as the following:

$$U_k(y) = \lambda_k^{\frac{2}{p-1}} u_k(P_k + \lambda_k y), \qquad \lambda_k = M_k^{-\frac{p-1}{2}}.$$

Therefore, we deduce that,

$$\begin{aligned}-\Delta U_k(y) &= -\lambda_k^{\frac{2p}{p-1}}\Delta u_k(P_k+\lambda_k y)\\ &= \lambda_k^{\frac{2p}{p-1}}\left(\frac{u_k^p(P_k+\lambda_k y)}{|P_k+\lambda_k y|^a}+h\big(P_k+\lambda_k y, u_k(P_k+\lambda_k y), \nabla u_k(P_k+\lambda_k y)\big)\right)\\ &= \frac{U_k^p(y)}{|P_k+\lambda_k y|^a}+\lambda_k^{\frac{2p}{p-1}}h\big(P_k+\lambda_k y, \lambda_k^{-\frac{2}{p-1}}U_k(y), \lambda_k^{-\frac{p+1}{p-1}}\nabla U_k(y)\big).\end{aligned} \tag{4.8}$$

Then U_k satisfies that,

$$\begin{cases}-\Delta U_k = \frac{U_k^p}{|P_k+\lambda_k y|^a}+\lambda_k^{\frac{2p}{p-1}}h\Big(P_k+\lambda_k y, \lambda_k^{-\frac{2}{p-1}}U_k, \lambda_k^{-\frac{p+1}{p-1}}\nabla U_k\Big), & \text{in } \Omega_k,\\ U_k>0, & \text{in } \Omega_k,\\ U_k=0, & \text{on } \partial\Omega_k,\end{cases} \tag{4.9}$$

where $\Omega_k = \lambda_k^{-1}(\Omega-\{P_k\})$. Notice that $\lambda_k = M_k^{-\frac{p-1}{2}}$, we can deduce that,

$$\begin{aligned}&\left|\frac{U_k^p}{|P_k+\lambda_k y|^a}+\lambda_k^{\frac{2p}{p-1}}h(P_k+\lambda_k y, \lambda_k^{-\frac{2}{p-1}}U_k, \lambda_k^{-\frac{p+1}{p-1}}\nabla U_k)\right|\\ &\quad = \left|\frac{U_k^p}{|P_k+\lambda_k y|^a}+M_k^{-p}h(P_k+\lambda_k y, M_k U_k, M_k^{\frac{p+1}{2}}\nabla U_k)\right|\\ &\quad \le \left|\frac{U_k^p}{|P_k+\lambda_k y|^a}+C(M_k^{-p}+U_k^p+M_k^{\frac{(p+1)b}{2}-p}|\nabla U_k|^b)\right|\\ &\quad \le C,\end{aligned} \tag{4.10}$$

and so we find that U_k is a solution of the equation:

$$-\Delta U_k = \frac{U_k^p}{|P_k+\lambda_k y|^a}+\lambda_k^{\frac{2p}{p-1}}h\Big(P_k+\lambda_k y, \lambda_k^{-\frac{2}{p-1}}U_k, \lambda_k^{-\frac{p+1}{p-1}}\nabla U_k\Big)$$

in a rescaled domain Ω_k. Since $U_k(0) = 1, 0 < U_k \le 1$, by elliptic estimates and standard embedding similar as that in [10], up to a subsequence, without loss of generality, still denoted by U_k, we can deduce that $\{U_k\}$ is convergent in $C_{loc}(\mathbb{R}^N)$. Hence, by the Arzelà–Ascoli theorem and standard diagonal argument, up to a subsequence, there exists a subsequence of $\{U_k\}$ and function $v \in C(\Omega)$, such that $U_k \to v$ uniformly on compact sets of Ω. In addition, v satisfies the equation $-\Delta v = lv^p$, where $1 < p < \frac{N-a}{N-2}$, for some $l > 0$ either in the whole space $\mathbb{R}^N$, or in a half-space with 0 boundary conditions. Clearly, this contradicts with the Lemma 2.1 and Lemma 2.2.

Case 2. $P = 0$. In this case, we rescale the solution according to $U_k(y) = \lambda_k^{\frac{2-a}{p-1}}u_k(P_k+\lambda_k y)$, $\lambda_k = M_k^{-\frac{p-1}{2-a}}$. By a simple calculation, we infer that,

$$\begin{aligned}-\Delta U_k(y) &= -\lambda_k^{\frac{2p-a}{p-1}}\Delta u_k(P_k+\lambda_k y)\\ &= \lambda_k^{\frac{2p-a}{p-1}}\left(\frac{u_k^p(P_k+\lambda_k y)}{|P_k+\lambda_k y|^a}+h\big(P_k+\lambda_k y, u_k(P_k+\lambda_k y), \nabla u_k(P_k+\lambda_k y)\big)\right)\\ &= \frac{U_k^p(y)}{|\frac{P_k}{\lambda_k}+y|^a}+\lambda_k^{\frac{2p-a}{p-1}}h\big(P_k+\lambda_k y, \lambda_k^{-\frac{2-a}{p-1}}U_k(y), \lambda_k^{-\frac{p+1-a}{p-1}}\nabla U_k(y)\big),\end{aligned} \tag{4.11}$$

Then U_k satisfies that

$$\begin{cases} -\Delta U_k = \frac{U_k^p}{|\frac{P_k}{\lambda_k}+y|^a} + \lambda_k^{\frac{2p-a}{p-1}} h\Big(P_k + \lambda_k y, \lambda_k^{-\frac{2-a}{p-1}} U_k, \lambda_k^{-\frac{p+1-a}{p-1}} \nabla U_k\Big), & \text{in } \Omega_k, \\ U_k > 0, & \text{in } \Omega_k, \\ U_k = 0, & \text{on } \partial\Omega_k, \end{cases} \tag{4.12}$$

where $\Omega_k = \lambda_k^{-1}(\Omega - \{P_k\})$. Due to $\lambda_k = M_k^{-\frac{p-1}{2-a}}$, we can deduce that

$$\begin{aligned} &\left| \frac{U_k^p}{|\frac{P_k}{\lambda_k} + y|^a} + \lambda_k^{\frac{2p-a}{p-1}} h(P_k + \lambda_k y, \lambda_k^{-\frac{2-a}{p-1}} U_k, \lambda_k^{-\frac{p+1-a}{p-1}} \nabla U_k) \right| \\ &\quad = \left| \frac{U_k^p}{|\frac{P_k}{\lambda_k} + y|^a} + M_k^{-\frac{2p-a}{2-a}} h(P_k + \lambda_k y, M_k U_k, M_k^{\frac{p+1-a}{2-a}} \nabla U_k) \right| \\ &\quad \le \left| \frac{U_k^p}{|\frac{P_k}{\lambda_k} + y|^a} + C(M_k^{-\frac{2p-a}{2-a}} + M_k^{p-\frac{2p-a}{2-a}} U_k^p + M_k^{\frac{(p+1-a)b-(2p-a)}{2-a}} |\nabla U_k|^b) \right| \\ &\quad \le C, \end{aligned} \tag{4.13}$$

and thus we find that U_k is a solution of the following equation:

$$-\Delta U_k = \frac{U_k^p}{|\frac{P_k}{\lambda_k} + y|^a} + \lambda_k^{\frac{2p-a}{p-1}} h\left(P_k + \lambda_k y, \lambda_k^{-\frac{2-a}{p-1}} U_k, \lambda_k^{-\frac{p+1-a}{p-1}} \nabla U_k\right)$$

in a rescaled domain Ω_k containing $B(0, \rho\lambda_k^{-1})$ for some $\rho > 0$. Moreover, it follows from the estimate in Theorem 1.3 that the sequence $\frac{|P_k|}{\lambda_k} = |P_k| u_k^{\frac{p-1}{2-a}}(P_k)$ is bounded. We may thus assume that $\frac{P_k}{\lambda_k} \to x_0 \in \mathbb{R}^N$ as $k \to +\infty$. A similar limiting procedure as in Case 1 then produces a positive solution v of

$$-\Delta v = \frac{v^p}{|y + x_0|^a}, \qquad y \in \mathbb{R}^N, \tag{4.14}$$

where $0 < a < 2$, then by elliptic regularity, we obtain that u_k satisfy a local $W^{2,\widehat{q}}$ bound for $\frac{N}{2} < \widehat{q} < \frac{N}{|a|}$, so a local Hölder bound holds, and this is sufficient to pass the limit to obtain a solution of problem (4.14). After a space shift, this gives a contradiction with Lemma 2.4. Therefore, there exists a positive constant $\widehat{C}$ such that

$$\|u\|_{C(\Omega)} \le \widehat{C}. \tag{4.15}$$

(4.7) and (4.15) yield the desired conclusion of Theorem 1.4 and this completes the proof. □

5 Existence results

This section devotes to proving some existence results to problem (1.1). For the convenience of proving existence results, we consider the following problem with a parameter $t \in [0, 1]$,

$$\begin{cases} -\Delta u = \frac{u^p}{|x|^a} + h(x, u, \nabla u) + t(|u| + \lambda), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Fortunately, we have proved the boundedness of solutions firstly in Section 4, therefore, in this section, we only need to use the Leray–Schauder topological degree theory (see [3,6,17]) to prove the existence of solutions.

Proof of Theorem 1.5. We always assume that $h(x,u,\nabla u)$ satisfies (H_1) and (H_2). Let $X = C(\Omega)$, we denote that,

$$f(x,u) = \frac{u^p}{|x|^a} + h(x,u,\nabla u).$$

Given $u \in X$ and $t > 0$, let $\phi_t(u) = u$ be the unique solution of the problem (5.1). Then the solution to problem (1.1) is equivalent to a fixed point of the operator $\phi_0(u)$. Since $f \in L^k(\Omega)$ for $k < \min\{\frac{N}{a}, \frac{N(p-1)}{(p+1-a)b}\}$, we have $\phi_t(u) \in W^{2,r}(\Omega)$ for $r \in (\frac{N}{2}, \min\{\frac{N}{a}, \frac{N(p-1)}{(p+1-a)b}\})$. Therefore, $\phi_t : X \to X$ is compact. Observe that the right-hand sides in (5.1) are nonnegative for every $u \in X$, hence, ϕ_t has no fixed point beyond the nonnegative cone $K = \{u' \in X : u' > 0\}$ for any $t \geq 0$.

Let $\|u\|_X = \varepsilon$ for $\varepsilon > 0$ small. Assume $u = \phi_0(u)$, using L^p estimates, we have that,

$$\|u\|_\infty \leq C\|u\|_{2,r} \leq C\|f\|_r \leq C\|u\|_\infty^p,$$

where $\|\cdot\|_{2,r}$ denotes the norm in $W^{2,r}(\Omega)$. Furthermore, We can deduce that,

$$\|u\|_\infty \leq C\|u\|_\infty^p \leq C\varepsilon^{p-1}\|u\|_\infty.$$

This is a contradiction for ε sufficiently small due to the assumption $p > 1$. Hence $u \neq \phi_0(u)$ and the homotopy invariance of the topological degree implies

$$\deg(I - \phi_0, 0, B_\varepsilon) = \deg(I, 0, B_\varepsilon) = 1,$$

where I denotes the identity and $B_\varepsilon = \{u \in X : \|u\|_X < \varepsilon\}$.

Theorem 1.4 immediately implies $\phi_T(u) \neq u$ for T large and $u \in \overline{B_R} \bigcap K$, $\phi_t(u) \neq u$ for $t \in [0,T]$ and $u \in (\overline{B_R} \backslash B_R) \bigcap K$ (where $R > 0$ is lage enough), hence we have that,

$$\deg(I - \phi_0, 0, B_R) = \deg(I - \phi_T, 0, B_R) = 0.$$

Then we can obtain $\deg(I - \phi_0, 0, B_R \backslash \overline{B_\varepsilon}) = -1$, hence, there exist $u \in (B_R \backslash \overline{B_\varepsilon}) \bigcap K$ such that $\phi_0(u) = u$. Finally, the maximum principle implies the positivity of u. □

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