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Two-Point Boundary Value Problems For Strongly Singular Higher-Order Linear Differential Equations With Deviating Arguments

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Abstract

For strongly singular higher-order differential equations with deviating arguments, under two-point conjugated and right-focal boundary conditions, Agarwal-Kiguradze type theorems are established, which guarantee the presence of Fredholm's property for the above mentioned problems. Also we provide easily verifiable best possible conditions that guarantee the existence of a unique solution of the studied problems.

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1 Statement of the main results

1.1. Statement of the problems and the basic notations. Consider the differential equations with deviating arguments

$$u^{(n)}(t) = \sum_{j=1}^{m} p_j(t)u^{(j-1)}(\tau_j(t)) + q(t) \quad \text{for} \quad a < t < b, \tag{1.1}$$

with the two-point boundary conditions

$$u^{(i-1)}(a) = 0 \ (i = 1, \cdots, m), \quad u^{(j-1)}(b) = 0 \ (j = 1, \cdots, n-m), \tag{1.2}$$

$$u^{(i-1)}(a) = 0 \ (i = 1, \cdots, m), \quad u^{(j-1)}(b) = 0 \ (j = m+1, \cdots, n). \tag{1.3}$$

Here $n \geq 2$, m is the integer part of $n/2$, $-\infty < a < b < +\infty$, $p_j, q \in L_{loc}(]a, b[)$ $(j = 1, \cdots, m)$, and $\tau_j :]a, b[\to]a, b[$ are measurable functions. By $u^{(j-1)}(a)$ $(u^{(j-1)}(b))$ we denote the right (the left) limit of the function $u^{(j-1)}$ at the point $a\,(b)$. Problems (1.1), (1.2), and (1.1), (1.3) are said to be singular if some or all the coefficients of (1.1) are non-integrable on $[a, b]$, having singularities at the end-points of this segment.

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The linear ordinary differential equations and differential equations with deviating arguments with boundary conditions (1.2) and (1.3), and with the conditions

$$\int_a^b (s-a)^{n-1}(b-s)^{2m-1}[(-1)^{n-m}p_1(s)]_+ ds < +\infty,$$

$$\int_a^b (s-a)^{n-j}(b-s)^{2m-j}|p_j(s)|ds < +\infty \quad (j=2,\cdots,m), \tag{1.4}$$

$$\int_a^b (s-a)^{n-m-1/2}(b-s)^{m-1/2}|q(s)|ds < +\infty,$$

and

$$\int_a^b (s-a)^{n-1}[(-1)^{n-m}p_1(s)]_+ ds < +\infty,$$

$$\int_a^b (s-a)^{n-j}|p_j(s)|ds < +\infty \quad (j=2,\cdots,m), \tag{1.5}$$

$$\int_a^b (s-a)^{n-m-1/2}|q(s)|ds < +\infty,$$

respectively, were studied by I. Kiguradze, R. P. Agarwal and some other authors (see [1], [2], [4] - [22]).

The first step in studying the linear ordinary differential equations under conditions (1.2) or (1.3), in the case when the functions p_j and q have strong singularities at the points a and b, i.e. when conditions (1.4) and (1.5) are not fulfilled, was made by R. P. Agarwal and I. Kiguradze in the article [3].

In this paper the Agarwal-Kiguradze type theorems are proved which guarantee Fredholm's property for problems (1.1), (1.2), and (1.1), (1.3) (see Definition 1.1). Moreover, we establish optimal, in some sense, sufficient conditions for the solvability of problems (1.1), (1.2), and (1.1), (1.3).

Throughout the paper we use the following notation.

$R^+ = [0, +\infty[$;

$[x]_+$ is the positive part of number x, that is $[x]_+ = \frac{x+|x|}{2}$;

$L_{loc}(]a,b[)$ ($L_{loc}(]a,b])$) is the space of functions $y :]a, b[\to R$, which are integrable on $[a+\varepsilon, b-\varepsilon]$; ($[a+\varepsilon, b]$) for arbitrary small $\varepsilon > 0$;

$L_{\alpha,\beta}(]a,b[)$ ($L^2_{\alpha,\beta}(]a,b[)$) is the space of integrable (square integrable) with the weight $(t-a)^\alpha(b-t)^\beta$ functions $y :]a, b[\to R$, with the norm

$$||y||_{L_{\alpha,\beta}} = \int_a^b (s-a)^\alpha(b-s)^\beta|y(s)|ds \quad \Big(||y||_{L^2_{\alpha,\beta}} = \Big(\int_a^b (s-a)^\alpha(b-s)^\beta y^2(s)ds\Big)^{1/2}\Big);$$

$L([a,b]) = L_{0,0}(]a,b[)$, $L^2([a,b]) = L^2_{0,0}(]a,b[)$;

$M(]a,b[)$ is the set of measurable functions $\tau :]a,b[\to]a,b[$;

$\widetilde{L}^2_{\alpha,\beta}(]a,b[)$ $(\widetilde{L}^2_{\alpha}(]a,b])$ is the Banach space of functions $y \in L_{loc}(]a,b[)$ $(L_{loc}(]a,b]))$, satisfying

$$\mu_1 \equiv \max\Big\{\Big[\int\limits_a^t (s-a)^{\alpha}\Big(\int\limits_s^t y(\xi)d\xi\Big)^2 ds\Big]^{1/2} : a \le t \le \frac{a+b}{2}\Big\}+$$

$$+\max\Big\{\Big[\int\limits_t^b (b-s)^{\beta}\Big(\int\limits_t^s y(\xi)d\xi\Big)^2 ds\Big]^{1/2} : \frac{a+b}{2} \le t \le b\Big\} < +\infty,$$

$$\mu_2 \equiv \max\Big\{\Big[\int\limits_a^t (s-a)^{\alpha}\Big(\int\limits_s^t y(\xi)d\xi\Big)^2 ds\Big]^{1/2} : a \le t \le b\Big\} < +\infty.$$

The norm in this space is defined by the equality $||\cdot||_{\widetilde{L}^2_{\alpha,\beta}} = \mu_1$ $(||\cdot||_{\widetilde{L}^2_{\alpha}} = \mu_2)$.

$\widetilde{C}^{n-1,\,m}(]a,\,b[)$ $(\widetilde{C}^{n-1,\,m}(]a,\,b]))$ is the space of functions $y \in \widetilde{C}^{n-1}_{loc}(]a,\,b[)$ $(y \in \widetilde{C}^{n-1}_{loc}(]a,\,b]))$, satisfying

$$\int\limits_a^b |y^{(m)}(s)|^2 ds < +\infty. \tag{1.6}$$

When problem (1.1), (1.2) is discussed, we assume that for $n = 2m$, the conditions

$$p_j \in L_{loc}(]a,\,b[) \;\; (j = 1,\cdots,m) \tag{1.7}$$

are fulfilled, and for $n = 2m+1$, along with (1.7), the conditions

$$\limsup_{t\to b}\Big|(b-t)^{2m-1}\int\limits_{t_1}^t p_1(s)ds\Big| < +\infty \;\; (t_1 = \frac{a+b}{2}) \tag{1.8}$$

are fulfilled. Problem (1.1), (1.3) is discussed under the assumptions

$$p_j \in L_{loc}(]a,\,b]) \;\; (j = 1,\cdots,m). \tag{1.9}$$

A solution of problem (1.1), (1.2) ((1.1), (1.3)) is sought in the space $\widetilde{C}^{n-1,\,m}(]a,\,b[)$ $(\widetilde{C}^{n-1,\,m}(]a,\,b]))$.

By $h_j :]a,\,b[\times]a,\,b[\to R_+$ and $f_j : R \times M(]a,b[) \to C_{loc}(]a,b[\times]a,b[)$ $(j = 1,\ldots,m)$ we denote the functions and the operators, respectively, defined by the equalities

$$\begin{aligned} h_1(t,s) &= \Big|\int\limits_s^t (\xi-a)^{n-2m}[(-1)^{n-m}p_1(\xi)]_+ d\xi\Big|, \\ h_j(t,s) &= \Big|\int\limits_s^t (\xi-a)^{n-2m}p_j(\xi)d\xi\Big| \quad (j = 2,\cdots,m), \end{aligned} \tag{1.10}$$

and,

$$f_j(c,\tau_j)(t,s)=\Big|\int\limits_s^t(\xi-a)^{n-2m}|p_j(\xi)|\Big|\int\limits_\xi^{\tau_j(\xi)}(\xi_1-c)^{2(m-j)}d\xi_1\Big|^{1/2}d\xi\Big|\quad(j=1,\cdots,m).\quad(1.11)$$

Let, moreover,

$$m!!=\begin{cases}1 & \text{for}\ \ m\le 0\\ 1\cdot 3\cdot 5\cdots m & \text{for}\ \ m\ge 1\end{cases},$$

if $m=2k+1$.

1.2. Fredholm type theorems.

Along with (1.1), we consider the homogeneous equation

$$v^{(n)}(t)=\sum_{j=1}^m p_j(t)v^{(j-1)}(\tau_j(t))\quad\text{for}\quad a<t<b.\qquad(1.1_0)$$

In the case where conditions (1.4) and (1.5) are violated, the question on the presence of the Fredholm's property for problem (1.1), (1.2) ((1.1), (1.3)) in some subspace of the space $\widetilde{C}^{n-1,m}_{loc}(]a,\,b[)$ ($\widetilde{C}^{n-1,m}_{loc}(]a,\,b])$) remains so far open. This question is answered in Theorem 1.1 (Theorem 1.2) formulated below which contains optimal in a certain sense conditions guaranteeing the Fredholm's property for problem (1.1), (1.2) ((1.1), (1.3)) in the space $\widetilde{C}^{n-1,\,m}(]a,\,b[)$ ($\widetilde{C}^{n-1,\,m}(]a,\,b])$).

Definition 1.1. We will say that problem (1.1), (1.2) ((1.1), (1.3)) has the Fredholm's property in the space $\widetilde{C}^{n-1,m}(]a,\,b[)$ ($\widetilde{C}^{n-1,m}(]a,\,b])$), if the unique solvability of the corresponding homogeneous problem (1.1_0), (1.2) ((1.1_0), (1.3)) in that space implies the unique solvability of problem (1.1), (1.2) ((1.1), (1.3)) for every $q\in\widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,b[)$ ($q\in\widetilde{L}^2_{2n-2m-2}(]a,b])$).

Theorem 1.1. *Let there exist $a_0\in]a,\,b[$, $b_0\in]a_0,\,b[$, numbers $l_{kj}>0$, $\gamma_{kj}>0$, and functions $\tau_j\in M(]a,\,b[)$ $(k=0,1,\ j=1,\dots,m)$ such that*

$$\begin{gathered}(t-a)^{2m-j}h_j(t,s)\le l_{0j}\quad\textit{for}\quad a<t\le s\le a_0,\\ \limsup_{t\to a}(t-a)^{m-\frac12-\gamma_{0j}}f_j(a,\tau_j)(t,s)<+\infty,\end{gathered}\qquad(1.12)$$

$$\begin{gathered}(b-t)^{2m-j}h_j(t,s)\le l_{1j}\quad\textit{for}\quad b_0\le s\le t<b,\\ \limsup_{t\to b}(b-t)^{m-\frac12-\gamma_{1j}}f_j(b,\tau_j)(t,s)<+\infty,\end{gathered}\qquad(1.13)$$

and

$$\sum_{j=1}^m\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\,l_{kj}<1\quad(k=0,1).\qquad(1.14)$$

Let, moreover, (1.1_0), (1.2) *have only the trivial solution in the space* $\widetilde{C}^{n-1,m}(]a,\,b[)$. *Then problem* (1.1), (1.2) *has the unique solution* u *for every* $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,\ b[)$, *and there exists a constant* r, *independent of* q, *such that*

$$||u^{(m)}||_{L^2} \le r||q||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}}. \tag{1.15}$$

Corollary 1.1. *Let numbers* $\kappa_{kj}, \nu_{kj} \in R^+$ *be such that*

$$\nu_{k1} > 2n+2-2k(2m-n), \quad \nu_{kj} > 2 \quad (k=0,1;\ j=2,\dots,m), \tag{1.16}$$

$$\limsup_{t\to a} \frac{|\tau_j(t)-t|}{(t-a)^{\nu_{0j}}} < +\infty, \quad \limsup_{t\to b} \frac{|\tau_j(t)-t|}{(b-t)^{\nu_{1j}}} < +\infty, \tag{1.17}$$

and

$$\sum_{j=1}^{m} \frac{2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\,\kappa_{kj} < 1 \ (k=0,1). \tag{1.18}$$

Moreover, let $\kappa \in R^+$, $p_{0j} \in L_{n-j,\,2m-j}(]a,b[;R^+)$, *and*

$$\begin{aligned} -\frac{\kappa}{[(t-a)(b-t)]^{2n}} - p_{01}(t) \le (-1)^{n-m}p_1(t) \le \\ \le \frac{\kappa_{01}}{(t-a)^n} + \frac{\kappa_{11}}{(t-a)^{n-2m}(b-t)^{2m}} + p_{01}(t), \end{aligned} \tag{1.19}$$

$$|p_j(t)| \le \frac{\kappa_{0j}}{(t-a)^{n-j+1}} + \frac{\kappa_{1j}}{(t-a)^{n-2m}(b-t)^{2m-j+1}} + p_{0j}(t) \quad (j=2,\dots,m). \tag{1.20}$$

Let, moreover, (1.1_0), (1.2) *have only the trivial solution in the space* $\widetilde{C}^{n-1,m}(]a,\,b[)$. *Then problem* (1.1), (1.2) *has the unique solution* u *for every* $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,\ b[)$, *and there exists a constant* r, *independent of* q, *such that* (1.15) *holds.*

Theorem 1.2. *Let there exist* $a_0 \in]a,b[$, *numbers* $l_{0j} > 0$, $\gamma_{0j} > 0$, *and functions* $\tau_j \in M(]a,\,b[)$ *such that condition* (1.12) *is fulfilled and*

$$\sum_{j=1}^{m} \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\,l_{0j} < 1. \tag{1.21}$$

Let, moreover, problem (1.1_0), (1.3) *have only the trivial solution in the space* $\widetilde{C}^{n-1,m}(]a,\,b])$. *Then problem* (1.1), (1.3) *has the unique solution* u *for every* $q \in \widetilde{L}^2_{2n-2m-2}(]a,\ b])$, *and there exists a constant* r, *independent of* q, *such that*

$$||u^{(m)}||_{L^2} \le r||q||_{\widetilde{L}^2_{2n-2m-2}}. \tag{1.22}$$

Corollary 1.2. *Let numbers* $\kappa_{0j}, \nu_{0j} \in R^+$ *be such that*

$$\nu_{01} > 2n+2, \qquad \nu_{0j} \ge 2 \quad (j=2,\dots,m), \tag{1.23}$$

$$\limsup_{t\to a} \frac{|\tau_j(t)-t|}{(t-a)^{\nu_{0j}}} < +\infty, \tag{1.24}$$

and

$$\sum_{j=1}^{m} \frac{2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\kappa_{0j} < 1. \tag{1.25}$$

Let, moreover, $\kappa \in R^+$, $p_{0j} \in L_{n-j,0}(]a,b];R^+)$, *and*

$$-\frac{\kappa}{(t-a)^{2n}} - p_{01}(t) \le (-1)^{n-m}p_1(t) \le \frac{\kappa_{01}}{(t-a)^n} + p_{01}(t), \tag{1.26}$$

$$|p_j(t)| \le \frac{\kappa_{0j}}{(t-a)^{n-j+1}} + p_{0j}(t) \quad (j=2,\dots,m). \tag{1.27}$$

Let, moreover, problem (1.1_0), (1.3) *have only the trivial solution in the space* $\widetilde{C}^{n-1,m}(]a,\, b])$. *Then problem* (1.1), (1.3) *has the unique solution* u *for every* $q \in \widetilde{L}^2_{2n-2m-2}(]a,\, b])$, *and there exists a constant* r, *independent of* q, *such that* (1.22) *holds.*

Theorem 1.3. *Let* $c_1 = a, c_2 = b$,

$$\operatorname*{ess\,sup}_{a<t<b} \frac{1}{|t-c_i|^{m+1-j}} \Big| \int_t^{\tau_j(t)} |\xi - c_i|^{m-j-1} d\xi \Big| < +\infty \ (j=1,\dots,m) \tag{1.28}$$

if $i = 1,2$ (*if* $i = 1$),

$$p_j \in L_{n-j,\,2m-j}(]a,\, b[) \quad \Big(p_j \in L_{n-j,0}(]a,\, b])\Big) \ (j=1,\dots,m), \tag{1.29}$$

and let problem (1.1), (1.2) ((1.1), (1.3)) *be uniquely solvable in the space* $\widetilde{C}^{n-1,\,m}(]a,\, b[)$ (*in the space* $\widetilde{C}^{n-1,\,m}(]a,\, b])$. *Then this problem is uniquely solvable in the space* $\widetilde{C}^{n-1}(]a,\, b[)$ (*in the space* $\widetilde{C}^{n-1}(]a,\, b])$ *as well.*

Remark 1.1. In [3], an example is constructed which demonstrates that if condition (1.29) is violated, then problem (1.1), (1.2) (problem (1.1), (1.3)) with $\tau_j(t) \equiv t\, (j = 1,\dots,m)$ may be uniquely solvable in the space $\widetilde{C}^{n-1,m}(]a,\, b[)$ (in the space $\widetilde{C}^{n-1,m}(]a,\, b])$) and this problem may have infinite set of solutions in the space $\widetilde{C}^{loc}(]a,\, b[)$ (in the space $\widetilde{C}^{loc}(]a,\, b])$).

Also, in [3] it is demonstrated that strict inequalities (1.14), (1.21), (1.18), (1.25) are sharp because they cannot be replaced by nonstrict ones.

1.2. Existence and uniqueness theorems.

Theorem 1.4. *Let there exist numbers* $t^* \in]a,\, b[$, $\ell_{kj} > 0$, $\bar{l}_{kj} \ge 0$, *and* $\gamma_{kj} > 0$ $(k = 0,1;\ j = 1,\dots,m)$ *such that along with*

$$\sum_{j=1}^{m} \Big(\frac{(2m-j)2^{2m-j+1}\, l_{0j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}\, \bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}} \Big) < \frac{1}{2}, \tag{1.30}$$

$$\sum_{j=1}^{m} \Big(\frac{(2m-j)2^{2m-j+1}\, l_{1j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(b-t^*)^{\gamma_{0j}}\, \bar{l}_{1j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}} \Big) < \frac{1}{2}, \tag{1.31}$$

the conditions

$$(t-a)^{2m-j}h_j(t,s) \le l_{0j}, \ \ (t-a)^{m-\gamma_{0j}-1/2}f_j(a,\tau_j)(t,s) \le \bar{l}_{0j} \ \textit{for} \quad a<t\le s\le t^*, \tag{1.32}$$

$$(b-t)^{2m-j}h_j(t,s) \le l_{1j}, \ \ (b-t)^{m-\gamma_{1j}-1/2}f_j(b,\tau_j)(t,s) \le \bar{l}_{1j} \quad \textit{for} \quad t^*\le s\le t<b \tag{1.33}$$

hold. Then for every $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,b[)$ *problem* (1.1), (1.2) *is uniquely solvable in the space* $\widetilde{C}^{n-1,\,m}(]a,\,b[)$.

To illustrate this theorem, we consider the second order differential equation with a deviating argument

$$u''(t) = p(t)u(\tau(t)) + q(t), \tag{1.34}$$

under the boundary conditions

$$u(a) = 0, \ \ u(b) = 0. \tag{1.35}$$

From Theorem 1.4, with $n = 2, \ m = 1, \ t^* = (a+b)/2, \ \gamma_{01} = \gamma_{11} = 1/2, \ l_{01} = l_{11} = \kappa_0, \ \bar{l}_{01} = \bar{l}_{11} = \sqrt{2}\kappa_1/\sqrt{b-a}$, we get

Corollary 1.3. *Let function* $\tau \in M(]a,b[)$ *be such that*

$$\begin{gathered} 0 \le \tau(t) - t \le \frac{2^6}{(b-a)^6}(t-a)^7 \quad \textit{for} \quad a < t \le \frac{a+b}{2}, \\ -\frac{2^6}{(b-a)^6}(b-t)^7 \le t - \tau(t) \le 0 \quad \textit{for} \quad \frac{a+b}{2} \le t < b. \end{gathered} \tag{1.36}$$

Moreover, let function $p :]a,b[\to R$ *and constants* $\kappa_0, \ \kappa_1$ *be such that*

$$-\frac{2^{-2}(b-a)^2\kappa_0}{[(b-t)(t-a)]^2} \le p_1(t) \le \frac{2^{-7}(b-a)^6\kappa_1}{[(b-t)(t-a)]^4} \quad \textit{for} \quad a < t \le b \tag{1.37}$$

and

$$4\kappa_0 + \kappa_1 < \frac{1}{2}. \tag{1.38}$$

Then for every $q \in \widetilde{L}^2_{0,\,0}(]a,b[)$ *problem* (1.34), (1.35) *is uniquely solvable in the space* $\widetilde{C}^{1,\,1}(]a,\,b[)$.

Theorem 1.5. *Let there exist numbers* $t^* \in]a,\,b[, \ \ \ell_{0j} > 0, \ \bar{\ell}_{0j} \ge 0$, *and* $\gamma_{0j} > 0$ $(j = 1,\ldots,m)$ *such that conditions*

$$(t-a)^{2m-j}h_j(t,s) \le l_{0j}, \ \ (t-a)^{m-\gamma_{0j}-1/2}f_j(a,\tau_j)(t,s) \le \bar{l}_{0j} \ \textit{for} \quad a<t\le s\le b, \tag{1.39}$$

and

$$\sum_{j=1}^{m}\Big(\frac{(2m-j)2^{2m-j+1}\,l_{0j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}\,\bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}}\Big) < 1 \tag{1.40}$$

hold. Then for every $q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$ *problem* (1.1), (1.3) *is uniquely solvable in the space* $\widetilde{C}^{n-1,\,m}(]a,\,b])$.

Theorem 1.6. *Let there exist numbers* $t^* \in]a, b[$, $\ell_{kj} > 0$, $\bar{l}_{kj} \geq 0$, *and* $\gamma_{kj} > 0$ $(k = 0, 1;\ j = 1, \dots, m)$ *such that along with* (1.40) *and*

$$\sum_{j=1}^{m} \Big(\frac{(2m-j)2^{2m-j+1}\, l_{1j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(b-t^*)^{\gamma_{0j}}\, \bar{l}_{1j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}} \Big) < 1, \tag{1.41}$$

conditions (1.32), (1.33) *hold. Moreover, let* $\tau_j \in M(]a, b[)$ $(j = 1, \dots, n)$ *and*

$$\operatorname{sign}[(\tau_j(t) - t^*)(t - t^*)] \geq 0 \quad \textit{for} \quad a < t < b. \tag{1.42}$$

Then for every $q \in \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a, b[)$ *problem* (1.1), (1.2) *is uniquely solvable in the space* $\widetilde{C}^{n-1,\, m}(]a,\, b[)$.

Also, from Theorem 1.6, with $n = 2,\ m = 1,\ t^* = (a + b)/2,\ \gamma_{01} = \gamma_{11} = 1/2,\ l_{01} = l_{11} = \kappa_0,\ \bar{l}_{01} = \bar{l}_{11} = \sqrt{2}\kappa_1/\sqrt{b-a}$, we get

Corollary 1.4. *Let functions* $p :]a, b[\to R$, $\tau \in M(]a, b[)$ *and constants* $\kappa_0 > 0$, $\kappa_1 > 0$ *be such that along with* (1.36) *and* (1.37) *the inequalities*

$$\operatorname{sign}[(\tau(t) - \frac{a+b}{2})(t - \frac{a+b}{2})] \geq 0 \quad \textit{for} \quad a < t < b \tag{1.43}$$

and

$$4\kappa_0 + \kappa_1 < 1 \tag{1.44}$$

hold. Then for every $q \in \widetilde{L}^2_{0,\,0}(]a, b[)$ *problem* (1.34), (1.35) *is uniquely solvable in the space* $\widetilde{C}^{1,\,1}(]a,\, b[)$.

2 Auxiliary propositions

2.1. Lemmas on integral inequalities. Now we formulate two lemmas which are proved in [3].

Lemma 2.1. *Let* $\in \widetilde{C}^{m-1}_{loc}(]t_0,\, t_1[)$ *and*

$$u^{(j-1)}(t_0) = 0 \quad (j = 1, \dots, m), \qquad \int_{t_0}^{t_1} |u^{(m)}(s)|^2 ds < +\infty. \tag{2.1}$$

Then

$$\int_{t_0}^{t} \frac{(u^{(j-1)}(s))^2}{(s-t_0)^{2m-2j+2}} ds \leq \Big(\frac{2^{m-j+1}}{(2m-2j+1)!!} \Big)^2 \int_{t_0}^{t} |u^{(m)}(s)|^2 ds \quad \textit{for} \quad t_0 \leq t \leq t_1. \tag{2.2}$$

Lemma 2.2. *Let* $u \in \widetilde{C}^{m-1}_{loc}(]t_0,\, t_1[)$, *and*

$$u^{(j-1)}(t_1) = 0 \quad (j = 1, \dots, m), \qquad \int_{t_0}^{t_1} |u^{(m)}(s)|^2 ds < +\infty. \tag{2.3}$$

Then

$$\int_t^{t_1} \frac{(u^{(j-1)}(s))^2}{(t_1 - s)^{2m-2j+2}} ds \le \Big(\frac{2^{m-j+1}}{(2m-2j+1)!!} \Big)^2 \int_t^{t_1} |u^{(m)}(s)|^2 ds \quad \textit{for} \quad t_0 \le t \le t_1. \tag{2.4}$$

Let $t_0, t_1 \in]a, b[,\ u \in \widetilde{C}^{m-1}_{loc}(]t_0,\, t_1[)$ and $\tau_j \in M(]a, b[)$ $(j = 1, \dots, m)$. Then we define the functions $\mu_j : [a,\ (a+b)/2] \times [(a+b)/2,\ b] \times [a, b] \to [a, b]$, $\rho_k : [t_0,\, t_1] \to R_+ \,(k = 0, 1)$, $\lambda_j : [a, b] \times]a,\ (a+b)/2] \times [(a+b)/2,\ b[\times]a, b[\to R_+$, by the equalities

$$\mu_j(t_0, t_1, t) = \begin{cases} \tau_j(t) & \text{for } \ \tau_j(t) \in [t_0,\ t_1] \\ t_0 & \text{for } \ \tau_j(t) < t_0 \\ t_1 & \text{for } \ \tau_j(t) > t_1 \end{cases},$$

$$\rho_k(t) = \Big| \int_t^{t_k} |u^{(m)}(s)|^2 ds \Big|, \qquad \lambda_j(c, t_0, t_1, t) = \Big| \int_t^{\mu_j(t_0, t_1, t)} (s - c)^{2(m-j)} ds \Big|^{1/2}. \tag{2.5}$$

Let also functions $\alpha_j : R^3_+ \times [0, 1[\to R_+$ and $\beta_j \in R_+ \times [0, 1[\to R_+$ $(j = 1, \cdots, m)$ be defined by the equalities

$$\alpha_j(x, y, z, \gamma) = x + \frac{2^{m-j}\, y\, z^{\gamma}}{(2m-2j-1)!!}, \ \beta_j(y, \gamma) = \frac{2^{2m-j-1}}{(2m-2j-1)!!(2m-3)!!} \frac{y^{\gamma}}{\sqrt{2\gamma}}. \tag{2.6}$$

Lemma 2.3. *Let* $a_0 \in]a, b[,\ t_0 \in]a,\ a_0[,\ t_1 \in]a_0,\ b[$, *and the function* $u \in \widetilde{C}^{m-1}_{loc}(]t_0,\, t_1[)$ *be such that conditions* (2.1) *hold. Moreover, let constants* $l_{0\,j} > 0$, $\overline{l}_{0\,j} \ge 0$, $\gamma_{0j} > 0$, *and functions* $\overline{p}_j \in L_{loc}(]t_0,\, t_1[)$, $\tau_j \in M(]a, b[)$ *be such that the inequalities*

$$(t - t_0)^{2m-1} \int_t^{a_0} [\overline{p}_1(s)]_+ ds \le l_{0\,1}, \tag{2.7}$$

$$(t - t_0)^{2m-j} \Big| \int_t^{a_0} \overline{p}_j(s) ds \Big| \le l_{0\,j} \ \ (j = 2, \dots, m), \tag{2.8}$$

$$(t - t_0)^{m - \frac{1}{2} - \gamma_{0j}} \Big| \int_t^{a_0} \overline{p}_j(s) \lambda_j(t_0, t_0, t_1, s) ds \Big| \le \overline{l}_{0\,j} \ \ (j = 1, \dots, m) \tag{2.9}$$

hold for $t_0 < t \le a_0$. *Then*

$$\begin{aligned} &\int_t^{a_0} \overline{p}_j(s) u(s) u^{(j-1)}(\mu_j(t_0, t_1, s)) ds \le \\ &\le \alpha_j(l_{0j}, \overline{l}_{0j}, a_0 - a, \gamma_{0j}) \rho_0^{1/2}(\tau^*) \rho_0^{1/2}(t) + \overline{l}_{0j} \beta_j(a_0 - a, \gamma_{0j}) \rho_0^{1/2}(\tau^*) \rho_0^{1/2}(a_0) + \\ &+ l_{0j} \frac{(2m-j) 2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} \rho_0(a_0) \quad \textit{for} \quad t_0 < t \le a_0, \end{aligned} \tag{2.10}$$

where $\tau^* = \sup\{\mu_j(t_0,t_1,t) : t_0 \le t \le a_0, j = 1,\dots,m\} \le t_1$.

Proof. In view of the formula of integration by parts, for $t \in [t_0, a_0]$ we have

$$\int_t^{a_0} \overline{p}_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))ds = \int_t^{a_0} \overline{p}_j(s)u(s)u^{(j-1)}(s)ds +$$
$$+\int_t^{a_0} \overline{p}_j(s)u(s)\Big(\int_s^{\mu_j(t_0,t_1,s)} u^{(j)}(\xi)d\xi\Big)ds = u(t)u^{(j-1)}(t)\int_t^{a_0} \overline{p}_j(s)ds+ \tag{2.11}$$
$$+\sum_{k=0}^{1}\int_t^{a_0}\Big(\int_s^{a_0} \overline{p}_j(\xi)d\xi\Big)u^{(k)}(s)u^{(j-k)}(s)ds + \int_t^{a_0} \overline{p}_j(s)u(s)\Big(\int_s^{\mu_j(t_0,t_1,s)} u^{(j)}(\xi)d\xi\Big)ds$$

$(j = 2,\dots,m)$, and

$$\int_t^{a_0} \overline{p}_1(s)u(s)u(\mu_1(t_0,t_1,s))ds \le \int_t^{a_0} [\overline{p}_1(s)]_+u^2(s)ds+$$
$$+\int_t^{a_0} |\overline{p}_1(s)u(s)|\Big|\int_s^{\mu_1(t_0,t_1,s)} u'(\xi)d\xi\Big|ds \le u^2(t)\int_t^{a_0}[\overline{p}_1(s)]_+ds+ \tag{2.12}$$
$$+2\int_t^{a_0}\Big(\int_s^{a_0}[\overline{p}_1(\xi)]_+d\xi\Big)|u(s)u'(s)|ds + \int_t^{a_0} |\overline{p}_1(s)u(s)|\Big|\int_s^{\mu_1(t_0,t_1,s)} u'(\xi)d\xi\Big|ds.$$

On the other hand, by conditions (2.1), the Schwartz inequality and Lemma 2.1, we deduce that

$$|u^{(j-1)}(t)| = \frac{1}{(m-j)!}\Big|\int_{t_0}^{t}(t-s)^{m-j}u^{(m)}(s)ds\Big| \le (t-t_0)^{m-j+1/2}\,\rho_0^{1/2}(t) \tag{2.13}$$

for $t_0 \le t \le a_0$ $(j = 1,\dots,m)$. If along with this, in the case $j > 1$, we take into account inequality (2.8), and lemma 2.1, for $t \in [t_0, a_0]$, we obtain the estimates

$$\Big|u(t)u^{(j-1)}(t)\int_t^{a_0} \overline{p}_j(s)ds\Big| \le (t-t_0)^{2m-j}\Big|\int_t^{a_0} \overline{p}_j(s)ds\Big|\rho_0(t) \le l_{0j}\rho_0(t), \tag{2.14}$$

and

$$\sum_{k=0}^{1}\int_t^{a_0}\Big(\int_s^{a_0} \overline{p}_j(\xi)d\xi\Big)u^{(k)}(s)u^{(j-k)}(s)ds \le l_{0j}\sum_{k=0}^{1}\int_t^{a_0}\frac{|u^{(k)}(s)u^{(j-k)}(s)|}{(s-t_0)^{2m-j}}ds \le$$
$$\le l_{0j}\sum_{k=0}^{1}\Big(\int_t^{a_0}\frac{|u^{(k)}(s)|^2ds}{(s-t_0)^{2m-2k}}\Big)^{1/2}\Big(\int_t^{a_0}\frac{|u^{(j-k)}(s)|^2ds}{(s-t_0)^{2m+2k-2j}}\Big)^{1/2} \le \tag{2.15}$$
$$\le l_{0j}\rho_0(a_0)\sum_{k=0}^{1}\frac{2^{2m-j}}{(2m-2k-1)!!(2m+2k-2j-1)!!}.$$

Analogously, if $j = 1$, by (2.7) we obtain

$$u^2(t) \int_t^{a_0} [\overline{p}_1(s)]_+ ds \le l_{01}\rho_0(t),$$
$$2\int_t^{a_0} \Big(\int_s^{a_0} [\overline{p}_1(\xi)]_+ d\xi \Big) |u(s)u'(s)| ds \le l_{01}\rho_0(a_0) \frac{(2m-1)2^{2m}}{[(2m-1)!!]^2} \tag{2.16}$$

for $t_0 < t \le a_0$.

By the Schwartz inequality, Lemma 2.1, and the fact that ρ_0 is nondecreasing function, we get

$$\Big| \int_s^{\mu_j(t_0,t_1,s)} u^{(j)}(\xi) d\xi \Big| \le \frac{2^{m-j}}{(2m-2j-1)!!} \lambda_j(t_0, t_0, t_1, s)\, \rho_0^{1/2}(\tau^*) \tag{2.17}$$

for $t_0 < s \le a_0$. Also, due to (2.2), (2.9) and (2.13), we have

$$|u(t)| \int_t^{a_0} |\overline{p}_j(s)| \lambda_j(t_0, t_0, t_1, s) ds = (t-t_0)^{m-1/2} \rho_0^{1/2}(t) \int_t^{a_0} |\overline{p}_j(s)| \lambda_j(t_0, t_0, t_1, s) ds \le$$
$$\le \overline{l}_{0j}\, (t-t_0)^{\gamma_{0j}}\, \rho_0^{1/2}(t),$$
$$\int_t^{a_0} |u'(s)| \Big(\int_s^{a_0} |\overline{p}_j(\xi)| \lambda_j(t_0, t_0, t_1, \xi) d\xi \Big) ds \le \overline{l}_{0j} \int_t^{a_0} \frac{|u'(s)|}{(s-t_0)^{m-\frac{1}{2}-\gamma_{0j}}} ds \le$$
$$\le \overline{l}_{0j} \frac{2^{m-1}(a_0-a)^{\gamma_{0j}}}{(2m-3)!!\sqrt{2\gamma_{0j}}} \rho_0^{1/2}(a_0)$$

for $t_0 < t \le a_0$. From the last three inequalities it is clear that

$$\Big| \frac{(2m-2j-1)!!}{2^{m-j}\rho_0^{1/2}(\tau^*)} \int_t^{a_0} \overline{p}_j(s) u(s) \Big(\int_s^{\mu_j(t_0,t_1,s)} u^{(j)}(\xi) d\xi \Big) ds \Big| \le \int_t^{a_0} |\overline{p}_j(s) u(s)| \lambda_j(t_0, t_0, t_1, s) ds \le$$
$$\le |u(t)| \int_t^{a_0} |\overline{p}_j(s)| \lambda_j(t_0, t_0, t_1, s) ds + \int_t^{a_0} |u'(s)| \Big(\int_s^{a_0} |\overline{p}_j(\xi)| \lambda_j(t_0, t_0, t_1, \xi) d\xi \Big) ds \le$$
$$\le \overline{l}_{0j}\, (t-t_0)^{\gamma_{0j}} \rho_0^{1/2}(t) + \overline{l}_{0j} \frac{2^{m-1}(a_0-a)^{\gamma_{0j}}}{(2m-3)!!\sqrt{2\gamma_{0j}}} \rho_0^{1/2}(a_0) \tag{2.18}$$

for $t_0 < t \le a_0$. Now, note that from (2.11) and (2.12) by (2.14)-(2.16) and (2.18), it immediately follows inequality (2.10). □

The following lemma can be proved similarly to Lemma 2.3.

Lemma 2.4. *Let $b_0 \in]a,\, b[,\ t_1 \in]b_0,\, b[,\ \ t_0 \in]a,\, b_0[,$ and the function $u \in \widetilde{C}^{m-1}_{loc}(]t_0,\, t_1[)$ be such that conditions* (2.3) *hold. Moreover, let constants $l_{1\,j} > 0,\ \bar{l}_{1\,j} \ge 0,\ \gamma_{1j} > 0$, and functions $\overline{p}_j \in L_{loc}(]t_0,\, t_1[),\ \tau_j \in M(]a,b[)$ be such that the inequalities*

$$(t_1 - t)^{2m-1} \int\limits_{b_0}^{t} [\overline{p}_1(s)]_+ ds \le l_{1\,1}, \tag{2.19}$$

$$(t_1 - t)^{2m-j} \Big| \int\limits_{b_0}^{t} \overline{p}_j(s) ds \Big| \le l_{1\,j} \ \ (j = 2, \dots, m), \tag{2.20}$$

$$(t_1 - t)^{m-\frac{1}{2}-\gamma_{1j}} \Big| \int\limits_{b_0}^{t} \overline{p}_j(s)\lambda_j(t_1, t_0, t_1, s) ds \Big| \le \bar{l}_{1\,j} \ \ (j = 1, \dots, m) \tag{2.21}$$

hold for $b_0 < t \le t_1$. Then

$$\begin{gathered}\int\limits_{b_0}^{t} \overline{p}_j(s)u(s)u^{(j-1)}(\mu_j(t_0, t_1, s))ds \le \\ \le \alpha_j(l_{1j}, \bar{l}_{1j}, b - b_0, \gamma_{1j})\rho_1^{1/2}(\tau_*)\rho_1^{1/2}(t) + \bar{l}_{1j}\beta_j(b - b_0, \gamma_{1j})\rho_1^{1/2}(\tau_*)\rho_1^{1/2}(b_0) + \\ + l_{1j}\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\rho_1(b_0) \quad \textit{for} \quad b_0 \le t < t_1,\end{gathered} \tag{2.22}$$

where $\tau_* = \inf\{\mu_j(t_0, t_1, t) : b_0 \le t \le t_1, j = 1, \dots, m\} \ge t_0$.

2.2. Lemma on the property of functions from the space $\widetilde{C}^{n-1,m}(]a,\, b[)$.

Lemma 2.5. *Let*

$$w(t) = \sum_{i=1}^{n-m} \sum_{k=i}^{n-m} c_{ik}(t)u^{(n-k)}(t)u^{(i-1)}(t),$$

where $\widetilde{C}^{n-1,m}(]a,\, b[)$, and each $c_{ik} : [a,b] \to R$ is an $(n-k-i+1)$ -times continuously differentiable function. Moreover, if

$$u^{(i-1)}(a) = 0 \ (i = 1, \dots, m), \quad \limsup_{t \to a} \frac{|c_{ii}(t)|}{(t-a)^{n-2m}} < +\infty \ \ (i = 1, \dots, n-m),$$

then

$$\liminf_{t \to a} |w(t)| = 0,$$

and if $u^{(i-1)}(b) = 0 \ \ (i = 1, \dots, n-m)$, then

$$\liminf_{t \to b} |w(t)| = 0.$$

The proof of this lemma is given in [9].

2.3. Lemmas on the sequences of solutions of auxiliary problems. Now for every natural k we consider the auxiliary boundary problems

$$u^{(n)}(t)=\sum_{j=1}^{m}p_j(t)u^{(j-1)}(\mu_j(t_{0k},t_{1k},t))+q_k(t)\quad\text{for}\quad t_{0k}\le t\le t_{1k}, \tag{2.23}$$

$$u^{(i-1)}(t_{0k})=0\ (i=1,\dots,m),\quad u^{(j-1)}(t_{1k})=0\ (j=1,\dots,n-m), \tag{2.24}$$

where

$$a<t_{0k}<t_{1k}<b\ (k\in N),\qquad \lim_{k\to+\infty}t_{0k}=a,\quad \lim_{k\to+\infty}t_{1k}=b, \tag{2.25}$$

and

$$u^{(n)}(t)=\sum_{j=1}^{m}p_j(t)u^{(j-1)}(\mu_j(t_{0k},b,t))+q_k(t)\quad\text{for}\quad t_{0k}\le t\le b, \tag{2.26}$$

$$u^{(i-1)}(t_{0k})=0\ (i=1,\dots,m),\quad u^{(j-1)}(b)=0\ (j=1,\dots,n-m), \tag{2.27}$$

where

$$a<t_{0k}<b\ (k\in N),\qquad \lim_{k\to+\infty}t_{0k}=a. \tag{2.28}$$

Throughout this section, when problems (1.1), (1.2) and (2.23), (2.24) are discussed we assume that

$$p_j\in L_{loc}(]a,b[)\ \ (j=1,...,m),\quad q,q_k\in\widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,\,b[), \tag{2.29}$$

and for an arbitrary $(m-1)$-times continuously differentiable function $x:]a,b[\to R$, we set

$$\Lambda_k(x)(t)=\sum_{j=1}^{m}p_j(t)x^{(j-1)}(\mu_j(t_{0k},t_{1k},t)),\quad \Lambda(x)(t)=\sum_{j=1}^{m}p_j(t)x^{(j-1)}(\tau_j(t)). \tag{2.30}$$

Problems (1.1), (1.3) and (2.26), (2.27) are considered in the case

$$p_j\in L_{loc}(]a,b])\ \ (j=1,...,m),\quad q,q_k\in\widetilde{L}^2_{2n-2m-2,\,0}(]a,\,b[), \tag{2.31}$$

and for an arbitrary $(m-1)$-times continuously differentiable function $x:]a,b]\to R$, we set

$$\Lambda_k(x)(t)=\sum_{j=1}^{m}p_j(t)x^{(j-1)}(\mu_j(t_{0k},b,t)),\quad \Lambda(x)(t)=\sum_{j=1}^{m}p_j(t)x^{(j-1)}(\tau_j(t)). \tag{2.32}$$

Remark 2.1. From the definition of the functions μ_j $(j=1,\dots,m)$, the estimate

$$|\mu_j(t_{0k},t_{1k},t)-\tau_j(t)|\le\begin{cases}0 & \text{for}\ \ \tau_j(t)\in]t_{0k},\ t_{1k}[\\ \max\{b-t_{1k},\ \ t_{0k}-a\} & \text{for}\ \ \tau_j(t)\notin]t_{0k},\ t_{1k}[\end{cases}$$

follows. Thus, if conditions (2.25) hold, then

$$\lim_{k\to+\infty}\mu_j(t_{0k},t_{1k},t)=\tau_j(t)\quad(j=1,\dots,m)\qquad\text{uniformly in}\quad]a,b[. \tag{2.33}$$

Lemma 2.6. *Let conditions* (2.25) *hold and the sequence of the* $(m-1)$*-times continuously differentiable functions* $x_k :]t_{0k}, t_{1k}[\to R$*, and functions* $x^{(j-1)} \in C([a,b])$ $(j = 1, \dots, m)$ *be such that*

$$\lim_{k\to+\infty} x_k^{(j-1)}(t) = x^{(j-1)}(t) \quad (j = 1, \dots, m) \quad \textit{uniformly in} \quad]a,b[\ \ (]a,b]). \tag{2.34}$$

Then for any nonnegative function $w \in C([a,b])$ *and* $t^* \in]a,b[$,

$$\lim_{k\to+\infty} \int_{t^*}^{t} w(s)\Lambda_k(x_k)(s)ds = \int_{t^*}^{t} w(s)\Lambda(x)(s)ds \tag{2.35}$$

uniformly in $]a,b[$, *where* Λ_k *and* Λ *are defined by equalities* (2.30).

Proof. We have to prove that for any $\delta \in]0, \ \min\{b - t^*, \ t^* - a\}[$, and $\varepsilon > 0$, there exists a constant $n_0 \in N$ such that

$$\Big| \int_{t^*}^{t} w(s)(\Lambda_k(x_k)(s) - \Lambda(x)(s))ds \Big| \le \varepsilon \quad \text{for} \quad t \in [a+\delta, \, b-\delta], \ k > n_0. \tag{2.36}$$

Let, now $w(t_*) = \max\limits_{a\le t\le b} w(t)$, and $\varepsilon_1 = \varepsilon\Big(2w(t_*)\sum\limits_{j=1}^{m} \int_{a+\delta}^{b-\delta} |p_j(s)|ds\Big)^{-1}$. Then from the inclusions $x_k^{(j-1)} \in C([a+\delta, \, b-\delta])$, $x^{(j-1)} \in C([a,b])$ $(j = 1, \dots, m)$, conditions (2.33) and (2.34), it follows the existence of such constant $n_0 \in N$ that

$$|x_k^{(j-1)}(\mu_j(t_{0k}, t_{1k}, s)) - x^{(j-1)}(\mu_j(t_{0k}, t_{1k}, s))| \le \varepsilon_1, |x^{(j-1)}(\mu_j(t_{0k}, t_{1k}, s)) - x^{(j-1)}(\tau_j(s))| \le \varepsilon_1$$

for $t \in [a+\delta, \, b-\delta]$, $\ k > n_0$, $\ j = 1, \dots, m$. Thus from the inequality

$$|\Lambda_k(x_k)(s) - \Lambda(x)(s)| \le |\Lambda_k(x_k)(s) - \Lambda_k(x)(s)| + |\Lambda_k(x)(s) - \Lambda(x)(s)| \le 2\varepsilon_1 \sum_{j=1}^{m} |p_j(t)|,$$

we have (2.36). □

The proof of the following lemma is analogous to that of Lemma 2.6.

Lemma 2.7. *Let conditions* (2.28) *hold and the sequence of the* $(m-1)$*-times continuously differentiable functions* $x_k :]t_{0k}, b] \to R$*, and functions* $x^{(j-1)} \in C([a,b])$ $(j = 1, \dots, m)$ *be such that* $\lim\limits_{l\to+\infty} x_k^{(j-1)}(t) = x^{(j-1)}(t) \quad (j = 1, \dots, m) \quad$ *uniformly in* $\ ]a,b]$. *Then for any nonnegative function* $w \in C([a,b])$, *and* $t^* \in]a,b]$, *condition* (2.35) *holds uniformly in* $]a,b]$, *where* Λ_k *and* Λ *are defined by equalities* (2.32).

Lemma 2.8. *Let condition* (2.25) *hold, and for every natural* k, *problem* (2.23), (2.24) *have a solution* $u_k \in \widetilde{C}_{loc}^{n-1}(]a,b[)$, *and there exist a constant* $r_0 > 0$ *such that*

$$\int_{t_{0k}}^{t_{1k}} |u_k^{(m)}(s)|ds \le r_0^2 \quad (k \in N) \tag{2.37}$$

holds, and if $n = 2m+1$, *let there exist constants* $\rho_j \geq 0,\ \overline{\rho}_j \geq 0,\ \gamma_{1j} > 0$ *such that*

$$\rho_j = \sup\Big\{(b-t)^{2m-j}\Big|\int\limits_{t_1}^{t}(s-a)p_j(s)ds\Big| : t_0 \leq t < b\Big\} < +\infty,$$
$$\overline{\rho}_j = \sup\Big\{(b-t)^{m-\gamma_{1j}-1/2}\int\limits_{t_1}^{t}(s-a)\Big|p_j(s)\Big|\lambda_j(b,t_{0k},t_{1k},s)ds : t_0 \leq t < b\Big\} < +\infty, \tag{2.38}$$

for $t_1 = \frac{a+b}{2}$, $(j = 1,...,m)$. *Moreover, let*

$$\lim_{k\to+\infty}||q_k - q||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} = 0, \tag{2.39}$$

and the homogeneous problem (1.1_0), (1.2) *have only the trivial solution in the space* $\widetilde{C}^{n-1,m}(]a,b[)$. *Then nonhomogeneous problem* (1.1), (1.2) *has a unique solution* u *such that*

$$||u^{(m)}||_{L^2} \leq r_0, \tag{2.40}$$

and

$$\lim_{k\to+\infty}u_k^{(j-1)}(t) = u^{(j-1)}(t) \quad (j = 1,\dots,n) \quad \text{uniformly in} \quad]a,b[\tag{2.41}$$

(that is, uniformly on $[a+\delta, b-\delta]$ *for an arbitrarily small* $\delta > 0$*).*

Proof. Suppose $t_1,\dots,t_n$ are the numbers such that

$$\frac{a+b}{2} = t_1 < \cdots < t_n < b, \tag{2.42}$$

and $g_i(t)$ are the polynomials of $(n-1)$-th degree, satisfying the conditions

$$g_j(t_j) = 1, \quad g_j(t_i) = 0 \ \ (i \neq j;\ \ i,j = 1,\dots,n). \tag{2.43}$$

Then for every natural k, for the solution u_k of problem (2.23), (2.24) the representation

$$u_k(t) = \sum_{j=1}^{n}\Big(u_k(t_j) - \frac{1}{(n-1)!}\int\limits_{t_1}^{t_j}(t_j-s)^{n-1}(\Lambda_k(u_k)(s) + q_k(s))ds\Big)g_j(t)+$$
$$+\frac{1}{(n-1)!}\int\limits_{t_1}^{t}(t-s)^{n-1}(\Lambda_k(u_k)(s) + q_k(s))ds \tag{2.44}$$

is valid. For an arbitrary $\delta \in]0, \frac{a+b}{2}[$, we have

$$\Big|\int\limits_{t}^{t_1}(s-t)^{n-j}(q_k(s) - q(s))ds\Big| = (n-j)\Big|\int\limits_{t}^{t_1}(s-t)^{n-j-1}\Big(\int\limits_{s}^{t_1}(q_k(\xi) - q(\xi))d\xi\Big)ds\Big| \leq$$
$$\leq n\Big(\int\limits_{t}^{t_1}(s-a)^{2m-2j}ds\Big)^{1/2}\Big(\int\limits_{t}^{t_1}(s-a)^{2n-2m-2}\Big(\int\limits_{s}^{t_1}(q_k(\xi) - q(\xi))d\xi\Big)^2ds\Big)^{1/2} \leq$$

$$\leq n\Big|(t_1-a)^{2m-2j+1}-\delta^{2m-2j+1}\Big|^{1/2}||q_k-q||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} \quad \text{for } \ a+\delta\leq t\leq t_1,$$

$$\Big|\int\limits_{t_1}^{t}(t-s)^{n-j}(q_k(s)-q(s))ds\Big|\leq n\Big|(b-t_1)^{2n-2m-2j+1}-\delta^{2n-2m-2j+1}\Big|^{1/2}\times \tag{2.45}$$

$$\times||q_k-q||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} \qquad \text{for } \ t_1\leq t\leq b-\delta \ \ (j=1,\dots,n-1).$$

Hence, by condition (2.39), we find

$$\lim_{k\to+\infty}\int\limits_{t}^{t_1}(s-t)^{n-j}(q_k(s)-q(s))ds=0 \quad \text{ uniformly in } \ ]a,b[\ \ (j=1,\dots,n-1). \tag{2.46}$$

Analogously one can show that if $t_0\in]a,b[$, then

$$\lim_{k\to+\infty}\int\limits_{t_0}^{t}(s-t_0)(q_k(s)-q(s))ds=0 \quad \text{ uniformly on } \ I(t_0), \tag{2.47}$$

where $I(t_0)=[t_0,(a+b)/2]$ for $t_0<(a+b)/2$ and $I(t_0)=[(a+b)/2,t_0]$ for $t_0>(a+b)/2$.

In view of inequalities (2.37), the identities

$$u_k^{(j-1)}(t)=\frac{1}{(m-j)!}\int\limits_{t_{ik}}^{t}(t-s)^{m-j}u_k^{(m)}(s)ds \tag{2.48}$$

for $i=0,1;\ j=1,\dots,m;\ \ k\in N$, yield

$$|u_k^{(j-1)}(t)|\leq r_j[(t-a)(b-t)]^{m-j+1/2} \tag{2.49}$$

for $t_{0k}\leq t\leq t_{1k}$ $(j=1,\dots,m;\ \ k\in N)$, where

$$r_j=\frac{r_0}{(m-j)!}(2m-2j+1)^{-1/2}\Big(\frac{2}{b-a}\Big)^{m-j+1/2} \qquad (j=1,\dots,m). \tag{2.50}$$

By virtue of the Arzela-Ascoli lemma and conditions (2.37) and (2.49), the sequence $\{u_k\}_{k=1}^{+\infty}$ contains a subsequence $\{u_{k_l}\}_{l=1}^{+\infty}$ such that $\{u_{k_l}^{(j-1)}\}_{l=1}^{+\infty}$ $(j=1,\dots,m)$ are uniformly convergent in $]a,b[$. Suppose

$$\lim_{l\to+\infty}u_{k_l}(t)=u(t). \tag{2.51}$$

Then in view of (2.49), $u^{(j-1)}\in C([a,b])$ $(j=1,\dots,m)$, and

$$\lim_{l\to+\infty}u_{k_l}^{(j-1)}(t)=u^{(j-1)}(t) \quad (j=1,\dots,m) \quad \text{uniformly in } \ \ ]a,b[. \tag{2.52}$$

If along with this we take into account conditions (2.25) and (2.46), from (2.44) by lemma 2.6 we find

$$\begin{gathered} u(t)=\sum_{j=1}^{n}\Big(u(t_j)-\frac{1}{(n-1)!}\int\limits_{t_1}^{t_j}(t_j-s)^{n-1}(\Lambda(u)(s)+q(s))ds\Big)g_j(t)+ \\ +\frac{1}{(n-1)!}\int\limits_{t_1}^{t}(t-s)^{n-1}(\Lambda(u)(s)+q(s))ds \quad \text{for} \quad a<t<b, \end{gathered} \tag{2.53}$$

$$|u^{(j-1)}(t)| \le r_j[(t-a)(b-t)]^{m-j+1/2} \quad \text{for} \quad a<t<b \ (j=1,\dots,m), \tag{2.54}$$

$u \in \widetilde{C}^{n-1}_{loc}(]a,b[)$, and

$$\lim_{l\to+\infty} u^{(j-1)}_{k_l}(t) = u^{(j-1)}(t) \quad (j=1,\dots,n-1) \quad \text{uniformly in} \quad]a,b[. \tag{2.55}$$

On the other hand, for any $t_0 \in]a,b[$ and natural l, we have

$$(t-t_0)u^{(n-1)}_{k_l}(t) = u^{(n-2)}_{k_l}(t) - u^{(n-2)}_{k_l}(t_0) + \int_{t_0}^{t} (s-t_0)(\Lambda_k(u_{k_l})(s) + q_{k_l}(s))ds. \tag{2.56}$$

Hence, due to (2.25), (2.47), (2.55), and Lemma 2.6 we get

$$\lim_{l\to+\infty} u^{(n-1)}_{k_l}(t) = u^{(n-1)}(t) \quad \text{uniformly in} \quad]a,b[. \tag{2.57}$$

Now it is clear that (2.55), (2.57), and (2.37) results in (2.40) and (2.41). Therefore, $u \in \widetilde{C}^{n-1,\,m}_{loc}(]a,b[)$. On the other hand, from (2.53) it is obvious that u is a solution of (1.1). In the case where $n=2m$, from (2.54) equalities (1.2) follow, that is, u is a solution of problem (1.1), (1.2).

Let us show that u is the solution of that problem in the case $n=2m+1$ as well. In view of (2.54), it suffice to prove that $u^{(m)}(b)=0$. First we find an estimate for the sequence $\{u_k\}_{k=1}^{+\infty}$. For this, without loss of generality we assume that

$$t_1 \le t_{1k} \qquad (k \in N). \tag{2.58}$$

From (2.44), by (2.39) and (2.49), it follows the existence of a positive constant ρ_0, independent of k, such that

$$\begin{gathered} |u^{(m+1)}_k(t)| \le \\ \le \rho_0 + \frac{1}{(m-1)!}\Big(\Big|\int_{t_1}^{t}(t-s)^{m-1}\Lambda_k(u_k)(s)ds\Big| + \Big|\int_{t_1}^{t}(t-s)^{m-1}q_k(s)ds\Big|\Big) \end{gathered} \tag{2.59}$$

for $t_1 \le t \le t_{1k}$, and

$$||q_k||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} \le \rho_0, \tag{2.60}$$

for $k \in N$. On the other hand, it is evident that

$$\begin{gathered} \Big|\int_{t_1}^{t}(t-s)^{m-1}\Lambda_k(u_k)(s)ds\Big| \le \sum_{j=1}^{m}\Big|\int_{t_1}^{t}(t-s)^{m-1}p_j(s)u^{(j-1)}_k(s)ds\Big| + \\ + \sum_{j=1}^{m}\Big|\int_{t_1}^{t}(t-s)^{m-1}p_j(s)\Big(\int_{s}^{\mu_j(t_{0k},t_{1k},s)} u^{(j)}_k(\xi)d\xi\Big)ds\Big| \end{gathered} \tag{2.61}$$

for $t_1 \le t \le t_{1k}$ $(k \in N)$.

Let, now $m > 1$. From Lemma 2.2 and condition (2.37) we get the estimates

$$\int\limits_{t_1}^{t} \frac{|u_k^{(j)}(s)|^2}{(b-s)^{2m-2j}} ds \le \int\limits_{t_0}^{t_{1k}} \frac{|u_k^{(j)}(s)|^2}{(t_{1k}-s)^{2m-2j}} ds \le 2^{2m} r_0^2 \tag{2.62}$$

for $t_1 \le t \le t_{1k}$ $(j = 1, \dots, m)$. Then by conditions (2.38) we find

$$\Big| \int\limits_{t_1}^{t} (t-s)^{m-1} p_j(s) u_k^{(j-1)}(s) ds \Big| =$$

$$= \Big| \int\limits_{t_1}^{t} \frac{1}{(b-s)^{2m-j}} \Big(\frac{\partial}{\partial s} \frac{(t-s)^{m-1} u_k^{(j-1)}(s)}{s-a} \Big) \Big((b-s)^{2m-j} \int\limits_{t_1}^{s} (\xi - a) p_j(\xi) d\xi \Big) ds \Big| \le$$

$$\le \frac{4m\rho_j}{b-a} \Big(\int\limits_{t_1}^{t} \frac{|u_k^{(j-1)}(s)|}{(b-s)^{m-j+2}} ds + \int\limits_{t_1}^{t} \frac{|u_k^{(j)}(s)|}{(b-s)^{m-j+1}} ds \Big) \le$$

$$\le \frac{4m\rho_j}{b-a} \Big[\Big(\int\limits_{t_1}^{t} \frac{(u_k^{(j-1)}(s))^2}{(b-s)^{2m-2j+2}} ds \Big)^{1/2} + \Big(\int\limits_{t_1}^{t} \frac{(u_k^{(j)}(s))^2}{(b-s)^{2m-2j}} ds \Big)^{1/2} \Big] \times \tag{2.63}$$

$$\times \Big(\int\limits_{t_1}^{t} (b-s)^{-2} ds \Big)^{1/2} \le \frac{2^m m r_0 \rho_j}{b-a} (b-t)^{-1/2}$$

for $t_1 \le t \le t_{1k}$ $(j = 1, \dots, m)$. On the other hand, by the Schwartz inequality, the definition of the functions μ_j and (2.4) it is clear that

$$\Big| \int\limits_{s}^{\mu_j(t_{0k}, t_{1k}, s)} u_k^{(j)}(\xi) d\xi \Big| \le \frac{2^{m-j}}{(2m-2j-1)!!} \lambda_j(b, t_{0k}, t_{1k}, s) \Big(\int\limits_{t_{0k}}^{t_{1k}} |u_k^{(m)}(\xi)|^2 d\xi \Big)^{1/2} \le \tag{2.64}$$

$$\le 2^m r_0 \lambda_j(b, t_{0k}, t_{1k}, s)$$

for $t_1 < s \le t_{1k}$ $(j = 1, \dots, m)$. Then by the integration by parts and (2.38), (2.64) we get

$$\Big| \int\limits_{t_1}^{t} (t-s)^{m-1} p_j(s) \Big(\int\limits_{s}^{\mu_j(t_{0k}, t_{1k}, s)} u_k^{(j)}(\xi) d\xi \Big) ds \Big| \le$$

$$\le 2^m r_0 \Big| \int\limits_{t_1}^{t} \Big| \frac{\partial}{\partial s} \frac{(t-s)^{m-1}}{s-a} \Big| \Big(\int\limits_{t_1}^{s} (\xi - a) |p_j(\xi)| \lambda_j(b, t_{0k}, t_{1k}, \xi) d\xi \Big) ds \Big| \le 2^m r_0 \times \tag{2.65}$$

$$\times \overline{\rho}_j \int\limits_{t_1}^{t} \Big| \frac{\partial}{\partial s} \frac{(t-s)^{m-1}}{s-a} \Big| (b-s)^{\gamma_{1j} - m + 1/2} ds \le 2^m r_0 \overline{\rho}_j \times$$

$$\times \int\limits_{t_1}^{t} \Big(\frac{m-1}{s-a} + \frac{t-a}{(s-a)^2}\Big)(b-s)^{\gamma_{1j}-3/2} ds \leq \frac{(m+1)2^{m+1} r_0 \overline{\rho}_j (b-a)^{\gamma_{1j}}}{b-a} \times$$

$$\times \int\limits_{t_1}^{t} (b-s)^{-3/2} ds \leq \frac{(m+1)2^{m+2} r_0 (b-a)^{\gamma_{1j}} \overline{\rho}_j}{b-a} (b-t)^{-1/2}$$

for $t_1 < s \leq t_{1k}$ $(j = 1, \dots, m)$.

Thus from (2.61), by (2.63) and (2.65) we have

$$\Big| \int\limits_{t_1}^{t} (t-s)^{m-1} \Lambda_k(u_k)(s) ds \Big| \leq \kappa_0 (b-t)^{-1/2} \tag{2.66}$$

for $t_1 \leq t \leq t_{1k},\ m > 1$, where $\kappa_0 = \frac{r_0(m+1)2^{m+2}}{b-a} \sum\limits_{j=1}^{m} (\rho_j + \overline{\rho}_j (b-a)^{\gamma_{1j}})$.

Let, now $m = 1$, then due to (2.37), (2.38), and (2.64) we obtain

$$\Big| \int\limits_{t_1}^{t} (t-s)^{m-1} \Lambda_k(u_k)(s) ds \Big| = \Big| \int\limits_{t_1}^{t} p_1(s) u_k(s) ds +$$

$$+ \int\limits_{t_1}^{t} p_1(s) \Big(\int\limits_{s}^{\mu_1(t_{01}, t_{1k}, s)} u_k'(\xi) d\xi \Big) ds \Big| \leq \frac{|u_k(t)|}{(t-a)} \Big| \int\limits_{t_1}^{t} (s-a) p_1(s) ds \Big| +$$

$$+ \Big| \int\limits_{t_1}^{t} \Big(\frac{|u_k'(s)|}{(s-a)(b-s)} + \frac{|u_k(s)|}{(s-a)^2 (b-s)} \Big) \Big((b-s) \int\limits_{t_1}^{s} (\xi - a) p_1(\xi) d\xi \Big) ds \Big| +$$

$$+ \frac{2r_0}{t_1 - a} \int\limits_{t_1}^{t} (s-a) |p_1(s)| \lambda_1(b, t_{01}, t_{1k}, s) ds \leq \frac{2\rho_1}{b-a} \Big[\frac{|u_k(t)|}{b-t} +$$

$$+ r_0 \Big(\int\limits_{t_1}^{t} \frac{1}{(b-s)^2} ds \Big)^{1/2} + \frac{2}{b-a} (t-t_1)^{1/2} \Big(\int\limits_{t_1}^{t} \frac{u_k^2(s)}{(b-s)^2} ds \Big)^{1/2} \Big] + \tag{2.67}$$

$$+ \frac{4 r_0 \overline{\rho}_1}{b-a} (b-t)^{\gamma_{11} - 1/2} \quad \text{for} \quad t_1 \leq t \leq t_{1k}.$$

On the other hand, from (2.24), (2.37), and Lemma 2.2 it follow the estimates

$$|u_k(t)| = \Big| \int\limits_{t}^{t_{1k}} u_k'(s) ds \Big| \leq \Big((t_{1k} - t) \int\limits_{t}^{t_{1k}} u_k'^2(s) ds \Big)^{1/2} \leq r_0 (b-t)^{1/2},$$

$$\int\limits_{t}^{t_{1k}} \frac{u_k^2(s)}{(b-s)^2} ds \leq \int\limits_{t}^{t_{1k}} \frac{u_k^2(s)}{(t_{1k} - s)^2} ds \leq 2r_0,$$

for $t_1 \le t \le t_{1k}$. Then from (2.67) by these inequalities we get

$$\begin{gathered}\Big|\int\limits_{t_1}^{t}(t-s)^{m-1}\Lambda_k(u_k)(s)ds\Big| \le \frac{2\rho_1}{b-a}\Big(\frac{2r_0}{(b-t)^{1/2}}+\frac{4r_0}{(b-a)^{1/2}}\Big)+\\ +\frac{4r_0\overline{\rho}_1}{(b-a)}(b-t)^{\gamma_{11}-1/2} \le \kappa_1((b-t)^{-1/2}+(b-t)^{\gamma_{11}-1/2})+\kappa_2,\end{gathered} \tag{2.68}$$

where $\kappa_1 = \frac{4r_0}{b-a}(\rho_1+\overline{\rho}_1)$, $\kappa_2 = \frac{8r_0}{(b-a)^{3/2}}\rho_1$.

If $m > 1$, due to conditions (2.60) and the fact that $n = 2m+1$, we have

$$\Big|\int\limits_{t_1}^{t}(t-s)^{m-1}q_k(s)ds\Big| = (m-1)\Big|\int\limits_{t_1}^{t}(t-s)^{2m-n-1}\Big((t-s)^{n-m-1}\int\limits_{t_1}^{s}|q_k(\xi)|d\xi\Big)ds\Big| \le$$

$$\le m(b-t)^{-1/2}||q_k||_{\widetilde{L}^2_{2n-2m-2,2m-2}} \le m\rho_0(b-t)^{-1/2} \quad \text{for} \quad t_1 \le t < b, \tag{2.69}$$

and if $m = 1$,

$$\int\limits_{t_1}^{t}\Big|\int\limits_{t_1}^{s}q_k(\xi)d\xi\Big|ds \le (b-t)^{1/2}||q_k||_{\widetilde{L}^2_{0,0}} \le \rho_0(b-t)^{1/2} \quad \text{for} \quad t_1 \le t < b. \tag{2.70}$$

Also it is clear that

$$u_k^{(m)}(t) = \int\limits_{t_{1k}}^{t}u_k^{(m+1)}(s)ds, \tag{2.71}$$

since $u_k^{(m)}(t_{1k}) = 0$.

Now, from (2.59), by (2.66) and (2.69) if $m > 1$, and by (2.68) if $m = 1$, we have, respectively,

$$\begin{gathered}|u^{(m+1)}(t)| \le \rho_0 + (m\rho_0+\kappa_0)(b-t)^{-1/2},\\ |u^{(m+1)}(t)| \le \rho_0+\kappa_2+\kappa_1[(b-t)^{-1/2}+(b-t)^{\gamma_{11}-1/2}]+\int\limits_{t_1}^{t}|q_k(s)|ds,\end{gathered} \tag{2.72}$$

for $t_1 \le t \le t_{1k}$. From (2.71), by (2.72), and (2.70), it follows the existence of a constant $\rho^* > 0$ such that

$$|u_k^{(m)}(t)| \le \rho^*[(b-t)^{1/2}+(b-t)^{\gamma_{11}+1/2}] \quad \text{for} \quad t_1 \le t < t_{1k},\ m \ge 1,$$

from which, in view of (2.25), (2.55), and (2.57), it is evident that $u^{(m)}(b) = 0$. Thus we have proved that u is the solution of problem (1.1), (1.2) also in the case $n = 2m+1$.

To complete the proof of the lemma, it remains to show that equality (2.41) is satisfied. First note that in the space $\widetilde{C}^{n-1,m}(]a,b[)$ problem (1.1), (1.2) does not have another solution since in that space the homogeneous problem (1.1_0), (1.2) has only the trivial solution. Now assume the contrary. Then there exist $\delta \in]0, \frac{b-a}{2}[$, $\varepsilon > 0$, and an increasing sequence of natural numbers $\{k_l\}_{l=1}^{+\infty}$ such that

$$\max\Big\{\sum_{j=1}^{n} |u_{k_l}^{(j-1)}(t) - u^{(j-1)}(t)| : a+\delta \le t \le b-\delta\Big\} > \varepsilon \quad (l \in N). \tag{2.73}$$

By virtue of the Arzela-Ascoli lemma and condition (2.37) the sequence $\{u_{k_l}^{(j-1)}\}_{l=1}^{+\infty}$ $(j = 1,\dots,m)$, without loss of generality, can be assumed to be uniformly converging in $]a,b[$. Then, in view of what we have shown above, conditions (2.55) and (2.57) hold. But this contradicts condition (2.73). The obtained contradiction proves the validity of the lemma. □

Analogously we can prove the following lemma if we apply Lemma 2.7 instead of Lemma 2.6.

Lemma 2.9. *Let condition* (2.28) *hold, for every natural k problem* (2.26), (2.27) *have a solution $u_k \in \widetilde{C}_{loc}^{n-1}(]a,b])$, and let there exist a constant $r_0 > 0$ such that*

$$\int_{t_{0k}}^{b} |u_k^{(m)}(s)|ds \le r_0^2 \quad (k \in N), \tag{2.74}$$

$$\lim_{k\to+\infty} ||q_k - q||_{\widetilde{L}^2_{2n-2m-2}} = 0, \tag{2.75}$$

and the homogeneous problem (1.1_0), (1.3) *has only the trivial solution in the space $\widetilde{C}^{n-1,m}(]a,b])$. Then the nonhomogeneous problem* (1.1), (1.3) *has a unique solution u such that inequality* (2.40) *holds, and*

$$\lim_{k\to+\infty} u_k^{(j-1)}(t) = u^{(j-1)}(t) \quad (j = 1,\dots,n) \quad \textit{uniformly in} \quad]a,b] \tag{2.76}$$

(that is, uniformly on $[a+\delta, b]$ for an arbitrarily small $\delta > 0$).

To prove Lemma 2.11 we need the following proposition, which is a particular case of Lemma 4.1 in [8].

Lemma 2.10. *If $u \in C_{loc}^{n-1}(]a,b[)$, then for any $s, t \in]a,b[$ the equality*

$$(-1)^{n-m}\int_{s}^{t} (\xi - a)^{n-2m} u^{(n)}(\xi)u(\xi)d\xi = w_n(t) - w_n(s) + \nu_n \int_{s}^{t} |u^{(m)}(\xi)|^2 d\xi \tag{2.77}$$

is valid, where $\nu_{2m} = 1$, $\nu_{2m+1} = \frac{2m+1}{2}$, $w_{2m}(t) = \sum_{j=1}^{m} (-1)^{m+j-1} u^{(2m-j)}(t)u(t)$,

$$w_{2m+1}(t) = \sum_{j=1}^{m} (-1)^{m+j}[(t-a)u^{(2m+1-j)}(t) - ju^{(2m-j)}(t)]u^{(j-1)}(t) - \frac{t-a}{2}|u^{(m)}(t)|^2.$$

Lemma 2.11. *Let $a_0 \in]a,b[$, $b_0 \in]a_0,b[$, the functions h_j and the operators f_j be given by equalities* (1.10) *and* (1.11). *Let, moreover, $\tau_j \in M(]a,b[)$, and the constants $l_{k,j} > 0$, $\gamma_{kj} > 0$ $(k = 0, 1;\ j = 1, \dots, m)$ be such that conditions* (1.12)-(1.14) *are fulfilled. Then there exist positive constants δ and r_1 such that if $a_0 \in]a, a+\delta[$, $b_0 \in]b-\delta, b[$, $t_0 \in]a, a_0[$, $t_1 \in]b_0, b[$, and $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,b[)$, an arbitrary solution $u \in C^{n-1}_{loc}(]a,b[)$ of the problem*

$$u^{(n)}(t) = \sum_{j=1}^{m} p_j(t) u^{(j-1)}(\mu_j(t_0,t_1,t)) + q(t), \tag{2.78}$$

$$u^{(i-1)}(t_0) = 0 \ \ (i = 1, \dots, m), \qquad u^{(j-1)}(t_1) = 0 \ \ (j = 1, \dots, n-m) \tag{2.79}$$

satisfies the inequality

$$\int_{t_0}^{t_1} |u^{(m)}(s)|^2 ds \le \tag{2.80}$$
$$\le r_1 \Big(\Big| \sum_{j=1}^{m} \int_{a_0}^{b_0} (s-a)^{n-2m} p_j(s) u(s) u^{(j-1)}(\mu_j(t_0,t_1,s)) ds \Big| + ||q||^2_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} \Big).$$

Proof. From conditions (1.12) and (1.13) it follows the existence of constants $\overline{\ell}_{kj} \ge 0$ such that

$$(t-a)^{m-\frac{1}{2}-\gamma_{0j}} f_j(a,\tau_j)(t,s) \le \overline{\ell}_{0j} \quad \text{for} \quad a < t \le s \le a_0,$$
$$(b-t)^{m-\frac{1}{2}-\gamma_{1j}} f_j(b,\tau_j)(t,s) \le \overline{\ell}_{1j} \quad \text{for} \quad b_0 \le s \le t < b.$$

Consequently, all the requirements of Lemma 2.3 with $\overline{p}_j(t) = (t-a)^{n-2m}(-1)^{n-m} p_j(t)$, $a < t_0 < a_0$, and Lemma 2.4 with $\overline{p}_j(t) = (b-t)^{n-2m}(-1)^{n-m} p_j(t)$, $b_0 < t_1 < b$, are fulfilled. Also from condition (1.14) and the definition of a constant ν_n, it follows the existence of $\nu \in]0,1[$ such that

$$\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} \ell_{kj} < \nu_n - 2\nu \ \ (k = 0, 1). \tag{2.81}$$

On the other hand, without loss of generality we can assume that $a_0 \in]a, a+\delta[$ and $b_0 \in]b-\delta, b[$, where δ is a constant such that

$$\sum_{j=1}^{m} (\overline{l}_{0j} \beta_j(\delta, \gamma_{0j}) + \overline{l}_{1j} \beta_j(\delta, \gamma_{1j})) < \nu, \tag{2.82}$$

where the functions β_j are defined by (2.6). Let now $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,b[)$, u be a solution of problem (2.78), (2.79), and

$$r_1 = 2^{2m+1}(1+b-a)^2 \nu^{-2}. \tag{2.83}$$

Multiplying both sides of (2.78) by $(-1)^{n-m}(t-a)^{n-2m}u(t)$ and then integrating from t_0 to t_1, by Lemma 2.10 we obtain

$$(n-2m)\frac{t_0-a}{2}|u^{(m)}(t_0)|^2+\nu_n\int\limits_{t_0}^{t_1}|u^{(m)}(s)|^2ds=$$

$$=(-1)^{n-m}\sum_{j=1}^{m}\int\limits_{t_0}^{t_1}(s-a)^{n-2m}p_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))ds+ \tag{2.84}$$

$$+(-1)^{n-m}\int\limits_{t_0}^{t_1}(s-a)^{n-2m}q(s)u(s)ds.$$

From Lemma 2.3 with $\overline{p}_j(t)=(t-a)^{n-2m}(-1)^{n-m}p_j(t)$, Lemma 2.4 with $\overline{p}_j(t)=(b-t)^{n-2m}(-1)^{n-m}p_j(t)$, and the equalities $\rho_0(t_0)=\rho_1(t_1)=0$, by (2.81) we get

$$(-1)^{n-m}\sum_{j=1}^{m}\int\limits_{t_0}^{a_0}(s-a)^{n-2m}p_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))ds\leq$$

$$\leq\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}l_{0j}\rho_0(a_0)+\sum_{j=1}^{m}\overline{l}_{0j}\beta_j(a-a_0,\gamma_{0j})\rho_0(\tau^*)\leq \tag{2.85}$$

$$\leq(\nu_n-2\nu)\rho_0(a_0)+\sum_{j=1}^{m}\overline{l}_{0j}\beta_j(\delta,\gamma_{0j})\int\limits_{t_0}^{t_1}|u^{(m)}(s)|^2ds,$$

$$(-1)^{n-m}\sum_{j=1}^{m}\int\limits_{b_0}^{t_1}(s-a)^{n-2m}p_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))ds\leq$$

$$\leq\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}l_{1j}\rho_1(b_0)+\sum_{j=1}^{m}\overline{l}_{1j}\beta_j(b_0-b,\gamma_{1j})\rho_1(\tau_*)\leq \tag{2.86}$$

$$\leq(\nu_n-2\nu)\rho_1(b_0)+\sum_{j=1}^{m}\overline{l}_{1j}\beta_j(\delta,\gamma_{1j})\int\limits_{t_0}^{t_1}|u^{(m)}(s)|^2ds.$$

If along with this we take into account inequalities (2.82) and $a_0\leq b_0$, we find

$$(-1)^{n-2m}\sum_{j=1}^{m}\int\limits_{t_0}^{t_1}(s-a)^{n-2m}p_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))ds\leq$$

$$
\begin{gathered}
\leq \Big|\sum_{j=1}^{m}\int\limits_{a_0}^{b_0}(s-a)^{n-2m}p_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))ds\Big|+\\
+(\nu_n-2\nu)\Big(\rho_0(a_0)+\rho_1(b_0)\Big)+\nu\int\limits_{t_0}^{t_1}|u^{(m)}(s)|^2ds\leq(\nu_n-\nu)\int\limits_{t_0}^{t_1}|u^{(m)}(s)|^2ds+ \qquad (2.87)\\
+\Big|\sum_{j=1}^{m}\int\limits_{a_0}^{b_0}(s-a)^{n-2m}p_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))ds\Big|.
\end{gathered}
$$

On the other hand, if we put $c=(a+b)/2$, then again on the basis of Lemmas 2.1, 2.2, and Young's inequality we get

$$
\begin{gathered}
\Big|\int\limits_{t_0}^{t_1}(s-a)^{n-2m}q(s)u(s)ds\Big|\leq\Big|\int\limits_{t_0}^{c}(s-a)^{n-2m}q(s)u(s)ds\Big|+\Big|\int\limits_{c}^{t_1}(s-a)^{n-2m}q(s)u(s)ds\Big|=\\
=\Big|\int\limits_{t_0}^{c}[(n-2m)u(s)+(s-a)^{n-2m}u'(s)]\Big(\int\limits_{s}^{c}q(\xi)d\xi\Big)ds\Big|+\\
+\Big|\int\limits_{c}^{t_1}[(n-2m)u(s)+(s-a)^{n-2m}u'(s)]\Big(\int\limits_{c}^{s}q(\xi)d\xi\Big)ds\Big|\leq\\
\leq\Big[(n-2m)\Big(\int\limits_{t_0}^{c}\frac{u^2(s)}{(s-a)^{2m}}ds\Big)^{1/2}+\Big(\int\limits_{t_0}^{c}\frac{u'^2(s)}{(s-a)^{2m-2}}ds\Big)^{1/2}\Big]\times\\
\times\Big(\int\limits_{t_0}^{c}(s-a)^{2n-2m-2}\Big(\int\limits_{s}^{c}q(\xi)d\xi\Big)^2ds\Big)^{1/2}+\\
+(1+b-a)\Big[(n-2m)\Big(\int\limits_{c}^{t_1}\frac{u^2(s)}{(b-s)^{2m}}ds\Big)^{1/2}+\Big(\int\limits_{c}^{t_1}\frac{u'^2(s)}{(b-s)^{2m-2}}ds\Big)^{1/2}\Big]\times\\
\times\Big(\int\limits_{c}^{t_1}(b-s)^{2m-2}\Big(\int\limits_{c}^{s}q(\xi)d\xi\Big)^2ds\Big)^{1/2}\leq 2^{m+1}(1+b-a)||q||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}}\times\\
\times\Big[\Big(\int\limits_{t_0}^{c}|u^{(m)}(s)|^2ds\Big)^{1/2}+\Big(\int\limits_{c}^{t_1}|u^{(m)}(s)|^2ds\Big)^{1/2}\Big]\leq\\
\leq\frac{\nu}{2}\int\limits_{t_0}^{t_1}|u^{(m)}(s)|^2ds+2^{2m+3}(1+b-a)^2\nu^{-1}||q||^2_{\widetilde{L}^2_{2n-2m-2,\,2m-2}}. \qquad (2.88)
\end{gathered}
$$

In view of inequalities (2.87), (2.88) and notation (2.83), equality (2.84) results in estimate (2.80). ☐

The proof of the following lemma is analogous to that of Lemma 2.11.

Lemma 2.12. *Let $a_0 \in]a,b[$, the functions h_j and the operators f_j be given by equalities* (1.10) *and* (1.11). *Let, moreover, $\tau_j \in M(]a,b])$, constants $l_{0,j} > 0$, $\gamma_{0j} > 0$, $(j = 1,\dots,m)$ be such that conditions* (1.12) *and* (1.21) *are fulfilled. Then there exists a positive constant r_1 such that for any $t_0 \in]a, a_0[$, and $q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$, an arbitrary solution $u \in C^{n-1}_{loc}(]a,b])$ of the problem*

$$u^{(n)}(t) = \sum_{j=1}^{m} p_j(t) u^{(j-1)}(\mu_j(t_0,b,t)) + q(t), \tag{2.89}$$

$$u^{(i-1)}(t_0) = 0 \ \ (i = 1,\dots,m), \qquad u^{(j-1)}(b) = 0 \ \ (j = m+1,\dots,n) \tag{2.90}$$

satisfies the inequality

$$\int_{t_0}^{b} |u^{(m)}(s)|^2 ds \le r_1 \Big(\Big| \sum_{j=1}^{m} \int_{a_0}^{b} (s-a)^{n-2m} p_j(s) u(s) u^{(j-1)}(\mu_j(t_0,b,s)) ds \Big| + ||q||^2_{\widetilde{L}^2_{2n-2m-2}} \Big).$$

Lemma 2.13. *Let $\tau_j \in M(]a,b[)$, $a_0 \in]a,b[$, $b_0 \in]a_0,b[$, conditions* (1.7), (1.12)- (1.14), *hold, and let in the case when n is odd, in addition* (1.8) *be fulfilled, where the functions h_j, β_j and the operators f_j are given by equalities* (1.10)-(1.11), *and l_{kj}, $\bar{l}_{kj}$, γ_{kj} $(k = 0,1;\ j = 1,\dots,m)$ are nonnegative numbers. Moreover, let the homogeneous problem* (1.1$_0$), (1.2) *in the space $\widetilde{C}^{n-1,m}(]a,b[)$ have only the trivial solution. Then there exist $\delta \in]0, \frac{b-a}{2}[$ and $r > 0$ such that for any $t_0 \in]a, a+\delta]$, $t_1 \in]b+\delta, b]$, and $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,b[)$ problem* (2.78), (2.79) *is uniquely solvable in the space $\widetilde{C}^{n-1}(]a,b[)$, and its solution admits the estimate*

$$\Big(\int_{t_0}^{t_1} |u^{(m)}(s)|^2 ds \Big)^{1/2} \le r ||q||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}}. \tag{2.91}$$

Proof. First note that all the requirements of Lemma 2.11 are fulfilled, and in view of (1.8) and (1.13), conditions (2.38) of Lemma 2.8 hold.

Let, now $\delta \in]0, \min\{b-b_0, a_0-a\}]$ be such as in Lemma 2.11 and assume that estimate (2.91) is invalid. Then for an arbitrary natural k there exist

$$t_{0k} \in]a,\, a+\delta/k[, \qquad t_{1k} \in]b-\delta/k,\, b[, \tag{2.92}$$

and a function $q_k \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,b[)$ such that problem (2.23), (2.24) has a solution $u_k \in \widetilde{C}^{n-1}(]a,b[)$, satisfying the inequality

$$\Big(\int_{t_{0k}}^{t_{1k}} |u_k^{(m)}(s)| ds \Big)^{1/2} > k ||q_k||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}}. \tag{2.93}$$

In the case when the homogeneous equation

$$u^{(n)}(t) = \sum_{j=1}^{m} p_j(t) u^{(j-1)}(\mu_j(t_{0k}, t_{1k}, t)) \tag{2.33$_0$}$$

under the boundary conditions (2.24) has a nontrivial solution, in (2.23) we put that $q_k(t) \equiv 0$ and assume that u_k is that nontrivial solution of problem (2.33_0), (2.24).

Let now

$$v_k(t) = \Big(\int_{t_{0k}}^{t_{1k}} |u_k^{(m)}(s)|ds \Big)^{-1/2} u_k(t), \quad q_{0k}(t) = \Big(\int_{t_{0k}}^{t_{1k}} |u_k^{(m)}(s)|ds \Big)^{-1/2} q_k(t). \tag{2.94}$$

Then v_k is a solution of the problem

$$\begin{gathered} v^{(n)}(t) = \sum_{i=1}^{m} p_i(t) v^{(i-1)}(\mu_i(t_{0k}, t_{1k}, t)) + q_{0k}(t) \quad \text{for} \quad t_{0k} \le t \le t_{1k}, \\ v^{(i-1)}(t_{0k}) = 0 \ (i = 1, \dots, m), \qquad v^{(i-1)}(t_{1k}) = 0 \ (i = 1, \dots, n-m). \end{gathered} \tag{2.95}$$

Moreover, in view of (2.93), it is clear that

$$\int_{t_{0k}}^{t_{1k}} |v_k^{(m)}(s)|^2 ds = 1, \quad ||q_{0k}||_{\widetilde{L}^2_{2n-2m-2,\, 2m-2}} < \frac{1}{k} \ \ (k \in N). \tag{2.96}$$

On the other hand, in view of the fact that problem (1.1_0), (1.2) has only the trivial solution in the space $\widetilde{C}^{n-1,m}(]a, b[)$, by Lemmas 2.8, 2.11, and (2.96) we have

$$\begin{gathered} \lim_{t \to +\infty} v_k^{(j-1)}(t) = 0 \quad \text{uniformly in} \quad]a, b[\ \ (j = 1, \dots n), \\ 1 < r_0 \Big(\Big| \int_{a_0}^{b_0} (s-a)^{n-2m} \Lambda_k(v_k)(s) ds \Big| + k^{-2} \Big) \ \ (k \in N), \end{gathered} \tag{2.97}$$

where r_0 is a positive constant independent of k. Now, if we pass to the limit in (2.97) as $k \to +\infty$, by Lemma 2.6 we obtain the contradiction $1 < 0$. Consequently, for any solution of problem (2.78), (2.79), with arbitrary $q \in \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a, b[)$, estimate (2.91) holds. Thus the homogeneous equation

$$v^{(n)}(t) = \sum_{j=1}^{m} p_j(t) v^{(j-1)}(\mu_j(t_0, t_1, t)) \quad \text{for} \quad t_0 \le t \le t_1, \tag{2.82_0}$$

under conditions (2.79), has only the trivial solution. But for arbitrarily fixed $t_0 \in]a, a+\delta[$, $t_1 \in]b-\delta, b[$, and $q \in L([t_0, t_1])$ problem (2.78), (2.79) is regular and has the Fredholm property in the space $\widetilde{C}^{n-1}(]t_0, t_1[)$. Thus problem (2.78), (2.79) is uniquely solvable. □

Analogously we can prove the following lemma if we apply Lemmas 2.7 and 2.12 instead of Lemmas 2.6 and 2.11.

Lemma 2.14. *Let $\tau_j \in M(]a, b[)$, $a_0 \in]a, b[$, conditions* (1.9), (1.12) *and* (1.21) *hold, where the functions h_j, β_j and the operators f_j are given by equalities* (1.10)-(1.11), *and l_{0j}, $\bar{l}_{0j}$ γ_{0j} $(j = 1, \dots, m)$ are nonnegative numbers. Let, moreover, the homogeneous problem* (1.1_0), (1.3) *in the space $\widetilde{C}^{n-1}(]a, b])$ have only the trivial solution. Then there exist* *positive constants δ and r such that if $a_0 \in]a, a+\delta[$, and $q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$, problem* (2.89), (2.90) *is uniquely solvable in the space $\widetilde{C}^{n-1}(]a,b])$, and its solution admits the estimate $\int\limits_{t_0}^{b} |u^{(m)}(s)|^2 ds \le r||q||_{\widetilde{L}^2_{2n-2m-2}}$.*

Lemma 2.15. *Let $\tau_j \in M(]a,b[)$, $\alpha \ge 0$, $\beta \ge 0$, and let there exist $\delta \in]0, b-a[$ such that*

$$|\tau_j(t) - t| \le k_1(t-a)^\beta \quad \textit{for} \quad a < t \le a+\delta. \tag{2.98}$$

Then

$$\Big|\int\limits_t^{\tau(t)} (s-a)^\alpha ds\Big| \le \begin{cases} k_1[1+k_1\delta^{\beta-1}]^\alpha (t-a)^{\alpha+\beta} & \textit{for} \ \ \beta \ge 1 \\ k_1[\delta^{1-\beta}+k_1]^\alpha (t-a)^{\alpha\beta+\beta} & \textit{for} \ \ 0 \le \beta < 1 \end{cases},$$

for $a < t \le a+\delta$.

Proof. First note that

$$\Big|\int\limits_t^{\tau(t)} (s-a)^\alpha ds\Big| \le (\max\{\tau(t), t\} - a)^\alpha |\tau(t) - t| \quad \text{for} \quad a \le t \le a+\delta,$$

and $\max\{\tau(t), t\} \le t + |\tau(t) - t|$ for $a \le t \le a+\delta$. Then in view of condition (2.98) we get

$$\Big|\int\limits_t^{\tau(t)} (s-a)^\alpha ds\Big| \le k_1[(t-a) + k_1(t-a)^\beta]^\alpha (t-a)^\beta \quad \text{for} \quad a \le t \le a+\delta.$$

From this inequality it immediately follows the validity of the lemma. □

Analogously, one can prove

Lemma 2.16. *Let $\tau_j \in M(]a,b[)$, $\alpha \ge 0$, $\beta \ge 0$ and let there exist $\delta \in]0, b-a[$ such that*

$$|\tau_j(t) - t| \le k_1(b-t)^\beta \quad \textit{for} \quad b-\delta \le t < b. \tag{2.99}$$

Then

$$\Big|\int\limits_t^{\tau(t)} (b-t)^\alpha ds\Big| \le \begin{cases} k_1[1+k_1\delta^{\beta-1}]^\alpha (b-t)^{\alpha+\beta} & \textit{for} \ \ \beta \ge 1 \\ k_1[\delta^{1-\beta}+k_1]^\alpha (b-t)^{\alpha\beta+\beta} & \textit{for} \ \ 0 \le \beta < 1 \end{cases},$$

for $b-\delta \le t < b$.

3 Proofs

Proof of Theorem 1.1 (Theorem 1.2). Suppose problem (1.1_0), (1.2) (problem (1.1_0), (1.3)) has only the trivial solution, and r and δ are the numbers appearing in Lemma 2.13 (Lemma 2.14). Set

$$t_{0k} = a + \delta/k \qquad t_{1k} = b - \delta/k \quad (k \in N). \tag{3.1}$$

By Lemma 2.13 (Lemma 2.14), for every natural k, problem (2.78), (2.79) in the space $\widetilde{C}^{n-1}_{loc}(]a,\,b[)$ (problem (2.89), (2.90) in the space $\widetilde{C}^{n-1}_{loc}(]a,\,b]))$ has a unique solution u_k, and

$$\Big(\int\limits_{t_{0k}}^{t_{1k}} |u_k^{(m)}(s)|^2 ds\Big)^{1/2} \le r||q||_{\widetilde{L}^2_{2n-2m-2,2m-2}} \quad \Big(\Big(\int\limits_{t_{0k}}^{b} |u_k^{(m)}(s)|^2 ds\Big)^{1/2} \le r||q||_{\widetilde{L}^2_{2n-2m-2}}\Big), \tag{3.2}$$

where the constant r does not depend on q. From (3.2), by Lemma 2.8 with $r_0 = r||q||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}}$ (by Lemma 2.9 with $r_0 = r||q||_{\widetilde{L}^2_{2n-2m-2}}$), it follows that problem (1.1), (1.2) (problem (1.1), (1.3)) in the space $\widetilde{C}^{n-1}_{loc}(]a,\,b[)$ $(\widetilde{C}^{n-1}_{loc}(]a,\,b]))$ is uniquely solvable for an arbitrary $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,\,b[)$ $(q \in \widetilde{L}^2_{2n-2m-2}(]a,\,b]))$. Thus that problem has Fredholm's property, and its solution admits estimate (1.15) (estimate (1.22)). □

Proof of Corollary 1.1. In view of conditions (1.18), there exists a number $\varepsilon > 0$ such that

$$\sum_{j=1}^{m} \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\Big(\frac{\kappa_{kj}}{2m-j}+\varepsilon\Big) < 1 \ (k=0,1). \tag{3.3}$$

On the other hand, in view of conditions (1.19) and (1.20) we have

$$\begin{gathered}(t-a)^{2m-j}h_j(t,s) \le \frac{\kappa_{0j}}{2m-j} + \kappa_{1j}\int\limits_a^{a_0} \frac{(\xi-a)^{2m-j}}{(b-\xi)^{2m+1-j}}d\xi + \int\limits_a^{a_0}(\xi-a)^{n-j}p_{0j}(\xi)d\xi \\ \text{for} \quad a < t \le s \le a_0, \\ (b-t)^{2m-j}h_j(t,s) \le \frac{\kappa_{1j}}{2m-j} + \kappa_{0j}\int\limits_{b_0}^{b} \frac{(b-\xi)^{2m-j}}{(\xi-a)^{2m-j+1}}d\xi + \\ +(b-a)^{n-2m}\int\limits_{b_0}^{b}(b-\xi)^{2m-j}p_{0j}(\xi)d\xi \quad \text{for} \quad b_0 \le s \le t < b.\end{gathered} \tag{3.4}$$

Let δ be the constant defined in Lemmas 2.15, 2.16. From (1.19) it follows the existence of $a_0 \in]a, a+\delta[$ and $b_0 \in]b-\delta, b[$ such that

$$|p_1(t)| \le \frac{\kappa}{[(t-a)(b-t)]^{2n}} + p_{01}(t) \quad \text{for} \quad t \in [a, a_0] \cup [b_0, b]. \tag{3.5}$$

On the other hand, from lemmas 2.15, and 2.16 by the condition (1.17) it follows the existence of a constant k_0 such that

$$\begin{gathered}\Big|\int\limits_t^{\tau_j(t)} (s-a)^{2(m-j)}ds\Big|^{1/2} \le k_0^{1/2}(s-a)^{m-j+\nu_{0j}/2} \quad \text{for} \quad a \le t \le a_0, \\ \Big|\int\limits_t^{\tau_j(t)} (b-s)^{2(m-j)}ds\Big|^{1/2} \le k_0^{1/2}(b-s)^{m-j+\nu_{1j}/2} \quad \text{for} \quad b_0 \le t \le b.\end{gathered} \tag{3.6}$$

Consequently, if $p_{01} \in L_{n-j,\,2m-j}(]a,b[)$, then by (1.16) and (3.6), from (1.19) and (1.20) it follows the existence of a nonnegative constant k_2 such that

$$\begin{aligned} &(t-a)^{m-1} f_j(a,\tau_1)(t,s) \le k_2(a_0-a)^{\varepsilon_0} \quad \text{for} \quad a \le t < s \le a_0, \\ &(b-t)^{m-1} f_j(b,\tau_1)(t,s) \le k_2(b-b_0)^{\varepsilon_0} \quad \text{for} \quad b_0 \le s < t \le b, \end{aligned} \tag{3.7}$$

where $0 < \varepsilon_0 = \min\{\nu_{k1}-2n-2+2k(2m-n),\, \nu_{kj}-2 : k=0,1;\ j=2,\dots,m\}$. Now, from (3.4), and (3.7) it is clear that we can choose $\delta_1 \le \delta$ so that if $\max\{b-b_0,\, a_0-a\} \le \delta_1$, then

$$\begin{aligned} (t-a)^{2m-j} h_j(t,s) &\le \frac{\kappa_{0j}}{2m-j} + \varepsilon \quad \text{for} \quad a < t \le s \le a_0, \\ (b-t)^{2m-j} h_j(t,s) &\le \frac{\kappa_{1j}}{2m-j} + \varepsilon \quad \text{for} \quad b_0 \le s \le t < b, \end{aligned}$$

$j \in \{1,\dots,m\}$. From (3.7), the last inequalities and (3.3), it is clear that all the assumptions of Theorem 1.1, with $\ell_{kj} = \frac{\kappa_{kj}}{2m-j} + \varepsilon$, $\gamma_{kj} = 1/2$, and $\max\{b-b_0,\, a_0-a\} \le \delta_1$, are fulfilled, and thus the corollary is valid. □

Proof of Theorem 1.3. It suffice to show that if $u \in \widetilde{C}^{n-1}_{loc}(]a,\,b[)$ $\quad (u \in \widetilde{C}^{n-1}_{loc}(]a,\,b]))$ is a solution of problem (1.1_0), (1.2) $\quad ((1.1_0)$, (1.3)), then

$$\int_a^b |u^{(m)}(s)|^2 ds < +\infty. \tag{3.8}$$

For an arbitrary $t_0 \in]a,\,b[$ we have

$$\begin{aligned} u^{(m)}(t) = w(t_0) &+ \frac{1}{(n-m-1)!} \int_{t_0}^{t} (t-s)^{n-m-1} \Big(\sum_{j=1}^{m} p_j(s) u^{(j-1)}(s) \Big) ds, + \\ &+ \frac{1}{(n-m-1)!} \int_{t_0}^{t} (t-s)^{n-m-1} \Big(\sum_{j=1}^{m} p_j(s) \int_s^{\tau_j(s)} u^{(j)}(\xi) d\xi \Big) ds, \end{aligned} \tag{3.9}$$

where $w(t_0) = \sum\limits_{j=m+1}^{n} \frac{(t_0-a)^{j-m-1}}{(j-m-1)!} u^{(j-1)}(t_0)$. Now note that by the equalities

$$|u^{(i)}(t)| = \frac{1}{(k-i-1)!} \Big| \int_c^t (t-s)^{k-i-1} u^{(k)}(s) ds \Big| \quad \text{for} \quad a < t < b, \tag{3.10}$$

$k = 1,\dots,m,\ i = 0,\dots,k-1,$ with $c = a$, from (3.9) we get the estimate

$$\begin{aligned} |u^{(m)}(t)| \le |w(t_0)| + (1-\delta_{1m}) ||u^{(m-1)}||_C \sum_{j=1}^{m-1} \Big(&\int_t^{t_0} (s-a)^{n-j-1} |p_j(s)| ds + \\ &+ \int_t^{t_0} (s-a)^{n-m-1} |p_j(s)| \Big| \int_s^{\tau_j(s)} (\xi-a)^{m-j-1} d\xi \Big| ds \Big) + \\ &+ ||u^{(m-1)}||_C \int_t^{t_0} (s-a)^{n-m-1} |p_m(s)| ds \quad \text{for} \quad a < t < t_0, \end{aligned} \tag{3.11}$$

where δ_{ij} is Kronecker's delta. Then conditions (1.28) yield

$$|u^{(m)}(t)| \leq |w(t_0)| + (1-\delta_{1m})||u^{(m-1)}||_C \int_t^{t_0} (s-a)^{-1} p(s)ds +$$

$$+\gamma ||u^{(m-1)}||_C \int_t^{t_0} p(s)ds + ||u^{(m-1)}||_C \int_t^{t_0} (s-a)^{n-m-1}|p_m(s)|ds \quad \text{for} \quad a < t < t_0,$$

where $p(t) = \sum\limits_{j=1}^{m} (t-a)^{n-j}|p_j(t)|$,

$$\gamma_j = \operatorname*{ess\,sup}_{a<t<b} \frac{1}{|t-a|^{m+1-j}} \Big| \int_t^{\tau_j(t)} (\xi - a)^{m-j-1} d\xi \Big|, \quad \gamma = \max\{\gamma_1, \dots, \gamma_m\}.$$

Consequently, in view of condition (1.29), $u^{(m)} \in L([a, t_0])$. Analogously, by (3.10) with $c = b$, we can show that $u^{(m)} \in L([t_0, b])$. Finally $u^{(m)} \in L([a, b])$ and if we put $v(t) = \int\limits_a^t |u^{(m)}(s)|ds$, then

$$v \in C([a, b]), \tag{3.12}$$

and from (3.10) it is clear that

$$|u^{(i)}(t)| \leq (t-a)^{m-i-1} v(t) \ (i = 1, \dots, m-1) \quad \text{for} \quad a < t < t_0. \tag{3.13}$$

In view of condition (1.29) we can choose $\delta > 0$ such that

$$\int_a^{a+\delta} p(s)ds < \frac{1}{2}. \tag{3.14}$$

From (3.9), by conditions (1.28), (3.12) and inequality (3.13), we get

$$|u^{(m)}(t)| \leq |w(t_0)| + \int_t^{t_0} \frac{p(s)v(s)}{s-a} ds + \sum_{j=1}^{m} \int_t^{t_0} (s-a)^{n-m-1}|p_j(s)| \Big| \int_s^{\tau_j(s)} (\xi - a)^{m-j-1} v(\xi) d\xi \Big| ds \leq$$

$$\leq |w(t_0)| + \int_t^{t_0} \frac{p(s)v(s)}{s-a} ds + \gamma ||v||_C \int_a^{a_0} p(s)ds, \quad \text{for} \quad a < t < a + \delta.$$

Consequently, if $w_0 = |w(t_0)| + \gamma ||v||_C \int\limits_a^{a_0} p(s)ds$, then

$$|u^{(m)}(t)| \leq w_0 + \int_t^{t_0} \frac{p(s)v(s)}{s-a} ds \quad \text{for} \quad a < t < a + \delta. \tag{3.15}$$

From the last inequality, by the integration by parts and (3.14), we get

$$v(t) \le w_0(t-a) + (t-a)\int\limits_t^{t_0} \frac{p(s)v(s)}{s-a}ds + \frac{1}{2}v(t) \quad \text{for} \quad a<t<a+\delta.$$

The last inequality, by the Gronwall-Bellman lemma, results in

$$\frac{v(t)}{t-a} \le 2w_0 e^{2\int_t^{t_0} p(s)ds} \le 2w_0 e \quad \text{for} \quad a<t<a+\delta.$$

Due to this inequality, from (3.15) by (3.14) we get $|u^{(m)}(t)| \le w_0(1+e)$ for $a<t<a+\delta$. Analogously we can show that $u^{(m)}$ is bounded in the neighborhood of the point b. Therefore, condition (3.8) is satisfied. □

Proof of Theorem 1.4. From Theorem 1.1 by conditions (1.30)-(1.33) it is obvious that problem (1.1), (1.2) has Fredholm's property. Thus to prove Theorem 1.4, it suffice to show that the homogeneous problem (1.1_0), (1.2) has only the trivial solution in the space $\widetilde{C}^{n-1,m}(]a,b[)$. Suppose $u \in \widetilde{C}^{n-1,m}(]a,b[)$ is a solution of problem (1.1_0), (1.2). Then from Theorem 1.1 it is clear that

$$\rho = \int\limits_a^b |u^{(m)}(s)|^2 ds < +\infty. \tag{3.16}$$

Multiplying both sides of (1.1_0) by $(-1)^{n-m}(t-a)^{n-2m}u(t)$ and integrating from t_0 to t_1, by Lemma 2.10 we obtain

$$w_n(t) - w_n(s) + \nu_n \int\limits_s^t |u^{(m)}(\xi)|^2 d\xi = (-1)^{n-m}\sum_{j=1}^m \int\limits_s^t (\xi-a)^{n-2m}p_j(\xi)u^{(j-1)}(\tau_j(\xi))u(\xi)d\xi.$$

Moreover, from Lemma 2.5 it is evident that

$$\liminf_{s\to a}|w_n(s)| = 0, \quad \liminf_{t\to b}|w_n(t)| = 0.$$

Then by (3.16) we get

$$\nu_n\rho = (-1)^{n-m}\sum_{j=1}^m \int\limits_a^b (\xi-a)^{n-2m}p_j(\xi)u^{(j-1)}(\tau_j(\xi))u(\xi)d\xi. \tag{3.17}$$

According to (1.32), (1.33) and (3.16), all the conditions of Lemmas 2.3 and 2.4 with $\overline{p}_j(t) = (-1)^{n-m}(t-a)^{n-2m}p_j(t)$, $a_0 = b_0 = t^*$, $t_0 = a$, $t_1 = b$ and $\mu_j(t_0,t_1,t) = \tau_j(t)$ hold. Consequently, due to equalities $\rho_0(a) = \rho_1(b) = 0$, we have

$$\begin{aligned}(-1)^{n-m}\int\limits_a^b (\xi-a)^{n-2m}p_j(\xi)u^{(j-1)}(\tau_j(\xi))u(\xi)d\xi \le \\ \le \bar{l}_{0j}\beta_j(t^*-a,\gamma_{0j})\rho_0^{1/2}(\tau^*)\rho_0^{1/2}(t^*) + l_{0j}\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\rho_0(t^*) + \\ +\bar{l}_{1j}\beta_j(b-t^*,\gamma_{1j})\rho_1^{1/2}(\tau_*)\rho_1^{1/2}(t^*) + l_{1j}\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\rho_1(t^*)\end{aligned} \tag{3.18}$$

for $a < t^* < b$. On the other hand, due to conditions (1.30) and (1.31), the number $\nu \in]0,1[$ can be chosen such that inequalities

$$\begin{aligned}
&\sum_{j=1}^{m}\Big(l_{0j}\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}+\bar{l}_{0j}\beta_j(t^*-a,\gamma_{0j})\Big)<\frac{\nu_n-\nu}{2},\\
&\sum_{j=1}^{m}\Big(l_{1j}\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}+\bar{l}_{1j}\beta_j(b-t^*,\gamma_{1j})\Big)<\frac{\nu_n-\nu}{2}
\end{aligned} \tag{3.19}$$

are satisfied. Thus according to (3.18), (3.19), and inequalities $\rho_0^{1/2}(\tau^*)\rho_0^{1/2}(t^*) \leq \rho$, $\rho_1^{1/2}(\tau_*)\rho_1^{1/2}(t^*) \leq \rho$, (3.17) implies the inequality $\nu_n\rho \leq (\nu_n-\nu)\rho$, and consequently, $\rho = 0$. Hence, by

$$|u(t)| = \frac{1}{(k-1)!}\Big|\int_a^t (t-s)^{m-1}u^{(m)}(s)ds\Big| \leq (t-a)^{m-1/2}\rho \quad \text{for} \quad a<t<b,$$

we have $u(t) \equiv 0$. □

Proof of Theorem 1.5. The proof is analogous to that of Theorem 1.4. The only difference is that instead of Theorem 1.1, Theorem 1.2 is applied. □

Proof of Theorem 1.6. Let u be a nonzero solution of the problem (1.1_0), (1.2). Then analogously to Theorem 1.4, from conditions (1.40),(1.41), (1.32) and (1.33) it follow the validity of relations (3.16), (3.17), (3.18) and the existence of $\nu \in]0,1[$ such that

$$\begin{aligned}
&\sum_{j=1}^{m}\Big(l_{0j}\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}+\bar{l}_{0j}\beta_j(t^*-a,\gamma_{0j})\Big)<\nu_n-\nu,\\
&\sum_{j=1}^{m}\Big(l_{1j}\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}+\bar{l}_{1j}\beta_j(b-t^*,\gamma_{1j})\Big)<\nu_n-\nu.
\end{aligned} \tag{3.20}$$

For the constants τ^* and τ_*, appearing in inequality (3.18), which are defined in Lemmas 2.3 and 2.4 (with $t_0 = a$, $t_1 = b$, $a_0 = b_0 = t^*$, and $\mu_j(t_0,t_1,t) = \tau_j(t)$), from the condition (1.42) we have the estimates

$$\tau^* \leq t^* \quad \text{for} \quad a<t\leq t^*, \qquad t^* \leq \tau_* \quad \text{for} \quad t^* \leq t < b.$$

By the last estimates, from (3.18) it immediately follows the inequality $\nu_n\rho \leq (\nu_n-\nu)\rho$. Thus $u \equiv 0$.

□

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