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Multi-bump solutions of a Schrödinger–Bopp–Podolsky system with steep potential well

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Abstract. In this paper, we study the existence of multi-bump solutions for the following Schrödinger–Bopp–Podolsky system with steep potential well:

$$\begin{cases} -\Delta u + (\lambda V(x) + V_0(x))u + K(x)\phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi + a^2 \Delta^2 \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $p \in (4,6), a > 0$ and λ is a parameter. We require that $V(x) \geq 0$ and has a bounded potential well $\Omega = V^{-1}(0)$. Combining this with other suitable assumptions on Ω, V_0 and K, when λ is large enough, we obtain the existence of multi-bump-type solutions u_λ by using variational methods.

Keywords: Schrödinger–Bopp–Podolsky system, penalization method, variational methods.

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1 Introduction and main results

In this paper, we investigate the existence of multi-bump solutions for the following problem with steep potential well:

$$\begin{cases} -\Delta u + (\lambda V(x) + V_0(x))u + K(x)\phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi + a^2 \Delta^2 \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $p \in (4,6), a > 0$ and λ is a parameter.

To illustrate the significance of this article, we first introduce some background about Schrödinger–Bopp–Podolsky system. As mentioned in [10], problem (1.1) is a version of

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the Schrödinger–Bopp–Podolsky system, which is a Schrödinger equation coupled with a Bopp–Podolsky equation. It is worth mentioning that, Podolsky's theory is a second-order gauge theory for the electromagnetic field developed by Bopp [7], independently by Podolsky–Schwed [14]. For some more details about the Bopp–Podolsky equation, we refer to [5, 6, 15] and the references therein.

If $a = V_0(x) = 0, \lambda = K(x) = 1$, system (1.1) gives back the classical Schrödinger–Poisson system as follows:

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

which has been first introduced by D'Aprile–Mugnai [9]. The authors studied the existence of radially symmetric solitary waves by using the variational approach method for the above question when $V(x)$ is a constant. In this system, the potential function V is regarded as an external potential, u and ϕ represent the wave functions associated with the particle and electric potential respectively. For more details on the physical aspects of this system, we refer the readers to [4, 8] and the references therein.

In the last decades, the classical Schrödinger–Poisson system has been widely studied under variant assumptions on V and f. By using variational methods, the existence, nonexistence, and multiplicity results are obtained in many papers. For example, when $f(u) = |u|^{p-1}u$ with $p \in (3,5)$, Cerami and Vaira in [8] studied the following Schrödinger–Poisson system:

$$\begin{cases} -\Delta u + u + K(x)\phi(x)u = a(x)f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3. \end{cases}$$

Without requiring any symmetry property on $K(x)$ and $a(x)$, they proved the existence of the positive ground state and bound state solutions by minimizing energy functional restricted to a Nehari manifold when $K(x)$ and $a(x)$ satisfy different assumptions. After that, Sun et al. in [18] extended the result to a general nonlinear term.

Note that, the steep potential well has been introduced by Bartsch and Wang [3] in the study of nonlinear Schrödinger equation. Our assumptions on V are similar to [11], in which Ding and Tanaka have proven the existence of multi-bump-type solutions for nonlinear Schrödinger equations. After that, more and more researchers have studied multi-bump-type solutions, we refer the readers to the papers [1, 12, 19]. In particular, Zhang and Ma in [21] considered the following system with steep potential well

$$\begin{cases} -\Delta u + (\lambda a(x) + a_0(x))u + K(x)\phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

they obtained the existence of multi-bump solutions for (1.2) by using variational methods. Compared with [21], although our paper also studies the existence of multi-bump solutions, it studies a new system which has great significance.

If $a \neq 0$, system (1.1) is a Schrödinger–Bopp–Podolsky system. Based on variational methods, D'Avenia–Siciliano [10] first proved the existence and nonexistence results which depended on the parameters p and q to system

$$\begin{cases} -\Delta u + \omega u + q^2 \phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi + \varepsilon^2 \Delta^2 \phi = 4\pi u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.3}$$

Later, for $p \in (2, 3]$, Siciliano–Silva [17] obtained the existence and nonexistence of solutions to system (1.3) by means of the fiber map approach and the Implicit Function Theorem. Note that, the authors in [10] and [17] merely considered system (1.3) with subcritical growth, so Liu and Chen in [13] filled the gaps. More precisely, they studied the existence, nonexistence, and asymptotic behavior of ground state solutions to system (1.3) which involves a critical nonlinearity.

Recently, Wang et al. in [20] considered Schrödinger–Bopp–Podolsky system with general nonlinear term:

$$\begin{cases} -\Delta u + \omega u + q^2\phi u = f(u), & x \in \mathbb{R}^3, \\ -\Delta\phi + \varepsilon^2\Delta^2\phi = 4\pi u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.4}$$

where f is a continuous, superlinear, and subcritical nonlinearity. They proved the existence and multiplicity of sign-changing solutions of system (1.4) by using the method of invariant sets of descending flow incorporated with minimax arguments. In addition, the asymptotic behavior of sign-changing solutions was also established.

Motivated by all results mentioned above, it is quite natural to ask, does the system (1.1) have multi-bump solutions? In the present paper, we give an affirmative answer.

In this paper, we make the following assumptions:

(V_1) $V(x) \in C\left(\mathbb{R}^3, \mathbb{R}^+\right)$ and $\Omega := \operatorname{int} V^{-1}(0)$ is a non-empty bounded set with smooth boundary. Moreover, there is a positive constant M_0 such that the measure of the set $A = \left\{x \in \mathbb{R}^3 : V(x) \le M_0\right\}$ is finite.

(V_2) There is a $V_0(x) \in C\left(\mathbb{R}^3, \mathbb{R}\right)$ and a constant $M_1 > 1$ such that $|V_0(x)| \le M_1(V(x) + 1)$.

(V_3) Ω possesses m connected components $\Omega_1, \ldots, \Omega_m$ such that $\overline{\Omega_j} \cap \overline{\Omega \backslash \Omega_j} = \emptyset$, and $\inf_{u \in H_0^1(\Omega_j), |u|_2 = 1} \int_\Omega \left[|\nabla u|^2 + V_0(x)u^2\right] \mathrm{d}x > 0$ for $j = 1, 2, \ldots, m$.

Now, we say something about (V_1): although A and M_0 in (V_1) are not explicitly mentioned in the article, they are used in the proof of Proposition 2.4. Note that the proof of Proposition 2.4 is very similar to Corollary 1.4 in [11], so it is omitted. In [11], Corollary 1.4 is proven by using Proposition 1.1, but the proof of Proposition 1.1 requires the use of A and M_0 to ensure the vanishing of the energy outside the sphere. Please see [11] for details. Therefore, the role of (V_1) is to ensure that Proposition 2.4 holds in our manuscript.

We also assume that

(K) $K \in L^\infty\left(\mathbb{R}^3\right), K(x) \ge 0$ and $K \not\equiv 0$.

The main result of this paper reads as follows:

Theorem 1.1. *Assume that $(V_1), (V_2), (V_3)$ and (K) hold. Then, for any small $\nu > 0$ and any non-empty subset J of $\{1, 2, \ldots, m\}$, there exist $\Lambda = \Lambda(\nu)$ and $k(\nu) > 0$ such that, when $\lambda > \Lambda$ and $|K|_\infty \le k(\nu)$, (1.1) has a solution $u_\lambda \in H^1\left(\mathbb{R}^3\right)$ satisfying*

$$\left| \int_{\Omega_j} \left[|\nabla u_\lambda|^2 + (\lambda V(x) + V_0(x))\, u_\lambda^2\right] \mathrm{d}x - \left(\frac{1}{2} - \frac{1}{p}\right)^{-1} c\left(\Omega_j\right) \right| \le \nu, \qquad j \in J$$

and

$$\int_{\mathbb{R}^3 \backslash \Omega_J} \left[|\nabla u_\lambda|^2 + (\lambda V(x) + V_0(x))\, u_\lambda^2\right] \mathrm{d}x \le \nu,$$

where $\Omega_J = \bigcup_{j \in J} \Omega_j, c\left(\Omega_j\right)$ are some constants. Moreover, for any sequence of solutions $\{u_{\lambda_n}\}$ with $\lambda_n \to \infty$, going if necessary to a subsequence, u_{λ_n} converges strongly in $H^1\left(\mathbb{R}^3\right)$ to a function u satisfying $u(x) = 0$ for $x \in \mathbb{R}^3 \backslash \Omega_J$.

Remark 1.2. The constant $c\left(\Omega_j\right)$ in Theorem 1.1 is the least energy of all the nontrivial solutions for the following boundary value problem

$$-\Delta u + V_0(x)u = |u|^{p-2}u \quad \text{in } \Omega_j,\, u|_{\partial\Omega_j} = 0.$$

Hence under the assumption of $(V_3)\,, c\left(\Omega_j\right) > 0$.

This paper is organized as follows. In Section 2, we give some variational frameworks. After that, we introduce a modified functional and verify the Palais–Smale condition. In Sections 4 and 5, we give some results on the Nehari manifold and the proof of Theorem 1.1 respectively.

2 Variational frameworks

We consider the following functional space

$$E := \left\{u \in H^1\left(\mathbb{R}^3\right) : \int_{\mathbb{R}^3} V(x)u^2 \mathrm{d}x < \infty\right\}$$

with the inner product

$$(u,v)_E := \int_{\mathbb{R}^3} \left[\nabla u \nabla v + (V(x)+1)uv\right] \mathrm{d}x,$$

and the corresponding norm is $\|u\|_E = (u,u)_E^{1/2}$. It is easy to see that $(E, \|\cdot\|_E)$ is a Hilbert space and the embedding $E \hookrightarrow H^1\left(\mathbb{R}^3\right)$ is continuous. For any open set $D \subset \mathbb{R}^3$, we also define

$$E(D) = \left\{u \in H^1(D) : \int_D V(x)u^2 \mathrm{d}x < \infty\right\},$$

$$\|u\|_{E(D)} = \int_D \left[|\nabla u|^2 + (V(x)+1)u^2\right] \mathrm{d}x.$$

Note that $\|\cdot\|_{E(D)}$ is equivalent to $\|\cdot\|_{H^1(D)}$ when D is bounded.

Now, we define $\mathcal{D}$ be the completion of $C_0^\infty\left(\mathbb{R}^3\right)$ with respect to the norm $\|\cdot\|_{\mathcal{D}}$ induced by the scalar product

$$(u,v)_{\mathcal{D}} = \int_{\mathbb{R}^3} \left(\nabla u \nabla v + a^2 \Delta u \Delta v\right) \mathrm{d}x.$$

Then $\mathcal{D}$ is a Hilbert space, which is continuously embedded into $D^{1,2}\left(\mathbb{R}^3\right)$ and consequently into $L^6(\mathbb{R}^3)$. We denote that $L^q(\mathbb{R}^3)$ is the usual Lebesgue space with the standard norm $\|u\|_q := \left(\int_{\mathbb{R}^3} |u|^q \mathrm{d}x\right)^{\frac{1}{q}}$, $1 \leq q < \infty$.

Proposition 2.1 (see [10]). *The space $\mathcal{D}$ is continuously embedded into $L^\infty(\mathbb{R}^3)$.*

By using the Lax–Milgram theorem, for every fixed $u \in E$, there exists a unique solution $\phi_u^a \in \mathcal{D}$ of the second equation in system (1.1). In order to explicitly write such solution (see [15]), we consider that

$$\mathcal{K}(x) = \frac{1 - e^{\frac{-|x|}{a}}}{|x|}.$$

As for $\mathcal{K}$, we have the following fundamental properties from [10].

Proposition 2.2 (see [10]). *For all $y \in \mathbb{R}^3, \mathcal{K}(\cdot - y)$ solves in the sense of distributions*

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi\delta_y.$$

Moreover,

(i) *if $f \in L^1_{loc}\left(\mathbb{R}^3\right)$ and for a.e. $x \in \mathbb{R}^3$, the map $y \in \mathbb{R}^3 \to \frac{f(y)}{|x-y|}$ is summated, then $\mathcal{K} * f \in L^1_{loc}\left(\mathbb{R}^3\right)$;*

(ii) *if $f \in L^p\left(\mathbb{R}^3\right)$ with $1 \leqslant p < \frac{3}{2}$, then $\mathcal{K} * f \in L^q\left(\mathbb{R}^3\right)$ for $q \in \left(\frac{3p}{3-2p}, +\infty\right]$.*

In both cases $\mathcal{K} * f$ solves

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi f.$$

Then if we fix $u \in E$, the unique solution in $\mathcal{D}$ of the second equation in system (1.1) can be expressed by

$$\phi_u^a = \mathcal{K} * (Ku^2) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1 - e^{\frac{-|x-y|}{a}}}{|x-y|} K(y)u^2(y)\mathrm{d}y.$$

Now, let us summarize some properties of ϕ_u^a.

Proposition 2.3 (see [10]). *For every $u, v \in E$, the following statements are correct.*

(i) $\phi_u^a \geqslant 0$.

(ii) *For each $t > 0, \phi_{tu}^a = t^2\phi_u^a$.*

(iii) *If $u_n \rightharpoonup u$ in E, then $\phi_{u_n}^a \rightharpoonup \phi_u^a$ in $\mathcal{D}$.*

(iv) $\|\phi_u^a\|_{\mathcal{D}} \leqslant C\|u\|^2_{\frac{12}{5}} \leqslant C\|u\|_E^2$ *and* $\int_{\mathbb{R}^3} \phi_u^a |u|^2 \,\mathrm{d}x \leqslant C\|u\|^4_{\frac{12}{5}} \leqslant C\|u\|_E^4$.

By using the classical reduction argument, system (1.1) can be reduced to a single equation:

$$-\Delta u + (\lambda V(x) + V_0(x))\, u + K(x)\phi_u^a u = |u|^{p-2}u, \qquad x \in \mathbb{R}^3. \tag{2.1}$$

From now on, the solutions of system (1.1) are equal to the solutions of equation (2.1). It is easy to see that the solutions of equation (2.1) can be regarded as critical points of the energy functional $I_\lambda : E \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + (\lambda V(x) + V_0(x))\, u^2\right) \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^a u^2 \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p \mathrm{d}x.$$

According to (V_1) and (V_3), it is easy to check that I_λ is a well defined C^1 functional in E. Moreover, $\forall \varphi \in E$, we have

$$\left\langle I'_\lambda(u), \varphi\right\rangle = \int_{\mathbb{R}^3} (\nabla u \nabla\varphi + (\lambda V(x) + V_0(x))\, u\varphi)\mathrm{d}x + \int_{\mathbb{R}^3} K(x)\phi_u^a u\varphi \mathrm{d}x - \int_{\mathbb{R}^3} |u|^{p-2}u\varphi \mathrm{d}x.$$

By assumption (V_3), there exist smoothly bounded open sets $\Omega'_1, \Omega'_2, \ldots, \Omega'_m \subset \mathbb{R}^3$ such that $\overline{\Omega_j} \subset \Omega'_j$ and $\overline{\Omega'_i} \cap \overline{\Omega'_j} = \emptyset$ for $i \neq j$. In the following proposition, which is one of the keys of our argument, we will give the positivity of the operator $-\Delta + (\lambda V(x) + V_0(x))$ acting on the space $E(D)$, where D is one of the following sets:

$$D = \mathbb{R}^3, \qquad \Omega'_j \ (j = 1, 2, \ldots, k), \quad \text{or} \quad \mathbb{R}^3 \backslash \bigcup_{j \in J} \Omega'_j \qquad (J \subset \{1, 2, \ldots, k\}).$$

Now, we define a norm $\|\cdot\|_{\lambda,D}$ on $E(D)$ for $\lambda \geq \Lambda_1$ by

$$\|u\|_{\lambda,D}^2 = \int_D \left[|\nabla u|^2 + (\lambda V(x) + V_0(x))\, u^2\right] \mathrm{d}x.$$

We write $\|\cdot\|_\lambda = \|\cdot\|_{\lambda,\mathbb{R}^3}$ for simplicity. From Corollary 1.3 in [11], we can get that there exist $C_{1,\lambda}, C'_{1,\lambda} > 0$ such that

$$C_{1,\lambda}\|u\|_{E(D)} \leq \|u\|_{\lambda,D} \leq C'_{1,\lambda}\|u\|_{E(D)} \quad \text{for } u \in E(D).$$

Proposition 2.4. (see [11]) *There exist $\delta_0, \nu_0 > 0$ such that for any set D and $u \in E(D)$*

$$\delta_0\|u\|_{\lambda,D}^2 \leq \|u\|_{\lambda,D}^2 - (p-1)\nu_0\|u\|_{L^2(D)}^2 \quad \textit{for } \lambda \geq \Lambda_1.$$

3 Compactness condition

Since I_λ given in Section 2 does not satisfy the Palais–Smale condition easily, we modify it and establish the compactness conditions in this section. For $t \in \mathbb{R}$ and ν_0 given in Proposition 2.4, set

$$f(t) = \begin{cases} |t|^{p-2}t, & \text{if } |t| \leq \nu_0^{\frac{1}{p-2}}, \\ \nu_0 t, & \text{if } |t| \geq \nu_0^{\frac{1}{p-2}}, \end{cases}$$

and $F(t) = \int_0^t f(s)\mathrm{d}s$. Let $J \subset \{1, 2, \ldots, k\}$ and $\chi_J : \mathbb{R}^3 \to [0,1]$ be the characteristic function of $\Omega'_J := \bigcup_{j\in J} \Omega'_j$. We consider the penalized nonlinearity

$$g(x,t) = \chi_J(x)|t|^{p-2}t + (1-\chi_J(x))\, f(t).$$

Setting $G(t) = \int_0^t g(s)\mathrm{d}s$, we define $J_\lambda : E \to \mathbb{R}$ by

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + \left(\lambda V(x) + V_0(x)u^2\right)\right) \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^a u^2 \mathrm{d}x - \int_{\mathbb{R}^3} G(x,u)\mathrm{d}x.$$

By using a standard method, one can see that J_λ is of class C^1 and its nontrivial critical points are nontrivial solutions of

$$-\Delta u + (\lambda V(x) + V_0(x))\, u + K(x)\phi_u^a(x)u = g(x,u) \quad \text{in } \mathbb{R}^3.$$

Since $f(t) = |t|^{p-2}t$ for $|t| \leq \nu_0^{\frac{1}{p-2}}$, a critical point u of J_λ solves the original problem (1.1) when it satisfies $|u(x)| \leq \nu_0^{\frac{1}{p-2}}$ for all $x \in \mathbb{R}^3 \backslash \Omega'_J$.

Next, we verify the Palais–Smale condition of J_λ. First of all, the following lemma can give the boundedness of the $(PS)_c$ sequence of J_λ.

Lemma 3.1. *For any $(PS)_c$ sequence $\{u_n\}_n \subset E$ of J_λ, there exists a positive constant $M(c)$ which is independent of $\lambda \geq \Lambda_1$ such that*

$$\limsup_{n\to\infty} \|u_n\|_\lambda^2 \leq M(c).$$

Proof. Due to $\{u_n\}_n$ is the $(PS)_c$ sequence of J_λ, we have

$$J_\lambda(u_n) - \frac{1}{p}\langle J'_\lambda(u_n), u_n\rangle = c + o(1) + \varepsilon_n \|u_n\|_\lambda,$$

where $\varepsilon_n \to \infty$ as $n \to \infty$. Then by using the fact $F(t) - \frac{1}{p}f(t)t \le (\frac{1}{2} - \frac{1}{p})\nu_0 t^2$ for $t \in \mathbb{R}$ and $\int_{\mathbb{R}^3} K(x)\phi_{u_n}^a u_n^2 \mathrm{d}x \ge 0$, we get

$$\begin{aligned}
c + o(1) + \varepsilon_n \|u_n\|_\lambda &= J_\lambda(u_n) - \frac{1}{p}\langle J_\lambda'(u_n), u_n\rangle \\
&= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_\lambda^2 + \left(\frac{1}{4} - \frac{1}{p}\right)\int_{\mathbb{R}^3} K(x)\phi_{u_n}^a u_n^2 \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3 \setminus \Omega_J'} \left(F(u_n) - \frac{1}{p}f(u_n)u_n\right)\mathrm{d}x - \int_{\Omega_J'} \left(F(u_n) - \frac{1}{p}f(u_n)u_n\right)\mathrm{d}x \\
&= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_\lambda^2 + \left(\frac{1}{4} - \frac{1}{p}\right)\int_{\mathbb{R}^3} K(x)\phi_{u_n}^a u_n^2 \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3 \setminus \Omega_J'} \left(F(u_n) - \frac{1}{p}f(u_n)u_n\right)\mathrm{d}x \\
&\ge \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_\lambda^2 - \left(\frac{1}{2} - \frac{1}{p}\right)\nu_0\|u_n\|_{L^2}^2.
\end{aligned}$$

Using Proposition 2.4, we obtain

$$\left(\frac{1}{2} - \frac{1}{p}\right)\delta_0\|u_n\|_\lambda^2 \le c + o(1) + \varepsilon_n\|u_n\|_\lambda.$$

Hence, $\|u_n\|_\lambda$ is bounded as $n \to \infty$ and

$$\limsup_{n\to\infty}\|u_n\|_\lambda^2 \le M(c). \qquad \square$$

Now we have the following fact.

Lemma 3.2. *When $c > 0$, there exists $\Lambda_1 > 0$, such that J_λ satisfies the Palais–Smale condition at level c on E for $\lambda \ge \Lambda_1$ large enough.*

Proof. By using Lemma 3.1, we know that any $(PS)_c$-sequence $\{u_n\}_n$ is bounded in E. So, going if necessary to a subsequence, we may assume that

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{in } E \text{ and } H^1(\mathbb{R}^3),\\
&u_n \to u \quad \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^3),\ 1 \le q < 6,\\
&u_n \to u \quad \text{a.e. in } \mathbb{R}^3.
\end{aligned}$$

Now we prove that $u_n \to u$ in E. Firstly, it is easy to check that $J_\lambda'(u) = 0$. In fact, by Proposition 2.3, we know that $\phi_{u_n}^a \rightharpoonup \phi_u^a$ in $\mathcal{D}$. For any $\varphi \in C_0^\infty(\mathbb{R}^3)$, since $K(x)u\varphi \in L^{\frac{6}{5}}(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} K(x)u\varphi\left(\phi_{u_n}^a - \phi_u^a\right)\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

Similarly,

$$\begin{aligned}
\int_{\mathbb{R}^3} K(x)\varphi\phi_{u_n}^a(u_n - u)\mathrm{d}x &\le |K|_\infty\|\varphi\|_3\|\phi_{u_n}^a\|_6\|u_n - u\|_{L^2(\Omega_\varphi)} \\
&\le C\|u_n - u\|_{L^2(\Omega_\varphi)} \\
&\to 0
\end{aligned}$$

as $n \to \infty$, where Ω_φ is the support of φ. Consequently,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left(K(x)\phi_{u_n}^a u_n\varphi - K(x)\phi_u^a u\varphi\right) \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^3} K(x)u\varphi \left(\phi_{u_n}^a - \phi_u^a\right) \mathrm{d}x + \int_{\mathbb{R}^3} K(x)\varphi\phi_{u_n}^a \left(u_n - u\right) \mathrm{d}x \\
&\quad \to 0
\end{aligned}$$

as $n \to \infty$, thus we see that

$$\begin{aligned}
\langle J_\lambda'(u_n) - J_\lambda'(u), \varphi\rangle &= \langle J_\lambda'(u_n), \varphi\rangle - \langle J_\lambda'(u), \varphi\rangle \\
&= \int_{\mathbb{R}^3} (\nabla u_n \nabla\varphi + (\lambda V(x) + V_0(x))\, u_n\varphi)\mathrm{d}x + \int_{\mathbb{R}^3} K(x)\phi_{u_n}^a u_n\varphi \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} (\nabla u \nabla\varphi - (\lambda V(x) + V_0(x))\, u\varphi)\mathrm{d}x - \int_{\mathbb{R}^3} K(x)\phi_u^a u\varphi \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} g(x, u_n)\varphi \mathrm{d}x + \int_{\mathbb{R}^3} g(x, u)\varphi \mathrm{d}x \\
&= o(1).
\end{aligned}$$

So $J_\lambda'(u) = 0$. Then we have

$$\begin{aligned}
&\langle J_\lambda'(u_n) - J_\lambda'(u), u_n - u\rangle \\
&\quad = \langle J_\lambda'(u_n), u_n - u\rangle - \langle J_\lambda'(u), u_n - u\rangle \\
&\quad = \|u_n - u\|_\lambda^2 + \int_{\mathbb{R}^3} \left(K(x)\phi_{u_n}^a u_n (u_n - u) - K(x)\phi_u^a u (u_n - u)\right) \mathrm{d}x \\
&\qquad - \int_{\mathbb{R}^3\setminus\Omega_J'} (f(u_n) - f(u))(u_n - u)\, \mathrm{d}x - \int_{\Omega_J'} \left(|u_n|^{p-2} u_n - |u|^{p-2}u\right)(u_n - u)\, \mathrm{d}x \\
&\quad = \|u_n - u\|_\lambda^2 + \int_{\mathbb{R}^3} K(x)\phi_{u_n}^a (u_n - u)^2 \mathrm{d}x + \int_{\mathbb{R}^3} K(x)\left(\phi_{u_n}^a - \phi_u^a\right) u (u_n - u)\, \mathrm{d}x \\
&\qquad - \int_{\mathbb{R}^3\setminus\Omega_J'} (f(u_n) - f(u))(u_n - u)\, \mathrm{d}x - \int_{\Omega_J'} \left(|u_n|^{p-2} u_n - |u|^{p-2}u\right)(u_n - u)\, \mathrm{d}x \\
&\quad = o(1)
\end{aligned}$$

as $n \to \infty$. Because of $\max_{x\in\mathbb{R}} |f'(x)| \le (p-1)\nu_0$, by using the Mean Value Theorem, we get that

$$\int_{\mathbb{R}^3\setminus\Omega_J'} (f(u_n) - f(u))(u_n - u)\, \mathrm{d}x \le (p-1)\nu_0 \|u_n - u\|_2^2.$$

Noting that $u_n \to u$ in $L_{\mathrm{loc}}^p(\mathbb{R}^3)$, so we have

$$\int_{\Omega_J'} \left(|u_n|^{p-2} u_n - |u|^{p-2}u\right)(u_n - u)\, \mathrm{d}x = o(1) \quad \text{as } n \to \infty.$$

We also remark that $u_n \rightharpoonup u$ in $L^3(\mathbb{R}^3)$. Thus, by the uniqueness of limit, we have $|u_n - u|^{\frac{6}{5}} \rightharpoonup 0$ in $L^{\frac{5}{2}}(\mathbb{R}^3)$. Then according to $K \in L^\infty(\mathbb{R}^3)$ and $|u|^{\frac{6}{5}} \in L^{\frac{5}{3}}(\mathbb{R}^3)$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^3} K(x)\left(\phi_{u_n}^a - \phi_u^a\right) u (u_n - u)\, \mathrm{d}x &\le |K|_\infty \|\phi_{u_n}^a - \phi_u^a\|_6 \left(\int_{\mathbb{R}^3} |u|^{\frac{6}{5}} |u_n - u|^{\frac{6}{5}} \mathrm{d}x\right)^{\frac{5}{6}} \\
&= o(1) \quad \text{as } n \to \infty.
\end{aligned}$$

Combining all these and the fact $\int_{\mathbb{R}^3} K(x)\phi_{u_n}^a (u_n - u)^2 \,\mathrm{d}x \geq 0$, by using Proposition 2.4, we have

$$\delta_0 \|u_n - u\|_\lambda^2 \leq \|u_n - u\|_\lambda^2 - (p-1)\nu_0 \|u_n - u\|_2^2 + \int_{\mathbb{R}^3} K(x)\phi_{u_n}^a (u_n - u)^2 \,\mathrm{d}x \leq o(1)$$

as $n \to \infty$, which completes the proof. □

Following the spirit of Lemma 3.2, we have

Lemma 3.3. *Suppose the sequences $\lambda_n \to \infty$ as $n \to \infty$ and $\{u_n\}_n$ in E satisfy*

$$J_{\lambda_n}(u_n) \leq c, \qquad \|\nabla J_{\lambda_n}(u_n)\|_{\lambda_n} \to 0.$$

Then, after passing to a subsequence, we have:

- (a) *$u_n \rightharpoonup u$ in E for some $u \in E$;*
- (b) *$u \equiv 0$ in $\mathbb{R}^3 \backslash \Omega_J$, and $u_j = u|_{\Omega_j} \in H_0^1(\Omega_j)$ solves $-\Delta v + V_0(x)v + K(x)\phi_u^a v = |v|^{p-2}v$ in Ω_j weakly for $j \in J$;*
- (c) *$\|u_n - u\|_{\lambda_n} \to 0$, consequently $u_n \to u$ in $H^1(\mathbb{R}^3)$;*
- (d) *For $n \to \infty$, u_n also satisfies:*
  - (1) $\int_{\mathbb{R}^3} \lambda_n V(x) u_n^2 \mathrm{d}x \to 0$;
  - (2) $\int_{\mathbb{R}^3 \backslash \Omega_J'} \left(|\nabla u_n|^2 + (\lambda_n V(x) + V_0(x)) u_n^2\right) \mathrm{d}x \to 0$;
  - (3) $\int_{\Omega_j'} \left(|\nabla u_n|^2 + (\lambda_n V(x) + V_0(x)) u_n^2\right) \mathrm{d}x \to \int_{\Omega_j} \left(|\nabla u|^2 + V_0(x)u^2\right) \mathrm{d}x, j = 1, \ldots, m.$

Proof. By a similar method of Lemma 3.1, we obtain that $\{u_n\}_n$ is bounded in E and $H^1(\mathbb{R}^3)$. So we could assume that for some $u \in E$,

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } E \text{ and } H^1(\mathbb{R}^3), \\ u_n &\to u \quad \text{in } L^q_{\text{loc}}(\mathbb{R}^3),\ 1 \leq q < 6, \\ u_n &\to u \quad \text{a.e in } \mathbb{R}^3. \end{aligned}$$

Let $C_m = \left\{x \in \mathbb{R}^3 : V(x) \geq \frac{1}{m}\right\}$. When n large enough such that $\lambda_n \leq 2(\lambda_n - \lambda_1)$, we have that

$$\begin{aligned} \int_{C_m} u_n^2 \mathrm{d}x &\leq \frac{m}{\lambda_n} \int_{\mathbb{R}^3} \lambda_n V(x) u_n^2 \mathrm{d}x \\ &\leq \frac{2m}{\lambda_n} \int_{\mathbb{R}^3} (\lambda_n - \lambda_1) V(x) u_n^2 \mathrm{d}x \\ &\leq \frac{2m}{\lambda_n} \int_{\mathbb{R}^3} (\lambda_n - \lambda_1) V(x) u_n^2 \mathrm{d}x + \frac{2m}{\lambda_n} \|u_n\|_{\lambda_1}^2 \\ &= \frac{2m}{\lambda_n} \int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + (\lambda_n V(x) + V_0(x)) u_n^2\right) \mathrm{d}x \\ &= \frac{2m}{\lambda_n} \|u_n\|_{\lambda_n}^2 \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Therefore, $u(x) = 0$ a.e. in $\bigcup_m C_m = \mathbb{R}^3 \backslash \overline{\Omega}$. For any $\varphi \in C_0^\infty(\Omega_j), j \in J$, we get

$$\langle J_{\lambda_n}'(u_n), \varphi \rangle = \int_{\Omega_j} \left(\nabla u_n \nabla \varphi + V_0(x) u_n \varphi + K(x)\phi_{u_n}^a u_n \varphi - |u_n|^{p-2} u_n \varphi\right) \mathrm{d}x.$$

Due to $K(x)u\varphi \in L^{\frac{6}{5}}\left(\mathbb{R}^3\right), \Phi\left(u_n\right) \rightharpoonup \Phi(u)$ in $\mathcal{D}$ and $u_n \to u$ in $L^q_{\mathrm{loc}}\left(\mathbb{R}^3\right)$ for $1 \le q < 6$, for $n \to \infty$, we have

$$\begin{aligned}\int_{\Omega_j}\left(K(x)\phi^a_{u_n}u_n\varphi - K(x)\phi^a_u u\varphi\right)\mathrm{d}x &= \int_{\Omega_j} K(x)\phi^a_{u_n}\left(u_n - u\right)\varphi\mathrm{d}x + \int_{\Omega_j} K(x)\left(\phi^a_{u_n} - \phi^a_u\right)u\varphi\mathrm{d}x \\ &\to 0.\end{aligned}$$

Similar to the proof of Lemma 3.2, we have $\langle J'_{\lambda_n}(u_n) - J'_{\lambda_n}(u), \varphi\rangle \to 0$. Thus it follows from $\langle J'_{\lambda_n}\left(u_n\right), \varphi\rangle \to 0$ that

$$\int_{\Omega_j}\left(\nabla u\nabla\varphi + V_0(x)u\varphi + K(x)\phi^a_u u\varphi - |u|^{p-2}u\varphi\right)\mathrm{d}x = 0.$$

As a result, for $j \in J, u_j = u|_{\Omega_j} \in H^1_0\left(\Omega_j\right)$ solves $-\Delta v + V_0(x)v + K(x)\phi^a_u v = |v|^{p-2}v$ in Ω_j weakly. When $j \in \{1, 2, \ldots, m\} \setminus J$, let $\varphi = u$, then we get

$$\int_{\Omega_j}\left(|\nabla u|^2 + V_0(x)u^2 + K(x)\phi^a_u u^2 - f(u)u\right)\mathrm{d}x = 0.$$

Because of $\varphi = u \in C^\infty_0\left(\Omega_j\right)$, we have

$$\int_{\Omega'_j}\left(|\nabla u|^2 + V_0(x)u^2 + K(x)\phi^a_u u^2 - f(u)u\right)\mathrm{d}x = 0.$$

From Proposition 2.4, $f(t)t \le \nu_0 t^2$ for $t \in \mathbb{R}$ and the fact that $K(x)\phi^a_u u^2 \ge 0$, we have

$$\begin{aligned}\delta_0\|u\|^2_{\Lambda_1,\Omega'_j} &\le \|u\|^2_{\Lambda_1,\Omega'_j} - (p-1)\nu_0\|u\|^2_{L^2\left(\Omega'_j\right)} \\ &\le \|u\|^2_{\Lambda_1,\Omega'_j} - \nu_0\|u\|^2_{L^2\left(\Omega'_j\right)} \\ &\le \int_{\Omega'_j}\left(|\nabla u|^2 + a_0(x)u^2 + K(x)\phi^a_u u^2 - f(u)u\right)\mathrm{d}x \\ &= 0.\end{aligned}$$

So that, $u = 0$ in Ω_j when $j \in \{1, 2, \ldots, m\} \setminus J$ and we get (b).

In order to prove (c), we use the following fact:

$$\begin{aligned}o(1) &= \langle J'_{\lambda_n}\left(u_n\right) - J'_{\lambda_n}(u), u_n - u\rangle \\ &= \langle J'_{\lambda_n}\left(u_n\right), u_n - u\rangle - \langle J'_{\lambda_n}(u), u_n - u\rangle \\ &= \|u_n - u\|^2_{\lambda_n} + \int_{\mathbb{R}^3}\left(K(x)\phi^a_{u_n}u_n\left(u_n - u\right) - K(x)\phi^a_u u\left(u_n - u\right)\right)\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3\setminus\Omega'_J}\left(f\left(u_n\right) - f(u)\right)\left(u_n - u\right)\mathrm{d}x - \int_{\Omega'_J}\left(\left|u_n\right|^{p-2}u_n - |u|^{p-2}u\right)\left(u_n - u\right)\mathrm{d}x \\ &= \|u_n - u\|^2_{\lambda_n} + \int_{\mathbb{R}^3} K(x)\phi^a_{u_n}\left(u_n - u\right)^2\mathrm{d}x + \int_{\mathbb{R}^3} K(x)\left(\phi^a_{u_n} - \phi^a_u\right)u\left(u_n - u\right)\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3\setminus\Omega'_J}\left(f\left(u_n\right) - f(u)\right)\left(u_n - u\right)\mathrm{d}x - \int_{\Omega'_J}\left(\left|u_n\right|^{p-2}u_n - |u|^{p-2}u\right)\left(u_n - u\right)\mathrm{d}x.\end{aligned}$$

Similar to the proof of Lemma 3.2, we also have

$$\int_{\mathbb{R}^3\setminus\Omega'_J}\left(f\left(u_n\right) - f(u)\right)\left(u_n - u\right)\mathrm{d}x \le (p-1)\nu_0\left\|u_n - u\right\|^2_2,$$

$$\int_{\Omega'_J}\left(\left|u_n\right|^{p-2}u_n - |u|^{p-2}u\right)\left(u_n - u\right)\mathrm{d}x = o(1) \quad \text{as } n \to \infty$$

and

$$\int_{\mathbb{R}^3} K(x)\left(\phi_{u_n}^a-\phi_u^a\right) u\left(u_n-u\right) \mathrm{d}x=o(1) \quad \text{as } n \rightarrow \infty.$$

So we have

$$\delta_0\left\|u_n-u\right\|_{\lambda_n}^2 \leq\left\|u_n-u\right\|_{\lambda_n}^2-(p-1) \nu_0\left\|u_n-u\right\|_2^2+\int_{\mathbb{R}^3} K(x) \phi_{u_n}^a\left(u_n-u\right)^2 \mathrm{d}x \leq o(1).$$

This completes the proof of (c).

For (d), we use (c) and for sufficiently large $n, \lambda_n \leq 2\left(\lambda_n-\lambda_1\right)$. Then as $n \rightarrow \infty$, we have

$$\begin{aligned}
\frac{1}{2} \int_{\mathbb{R}^3} \lambda_n V(x) u_n^2 \mathrm{d}x & \leq \int_{\mathbb{R}^3}\left(\lambda_n-\lambda_1\right) V(x) u_n^2 \mathrm{d}x=\int_{\mathbb{R}^3}\left(\lambda_n-\lambda_1\right) V(x)\left(u_n-u\right)^2 \mathrm{d}x \\
& \leq \int_{\mathbb{R}^3}\left(\lambda_n-\lambda_1\right) V(x)\left(u_n-u\right)^2 \mathrm{d}x+\left\|u_n-u\right\|_{\lambda_1}^2=\left\|u_n-u\right\|_{\lambda_n}^2 \rightarrow 0 .
\end{aligned}$$

Thus (1) in (d) is obtained. It is easy to show that (2), (3) in (d) also follows immediately from (1) in (d) and (c), and we obtain the conclusion. □

Lemma 3.4. *For any fixed $c>0$, there exists $\Lambda_c \geq \Lambda_1$ such that if u_λ is a critical point of J_λ satisfying $\left|J_\lambda\left(u_\lambda\right)\right| \leq c$ for $\lambda \geq \Lambda_c$, then $\left|u_\lambda\right| \leq \nu_0^{\frac{1}{p-2}}$ on $\mathbb{R}^3 \backslash \Omega_J^{\prime}$, ν_0 is defined in Proposition 2.4. In particular, u_λ solves the original problem (1.1).*

Proof. Since $u_\lambda \in E$ is a critical point of J_λ with $\left|J_\lambda\left(u_\lambda\right)\right| \leq c, u_\lambda$ is bounded in E uniformly for $\lambda \geq \Lambda_1$. And it satisfies the equation

$$-\Delta u_\lambda+\left(\lambda V(x)+V_0(x)\right) u_\lambda+K(x) \phi_{u_\lambda}^a u_\lambda=g\left(x, u_\lambda\right) \quad \text{in } \mathbb{R}^3 .$$

Due to Lemma 5.1 in [2], $H_\lambda^{-1}:=\left(-\Delta+\left(\lambda V(x)+V_0(x)\right)\right)^{-1}$ is a well-defined bounded operator from $L^s\left(\mathbb{R}^3\right)$ to $L^r\left(\mathbb{R}^3\right)$ provided $1 \leq s \leq r \leq+\infty$ and $\frac{1}{s}-\frac{1}{r} \leq \frac{2}{3}$. And there exists a constant $C_{r, s}>0$ (independent of λ sufficiently large) such that

$$\left\|H_\lambda^{-1} g\right\|_r \leq C_{r, s}\|g\|_s, \qquad g \in L^s\left(\mathbb{R}^3\right).$$

Let $\chi_{\lambda, 0}$ be the characteristic function of the set $\left\{x \in \mathbb{R}^3:\left|u_\lambda(x)\right| \leq 1\right\}$ and define $v_{\lambda, 0}=\chi_{\lambda, 0} u_\lambda$, $w_{\lambda, 0}=u_\lambda-v_{\lambda, 0}=\left(1-\chi_{\lambda, 0}\right) u_\lambda$. Setting $l_{\lambda, 0}=g\left(\cdot, v_{\lambda, 0}\right)-K(\cdot) \phi_{u_\lambda}^a v_{\lambda, 0}$ and $h_{\lambda, 0}=g\left(\cdot, w_{\lambda, 0}\right)-K(\cdot) \phi_{u_\lambda}^a w_{\lambda, 0}$, we have $g\left(\cdot, u_\lambda\right)=l_{\lambda, 0}+h_{\lambda, 0}$. Since u_λ is bounded in $E, \phi_{u_\lambda}^a$ is bounded in L^{∞}. Thus, $l_{\lambda, 0}$ is bounded in $L^{\infty}\left(\mathbb{R}^3\right)$ uniformly in λ. Moreover, $h_{\lambda, 0}$ is bounded uniformly for λ in $L^{\frac{6}{p-1}}\left(\mathbb{R}^3\right)$. In fact,

$$\begin{aligned}
\left|\phi_{u_\lambda}^a(x)\right| & \leq \frac{1}{4 \pi}\left|\int_{\mathbb{R}^3} \frac{K(y)}{|x-y|} u_\lambda^2(y) \mathrm{d}y\right| \\
& \leq c|K|_{\infty}\left(\int_{B_1(x)} \frac{u_\lambda^2(y)}{|x-y|} \mathrm{d}y+\int_{B_1^c(x)} \frac{u_\lambda^2(y)}{|x-y|} \mathrm{d}y\right) \\
& \leq c|K|_{\infty}\left(\left(\int_{B_1(x)} \frac{1}{|x-y|^2} \mathrm{~d}y\right)^{1 / 2}\left(\int_{B_1(x)} u_\lambda^4 \mathrm{~d}y\right)^{1 / 2}\right. \\
& \left.\quad+\left(\int_{B_1^c(x)} \frac{1}{|x-y|^4} \mathrm{~d}y\right)^{1 / 4}\left(\int_{B_1^c(x)}\left|u_\lambda\right|^{8 / 3} \mathrm{~d}y\right)^{4 / 3}\right) \\
& \leq c^{\prime}|K|_{\infty} .
\end{aligned}$$

In the set $|u_\lambda| \le 1$, we have $|w_{\lambda,0}| = 0$; and in the set $|u_\lambda| > 1$, we have $|w_{\lambda,0}| = |u_\lambda - v_{\lambda,0}| = |\left(1 - \chi_{\lambda,0}\right) u_\lambda| = |u_\lambda| > 1$. So we have

$$\begin{aligned}\left(\int_{\mathbb{R}^3} |w_{\lambda,0}|^{\frac{6}{p-1}} \, \mathrm{d}x\right)^{\frac{p-1}{6}} &= \left(\int_{\{x:|u_\lambda|\le 1\}} |w_{\lambda,0}|^{\frac{6}{p-1}} \, \mathrm{d}x + \int_{\{x:|u_\lambda|>1\}} |w_{\lambda,0}|^{\frac{6}{p-1}} \, \mathrm{d}x\right)^{\frac{p-1}{6}} \\ &\le \left(0 + \int_{\{x:|u_\lambda|>1\}} |w_{\lambda,0}|^{6} \, \mathrm{d}x\right)^{\frac{p-1}{6}} \\ &= \left(\int_{\mathbb{R}^3} |w_{\lambda,0}|^{6} \, \mathrm{d}x\right)^{\frac{p-1}{6}}.\end{aligned}$$

Therefore, combining this with Minkowski inequality, we have

$$\begin{aligned}\|h_{\lambda,0}\|_{\frac{6}{p-1}} &\le \|g\left(\cdot, w_{\lambda,0}\right)\|_{\frac{6}{p-1}} + \left\|K(\cdot)\phi^a_{u_\lambda} w_{\lambda,0}\right\|_{\frac{6}{p-1}} \\ &\le \left(\int_{\mathbb{R}^3} |u_\lambda|^6 \, \mathrm{d}x\right)^{\frac{p-1}{6}} + |K|_\infty \left|\phi^a_{u_\lambda}\right|_\infty \left(\int_{\mathbb{R}^3} |w_{\lambda,0}|^{\frac{6}{p-1}} \, \mathrm{d}x\right)^{\frac{p-1}{6}} \\ &\le \left(\int_{\mathbb{R}^3} |u_\lambda|^6 \, \mathrm{d}x\right)^{\frac{p-1}{6}} + |K|_\infty \left|\phi^a_{u_\lambda}\right|_\infty \left(\int_{\mathbb{R}^3} |w_{\lambda,0}|^{6} \, \mathrm{d}x\right)^{\frac{p-1}{6}} \\ &\le C \|u_\lambda\|_E^{p-1}.\end{aligned}$$

Now we define $v_{\lambda,1} = H_\lambda^{-1} l_{\lambda,0}$ and $w_{\lambda,1} = H_\lambda^{-1} h_{\lambda,0}$ so that $u_\lambda = v_{\lambda,1} + w_{\lambda,1}$. Then, there exists $C_2 > 0$ such that

$$|v_{\lambda,1}|_\infty \le C_2 \quad \text{and} \quad \|w_{\lambda,1}\|_{p_1} \le C_2$$

uniformly in λ; here $p_1 = \infty$ if $p_0 = \frac{6}{p-1} > \frac{3}{2}$, and p_1 is arbitrarily close to and less than $\frac{3p_0}{3-2p_0}$ if $p_0 \le \frac{3}{2}$. In the case $p_0 > \frac{3}{2}$ we are done. In the case $p_0 \le \frac{3}{2}$, we have $5 \le p < 6$. Thus, we can assume that there is a positive constant $\delta \le 1$ such that $p = 6 - \delta$. Let $\chi_{\lambda,1}$ be the characteristic function of the set

$$A_\lambda = \left\{x \in \mathbb{R}^3 : |w_{\lambda,1}(x)| \le C_2 + 1\right\}.$$

Setting

$$\begin{aligned}\bar{v}_{\lambda,1} &= \chi_{\lambda,1} u_\lambda = \chi_{\lambda,1}\left(v_{\lambda,1} + w_{\lambda,1}\right), \\ \bar{w}_{\lambda,1} &= u_\lambda - \bar{v}_{\lambda,1} = \left(1 - \chi_{\lambda,1}\right)\left(v_{\lambda,1} + w_{\lambda,1}\right), \\ l_{\lambda,1} &= g\left(\cdot, \bar{v}_{\lambda,1}\right) - K(\cdot)\phi^a_{u_\lambda} \bar{v}_{\lambda,1}, \\ h_{\lambda,1} &= g\left(\cdot, \bar{w}_{\lambda,1}\right) - K(\cdot)\phi^a_{u_\lambda} \bar{w}_{\lambda,1}.\end{aligned}$$

We see that $|l_{\lambda,1}|_\infty$ is bounded uniformly in λ. In addition, since the measure of the set A^c_λ is finite and $\|w_{\lambda,1}\|_{p_1} \le C_2$, we have $h_{\lambda,1}$ is bounded in $L^{\frac{p_1}{p-1}}(\mathbb{R}^3)$. Now repeating the above argument with $v_{\lambda,2} = H_\lambda^{-1} l_{\lambda,1}$ and $w_{\lambda,2} = H_\lambda^{-1} h_{\lambda,1}$, we obtain a constant $C_3 > 0$ such that

$$|v_{\lambda,2}|_\infty \le C_3 \quad \text{and} \quad \|w_{\lambda,1}\|_{p_2} \le C_3,$$

where $p_2 = \infty$ if $\bar{p}_1 = \frac{p_1}{p-1} > \frac{3}{2}$, and p_2 is arbitrarily close to and less than $\frac{3\bar{p}_1}{3-2\bar{p}_1}$ if $\bar{p}_1 \le \frac{3}{2}$. Using the assumption $p = 6 - \delta, 0 < \delta \le 1$ and after a finite number of such steps we get a uniform bounded for $|u_\lambda|_\infty$.

According to the definition of g and uniform boundedness of $\left|\phi^a_{u_\lambda}\right|_\infty$, we obtain that $A(x) = \frac{g(x,u_\lambda(x))}{u_\lambda(x)} + K(x)\phi^a_{u_\lambda}$ is bounded in $L^\infty(\mathbb{R})$. Moreover, the negative part of $W_\lambda =$

$\lambda V + V_0 - A$ is bounded uniformly in λ. It follows from Theorem A.2.1 in [16] that the norm of W_λ^- in the Kato class K_3 is bounded uniformly in λ. Therefore, Theorem C.1.2 in [16] implies that there is a constant $C(r)$ which is independent of λ such that

$$|u_\lambda(x)| \le C(r) \int_{B_r(x)} |u_\lambda| \, \mathrm{d}x,$$

where $B_r(x)$ is a ball in $\mathbb{R}^3$ centered at x with radius r. From Lemma 3.3(b), as $n \to \infty$

$$u_\lambda \to 0 \quad \text{in } L^2\left(\mathbb{R}^3 \backslash \overline{\Omega}\right).$$

Thus, choosing $r = \frac{1}{2}\operatorname{dist}\left(\Omega, \mathbb{R}^3 \backslash \Omega'\right)$, we have uniformly in $x \in \mathbb{R}^3 \backslash \Omega'$,

$$\begin{aligned} |u_\lambda(x)| &\le C(r) \int_{B_r(x)} |u_\lambda| \, \mathrm{d}x \\ &\le C(r) \left(\operatorname{meas} B_r(x)\right)^{\frac{1}{2}} |u_\lambda|_{2, B_r(x)} \\ &\le C(r) \left(\operatorname{meas} B_r(x)\right)^{\frac{1}{2}} |u_\lambda|_{2, \mathbb{R}^3 \backslash \Omega} \\ &\to 0 \quad \text{as } \lambda \to \infty. \end{aligned}$$

□

4 Nehari manifold and minimax arguments

Consider the following nonlinear problems for $j \in J$,

$$\begin{cases} -\Delta u + V_0(x) u = |u|^{p-2} u, & \text{in } \Omega_j, \\ u = 0, & \text{on } \partial \Omega_j \end{cases}$$

and

$$\begin{cases} -\Delta u + \left(\lambda V(x) + V_0(x)\right) u = |u|^{p-2} u, & \text{in } \Omega_j', \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial \Omega_j' \end{cases}$$

with their corresponding functionals

$$\begin{aligned} I_j(u) &= \frac{1}{2} \int_{\Omega_j} \left(|\nabla u|^2 + V_0(x) u^2\right) \mathrm{d}x - \frac{1}{p} \int_{\Omega_j} |u|^p \mathrm{d}x; \qquad H_0^1\left(\Omega_j\right) \to \mathbb{R}, \\ I_{\lambda,j}(u) &= \frac{1}{2} \int_{\Omega_j'} \left(|\nabla u|^2 + \left(\lambda V(x) + V_0(x)\right) u^2\right) \mathrm{d}x - \frac{1}{p} \int_{\Omega_j'} |u|^p \mathrm{d}x; \qquad H^1\left(\Omega_j'\right) \to \mathbb{R}. \end{aligned}$$

It is easy to check that both I_j and $I_{\lambda,j}$ possess the mountain pass geometry and satisfy the (PS) condition. On the other hand, the infimum of I_j and $I_{\lambda,j}$ on their Nehari manifold

$$\begin{aligned} \mathcal{N}_j &= \left\{ u \in H_0^1\left(\Omega_j\right) \backslash \{0\} : \left(\nabla I_j(u), u\right) = 0 \right\}, \\ \mathcal{N}_{\lambda,j} &= \left\{ u \in H^1\left(\Omega_j'\right) \backslash \{0\} : \left(\nabla I_{\lambda,j}(u), u\right) = 0 \right\} \end{aligned}$$

are achieved by some $\omega_j \in \mathcal{N}$ and $\omega_{\lambda,j} \in \mathcal{N}_{\lambda,j}$ respectively. By a standard argument, we can see that $\omega_j, \omega_{\lambda,j}$ are critical points of I_j and $I_{\lambda,j}$ separately. The critical values $c_j = I_j\left(\omega_j\right)$ and $c_{\lambda,j} = I_{\lambda,j}\left(\omega_{\lambda,j}\right)$ are equal to the mountain pass value of their corresponding functional. Moreover, we also have the following lemma (see Lemma 3.1 in [11] and (3.8) for details).

Lemma 4.1. *The following statements hold:*

(a) *there is a* $\rho > 0$ *such* $0 < \rho \leq c_{\lambda,j} \leq c_j$ *for* $\lambda \geq \Lambda_1$ *sufficiently large;*

(b) $c_j = \max\limits_{r>0} I_j\left(rw_j\right), c_{\lambda,j} = \max\limits_{r>0} I_{\lambda,j}\left(rw_{\lambda,j}\right)$*;*

(c) $c_{\lambda,j} \to c_j$ *as* $\lambda \to \infty$*;*

(d)

$$c_j = \inf\left\{I_j(v) : v \in H_0^1\left(\Omega_j\right), \int_{\Omega_j} |v|^p \mathrm{d}x = \left(\frac{1}{2} - \frac{1}{p}\right)^{-1} c_j\right\},$$
$$c_{\lambda,j} = \inf\left\{I_{\lambda,j}(v) : v \in H^1\left(\Omega_j'\right), \int_{\Omega_j'} |v|^p \mathrm{d}x = \left(\frac{1}{2} - \frac{1}{p}\right)^{-1} c_{\lambda,j}\right\}.$$

In the following, we give a minimax argument for $J_\lambda(u)$. First of all, we fix $R \geq 2$ such that

$$\begin{aligned} &I_j\left(R\omega_j\right) < 0, \\ &R^2 \left\|w_j\right\|^2_{\lambda,\Omega_j'} = R^p \left|w_j\right|_p^p \geq 2\left(\frac{1}{2} - \frac{1}{p}\right)^{-1} c_j \end{aligned} \tag{4.1}$$

for all $j \in J$. By relabeling the indices, we could assume $J = \{1, 2, \ldots, l\}$ $(l \leq m)$. We define $\gamma_0 : [0,1]^l \to E$,

$$\gamma_0\left(t_1, t_2, \ldots, t_l\right)(x) = \sum_{j=1}^{l} t_j R\omega_j(x), \tag{4.2}$$

$$\Gamma_J = \left\{\gamma \in C\left([0,1]^l, E\right); \gamma\left(t_1, t_2, \ldots, t_l\right) = \gamma_0\left(t_1, t_2, \ldots, t_l\right), \left(t_1, t_2, \ldots, t_l\right) \in \partial\left([0,1]^l\right)\right\}$$

and

$$b_{\lambda,J} = \inf_{\gamma \in \Gamma_J} \max_{t \in [0,1]^l} J_\lambda(\gamma(t)).$$

Obviously, $\Gamma_J \neq \emptyset$ since $\gamma_0 \in \Gamma_J$. Thus $b_{\lambda,j}$ is well defined.

According to Lemma 3.3 in [11], by using a topological degree argument we can get the following conclusion.

Lemma 4.2. *For any* $\gamma \in \Gamma_J$*, there is a* $t_\gamma \in [0,1]^l$ *such that for* $j \in J$

$$\int_{\Omega_j'} \left|\gamma\left(t_\gamma\right)(x)\right|^p \mathrm{d}x = \left(\frac{1}{2} - \frac{1}{p}\right)^{-1} c_{\lambda,j}.$$

Lemma 4.3. $\sum_{j=1}^{l} c_{\lambda,j} \leq b_{\lambda,J} \leq \sum_{j=1}^{l} c_j + \mu$*, where*

$$\mu = \frac{R^4}{4} \sum_{j=1}^{l} \int_{\Omega_j} K(x) \phi^a_{\sum_{j=1}^{l} \omega_j} \left(\omega_j\right)^2 \mathrm{d}x. \tag{4.3}$$

Proof. According to Lemma 4.2, for any $\gamma \in \Gamma_J$, we have

$$\max_{t \in [0,1]^l} J_\lambda(\gamma(t)) \geq J_\lambda\left(\gamma\left(t_\gamma\right)\right) \geq J_{\lambda, \mathbb{R}^3 \backslash \Omega_J'}\left(\gamma\left(t_\gamma\right)\right) + \sum_{j=1}^{l} J_{\lambda,\Omega_j'}\left(\gamma\left(t_\gamma\right)\right),$$

where $J_{\lambda,\Omega_j'}(u)$ is defined by

$$J_{\lambda,\Omega_j'}(u)=\frac{1}{2}\int_{\Omega_j'}\left(|\nabla u|^2+\left(\lambda V(x)+V_0(x)u^2\right)\right)\mathrm{d}x+\frac{1}{4}\int_{\Omega_j'}K(x)\phi_u^a u^2\mathrm{d}x-\int_{\Omega_j'}G(x,u)\mathrm{d}x.$$

And the definition of $J_{\lambda,\mathbb{R}^3\setminus\Omega_j'}(u)$ is similar. According to Proposition 2.4 and the fact that $|G(x,t)|\le\frac{1}{2}\nu_0t^2$ when $x\in\mathbb{R}^3\setminus\Omega_J'$, we get that

$$J_{\lambda,\mathbb{R}^3\setminus\Omega_j'}(u)\ge 0\quad\text{for } u\in E \text{ and } j\in J.$$

By using $\int_{\Omega_j'}K(x)\phi_u^a u^2\mathrm{d}x\ge 0$ and Lemma 4.1(d) we obtain

$$\begin{aligned}\max_{t\in[0,1]^l}J_\lambda(\gamma(t))&\ge\sum_{j=1}^{l}J_{\lambda,\Omega_j'}\left(\gamma\left(t_\gamma\right)\right)\ge\sum_{j=1}^{l}I_{\lambda,j}\left(\gamma\left(t_\gamma\right)\right)\\&\ge\sum_{j=1}^{l}\inf\left\{I_{\lambda,j}(v):v\in H^1\left(\Omega_j'\right),\int_{\Omega_j'}|v|^p\mathrm{d}x=\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}c_{\lambda,j}\right\}\\&=\sum_{j=1}^{l}c_{\lambda,j}.\end{aligned}$$

According to the arbitrary choice of γ, we have $\sum_{j=1}^{l}c_{\lambda,j}\le b_{\lambda,J}$. On the other hand,

$$\begin{aligned}b_{\lambda,J}&\le\max_{t\in[0,1]^l}J_\lambda\left(\gamma_0(t)\right)\\&=\max_{t\in[0,1]^l}\sum_{j=1}^{l}I_j\left(t_jR\omega_j\right)+\frac{1}{4}\sum_{j=1}^{l}\int_{\Omega_j}K(x)\phi^a_{\sum_{j=1}^{l}t_jR\omega_j}\left(t_jR\omega_j\right)^2\mathrm{d}x\\&\le\sum_{j=1}^{l}c_j+\frac{R^4}{4}\sum_{j=1}^{l}\int_{\Omega_j}K(x)\phi^a_{\sum_{j=1}^{l}\omega_j}\left(\omega_j\right)^2\mathrm{d}x.\end{aligned}$$

Thus, we get the conclusion. □

In the following, we denote $\sum_{j=1}^{l}c_j$ by c_J. It is easy to see that, for $\gamma\in\Gamma_J,\gamma(t)=\gamma_0(t)$ on $\partial[0,1]^l$. So, for $t=(t_1,t_2,\ldots,t_l)\in\partial[0,1]^l$, one has

$$J_\lambda(\gamma(t))=J_\lambda\left(\gamma_0(t)\right)\le\sum_{j=1}^{l}I_j\left(t_jR\omega_j\right)+\frac{R^4}{4}\sum_{j=1}^{l}\int_{\Omega_j}K(x)\phi^a_{\sum_{j=1}^{l}\omega_j}\left(\omega_j\right)^2\mathrm{d}x.$$

Choosing k small enough such that

$$\frac{R^4}{4}\sum_{j=1}^{l}\int_{\Omega_j}K(x)\phi^a_{\sum_{j=1}^{l}\omega_j}\left(\omega_j\right)^2\mathrm{d}x\le\frac{1}{2}\min_{j\in J}c_j$$

when $|K|_\infty\le k$. Due to Lemma 4.1(b), $I_j\left(t_jR\omega_j\right)\le c_j$ for $j\in J$. But on the other hand, because of $t=(t_1,t_2,\ldots,t_l)\in\partial[0,1]^l$, there must be some $j_0\in J,t_{j_0}\in\{0,1\}$. Thus $I_{j_0}\le 0$. Hence

$$J_\lambda(\gamma(t))\le\sum_{j=1}^{l}c_j-c_{j_0}+\frac{1}{2}\min_{j\in J}c_j\le\sum_{j=1}^{l}c_j-\frac{1}{2}\rho.$$

By Lemma 4.3 and $c_{\lambda,j}\to c_j$ for $j\in J$, we have $b_{\lambda,j}\ge\sum_{j=1}^{l}c_j-\frac{1}{4}\rho$ when λ is sufficiently large. Combining this and the Palais-Smale condition of J_λ, we conclude that $b_{\lambda,J}$ is a critical value of J_λ by using a standard deformation argument. Therefore, we have

Corollary 4.4. *There exists $k>0$ such that when $|K|_\infty\le k,b_{\lambda,J}$ is a critical value of J_λ for large λ.*

5 Proof of Theorem 1.1

In this section, we find the so-called multi-bump solution u_λ.

Firstly, we define

$$D_\lambda^\nu=\left\{u\in E:\|u\|_{\lambda,\mathbb{R}^3\backslash\Omega_J'}\le\nu,\left|\|u\|_{\lambda,\Omega_j'}-\sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}c_j}\right|\le\nu,\quad j\in J\right\},$$

and

$$J_\lambda^c=\{u\in E:J_\lambda(u)\le c\}.$$

Then we have

Lemma 5.1. *For $0<\nu<\frac{1}{3}\min_{j\in J}\sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}c_j}$, there exist $k_1(\nu)>0$ and $\sigma_0>0$, such that for $\lambda\ge\Lambda_1$ sufficiently large and $u\in\left(D_\lambda^{2\nu}\backslash D_\lambda^\nu\right)\cap J_\lambda^{c_J+\mu}$ we have*

$$\|\nabla J_\lambda(u)\|_\lambda\ge\sigma_0 \tag{5.1}$$

when $|K|_\infty<k(\nu)$. Here

$$\mu=\frac{R^4}{4}\sum_{j=1}^{l}\int_{\Omega_j}K(x)\phi^a_{\sum_{j=1}^{l}\omega_j}(\omega_j)^2\,\mathrm{d}x$$

is defined in (4.3).

Proof. If the conclusion is false, we can assume that there exists $u_n\in\left(D_{\lambda_n}^{2\nu}\backslash D_{\lambda_n}^\nu\right)\cap J_\lambda^{c_J+\mu}$ such that $\|\nabla J_{\lambda_n}(u_n)\|_{\lambda_n}\to0,\lambda_n\to\infty$.

Since $(u_n)\subset J_{\lambda_n}^{c_J+\mu}$, according to Lemma 3.3, we have for some $u\in E$, $\|u_n-u\|_{\lambda_n}\to0$ and

$$\begin{aligned}c_J+\mu&\ge\lim_{n\to\infty}J_{\lambda_n}(u_n)\\&=\frac{1}{2}\int_{\Omega_J}\left(|\nabla u|^2+V_0(x)u^2\right)\mathrm{d}x+\frac{1}{4}\int_{\Omega_J}K(x)\phi_u^a u^2-\frac{1}{p}\int_{\Omega_J}|u|^p\mathrm{d}x\\&=\sum_{j\in J}\frac{1}{2}\int_{\Omega_j}\left(|\nabla u|^2+V_0(x)u^2\right)\mathrm{d}x+\frac{1}{4}\int_{\Omega_j}K(x)\phi_u^a u^2\mathrm{d}x-\frac{1}{p}\int_{\Omega_j}|u|^p\mathrm{d}x,\end{aligned}$$

where $u\equiv0$ in $\mathbb{R}^3\backslash\Omega_J$, and $u_j=u|_{\Omega_j}\in H_0^1(\Omega_j)$ is the weak solutions of $-\Delta v+V_0(x)v+K(x)\phi_u^a v=|v|^{p-2}v$ in Ω_j for $j\in J$. Hence, if $u_j\ne0,j\in J$ and $t_ju_j\in\mathcal{N}_j$, we have

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_j}\left(|\nabla u_j|^2+V_0(x)u_j^2\right)\mathrm{d}x+\frac{1}{4}\int_{\Omega_j}K(x)\phi_u^a u_j^2\mathrm{d}x-\frac{1}{p}\int_{\Omega_j}|u_j|^p\,\mathrm{d}x\\&\quad=\max_{t>0}\frac{t^2}{2}\int_{\Omega_j}\left(|\nabla u_j|^2+V_0(x)u_j^2\right)\mathrm{d}x+\frac{t^4}{4}\int_{\Omega_j}K(x)\phi_u^a u_j^2\mathrm{d}x-\frac{t^p}{p}\int_{\Omega_j}|u_j|^p\,\mathrm{d}x\\&\quad\ge\frac{t_j^2}{2}\int_{\Omega_j}\left(|\nabla u_j|^2+V_0(x)u_j^2\right)\mathrm{d}x+\frac{t_j^4}{4}\int_{\Omega_j}K(x)\phi_u^a u_j^2\mathrm{d}x-\frac{t_j^p}{p}\int_{\Omega_j}|u_j|^p\,\mathrm{d}x\\&\quad\ge I_j(t_ju_j)\ge c_j.\end{aligned}$$

Thus, we have two possibilities:

(1) there exist some $j_0\in J$ such $u_{j_0}=u|_{\Omega_{j_0}}=0$;

(2) $\frac{1}{2}\int_{\Omega_j}\left(|\nabla u_j|^2+V_0(x)u_j^2\right)\mathrm{d}x+\frac{1}{4}\int_{\Omega_j}K(x)\phi_u^a u_j^2\mathrm{d}x-\frac{1}{p}\int_{\Omega_j}|u_j|^p\,\mathrm{d}x\in[c_j,c_j+\mu]$.

When (1) occurs, by Lemma 3.3(d) we obtain

$$\left|\|u_n\|_{\lambda_n,\Omega'_{j_0}}-\sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}c_{j_0}}\right|\to\sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}c_{j_0}}\geq 3\nu,$$

which contradicts to the assumption $u_n\in D_{\lambda_n}^{2\nu}\backslash D_{\lambda_n}^{\nu}$.

If (2) occurs, by Lemma 3.3(b), it is easy to check

$$\begin{aligned}&\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\Omega_j}\left(|\nabla u_j|^2+V_0(x)u_j^2\right)\mathrm{d}x+\left(\frac{1}{4}-\frac{1}{p}\right)\int_{\Omega_j}K(x)\phi_u^a u_j^2\mathrm{d}x\\&\quad=\frac{1}{2}\int_{\Omega_j}\left(|\nabla u_j|^2+V_0(x)u_j^2\right)\mathrm{d}x+\frac{1}{4}\int_{\Omega_j}K(x)\phi_u^a u_j^2\mathrm{d}x-\frac{1}{p}\int_{\Omega_j}|u_j|^p\,\mathrm{d}x.\end{aligned}$$

Thus,

$$\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\Omega_j}\left(|\nabla u|^2+V_0(x)u^2\right)\mathrm{d}x+\left(\frac{1}{4}-\frac{1}{p}\right)\int_{\Omega_j}K(x)\phi_u^a u^2\mathrm{d}x\in\left[c_j,c_j+\mu\right].$$

Since $\|u\|_E\leq M\,(c_J+\mu)$, we can choose $k_1(\nu)>0$ such that for $j\in J$,

(1) $\mu\leq 1$ and $\sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}(c_j+\mu)}\leq\sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}c_j}+\nu$;

(2) $\left[\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}c_j-\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}\left(\frac{1}{4}-\frac{1}{p}\right)\int_{\Omega_j}K(x)\phi_u^a u^2\mathrm{d}x\right]^{1/2}\geq\sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}c_j}-\nu$, when $|K|_\infty<k_1(\nu)$.

Hence we have $\left|\left(\int_{\Omega_j}(|\nabla u|^2+V_0(x)u^2)\mathrm{d}x\right)^{1/2}-\sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}c_j}\right|\leq\nu$. By Lemma 3.3 again we get that $u_n\in D_{\lambda_n}^{2\nu}$ as n is large, which is a contradiction. □

Lemma 5.2. *For $0<\nu<\frac{1}{3}\min_{j\in J}\left(\frac{1}{2}-\frac{1}{p}\right)c_j$, there exists $k(\nu)>0$, such that for $\lambda\geq\Lambda_1$ sufficiently large,* (1.1) *possesses a solution satisfying $u_\lambda\in D_\lambda^\nu$ when $|K|_\infty<k(\nu)$.*

Proof. If the conclusion is false, we assume that J_λ has no critical point in $D_\lambda^\nu\cap J_\lambda^{c_J+\mu}$, here μ is defined as that in Lemma 5.1. Since J_λ satisfies the Palais–Smale condition (see Lemma 3.2), there is a constant $\sigma_\lambda>0$ such that

$$\|\nabla J_\lambda(u)\|_\lambda\geq\sigma_\lambda,\qquad u\in D_\lambda^\nu\cap J_\lambda^{c_J+\mu}.$$

By (5.1) there holds, for $\lambda\geq\Lambda_1$ and $|K|_\infty\leq k_1(\nu)$,

$$\|\nabla J_\lambda(u)\|_\lambda\geq\sigma_0,\qquad u\in\left(D_\lambda^{2\nu}\backslash D_\lambda^\nu\right)\cap J_\lambda^{c_J+\mu}.$$

Combining these, we could define a Lipschitz continuous function $\theta:E\to[0,1]$ such that $\theta(u)=1$ for $u\in D_\lambda^{3\nu/2}$; $\theta(u)=0$ for $u\notin D_\lambda^{2\nu}$. Then, the vector field

$$V:J_\lambda^{c_J+\mu}\to E,V(u)=-\theta(u)\frac{\nabla J_\lambda(u)}{\|\nabla J_\lambda(u)\|_\lambda}$$

is well defined and Lipschitz continuous. And moreover

$$\|V(u)\|_\lambda\leq 1,\qquad u\in E.\tag{5.2}$$

Now we consider the associated gradient flow $\eta : [0,+\infty) \times J_\lambda^{c_J+\mu} \to J_\lambda^{c_J+\mu}$ defined by

$$\frac{\mathrm{d}}{\mathrm{d}s}\eta = V(\eta), \qquad \eta(0,u) = u.$$

By a standard argument, one can show that

$$\frac{\mathrm{d}}{\mathrm{d}s} J_\lambda(\eta(s,u)) = -\theta(u)\left\|\nabla J_\lambda(u)\right\|_\lambda \leq 0 \tag{5.3}$$

and

$$\eta(s,u) = u, \qquad s \geq 0, \quad u \in J_\lambda^{c_J+\mu} \backslash D_\lambda^{2\nu}. \tag{5.4}$$

Recalling $\gamma_0 \in \Gamma_J$, a path which is defined by (4.2). Because of (4.1), we have that

$$\gamma_0(t) \notin D_\lambda^{2\nu}, \qquad t \in \partial[0,1]^l.$$

Therefore, by using (5.4), we have

$$\eta\left(s, \gamma_0(t)\right) = \gamma_0(t), \qquad t \in \partial[0,1]^l.$$

Thus, $\eta\left(s,\gamma_0(\cdot)\right) \in \Gamma_J$ for any $s \geq 0$.

Since $\operatorname{supp}\gamma_0 \subset \bigcup_{j\in J}\overline{\Omega_j}$ for $t \in [0,1]^l$, thus $J_\lambda\left(\gamma_0(t)\right)$ and $\|\gamma_0(t)\|^2_{\lambda,\Omega_j'}$ do not depend on $\lambda \geq 0$. Considering about

$$m_0 = \max\left\{J_\lambda(u) : u \in \gamma_0\left([0,1]^l\right) \backslash D_\lambda^\nu\right\}, \tag{5.5}$$

we also have that m_0 does not depend on $\lambda \geq 0$. Furthermore, we claim that there exists $k(\nu) > 0$ such that

$$m_0 < c_J \tag{5.6}$$

when $|K|_\infty \leq k(\nu)$. In fact, for any $u = \sum_{j=1}^l t_j R\omega_j \in \gamma_0\left([0,1]^l\right) \backslash D_\lambda^\nu$, there must exists some $j_0 \in J$ such that

$$\left| t_{j_0} R \left\|\omega_{j_0}\right\|_{\lambda,\Omega_j'} - \sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1} c_{j_0}} \right| > \nu.$$

According to the definition of ω_{j_0}, we know that $\|\omega_{j_0}\|^2_{\lambda,\Omega_j'} = \left(\frac{1}{2}-\frac{1}{p}\right)^{-1} c_{j_0}$. Thus, $\left|t_{j_0}R - 1\right| > \left(\frac{1}{2}-\frac{1}{p}\right)^{\frac{1}{2}} c_{j_0}^{-\frac{1}{2}}\nu$. So there exists $\delta(\nu) > 0$ such that

$$\frac{t_{j_0}^2 R^2}{2}\int_{\Omega_j}\left(\left|\nabla\omega_{j_0}\right|^2 + V_0(x)\omega_{j_0}^2\right)\mathrm{d}x - \frac{t_{j_0}^p R^p}{p}\int_{\Omega_j}\left|\omega_{j_0}\right|^p \mathrm{d}x < c_{j_0} - \delta(\nu).$$

And consequently,

$$\begin{aligned} J_\lambda(u) &= \sum_{j=1}^l I_j\left(t_j R\omega_j\right) + \frac{1}{4}\sum_{j=1}^l \int_{\Omega_j} K(x)\phi_u^a\left(t_j R\omega_j\right)^2 \mathrm{d}x \\ &< \sum_{j=1}^l c_j - \delta(\nu) + \frac{R^4}{4}\sum_{j=1}^l \int_{\Omega_j} K(x)\phi^a_{\sum_{j=1}^l \omega_j}\omega_j^2 \mathrm{d}x. \end{aligned}$$

Obviously, there is a $k(\nu) > 0$ such that

$$\frac{R^4}{4}\sum_{j=1}^{l}\int_{\Omega_j} K(x)\phi^a_{\sum_{j=1}^{l}\omega_j}\omega_j^2 \mathrm{d}x < \frac{1}{2}\delta(\nu) \quad \text{for } |K|_\infty \le k(\nu).$$

Thus, $J_\lambda(u) < c_J - \frac{1}{2}\delta(\nu)$ and we show the claim.

Next, we prove that there is $k(\nu) > 0$ such that for some $S > 0$ and $|K|_\infty < k(\nu)$,

$$\max_{t\in[0,1]^l} J_\lambda\left(\eta\left(S,\gamma_0(t)\right)\right) \le \max\left\{m_0, c_J - \frac{1}{4}\sigma_0\nu\right\}. \tag{5.7}$$

If this is true, according to Lemma 4.3 and $\eta\left(S,\gamma_0(\cdot)\right) \in \Gamma_J$ we have

$$\sum_{j=1}^{l} c_{\lambda,j} \le b_{\lambda,J} \le \max_{t\in[0,1]^L} J_\lambda\left(\eta\left(S,\gamma_0(t)\right)\right) \le \max\left\{m_0, c_J - \frac{1}{4}\sigma_0\nu\right\} < c_J,$$

which contradicts to the fact $\sum_{j=1}^{l} c_{\lambda,j} \to c_J$. Thus, we obtain the lemma.

Next, we want to prove (5.7). Setting $u = \gamma_0(t) \in E$, if $u \notin D_\lambda^\nu$, because of (5.3) and (5.5), $J_\lambda(\eta(s,u)) \le J_\lambda(u) \le m_0$ for all $s \ge 0$. If $u \in D_\lambda^\nu$, we consider two possibilities:

(1) $\eta(s,u) \in D_\lambda^{3\nu/2}$ for all $s \in [0,S]$;

(2) $\eta(s,u) \in \partial D_\lambda^{3\nu/2}$ for some $s_0 \in [0,S]$.

When (1) occurs, we have $\theta(\eta(s,u)) = 1$ and $\|\nabla J_\lambda(\eta(s,u))\|_\lambda \ge \min\{\sigma_0,\sigma_\lambda\}$ when $|K|_\infty \le k_1(\nu)$ and $\lambda \ge \Lambda_1$ (see Lemma 5.1). Thus, setting $S = \frac{\sigma_0\nu}{2\min\{\sigma_0,\sigma_\lambda\}}$, by (5.3)

$$\begin{aligned} J_\lambda(\eta(S,u)) &= J_\lambda(u) + \int_0^S \frac{\mathrm{d}}{\mathrm{d}s} J_\lambda(\eta(s,u))\mathrm{d}s \\ &= J_\lambda(u) - \int_0^S \theta(\eta(s,u))\,\|\nabla J_\lambda(\eta(s,u))\|_\lambda \,\mathrm{d}s \\ &\le c_J + \mu - S\min\{\sigma_0,\sigma_\lambda\} \\ &= c_J + \mu - \frac{1}{2}\sigma_0\nu. \end{aligned} \tag{5.8}$$

When (2) occurs, there exist $0 < s_1 < s_2 \le S$ such that

$$\begin{aligned} &\eta\left(s_1,u\right) \in \partial D_\lambda^\nu, \\ &\eta\left(s_2,u\right) \in \partial D_\lambda^{3\nu/2}, \\ &\eta(s,u) \in D_\lambda^{3\nu/2}\backslash D_\lambda^\nu, s \in \left(s_1,s_2\right]. \end{aligned} \tag{5.9}$$

So we have, for some $j_0 \in J$,

$$\|\eta\left(s_2,u\right)\|_{\lambda,\mathbb{R}^3\backslash\Omega_J'} = \frac{3}{2}\nu \quad \text{or} \quad \left|\|\eta\left(s_2,u\right)\|_{\lambda,\Omega_{j_0}'} - \sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1} c_{j_0}}\right| = \frac{3}{2}\nu.$$

We only see the latter case and the former one can be dealt with by a similar method. Following from (5.9), we have

$$\left|\|\eta\left(s_1,u\right)\|_{\lambda,\Omega_{j_0}'} - \sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1} c_{j_0}}\right| \le \nu,$$

$$
\begin{aligned}
\|\eta(s_2,u)-\eta(s_1,u)\|_{\lambda,\Omega'_{j_0}} &\geq \left| \|\eta(s_2,u)\|_{\lambda,\Omega'_{j_0}} - \sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1} c_{j_0}} \right| \\
&\quad - \left| \|\eta(s_1,u)\|_{\lambda,\Omega'_{j_0}} - \sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1} c_{j_0}} \right| \\
&\geq \frac{1}{2}\nu.
\end{aligned}
$$

This implies $\|\eta(s_2,u)-\eta(s_1,u)\|_\lambda \geq \frac{1}{2}\nu$.

According to (5.2), $\left\|\frac{\mathrm{d}}{\mathrm{d}s}\eta\right\|_\lambda = \|V(\eta)\|_\lambda \leq 1$. Hence

$$
\frac{1}{2}\nu \leq \|\eta(s_2,u)-\eta(s_1,u)\|_\lambda \leq \left\| \int_{s_1}^{s_2} \frac{\mathrm{d}\eta}{\mathrm{d}s}\mathrm{d}s \right\|_\lambda \leq \int_{s_1}^{s_2} \left\| \frac{\mathrm{d}\eta}{\mathrm{d}s} \right\|_\lambda \mathrm{d}s \leq s_2 - s_1.
$$

According to (5.1), we have

$$
\begin{aligned}
J_\lambda(\eta(S,u)) &= J_\lambda(u) - \int_0^S \theta(\eta(s,u)) \|\nabla J_\lambda(\eta(s,u))\|_\lambda \,\mathrm{d}s \\
&\leq c_J + \mu - \int_{s_1}^{s_2} \sigma_0 \mathrm{d}s \\
&\leq c_J + \mu - \frac{1}{2}\sigma_0\nu.
\end{aligned} \tag{5.10}
$$

Then, we can choose $k(\nu) > 0$ such that $\mu \leq \frac{1}{4}\sigma_0\nu$ if $|K|_\infty \leq k(\nu)$. Combining with (5.8) and (5.10) we get (5.7). And hence J_λ possesses a critical point u_λ in D_λ^ν for $\lambda \geq \Lambda_1$ and $|K|_\infty \leq k(\nu)$. According to Lemma 3.4, we know that u_λ is a solution of (1.1). □

Proof of Theorem 1.1. Setting $u_{\lambda_n}(\lambda_n \to \infty)$ be a sequence of solutions of (1.1) obtained by the procedure above. Then, they are critical points of J_{λ_n} with critical value bounded by $c_J + \mu$. According to Lemma 3.3, we get the conclusion. □

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