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Lagrange stability for a class of impulsive Duffing-type equations with low regularity

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Abstract. We discuss the Lagrange stability for a class of impulsive Duffing equation with time-dependent polynomial potentials. More precisely, we prove that under suitable impulses, all the solutions of the impulsive Duffing equation (with low regularity in time) are bounded for all time and that there are many (positive Lebesgue measure) quasi-periodic solutions clustering at infinity.

Keywords: boundedness, quasi-periodic solution, Moser's twist theorem, impulsive Duffing equation.

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1 Introduction

The stability theory plays a central role in differential equations for its practical value in real world applications. It is well known that the longtime behavior of a time-dependent nonlinear ordinary differential equation can be very intricate. For instance, the well-known Duffing equation,

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t),$$

is an example of dynamical system that exhibits chaotic behavior.

The generalized Duffing-type equation arises in a large class of practically important applied problems in mathematics, physical science and engineering such as the cubic–quintic Duffing oscillatory [9] and the Helmholtz–Duffing oscillator [8], which takes the form of

$$\ddot{x} = \sum_{j\in K} a_j\, x^j(t), \quad K \subset \mathbf{N} \text{ is finite}.$$

See [35] and the references therein for more details.

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1.1 Lagrange stability of Duffing-type equation

In contrast to "Lyapunov stability" that is related to the chaotic nature of the system, we pay special interest in this paper to the Lagrange stability of nonlinear systems, which means that all the solutions stay bounded for all time. The Lagrange stability refers roughly to the stability against the escape of a body from the system. We refer to the classical monograph [14] for more details about the Lagrange stability.

The study of Lagrange stability of Duffing-type equation dates back to Littlewood [16], Moser [20,21] and Morris [19]. In 1987, Dieckerhoff and Zehnder studied the Lagrange stability for the generalized Duffing-type equation with time-dependent polynomial potentials

$$\ddot{x} + x^{2n+1} + \sum_{i=0}^{2n} x^i p_i(t) = 0, \quad n \geq 1, \tag{1.1}$$

where $p_i \in C^\nu(\mathbf{T^1})$ $(0 \leq i \leq 2n)$ are 1-periodic with $\mathbf{T^1} = \mathbf{R}/\mathbf{Z}$, and proved that every solution $x(t)$ of (1.1) is bounded for all time, i.e., the solution $x(t)$ exists for all $t \in \mathbf{R}$ and $\sup_{t\in\mathbf{R}}(|x(t)| + |\dot{x}(t)|) < \infty$, if ν is the smallest integer satisfying

$$v > 1 + \frac{4}{n} + [\log_2 n] \to \infty \quad \text{as } n \to \infty.$$

There exist a lot of papers [12,15,17,18,32–34] devoting to the relaxation of the smoothness of p_i in (1.1) with respect to the t-variable when studying the Lagrange stability. However, there is an example in [31] showing that a continuous coefficient would result in an unbounded solution.

As we know, a abrupt change at certain instant during the evolution process falls into the scope of the impulsive dynamical system [1,13], which appear widely in applied mathematics. The appearance of impulse forces may cause complicated dynamic phenomenons and bring difficulties to study. There are many studies on the existence of periodic solutions of impulsive differential equations [2,7,10,22–24] via different approaches. See also [11,26] for the periodic solution of impulsive Duffing-type equation.

However, there are only few results on the Lagrange stability and the existence of quasi-periodic solutions for impulsive differential equations (see [3–5,25,30]). Coming back to the Duffing-type equation (1.1), the term $\sum_{i=0}^{2n} x^i p_i(t)$ can be regarded as the perturbation of $\ddot{x} + x^{2n+1} = 0$ (up to some transformations). Then the Lagrange stability of (1.1) show that all solutions of nonlinear equation $\ddot{x} + x^{2n+1} = 0$ is bounded under a periodic perturbation. It is very natural to ask the following question:

"what happens when the nonlinear equation $\ddot{x} + x^{2n+1} = 0$ is subject to both periodic perturbation and an impulse at the same time?"

Choosing different impulsive functions may have different effects on the solutions. It is also not surprising that an offhand choice of impulse force would destroy the Lagrange stability even though the coefficients p_i are sufficiently smooth. To establish the Lagrange stability of impulsive Duffing-type equation, one needs to be careful on the impulse such that the Poincaré map can be well organized in order to apply Moser's twist theorem after some symplectic transformations. We mention some progress in this respect. In 2019, [30] proved the boundedness of solutions and the existence of quasi-periodic solutions for the impulsive

Duffing equation

$$\begin{cases} \ddot{x} + x^{2n+1} + \sum_{i=0}^{2n} x^i p_i(t) = 0, \quad t \neq t_j, \quad n \geq 1, \\ \Delta x(t_j) := x(t_j^+) - x(t_j^-) = I_j(x(t_j^-), \dot{x}(t_j^-)), \\ \Delta \dot{x}(t_j) := \dot{x}(t_j^+) - \dot{x}(t_j^-) = J_j(x(t_j^-), \dot{x}(t_j^-)), \quad j = \pm 1, \pm 2, \ldots \end{cases} \tag{1.2}$$

with the *low regularity in time*

$$p_i(t) \in C^{\gamma}(\mathbf{T^1}), \quad \gamma > 2 - \frac{1}{n},$$

and with the general sequences of impulsive functions $I_j, J_j : \mathbf{R^2} \to \mathbf{R}$, where $\mathbf{T^1} := \mathbf{R}/\mathbf{Z}$. Moreover, the following restricted conditions are needed: for $j = 1, \ldots, k$,

(i) the jumps $I_j(x, y) = o(1)$ as $x^2 + y^2 \to +\infty$;

(ii) the jump map $\Phi_j : (x, y) \to (x, y) + (I_j(x, y), J_j(x, y))$ is an area-preserving homeomorphism,

which enables us to apply Moser's twist theorem. See [30, Remark 2.1] for the comparison of different types of impulse forces and their roles when studying the Lagrange stability.

For the particular case of cubic Duffing-type equation, [25] extended the Morris's boundedness result [19] to the impulsive Duffing equation

$$\begin{cases} \ddot{x} + x^3 + p(t) = 0, \quad t \neq t_j, \\ \Delta x(t_j) := x(t_j^+) - x(t_j^-) = I(x(t_j^-), \dot{x}(t_j^-)), \\ \Delta \dot{x}(t_j) := \dot{x}(t_j^+) - \dot{x}(t_j^-) = J(x(t_j^-), \dot{x}(t_j^-)), \quad j = \pm 1, \pm 2, \ldots, \end{cases}$$

where $0 < t_1 < 1, t_{j+1} = t_j + 1$ for $j = \pm 1, \pm 2, \cdots$ and $p(t)$ is 1-periodic and integrable.

In 2020, [3] proposed some concrete and simple impulse forces, which do not satisfy the above conditions (i) and (ii), and proved the Lagrange stability and the existence of quasi-periodic solutions for the impulsive Duffing-type equation

$$\begin{cases} \ddot{x} + x^{2n+1} + \sum_{i=0}^{n} x^i p_i(t) = 0, \quad t \neq t_j, \quad n \geq 1, \\ \Delta x(t_j) = (\gamma_j - 1)\ x(t_j^-), \\ \Delta \dot{x}(t_j) = (\gamma_j^{n+1} - 1)\ \dot{x}(t_j^-)), \quad j = \pm 1, \pm 2, \ldots, \end{cases} \tag{1.3}$$

where $\gamma_j > 0$ are some constants and the coefficients $p_i \in C^{\infty}(\mathbf{T^1})$ for technical simplicity.

In this paper, we pay special attention to the *sharp regularity* of the coefficients $p_i(t)$ in the Duffing-type equation, together with the impulse forces given by (1.3), to establish the Lagrange stability. More precisely, we will prove the Lagrange stability and the existence of quasi-periodic solutions for (1.3) with low regularity in time

$$p_i(t) \in C^{\gamma}(\mathbf{T^1}) \quad (i = 0, \ldots, n), \quad \gamma > 1 - \frac{1}{n}.$$

1.2 Main result

To formulate our main result we have to introduce some notations and hypotheses. Let $\mathbf{R}, \mathbf{C}, \mathbf{N}$ and $\mathbf{Z}$ be the sets of all real numbers, complex numbers, natural numbers and integers, respectively. Denote by $\mathcal{T}$ the impulsive time sequence $\{t_j\}, j = \pm 1, \pm 2, \ldots$, and denote by $\mathcal{A}$ the set of indexes j. We assume that the following condition **(H)** holds true.

(H) There exists a positive integer k such that $0 < t_1 < t_2 < \cdots < t_k < 1$, and that t_j's, γ_j's are 1-periodic in j in the sense that $t_{j+k} = t_j + 1, \gamma_{j+k} = \gamma_j$ for $j \le -(k+1)$ or $j \ge 1$; $t_{j+k+1} = t_j + 1, \gamma_{j+k+1} = \gamma_j$ for $-k \le j \le -1$.

The main result in this paper is the following theorem.

Theorem 1.1. *Suppose that condition* (**H**) *holds and that for each* $0 \le i \le n$*, there is* $p_i(t) \in C^\gamma(\mathbf{T^1})$ *with* $\gamma > 1 - \frac{1}{n}$*. In addition, assume that*

$$\prod_{j=1}^{k} \gamma_j = 1. \tag{1.4}$$

*Then the time-*1 *map* $\widetilde{P} : (x, \dot{x})_{t=0} \mapsto (x, \dot{x})_{t=1}$ *of* (1.3) *possesses many* (*positive Lebesgue measure*) *invariant closed curves whose radiuses tend to infinity, and thus every solution* $x(t)$ *of* (1.3) *is bounded for all time, i.e. it exists for all* $t \in \mathbf{R}$ *and*

$$\sup_{t\in\mathbf{R}}(|x(t)| + |\dot{x}(t)|) \le \widetilde{C} < +\infty,$$

where $\widetilde{C} = \widetilde{C}(x(0), \dot{x}(0))$ *depends on the initial data* $(x(0), \dot{x}(0))$*.*

Remark 1.2. In equation (1.3), the jump maps $\Phi_j : (x, y) \mapsto (x, y) + ((\gamma_j - 1)x, (\gamma_j^{n+1} - 1)y)$ are homeomorphisms which are not area-preserving (when $\gamma_j \ne 1$), and $|I_j(x, y)| = |(\gamma_j - 1)x| = O(|x|)$ (when $\gamma_j \ne 1$). Thus, the conditions (i) and (ii) mentioned above in [30] are not satisfied.

Remark 1.3. Equation (1.1) can be written as a Hamiltonian system with the Hamiltonian function $H = h_0(x, y) + R(x, y, t)$. It is essential to regard R as a relatively small perturbation with respect to h_0. See [15] for the detail. Otherwise, the stability might have been violated even without the impulse. Note also that the Duffing-type equation in (1.3) is simpler than (1.1) since the terms $p_i(t)x^i$ $(n + 1 \le i \le 2n)$ are absent. For the general case of equation (1.1) under the impulse given by (1.3), we refer to [3] for some discussions on the obstruction when establishing the Lagrange stability.

Remark 1.4. When using KAM theory to (1.2), one of the main difficulties is the estimation of "small property condition" of Moser's twist theorem. In [30], the difficult was overcome when the smoothness in time $p_i \in C^\gamma(\mathbf{T^1})$ with $\gamma > 2 - \frac{1}{n}$ is used. However, for equation (1.3), we observe that the *smoothness* can be relaxed to $C^\gamma(\mathbf{T^1})$ with $\gamma > 1 - \frac{1}{n}$, which is closely related to the almost sharp result in [34]. Our method is also based on the approximation techniques used in [34].

The paper is organized as follows. In Section 2, we establish the global existence of solutions for impulsive differential equations (1.3) and construct the associated time-one map. In Section 3, we introduce the action-angle variables and apply a preliminary symplectic transformation such that (1.3) becomes a nearly integrable Hamiltonian system. In Section 4, we introduce the approximate lemma to approximate the smooth periodic function by a real analytic function. In Section 5, we take further symplectic transformations such that Moser's twist theorem can be applied. In Section 6, we establish some estimates for the impulsive functions after the transformation. Finally, in Section 7, we prove Theorem 1.1 by employing Moser's twist theorem.

2 Global existence of solutions and time-one map

In this section, we establish the global existence of solutions for impulsive differential equations (1.3) and construct the associated time-one map. We begin with the general impulsive differential equation

$$\begin{cases} \dot{u} = F(t,u), \quad t \neq t_j, \\ \Delta u(t_j) := u(t_j^+) - u(t_j) = L_j(u(t_j)), \quad j \in \mathcal{A} \end{cases} \tag{2.1}$$

with the initial value condition

$$u(\tau^+) = u_0, \tag{2.2}$$

where $\tau \in \mathbf{R}, u_0 \in \mathbf{R^m}, m \in \mathbf{N}$, and where $u(\tau^+) = u(\tau)$ if $\tau \notin \mathcal{T}$. Suppose that

($\mathbf{H_1}$) The function $F : \mathbf{R} \times \mathbf{R^m} \mapsto \mathbf{R^m}$ is continuous, locally Lipschitz in the second variable.

($\mathbf{H_2}$) The function F is 1-periodic in the first variable. There exists a positive integer k such that $0 < t_1 < t_2 < \cdots < t_k < 1$, $t_{j+k} = t_j + 1$, $L_{j+k}(\cdot) = L_j(\cdot)$ for $j \leq -(k+1)$ or $j \geq 1$; $t_{j+k+1} = t_j + 1$, $L_{j+k+1}(\cdot) = L_j(\cdot)$ for $-k \leq j \leq -1$.

($\mathbf{H_3}$) The impulsive functions $L_j : \mathbf{R^m} \to \mathbf{R^m}$ are continuous for all $j \in \mathcal{A}$.

Lemma 2.1 ([30, Lemma 3.2]). *Assume that the conditions* ($\mathbf{H_1}$)–($\mathbf{H_3}$) *hold and that every jump equation*

$$u = v + L_j(v), \quad u \in \mathbf{R^m}, \qquad j = 1, \ldots, k, \tag{2.3}$$

has a unique solution with respect to $v \in \mathbf{R^m}$. Assume in addition that all the solutions of the unforced equation $\dot{u} = F(t,u)$ exist for all $t \in \mathbf{R}$. Then the following conclusions hold true.

(a) *For any $\tau \in \mathbf{R}, u_0 \in \mathbf{R^m}$, there is a unique solution $u = u(t;\tau,u_0)$ of (2.1) satisfying the initial value condition (2.2), and it exists for all $t \in \mathbf{R}$.*

(b) *If the equation $\dot{u} = F(t,u)$ is conservative and the impulsive maps $\aleph_j : u \mapsto u + L_j(u) (j \in \mathcal{A})$ are homeomorphisms of $\mathbf{R^m}$, then for $t \in \mathbf{R} \setminus \mathcal{T}$, the map $Q_t : u_0 \mapsto u(t;\tau,u_0)$ is also a homeomorphism.*

(c) *The solution satisfies the elastic property. That is, for any $b_0 > 0$, there is $r_{b_0} > 0$ such that the inequality $|u_0| \geq r_{b_0}$ implies $|u(t;\tau,u_0)| \geq b_0$, for $t \in (\tau, \tau+1]$.*

In order to deduce a global existence result of the impulsive Duffing equation (1.3), by letting $y = \dot{x}$ and noting that $x(t_j^-) = x(t_j), y(t_j^-) = y(t_j)$, we can rewrite equation (1.3) as an equivalent system of the form

$$\begin{cases} \dot{x} = y, \quad t \neq t_j, \\ \dot{y} = -x^{2n+1} - \sum_{i=0}^{n} p_i(t) x^i, \quad t \neq t_j; \\ \Delta x(t_j) = I_j(x(t_j), y(t_j)) = (\gamma_j - 1) x(t_j), \\ \Delta y(t_j) = J_j(x(t_j), y(t_j)) = (\gamma_j^{n+1} - 1) y(t_j), \quad j = 1, 2, \ldots, k. \end{cases} \tag{2.4}$$

For (2.4), each jump map

$$\widetilde{\aleph}_j : \begin{cases} u = x + I_j(x,y), \\ v = y + J_j(x,y), \quad j = 1, \ldots, k \end{cases} \tag{2.5}$$

is a homeomorphism on $\mathbf{R}^2$. Note also that every solution $(x(t), y(t))$ of the unforced Duffing equation

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x^{2n+1} - \sum_{i=0}^{n} p_i(t)x^i \end{cases}$$

satisfying the initial value condition $(x(t_0), y(t_0)) = (x_0, y_0)$ is unique and exists for all $t \in \mathbf{R}$. Then using the implicit function theorem and Lemma 2.1, we obtain the following corollary.

Corollary 2.2. *Suppose that condition* $(\mathbf{H})$ *holds and that for each* $0 \le i \le n, p_i(t) \in C^\gamma(\mathbf{T}^1)$ *with* $\gamma > 1 - \frac{1}{n}$. *Then the following statements hold.*

(a) *For any* $\tau \in \mathbf{R}$, $(x_0, y_0) \in \mathbf{R}^2$, *there is a unique solution*

$$(x(t), y(t)) = (x(t; \tau, x_0, y_0), y(t; \tau, x_0, y_0))$$

of (2.4) *satisfying the initial condition* $x(\tau^+) = x_0, y(\tau^+) = y_0$, *which exists for all* $t \in \mathbf{R}$.

(b) *The map* $Q_t : (x_0, y_0) \mapsto (x(t; \tau, x_0, y_0), y(t; \tau, x_0, y_0))$ *is continuous in* (x_0, y_0) *for* $t \in \mathbf{R} \setminus \mathcal{T}$.

(c) *The solution satisfies the elastic property. More precisely, for any* $b_0 > 0$, *there is* $r_{b_0} > 0$ *such that the inequalities* $|x_0| \ge r_{b_0}, |y_0| \ge r_{b_0}$ *implies that* $|x(t; \tau, x_0, y_0)| \ge b_0$ *and* $|y(t; \tau, x_0, y_0)| \ge b_0$ *for* $t \in (\tau, \tau + 1]$.

In order to deduce the time-one map of impulsive Duffing equation (2.4), we denote by $(x(t), y(t)) = (x(t; x_0, y_0), y(t; x_0, y_0))$ the solution of (2.4) satisfying the initial condition $(x(0), y(0)) = (x_0, y_0)$. Let

$$\begin{aligned}
\widetilde{P}_0 &: (x_0, y_0) \mapsto (x(t_1), y(t_1)) := (x_1, y_1), \\
\Phi_1 &: (x_1, y_1) \mapsto (x_1 + I_1(x_1, y_1), y_1 + J_1(x_1, y_1)) = (x(t_1^+), y(t_1^+)) := (x_1^+, y_1^+), \\
\widetilde{P}_1 &: (x_1^+, y_1^+) \mapsto (x(t_2), y(t_2)) := (x_2, y_2), \\
\Phi_2 &: (x_2, y_2) \mapsto (x_2 + I_2(x_2, y_2), y_2 + J_2(x_2, y_2)) = (x(t_2^+), y(t_2^+)) := (x_2^+, y_2^+), \\
&\vdots \\
\widetilde{P}_{k-1} &: (x_{k-1}^+, y_{k-1}^+) \mapsto (x(t_k), y(t_k)) := (x_k, y_k), \\
\Phi_k &: (x_k, y_k) \mapsto (x_k + I_k(x_k, y_k), y_k + J_k(x_k, y_k)) = (x(t_k^+), y(t_k^+) := (x_k^+, y_k^+), \\
\widetilde{P}_k &: (x_k^+, y_k^+) \mapsto (x(1), y(1)).
\end{aligned}$$

Then the time-one map $\widetilde{P} : (x_0, y_0) \mapsto (x(1), y(1))$ of (2.4) can be expressed by

$$\widetilde{P} = \widetilde{P}_k \circ \Phi_k \circ \cdots \circ \widetilde{P}_1 \circ \Phi_1 \circ \widetilde{P}_0.$$

Remark 2.3. Under the condition $(\mathbf{H})$, since the impulsive maps $\Phi_j : (x, y) \mapsto (x, y) + (I_j(x, y), J_j(x, y))$, $(j = 1, 2, \ldots, k)$ are homeomorphisms on $\mathbf{R}^2$, the time-one map $\widetilde{P}$ of (2.4) is also a homeomorphism on $\mathbf{R}^2$.

From Corollary 2.2 and Remark 2.3, we have the following result.

Corollary 2.4. *Suppose that the condition* $(\mathbf{H})$ *holds and that for each* $0 \le i \le n$ *there is* $p_i(t) \in C^\gamma(\mathbf{T}^1)$ *with* $\gamma > 1 - \frac{1}{n}$. *Then the time-one map* $\widetilde{P}$ *of* (2.4) *is a homeomorphism on* $\mathbf{R}^2$. *Moreover, for any* $b_0 > 0$, *there is* $r_{b_0} > 0$ *such that the inequalities* $|x_0| \ge r_{b_0}, |y_0| \ge r_{b_0}$ *implies that* $|x(1; x_0, y_0)| \ge b_0$ *and* $|y(1; x_0, y_0)| \ge b_0$.

3 Action-angle variables

In this section, we introduce the action-angle variables and apply a preliminary symplectic transformation such that (1.3) becomes a nearly integrable Hamiltonian system. The transformations are standard for the Duffing equation and can be found in [6, 30, 34] for instance. Let

$$x = AX, \tag{3.1}$$

$$Y = A^{-n}\dot{X} = A^{-n-1}\dot{x} = A^{-n-1}y, \tag{3.2}$$

$$\Delta X(t_j) = X(t_j^+) - X(t_j), \quad \Delta Y(t_j) = Y(t_j^+) - Y(t_j). \tag{3.3}$$

Then we see from equation (1.3) that

$$\begin{cases} \dot{X} = \frac{\partial H^*}{\partial Y}, \quad t \neq t_j, \\ \dot{Y} = -\frac{\partial H^*}{\partial X}, \quad t \neq t_j, \\ \Delta X(t_j) = (\gamma_j - 1)X(t_j) := \tilde{I}_j(X(t_j), Y(t_j)), \\ \Delta Y(t_j) = (\gamma_j^{n+1} - 1)Y(t_j) := \tilde{J}_j(X(t_j), Y(t_j)), \end{cases} \tag{3.4}$$

where $j = 1, 2, \ldots, k$ and

$$H^*(X, Y, t) = A^n \left(\frac{1}{2}Y^2 + \frac{1}{2(n+1)}X^{2(n+1)} \right) + \sum_{i=0}^{n} \frac{p_i(t)}{i+1} A^{i-n-1} X^{i+1}. \tag{3.5}$$

The similar formulation of (3.4) can be also found in Section 5 in [30].

Consider the auxiliary Hamiltonian system

$$\dot{X} = \frac{\partial H_0^*}{\partial Y}, \quad \dot{Y} = -\frac{\partial H_0^*}{\partial X}, \tag{3.6}$$

where

$$H_0^*(X, Y) = \frac{1}{2}Y^2 + \frac{1}{2(n+1)}X^{2(n+1)}.$$

Let $(X_0(t), Y_0(t))$ be the solution of (3.6) with initial $(X_0(0), Y_0(0)) = (1, 0)$. Then this solution is clearly periodic. Let T_0 be its minimal positive period. By the energy conservation, we have

- (s_1) $(n+1)Y_0^2(t) + X_0^{2n+2}(t) \equiv 1$;
- (s_2) $X_0(-t) = X_0(t)$, $Y_0(-t) = -Y_0(t)$;
- (s_3) $\dot{X}_0(t) = Y_0(t)$, $\dot{Y}_0(t) = -X_0^{2n+1}(t)$;
- (s_4) $X_0(t + T_0) = X_0(t)$, $Y_0(t + T_0) = Y_0(t)$.

We construct the following symplectic transformation

$$\Psi_0 : X = c^\alpha \lambda^\alpha X_0(\theta T_0), \quad Y = c^\beta \lambda^\beta Y_0(\theta T_0), \tag{3.7}$$

where $\alpha = \frac{1}{n+2}$, $\beta = 1 - \alpha = \frac{n+1}{n+2}$, $c = \frac{1}{\alpha T_0}$ and $(\lambda, \theta) \in \mathbf{R}^+ \times \mathbf{T}^1$ is the action-angle variables. By calculation, the Jacobian determinant det $\frac{\partial(X,Y)}{\partial(\theta,\lambda)} = 1$. Then the transformation Ψ_0 is indeed symplectic.

By (3.7), we have

$$\lambda = \frac{1}{c}[X^{2n+2} + (n+1)Y^2]^{\frac{n+2}{2n+2}}. \tag{3.8}$$

We claim that there exists the inverse function $\widetilde{X}_0^{-1}$ such that $\theta = \widetilde{X}_0^{-1}(c^{-\alpha}\lambda^{-\alpha}X)$. Indeed, from (3.7) we have $X_0(\theta T_0) = c^{-\alpha}\lambda^{-\alpha}X$. In the case of $\theta \in [0, \frac{1}{2}]$, by ($s_3$) we get $\frac{dX_0(\theta T_0)}{d\theta} = T_0 Y_0(\theta T_0) < 0$. Thus, we have

$$\theta = T_0^{-1}X_0^{-1}(c^{-\alpha}\lambda^{-\alpha}X).$$

In the case of $\theta \in (\frac{1}{2}, 1)$, by using (3.7), ($s_2$) and ($s_4$), we have

$$X = c^{\alpha}\lambda^{\alpha}X_0(\theta T_0) = c^{\alpha}\lambda^{\alpha}X_0(-\theta T_0) = c^{\alpha}\lambda^{\alpha}X_0((1-\theta)T_0).$$

Let $\xi = 1-\theta$ and we have $\frac{dX_0(\xi T_0)}{d\xi} = T_0 Y_0(\xi T_0) < 0$ for $\xi \in (0, \frac{1}{2})$. Then we get $\xi = T_0^{-1}X_0^{-1}(c^{-\alpha}\lambda^{-\alpha}X)$ and thus

$$\theta = 1 - T_0^{-1}X_0^{-1}(c^{-\alpha}\lambda^{-\alpha}X).$$

From (3.4), we have that for $j = 1, 2, \dots, k$

$$\begin{cases} X(t_j^+) = X(t_j) + \tilde{I}_j(X(t_j), Y(t_j)) = \gamma_j X(t_j), \\ Y(t_j^+) = Y(t_j) + \tilde{J}_j(X(t_j), Y(t_j)) = \gamma_j^{n+1} Y(t_j). \end{cases} \tag{3.9}$$

Then using (1.3), (3.7)–(3.9), we have that

$$\begin{aligned} \Delta\lambda(t_j) &= \lambda(t_j^+) - \lambda(t_j) \\ &= \frac{1}{c}[X^{2n+2}(t_j^+) + (n+1)Y^2(t_j^+)]^{\frac{n+2}{2n+2}} - \lambda(t_j) \\ &= \frac{1}{c}\{[X(t_j)+\widetilde{I}_j(X(t_j),Y(t_j))]^{2n+2}+(n+1)[Y(t_j)+\widetilde{J}_j(X(t_j),Y(t_j))]^2\}^{\frac{n+2}{2n+2}} - \lambda(t_j) \\ &= \frac{1}{c}\{[\gamma_j X(t_j)]^{2n+2} + (n+1)[\gamma_j^{n+1}Y(t_j)]^2\}^{\frac{n+2}{2n+2}} - \lambda(t_j) \\ &= \frac{1}{c}\{\gamma_j^{2n+2}[X^{2n+2}(t_j) + (n+1)Y^2(t_j)]\}^{\frac{n+2}{2n+2}} - \lambda(t_j) \\ &= \gamma_j^{n+2}\lambda(t_j) - \lambda(t_j) = (\gamma_j^{n+2} - 1)\lambda(t_j) \\ &=: J_j^*(\lambda(t_j), \theta(t_j)) \end{aligned} \tag{3.10}$$

for $j = 1, 2, \dots, k$.

By using (3.7), we have that for $j = 1, 2, \dots, k$ there is

$$X(t_j) = c^{\frac{1}{n+2}}\lambda^{\frac{1}{n+2}}(t_j)X_0(\theta(t_j)T_0), \qquad Y(t_j) = c^{\frac{n+1}{n+2}}\lambda^{\frac{n+1}{n+2}}(t_j)X_0(\theta(t_j)T_0), \tag{3.11}$$

and

$$X(t_j^+) = c^{\frac{1}{n+2}}\lambda^{\frac{1}{n+2}}(t_j^+)X_0(\theta(t_j^+)T_0), \qquad Y(t_j^+) = c^{\frac{n+1}{n+2}}\lambda^{\frac{n+1}{n+2}}(t_j^+)X_0(\theta(t_j^+)T_0). \tag{3.12}$$

Then using (3.10) and (3.12) we have that for $j = 1, 2, \dots, k$,

$$\begin{aligned} X(t_j^+) &= c^{\frac{1}{n+2}}[\lambda(t_j) + J_j^*(\lambda(t_j), \theta(t_j))]^{\frac{1}{n+2}}X_0(\theta(t_j^+)T_0) \\ &= c^{\frac{1}{n+2}}[\lambda(t_j) + (\gamma_j^{n+2} - 1)(\lambda(t_j)]^{\frac{1}{n+2}}X_0(\theta(t_j^+)T_0) \\ &= \gamma_j c^{\frac{1}{n+2}}\lambda^{\frac{1}{n+2}}(t_j)X_0(\theta(t_j^+)T_0). \end{aligned} \tag{3.13}$$

Combining $\gamma_j > 0$, (3.9), (3.12) and (3.13), we have that for $j = 1, 2, \dots, k$,

$$X(t_j) = c^{\frac{1}{n+2}}\lambda^{\frac{1}{n+2}}(t_j)X_0(\theta(t_j^+)T_0). \tag{3.14}$$

Similarly, by (1.3), (3.9), (3.10) and (3.12), we have that for $j = 1, 2, \dots, k$,

$$Y(t_j^+) = \gamma_j^{n+1} c^{\frac{n+1}{n+2}} \lambda^{\frac{n+1}{n+2}}(t_j) Y_0(\theta(t_j^+)T_0) = \gamma_j^{n+1} Y(t_j),$$

and thus

$$Y(t_j) = c^{\frac{n+1}{n+2}} \lambda^{\frac{n+1}{n+2}}(t_j) Y_0(\theta(t_j^+)T_0). \tag{3.15}$$

By (3.11), (3.14) and (3.15), we have $\theta(t_j^+) = \theta(t_j)$. Then, for $j = 1, 2, \dots, k$,

$$\Delta\theta(t_j) = \theta(t_j^+) - \theta(t_j) = 0 := I_j^*(\lambda(t_j), \theta(t_j)). \tag{3.16}$$

As a result, under the transformation Ψ_0, system (3.4) is changed into

$$\begin{cases} \dot{\theta} = \frac{\partial H}{\partial \lambda}, \quad t \neq t_j, \\ \dot{\lambda} = -\frac{\partial H}{\partial \theta}, \quad t \neq t_j, \\ \Delta\theta(t_j) = I_j^*(\lambda(t_j), \theta(t_j)), \\ \Delta\lambda(t_j) = J_j^*(\lambda(t_j), \theta(t_j)), \quad j = 1, 2, \dots, k, \end{cases} \tag{3.17}$$

where $I_j^*(\lambda(t_j), \theta(t_j)) = 0, J_j^*(\lambda(t_j), \theta(t_j)) = (\gamma_j^{n+2} - 1)\lambda(t_j)$ and $H(\lambda, \theta, t) = H_0(\lambda) + R(\lambda, \theta, t)$ with

$$H_0(\lambda) = d \cdot A^n \cdot \lambda^{\frac{2(n+1)}{n+2}}, \qquad d = \frac{1}{2(n+1)} c^{\frac{2(n+1)}{n+2}}$$

and

$$R(\lambda, \theta, t) = \sum_{i=0}^{n} \frac{p_i(t)}{i+1} A^{i-n-1} (c^\alpha X_0(\theta T_0))^{i+1} \lambda^{\alpha(i+1)}.$$

4 Approximation lemma

In this section, we make use of the Jackson–Moser–Zehnder approximate lemma (see [28,29,34] for the detail) to approximate the smooth periodic function R by a real analytic periodic function R_ε. Some estimates of R_ε and the remainder $R^\varepsilon = R - R_\varepsilon$ are also given for the later application.

Let $\mathbf{T}^1_\varepsilon = \{t \in \mathbf{C}/\mathbf{Z} : |\mathrm{Im}\, t| < \varepsilon\}$ for any $\varepsilon > 0$. By the Jackson–Moser–Zehnder lemma (see Lemma 6.1 in [30]), for each $p_i \in C^\gamma(\mathbf{T^1})$, $i = 0, 1, \dots, n$, and any $\varepsilon > 0$, there is a real analytic function (a complex value function $f(t)$ of complex variable t in some domain in $\mathbf{C}$ is called real analytic if it is analytic in the domain and is real for real argument t) $p_{i,\varepsilon}(t)$ from $\mathbf{T}^1_\varepsilon$ to $\mathbf{C}$ such that

$$\sup_{t \in \mathbf{T^1}} |p_{i,\varepsilon}(t) - p_i(t)| \leq C\varepsilon^\gamma \|p_i\|_{C^\gamma}$$

and

$$\sup_{t \in \mathbf{T}^1_\varepsilon} |p_{i,\varepsilon}(t)| \leq C\|p_i\|_{C^\gamma}.$$

Write

$$R(\lambda, \theta, t) = R_\varepsilon(\lambda, \theta, t) + R^\varepsilon(\lambda, \theta, t),$$

where

$$R_\varepsilon(\lambda, \theta, t) = \sum_{i=0}^{n} \frac{1}{i+1} A^{i-n-1} c^{\frac{i+1}{n+2}} X_0^{i+1}(\theta T_0) \lambda^{\frac{i+1}{n+2}} p_{i,\varepsilon}(t),$$

$$R^{\varepsilon}(\lambda,\theta,t)=\sum_{i=0}^{n}\frac{1}{i+1}A^{i-n-1}c^{\frac{i+1}{n+2}}X_0^{i+1}(\theta T_0)\lambda^{\frac{i+1}{n+2}}(p_i(t)-p_{i,\varepsilon}(t)).$$

Then, we have

$$H=H_0(\lambda)+R_{\varepsilon}(\lambda,\theta,t)+R^{\varepsilon}(\lambda,\theta,t), \tag{4.1}$$

where

$$H_0(\lambda)=d\cdot A^n\cdot\lambda^{\frac{2(n+1)}{n+2}},\qquad d=\frac{1}{2(n+1)}c^{\frac{2(n+1)}{n+2}}. \tag{4.2}$$

We introduce two definitions.

Definition 4.1. Given constants p and q, for a complex valued function $f=f(\lambda,\theta,t,A)$: $(\lambda,\theta,t)\in[1,+\infty)\times\mathbf{T^1}\times\mathbf{T}^1_{\varepsilon}\to\mathbf{C}$, where $A\gg 1$ is a large constant, we say that

$$f=O_{\varepsilon}(A^p\lambda^q),$$

if f is C^{∞} in $(\lambda,\theta)\in[1,+\infty)\times\mathbf{T^1}$ and is analytic in $t\in\mathbf{T}^1_{\varepsilon}$ and for all nonnegative integers k and l, there is

$$\sup_{(\theta,t)\in\mathbf{T^1}\times\mathbf{T}^1_{\varepsilon}}|(D_\lambda)^k(D_\theta)^l f(\lambda,\theta,t,A)|<C_{k,l}A^p\lambda^{q-k},\quad \lambda\gg 1,$$

where $C_{k,l}$ is a constant depending on k and l.

Definition 4.2. Given constants p and q, for a function $f=f(\lambda,\theta,t,A)$: $(\lambda,\theta,t)\in[1,+\infty)\times\mathbf{T^1}\times\mathbf{T^1}\to\mathbf{R}$, where $A\gg 1$ is a large constant, we say that

$$f=O(A^p\lambda^q),$$

if f is C^{∞} in $(\lambda,\theta)\in[1,+\infty)\times\mathbf{T^1}$ and C^1 in $t\in\mathbf{T^1}$ and for all nonnegative integers k and l, there is

$$\sup_{(\theta,t)\in\mathbf{T^1}\times\mathbf{T^1}}|(D_\lambda)^k(D_\theta)^l f(\lambda,\theta,t,A)|<C_{k,l}A^p\lambda^{q-k},\quad \lambda\gg 1,$$

where $C_{k,l}$ is a constant depending on k and l.

Lemma 4.3.

(i) *If $f_1=O(A^{p_1}\lambda^{q_1}), f_2=O(A^{p_2}\lambda^{q_2})$, then $f_1\cdot f_2=O(A^{p_1+p_2}\lambda^{q_1+q_2})$;*

(ii) *If $f=O(A^p\lambda^{q_1}), g(\lambda)=O(\lambda^{q_2})$ satisfy $|g(\lambda)|\ge c\lambda^{q_2}$ for $\lambda\ge\lambda_0$, and $c>0, q_2>0$, then $f^*(\lambda,\theta,t):=f(g(\lambda),\theta,t)=O(A^p\lambda^{q_1q_2})$;*

(iii) *If $f=O(A^p\lambda^q), u=O(A^p\lambda^{q_1}), v=O(A^p\lambda^{q_2})$ and $q_1<1, q_2<0$, then $f^{**}(\lambda,\theta,t):=f(\lambda+u,\theta+v,t)=O(A^p\lambda^q)$.*

Proof. (i). Since

$$(D^k_\lambda D^l_\theta)(f_1\cdot f_2)=\sum_{i=0}^{k}\sum_{j=0}^{l}C^i_kC^j_l(D^{k-i}_\lambda D^{l-j}_\theta f_1)\cdot(D^i_\lambda D^j_\theta f_2),$$

by Definition 4.2, it follows that

$$f_1\cdot f_2=O(A^{p_1+p_2}\lambda^{q_1+q_2}).$$

(ii). Note that $(D_\lambda^k D_\theta^l) f(g(\lambda),\theta,t)$ is a sum of the terms

$$(D_\theta^l D_g^p f(g(\lambda),\theta,t)) \cdot (D_\lambda^{m_1} g) \cdot (D_\lambda^{m_2} g) \cdots (D_\lambda^{m_p} g)$$

with $\sum_{i=1}^p m_i = k$. Direct computation leads to the estimate

$$\sup_{(\theta,t)\in\mathbf{T^1}\times\mathbf{T^1}} |(D_\lambda^k)(D_\theta^l) f(g(\lambda),\theta,t))| \le C_{k,l} A^p \lambda^{q_1 q_2 - k},$$

and consequently

$$f^*(\lambda,\theta,t)) = f(g(\lambda),\theta,t)) = O(A^p \lambda^{q_1 q_2}).$$

(iii). We observe that $(D_\lambda^k)(D_\theta^l) f(\lambda+u,\theta+v,t)$ is a sum of the terms

$$D_\theta^l[(D_\phi^q D_\mu^p f(\mu,\phi))(D_\lambda^{m_1}\mu)(D_\lambda^{m_2}\mu)\cdots(D_\lambda^{m_p}\mu)(D_\lambda^{n_1}\phi)(D_\lambda^{n_2}\phi)\cdots(D_\lambda^{n_q}\phi)],$$

where $\mu = \lambda + u,\ \phi = \theta + v,\ 0 \le p+q \le k,\ \sum_{i=1}^p m_i + \sum_{i=1}^q n_i = k$, and

$$\begin{aligned} &D_\theta^l[(D_\phi^q D_\mu^p f(\mu,\phi))(D_\lambda^{m_1}\mu)(D_\lambda^{m_2}\mu)\cdots(D_\lambda^{m_p}\mu)(D_\lambda^{n_1}\phi)(D_\lambda^{n_2}\phi)\cdots(D_\lambda^{n_q}\phi)] \\ &= \sum_{i=1}^{l} C_l^i (D_\theta^i D_\phi^q D_\mu^p f(\mu,\phi)) \cdot D_\theta^{l-i}[D_\lambda^{m_1}\mu)\cdots(D_\lambda^{m_p}\mu)(D_\lambda^{n_1}\phi)\cdots(D_\lambda^{n_q}\phi)], \end{aligned}$$

where $D_\theta^i D_\phi^q D_\mu^p f(\mu,\phi)$ is a sum of the terms

$$(D_\phi^{q+\tilde{q}} D_\mu^{p+\tilde{p}} f(\mu,\phi))(D_\theta^{\tilde{m}_1}\mu)(D_\theta^{\tilde{m}_2}\mu)\cdots(D_\theta^{\tilde{m}_p}\mu)(D_\theta^{\tilde{n}_1}\phi)(D_\theta^{\tilde{n}_2}\phi)\cdots(D_\theta^{\tilde{n}_q}\phi),$$

with $0 \le \tilde{p} + \tilde{q} \le i$, $\sum_{j=1}^{\tilde{p}} \tilde{m}_j + \sum_{j=1}^{\tilde{q}} \tilde{n}_j = i$. Noting that $u = O(A^p\lambda^{q_1})$, $v = O(A^p\lambda^{q_2})$, $q_1 < 1, q_2 < 0$, we have

$$\sup_{(\theta,t)\in\mathbf{T^1}\times\mathbf{T^1}} |(D_\lambda^k)(D_\theta^l) f(\lambda+u,\theta+v,t)| \le C_{k,l} A^p \lambda^{q-k}.$$

As a result, we obtain

$$f^{**}(\lambda,\theta,t) = f(\lambda+u,\theta+v,t) = O(A^p\lambda^q). \qquad \square$$

Let $\varepsilon = A^{-\nu}$, where $\nu > 0$ will be specified later. By Definition 4.1 and Definition 4.2, we have

$$R_\varepsilon(\lambda,\theta,t) = O_\varepsilon\big(A^{-1}\lambda^{\frac{n+1}{n+2}}\big), \tag{4.3}$$

$$R^\varepsilon(\lambda,\theta,t) = O\big(A^{-1-\nu\gamma}\lambda^{\frac{n+1}{n+2}}\big). \tag{4.4}$$

In the following, we will omit the constant d in $H_0(\lambda)$ (see (4.2)) without loss of generality.

5 Some transformations

Firstly, we look for a series of symplectic transformations $\Psi_1,\ldots,\Psi_N$ such that $H^N := H \circ \Psi_1 \circ \cdots \circ \Psi_N = H_0^N + O(\varepsilon_0)$, $\varepsilon_0 = A^{-\delta}$, $\delta > 0$. The following lemma is similar to Lemma 7.1 in [30] and we refer to [30] for the proof.

Lemma 5.1. *Let $H(\lambda,\theta,t)$ be the same as (4.1). For $A \gg 1$, $\lambda \gg 1$, then there is a symplectic diffeomorphism Ψ_1 depending periodically on t of the form*

$$\Psi_1 : \begin{cases} \lambda = \tilde{\mu} + u_1(\tilde{\mu},\tilde{\phi},t), \\ \theta = \tilde{\phi} + v_1(\tilde{\mu},\tilde{\phi},t), \end{cases}$$

with $u_1 = O_\varepsilon\big(A^{-1-n}\tilde{\mu}^{\frac{1}{n+2}}\big)$ and $v_1 = O_\varepsilon\big(A^{-1-n}\tilde{\mu}^{-\frac{n+1}{n+2}}\big)$. Moreover the transformed Hamiltonian vector field $\Psi_1(X_H) = X_{H^1}$ is of the form

$$H^1(\tilde{\mu},\tilde{\phi},t) = H_0^1(\tilde{\mu},t) + \widetilde{R}_\varepsilon^1(\tilde{\mu},\tilde{\phi},t) + R^\varepsilon \circ \Psi_1(\tilde{\mu},\tilde{\phi},t),$$

where

$$H_0^1(\tilde{\mu},t) = H_0(\tilde{\mu}) + [R_\varepsilon](\tilde{\mu},t), \quad H_0 = dA^n \cdot \tilde{\mu}^{\frac{2(n+1)}{n+2}},$$

$$[R_\varepsilon] = O_\varepsilon\Big(A^{-1}\tilde{\mu}^{\frac{n+1}{n+2}}\Big), \quad \tilde{R}_\varepsilon^1(\tilde{\mu},\tilde{\phi},t) = O_{\frac{\varepsilon}{2}}\left(A^{-2-n}\right) + O_{\frac{\varepsilon}{2}}\left(A^{u-1-n}\tilde{\mu}^{\frac{1}{n+2}}\right),$$

$$R^\varepsilon \circ \Psi_1(\tilde{\mu},\tilde{\phi},t) = O\left(A^{-1-v\gamma}\tilde{\mu}^{\frac{n+1}{n+2}}\right).$$

Let

$$\tau > 0, \quad \nu < n(1+\tau), \tag{5.1}$$

and

$$\lambda \in \left[c_1 A^{(n+2)\tau}, c_2 A^{(n+2)\tau}\right], \quad c_2 > c_1 > 0.$$

Repeating the symplectic diffeomorphism in Lemma 5.1 for N times, we get N symplectic transformations $\Psi_1,\ldots,\Psi_N$ such that

$$H^N(\mu,\phi,t) = H \circ \Psi_1 \circ \cdots \circ \Psi_N = H_0^N(\mu,t) + R_\varepsilon^N(\mu,\phi,t),$$

where

$$\mu \in \left[c_1 A^{(n+2)\tau}, c_2 A^{(n+2)\tau}\right], \quad c_2 > c_1 > 0,$$

$$H_0^N(\mu,t) = H_0(\mu) + H_1(\mu,t),$$

$$H_0 = d \cdot A^n \cdot \mu^{\frac{2(n+1)}{n+2}}, \quad H_1 = O_{\frac{\varepsilon}{2^N}}\left(A^{-1}\mu^{\frac{n+1}{n+2}}\right),$$

$$R_\varepsilon^N = O_{\frac{\varepsilon}{2^N}}\left(A^{-1-N(1+n)}\mu^{\frac{n+1-N(n+1)}{n+2}}\right) + O_{\frac{\varepsilon}{2^N}}\left(A^{-1+N(v-n)}\mu^{\frac{n+1-Nn}{n+2}}\right) + O\Big(A^{-1-\nu\gamma}\mu^{\frac{n+1}{n+2}}\Big).$$

Now the corresponding unforced equation in (3.17) can be changed into

$$\begin{cases} \dot{\phi} = \frac{\partial H^N}{\partial \mu} = \frac{\partial H_0(\mu)}{\partial \mu} + \frac{\partial H_1(\mu,t)}{\partial \mu} + \frac{\partial R_\varepsilon^N(\mu,\phi,t)}{\partial \mu}, \\ \dot{\mu} = -\frac{\partial H^N}{\partial \phi} = -\frac{\partial R_\varepsilon^N(\mu,\phi,t)}{\partial \phi}, \end{cases} \tag{5.2}$$

where

$$\frac{\partial H_0(\mu)}{\partial \mu} = d \cdot \frac{2n+2}{n+2} A^n \mu^{\frac{n}{n+2}}$$

and we omit the constant $d \cdot \frac{2n+2}{n+2}$ for simplicity in the following arguments. Define the diffeomorphism

$$\Psi : \quad \rho = \frac{\partial \mu^{\frac{2n+2}{n+2}}}{\partial \mu} = \frac{2n+2}{n+2}\mu^{\frac{n}{n+2}}, \qquad \phi = \phi, \tag{5.3}$$

and we get

$$\dot{\rho}=\frac{n(2n+2)}{(n+2)^2}\mu^{\frac{-2}{n+2}}\dot{\mu}.$$

Then we have

$$\dot{\rho}=O\left(A^{-1-N(n+1)}\mu^{\frac{n-1-N(n+1)}{n+2}}\right)+O\left(A^{-1+N(\nu-n)}\mu^{\frac{n-1-Nn}{n+2}}\right)+O\left(A^{-1-\nu\gamma}\mu^{\frac{n-1}{n+2}}\right),$$
$$\dot{\phi}=\rho+r(\rho,t)+O\left(A^{-1-N(n+1)}\mu^{\frac{-1-N(n+1)}{n+2}}\right)+O\left(A^{-1-N(\nu-n)}\mu^{\frac{-1-Nn}{n+2}}\right)+O\left(A^{-1-\nu\gamma}\mu^{\frac{-1}{n+2}}\right),$$

where $r(\rho,t)=\frac{\partial H_1(\mu,t)}{\partial\mu}$ with $\mu=\left(\frac{n+2}{2n+2}\rho\right)^{\frac{n+2}{n}}$. Thus

$$r(\rho,t)=O\left(A^{-1}\mu^{\frac{-1}{n+2}}\right)=O\left(A^{-1}\left(\frac{n+2}{2n+2}\rho\right)^{-\frac{1}{n}}\right)=O\left(A^{-1}\rho^{-\frac{1}{n}}\right).$$

Noting that $\mu\in[c_1A^{(n+2)\tau},c_2A^{(n+2)\tau}]$, we have

$$\rho\in\left[c_1\frac{2n+2}{n+2}A^{n\tau},c_2\frac{2n+2}{n+2}A^{n\tau}\right]. \tag{5.4}$$

It follows that

$$\dot{\rho}=O\left(A^{-1-N(n+1)+[n-1-N(n+1)]\tau}\right)+O\left(A^{-1+N(\nu-n)+(n-1-Nn)\tau}\right)+O\left(A^{-1-\nu\gamma+(n-1)\tau}\right),$$
$$\dot{\phi}=\rho+r(\rho,t)+O\left(A^{[-1-N(n+1)](1+\tau)}\right)+O\left(A^{-1+N(\nu-n)+(-1-Nn)\tau}\right)+O\left(A^{-1-\nu\gamma-\tau}\right).$$

When $N\gg 1$ and $\nu<n(1+\tau)$, we have

$$-1+N(\nu-n)+(n-1-Nn)\tau=N[\nu-n(1+\tau)]+(n-1)\tau-1<0,$$
$$-1+N(\nu-n)+(-1-Nn)\tau=N[\nu-n((1+\tau)]-(1+\tau)<0.$$

When $N\gg 1$ and $\tau>0$, we have

$$-1-N(n+1)+[n-1-N(n+1)]\tau<0,\quad [-1-N(n+1)](1+\tau)<0.$$

Note that $-1-\nu\gamma-\tau<-1-\nu\gamma+(n-1)\tau<n-1-\nu\gamma+(n-1)\tau$. Let

$$n-1-\nu\gamma+(n-1)\tau<0, \tag{5.5}$$

Then, by (5.1) and (5.5), we have

$$\frac{(n-1)(1+\tau)}{\gamma}<\nu<n(1+\tau). \tag{5.6}$$

Since $\gamma>1-\frac{1}{n}$, we have $(n-1)/\gamma<n$. Then, when $\tau>0$ and $\nu\in\left(\frac{(n-1)(1+\tau)}{\gamma},n(1+\tau)\right)$, there is $\delta>0$ and (5.2) can be changed into

$$\begin{cases}\dot{\phi}=\rho+r(\rho,t)+f(\rho,\phi,t)=\rho+r(\rho,t)+O(A^{-\delta}),\\ \dot{\rho}=g(\rho,\phi,t)=O(A^{-\delta}),\end{cases} \tag{5.7}$$

where $\phi\in\mathbf{T^1}$, $r(\rho,t)=O(A^{-1}\rho^{-\frac{1}{n}})$ and $\rho\in[c_3A^{n\tau},c_4A^{n\tau}]$ for $c_4>c_3>0$ given by (5.4).

Next we compute the transformed impulsive forces in (3.17). Based on the symplectic transformation Ψ_1 in Lemma 5.1, we see from the implicit function theorem that

$$\tilde{\mu} = \lambda + u(\lambda,\theta,t), \qquad \tilde{\phi} = \theta + v(\lambda,\theta,t).$$

Under the symplectic transformation Ψ_1, we see that the jumps $\Delta\theta(t_j)$ and $\Delta\lambda(t_j)$ in (3.17) can be changed into

$$\begin{cases} \Delta\tilde{\phi}(t_j) := \tilde{\phi}(t_j^+) - \tilde{\phi}(t_j) = \tilde{I}_j^*(\tilde{\mu}(t_j),\tilde{\phi}(t_j)), \\ \Delta\tilde{\mu}(t_j) := \tilde{\mu}(t_j^+) - \tilde{\mu}(t_j) = \tilde{J}_j^*(\tilde{\mu}(t_j),\tilde{\phi}(t_j)), \end{cases} \tag{5.8}$$

where $j = 1,2,\dots,k$.

In the same way, under the symplectic transformation Ψ_2, the jumps $\Delta\tilde{\phi}(t_j)$ and $\Delta\tilde{\mu}(t_j)$ can be changed into new forms

$$\begin{cases} \Delta\bar{\phi}(t_j) := \bar{\phi}(t_j^+) - \bar{\phi}(t_j) = \bar{I}_j^*(\bar{\mu}(t_j),\bar{\phi}(t_j)), \\ \Delta\bar{\mu}(t_j) := \bar{\mu}(t_j^+) - \bar{\mu}(t_j) = \bar{J}_j^*(\bar{\mu}(t_j),\bar{\phi}(t_j)), \end{cases} \tag{5.9}$$

where $j = 1,2,\dots,k$. Repeating this procedure by the symplectic transformations $\Psi_1,\dots,\Psi_N$, the jumps in (3.17) are finally changed into

$$\begin{cases} \Delta\phi(t_j) := \phi(t_j^+) - \phi(t_j) = I_j^{**}(\mu(t_j),\phi(t_j)), \\ \Delta\mu(t_j) := \mu(t_j^+) - \mu(t_j) = J_j^{**}(\mu(t_j),\phi(t_j)), \end{cases} \tag{5.10}$$

where $j = 1,2,\dots,k$. Combining (5.2) and (5.10), we see that (3.17) can be transformed into

$$\begin{cases} \dot{\phi} = \frac{\partial H_0(\mu)}{\partial\mu} + \frac{\partial H_1(\mu,t)}{\partial\mu} + \frac{\partial R_\varepsilon^N(\mu,\phi,t)}{\partial\mu}, \\ \dot{\mu} = -\frac{\partial R_\varepsilon^N(\mu,\phi,t)}{\partial\phi}, \quad t \neq t_j; \\ \Delta\phi(t_j) = I_j^{**}(\mu(t_j),\phi(t_j)), \\ \Delta\mu(t_j) = J_j^{**}(\mu(t_j),\phi(t_j)), \quad j = 1,2,\dots,k. \end{cases} \tag{5.11}$$

Similarly, under Ψ defined by (5.3), system (5.11) can be transformed into

$$\begin{cases} \dot{\phi} = \rho + r(\rho,t) + f(\rho,\phi,t) = \rho + r(\rho,t) + O(A^{-\delta}), \\ \dot{\rho} = g(\rho,\phi,t) = O(A^{-\delta}), \quad t \neq t_j; \\ \Delta\phi(t_j) = I_j^{**1}(\rho(t_j),\phi(t_j)), \\ \Delta\rho(t_j) = J_j^{**1}(\rho(t_j),\phi(t_j)), \quad j = 1,2,\dots,k, \end{cases} \tag{5.12}$$

where $\phi \in \mathbf{T^1}, r(\rho,t) = O(A^{-1}\rho^{-\frac{1}{n}})$ and $\rho \in [c_3 A^{n\tau}, c_4 A^{n\tau}]$.

It should be pointed out that, although we have not been able to formulate explicitly $I_j^{**1}(\rho(t_j),\phi(t_j))$ and $J_j^{**1}(\rho(t_j),\phi(t_j))$, we can implicitly express them. We will calculate the estimates of the impulsive functions $I_j^{**1}(\rho(t_j),\phi(t_j))$ and $J_j^{**1}(\rho(t_j),\phi(t_j))$ in next section.

6 Some estimates

In this section, we will establish some estimates for impulsive functions $I_j^{**1}(\rho,\phi)$ and $J_j^{**1}(\rho,\phi)$. To this end, we first give the estimates of $I_j^{**}(\mu,\phi)$ and $J_j^{**}(\mu,\phi)$. In this whole section and in the sequel, all the occurrences of j mean $j = 1,2,\dots,k$.

Lemma 6.1. *Assume that the conditions in Theorem 1.1 are satisfied. Let $\mu(t_j)=\mu,\phi(t_j)=\phi$. We have the following estimates*

$$I_j^{**}(\mu,\phi)=O(A^{-1-n}\mu^{-\frac{n+1}{n+2}}),$$
$$J_j^{**}(\mu,\phi)=(\gamma_j^{n+2}-1)\mu+f_j(\mu,\phi)$$

with $f_j(\mu,\phi)=O(A^{-1-n}\mu^{\frac{1}{n+2}})$, where $I_j^{**}(\mu,\phi)$ and $J_j^{**}(\mu,\phi)$ are given by (5.10).

Proof. For $(\lambda,\theta)\in[c_1A^{(n+2)\tau},c_2A^{(n+2)\tau}]\times\mathbf{T^1}$, from Lemma 5.1, the symplectic diffeomorphism Ψ_1 is of the form

$$\Psi_1:\lambda=\tilde{\mu}+u_1(\tilde{\mu},\tilde{\phi},t),\qquad \theta=\tilde{\phi}+v_1(\tilde{\mu},\tilde{\phi},t), \tag{6.1}$$

where $(\tilde{\mu},\tilde{\phi})\in[c_1A^{(n+2)\tau},c_2A^{(n+2)\tau}]\times\mathbf{T^1},u_1=O(A^{-1-n}\tilde{\mu}^{\frac{1}{n+2}}),v_1=O(A^{-1-n}\tilde{\mu}^{-\frac{n+1}{n+2}})$. By the implicit function theorem, we have

$$\tilde{\mu}=\lambda+u(\lambda,\theta,t),\qquad \tilde{\phi}=\theta+v(\lambda,\theta,t), \tag{6.2}$$

where $|u|<CA^{-1-n}\lambda^{\frac{1}{n+2}}$ and $|v|<CA^{-1-n}\lambda^{-\frac{n+1}{n+2}}$.

Next we show that

$$u=O\left(A^{-1-n}\lambda^{\frac{1}{n+2}}\right),\qquad v=O\left(A^{-1-n}\lambda^{-\frac{n+1}{n+2}}\right). \tag{6.3}$$

Indeed, we see from Lemma 5.1 that

$$\begin{cases}\lambda=\tilde{\mu}+\frac{\partial S_1}{\partial\theta}=\tilde{\mu}+\nu(\tilde{\mu},\theta,t),\\ \tilde{\phi}=\theta+\frac{\partial S_1}{\partial\tilde{\mu}}=\theta+g(\tilde{\mu},\theta,t),\end{cases} \tag{6.4}$$

where

$$\nu(\tilde{\mu},\theta,t)=O\left(A^{-1-n}\tilde{\mu}^{\frac{1}{n+2}}\right),\qquad g(\tilde{\mu},\theta,t)=O\left(A^{-1-n}\tilde{\mu}^{-\frac{n+1}{n+2}}\right).$$

From (6.2) and (6.4) we know

$$u=-\nu(\lambda+u,\theta,t). \tag{6.5}$$

If $\tilde{\mu}$ and λ are large, then $|D_{\tilde{\mu}}\nu|\le 1/2$, so that u is uniquely determined by the contraction principle. Moreover, the implicit function theorem implies that u is C^∞ with respect to $(\lambda,\theta)\in[c_1A^{(n+2)\tau},c_2A^{(n+2)\tau}]\times\mathbf{T^1}$. We claim that

$$u=O\left(A^{-1-n}\lambda^{\frac{1}{n+2}}\right). \tag{6.6}$$

Indeed, applying $(D_\lambda)^l$ to equation (6.5), the right hand side is a sum of the terms

$$(D_{\tilde{\mu}}^p)(D_\lambda^{j_1}(\lambda+u))(D_\lambda^{j_2}(\lambda+u))\cdots(D_\lambda^{j_p}(\lambda+u)), \tag{6.7}$$

with $1\le p\le l$ and $\sum_{i=1}^p j_i=l$. The highest order term is the one with $p=1$, namely $(D_{\tilde{\mu}}\nu)D_\lambda^n u$. Note that $|u|<CA^{-1-n}\lambda^{\frac{1}{n+2}}$. Assuming that for $j\le n-1$ the estimates $|D_\lambda^j u|<CA^{-1-n}\lambda^{\frac{1}{n+2}-j}$ hold true, then inductively, from (6.4) and (6.5) we can conclude that the same estimate holds true for $j=n$. In fact, from (6.4) we have

$$|D_{\tilde{\mu}}^p\nu|<CA^{-1-n}\lambda^{\frac{1}{n+2}-p},$$

which yields

$$|(1-D_{\tilde{\mu}}\nu)D_\lambda^l u| \le CA^{-1-n}\lambda^{\frac{1}{n+2}-p}\lambda^{1-j_1}\cdots\lambda^{1-j_p} < CA^{-1-n}\lambda^{\frac{1}{n+2}-l}.$$

It follows that

$$|D_\lambda^l u| < CA^{-1-n}\lambda^{\frac{1}{n+2}-l}.$$

The estimates of $(D_\theta)^j(D_\lambda)^i u$ can be proved similarly. Thus, the claim (6.6) is valid. Similarly, one also has

$$v = O\left(A^{-1-n}\lambda^{-\frac{n+1}{n+2}}\right).$$

Under the symplectic transformation Ψ_1, the jumps $\Delta\theta(t_j)$ and $\Delta\lambda(t_j)$ in (3.17) can be changed into $\tilde{I}_j^*(\tilde{\mu}(t_j),\tilde{\phi}(t_j))$ and $\tilde{J}_j^*(\tilde{\mu}(t_j),\tilde{\phi}(t_j))$ (see (5.8)). Then using (3.17), (5.8), (6.1) and (6.2), we have

$$\begin{aligned}
\tilde{I}_j^*(\tilde{\mu}(t_j),\tilde{\phi}(t_j)) &= \tilde{\phi}(t_j^+) - \tilde{\phi}(t_j)\\
&= \theta(t_j^+) + v(\lambda(t_j^+),\theta(t_j^+),t_j) - \theta(t_j) - v(\lambda(t_j),\theta(t_j),t_j)\\
&= I_j^*(\lambda(t_j),\theta(t_j)) + v(\lambda(t_j) + J_j^*(\lambda(t_j),\theta(t_j)),\theta(t_j)\\
&\quad + I_j^*(\lambda(t_j),\theta(t_j)),t_j) - v(\lambda(t_j),\theta(t_j),t_j)\\
&= v[\gamma_j^{n+2}(\tilde{\mu}(t_j) + u_1(\tilde{\mu}(t_j),\tilde{\phi}(t_j),t_j)),\tilde{\phi}(t_j) + v_1(\tilde{\mu}(t_j),\tilde{\phi}(t_j),t_j),t_j]\\
&\quad - v[\tilde{\mu}(t_j) + u_1(\tilde{\mu}(t_j),\tilde{\phi}(t_j),t_j),\tilde{\phi}(t_j) + v_1(\tilde{\mu}(t_j),\tilde{\phi}(t_j),t_j),t_j],\\
\tilde{J}_j^*(\tilde{\mu}(t_j),\tilde{\phi}(t_j)) &= \tilde{\mu}(t_j^+) - \tilde{\mu}(t_j)\\
&= \lambda(t_j^+) + u(\lambda(t_j^+),\theta(t_j^+),t_j) - \lambda(t_j) - u(\lambda(t_j),\theta(t_j),t_j)\\
&= J_j^*(\lambda(t_j),\theta(t_j)) + u(\lambda(t_j) + J_j^*(\lambda(t_j),\theta(t_j)),\theta(t_j)\\
&\quad + I_j^*(\lambda(t_j),\theta(t_j)),t_j) - u(\lambda(t_j),\theta(t_j),t_j)\\
&= (\gamma_j^{n+2}-1)\lambda(t_j) + u(\gamma_j^{n+2}\lambda(t_j),\theta(t_j),t_j) - u(\lambda(t_j),\theta(t_j),t_j)\\
&= (\gamma_j^{n+2}-1)\tilde{\mu}(t_j) + (\gamma_j^{n+2}-1)u_1(\tilde{\mu}(t_j),\tilde{\phi}(t_j),t_j)\\
&\quad + u(\gamma_j^{n+2}(\tilde{\mu}(t_j) + u_1(\tilde{\mu}(t_j),\tilde{\phi}(t_j),t_j)),\tilde{\phi}(t_j) + v_1(\tilde{\mu}(t_j),\tilde{\phi}(t_j),t_j),t_j)\\
&\quad - u(\tilde{\mu}(t_j) + u_1(\tilde{\mu}(t_j),\tilde{\phi}(t_j),t_j),\tilde{\phi}(t_j) + v_1(\tilde{\mu}(t_j),\tilde{\phi}(t_j),t_j),t_j)\\
&=: (\gamma_j^{n+2}-1)\tilde{\mu}(t_j) + \tilde{f}_j(\tilde{\mu}(t_j),\tilde{\phi}(t_j)).
\end{aligned}$$

It follows from

$$\begin{aligned}
u_1 &= O\Big(A^{-1-n}\tilde{\mu}^{\frac{1}{n+2}}\Big), & v_1 &= O\Big(A^{-1-n}\tilde{\mu}^{-\frac{n+1}{n+2}}\Big),\\
u &= O\Big(A^{-1-n}\tilde{\mu}^{\frac{1}{n+2}}\Big), & v &= O\Big(A^{-1-n}\tilde{\mu}^{-\frac{n+1}{n+2}}\Big)
\end{aligned}$$

and Lemma 4.3 that

$$\tilde{I}_j^*(\tilde{\mu}(t_j),\tilde{\phi}(t_j)) = O\Big(A^{-1-n}\tilde{\mu}^{-\frac{n+1}{n+2}}\Big), \qquad \tilde{f}_j(\tilde{\mu}(t_j),\tilde{\phi}(t_j)) = O\Big(A^{-1-n}\tilde{\mu}^{\frac{1}{n+2}}\Big). \tag{6.8}$$

Similarly, under the symplectic transformation Ψ_2, the jumps $\tilde{I}_j^*(\tilde{\mu}(t_j),\tilde{\phi}(t_j))$ and $\tilde{J}_j^*(\tilde{\mu}(t_j),\tilde{\phi}(t_j))$ can be changed into $\bar{I}_j^*(\bar{\mu}(t_j),\bar{\phi}(t_j))$ and $\bar{J}_j^*(\bar{\mu}(t_j),\bar{\phi}(t_j))$ (see (5.9)). Moreover, there are

$$\bar{I}_j^*(\bar{\mu}(t_j),\bar{\phi}(t_j)) = O\Big(A^{-1-n}\bar{\mu}^{-\frac{n+1}{n+2}}\Big)$$

and

$$\bar{J}_j^*(\bar{\mu}(t_j), \bar{\phi}(t_j)) = (\gamma_j^{n+2} - 1)\bar{\mu}(t_j) + \bar{f}_j(\bar{\mu}(t_j), \bar{\phi}(t_j))$$

with

$$\bar{f}_j(\bar{\mu}(t_j), \bar{\phi}(t_j)) = O\Big(A^{-1-n}\bar{\mu}^{\frac{1}{n+2}}\Big).$$

Finally, by repeating this procedure and noting the fact that $\Psi = \Psi_m \circ \Psi_{m-1} \circ \cdots \circ \Psi_1$ transforms (3.17) into (5.11), we have

$$I_j^{**}(\mu(t_j), \phi(t_j)) = O\Big(A^{-1-n}\bar{\mu}^{-\frac{n+1}{n+2}}\Big)$$

and

$$J_j^{**}(\mu(t_j), \phi(t_j)) = (\gamma_j^{n+2} - 1)\mu(t_j) + f_j(\mu(t_j), \phi(t_j))$$

with $f_j(\mu(t_j), \phi(t_j)) = O\big(A^{-1-n}\mu^{\frac{1}{n+2}}\big)$. This completes the proof of Lemma 6.1. □

Lemma 6.2. *Under the assumptions of Theorem 1.1, we have*

$$I_j^{**1}(\rho(t_j), \phi(t_j)) = O\Big(A^{-1-n}\rho^{-\frac{n+1}{n}}\Big),$$

and

$$J_j^{**1}(\rho(t_j), \phi(t_j)) = (\gamma_j^n - 1)\rho(t_j) + \tilde{g}_j(\rho(t_j), \phi(t_j))$$

with $\tilde{g}_j(\rho(t_j), \phi(t_j)) = O\big(A^{-1-n}\rho^{-\frac{1}{n}}\big)$, *where* $I_j^{**1}(\rho(t_j), \phi(t_j))$ *and* $J_j^{**1}(\rho(t_j), \phi(t_j))$ *are given by* (5.12)*.*

Proof. By (5.3), (5.12), Lemma 6.1 and Taylor's formula, we have

$$\begin{aligned} I_j^{**1}(\rho(t_j), \phi(t_j)) &= \phi(t_j^+) - \phi(t_j) = I_j^{**}(\mu(t_j), \phi(t_j)) \\ &= I_j^{**}\left(\left(\frac{n+2}{2n+2}\rho(t_j)\right)^{\frac{n+2}{n}}, \phi(t_j)\right) \end{aligned}$$

and

$$\begin{aligned} J_j^{**1}(\rho(t_j), \phi(t_j)) &= \rho(t_j^+) - \rho(t_j) = \frac{2n+2}{n+2}\mu^{\frac{n}{n+2}}(t_j^+) - \rho(t_j) \\ &= \frac{2n+2}{n+2}[\mu(t_j) + J_j^{**}(\mu(t_j), \phi(t_j))]^{\frac{n}{n+2}} - \rho(t_j) \\ &= \frac{2n+2}{n+2}[\mu(t_j) + (\gamma_j^{n+2} - 1)\mu(t_j) + f_j(\mu(t_j), \phi(t_j))]^{\frac{n}{n+2}} - \rho(t_j) \\ &= \frac{2n+2}{n+2}[\gamma_j^{n+2}\mu(t_j)]^{\frac{n}{n+2}}\left(1 + \frac{f_j(\mu(t_j), \phi(t_j))}{\gamma_j^{n+2}\mu(t_j)}\right)^{\frac{n}{n+2}} - \rho(t_j) \\ &= \gamma_j^n\rho(t_j)\left[1 + \frac{n}{n+2}\frac{f_j(\mu(t_j), \phi(t_j))}{\gamma_j^{n+2}\mu(t_j)}\left(1 + \xi\frac{f_j(\mu(t_j), \phi(t_j))}{\gamma_j^{n+2}\mu(t_j)}\right)^{-\frac{2}{n+2}}\right] - \rho(t_j) \\ &= (\gamma_j^n - 1)\rho(t_j) + \frac{\rho^{-\frac{2}{n}}(t_j) f_j\left(\left(\frac{n+2}{2n+2}\rho(t_j)\right)^{\frac{n+2}{n}}, \phi(t_j)\right)}{\frac{n+2}{n}\left(\frac{n+2}{2n+2}\right)^{\frac{n+2}{n}}\gamma_j^2} \\ &\quad \times \left(1 + \xi\frac{f_j\left(\left(\frac{n+2}{2n+2}\rho(t_j)\right)^{\frac{n+2}{n}}, \phi(t_j)\right)}{\gamma_j^{n+2}\left(\frac{n+2}{2n+2}\rho(t_j)\right)^{\frac{n+2}{n}}}\right)^{-\frac{2}{n+2}} \end{aligned}$$

$$=: (\gamma_j^n - 1)\rho(t_j) + \tilde{g}_j(\rho(t_j), \phi(t_j))$$

by (6.8), where $\xi \in (0,1)$. Then by Lemma 4.3 and Lemma 6.1, we have

$$I_j^{**1}(\rho(t_j), \phi(t_j)) = O\Big(A^{-1-n}\rho^{\frac{n+2}{n}\cdot(-\frac{n+1}{n+2})}\Big) = O\Big(A^{-1-n}\rho^{-\frac{n+1}{n}}\Big),$$

$$\tilde{g}_j(\rho(t_j), \phi(t_j)) = O\Big(\rho^{-\frac{2}{n}}A^{-1-n}\rho^{\frac{n+2}{n}\cdot\frac{1}{n+2}}\Big) = O\Big(A^{-1-n}\rho^{-\frac{1}{n}}\Big).$$

This completes the proof of Lemma 6.2. □

7 Proof of Theorem 1.1

The following two lemmas are similar to Lemma 9.2 in [30] and Lemma 6.2 in [3], respectively. We refer to [30] and [3] for the proofs. Let $(\rho(t), \phi(t)) = (\rho(t,\rho,\phi), \phi(t,\rho,\phi))$ be the solution of (5.12) with the initial value $(\rho(0), \phi(0)) = (\rho, \phi)$. Let $\phi_1 = \phi(1), \rho_1 = \rho(1)$.

Lemma 7.1. *If all conditions of Theorem 1.1 hold, then the time one map Φ^1 of the flow Φ^t of (5.12) takes the form of*

$$\Phi^1 : \begin{cases} \phi_1 = \phi + \alpha(\rho) + F(\rho, \phi), \\ \rho_1 = \rho + G(\rho, \phi). \end{cases}$$

Moreover, $\dot{\alpha}(\rho) > 0$ and for any non-negative integers r, s with $r + s \le 5$,

$$\left|\frac{\partial^{r+s}F(\rho,\phi)}{\partial\rho^r\partial\phi^s}\right|, \quad \left|\frac{\partial^{r+s}G(\rho,\phi)}{\partial\rho^r\partial\phi^s}\right| < CA^{-\varepsilon_0},$$

where $\varepsilon_0 = \min(\tau, \delta) > 0$, $(\rho, \phi) \in [c_3A^{n\tau}, c_4A^{n\tau}] \times \mathbf{T^1}$, $c_4 > c_3 > 0$, $A \gg 1$, $\tau > 0$, $\delta > 0$.

Lemma 7.2. *Assume that the conditions of Theorem 1.1 are satisfied, then the time-1 map Φ^1 of (4.1) has the intersection property on $\Omega = \{(\rho, \phi) \mid \rho$ large enough, $\phi \in \mathbf{T^1}\}$, i.e. if Γ is an embedded circle in Ω homotopic to a circle $\rho = \text{const.}$ in Ω, then $\Phi^1(\Gamma) \bigcap \Gamma \neq \emptyset$. In particular, Φ^1 has the intersection property on $\Omega = \{(\rho, \phi) \mid c_3A^{n\tau} \le \rho \le c_4A^{n\tau}, \phi \in \mathbf{T^1}\}$, where $c_4 > c_3 > 0, \tau > 0$.*

Now let us state Moser's twist theorem. Let $\mathcal{D}$ be an annulus given by

$$\mathcal{D} \ : \ a \le r \le b, \quad 0 < a < b.$$

For convenience, we introduce for a function $h \in C^l(\mathcal{D})$ the norm

$$|h|_l = \sup_{\mathcal{D},\, m+n\le l} \left|\frac{\partial^{m+n}}{\partial r^m \partial\theta^n}\right|.$$

Theorem 7.3 (Moser's twist theorem)**.** *Let $\alpha(r) \in C^l$ and $|\partial_r\alpha(r)| \ge \nu > 0$ on the annulus $\mathcal{D}$ for some l with $l \ge 5$, and ε be a positive number. Then there exists a $\delta > 0$ depending on $\varepsilon, l, \alpha(r)$, such that any area-preserving mapping*

$$M : \begin{cases} \theta_1 = \theta + 2\pi\alpha(r) + f(r, \theta), \\ r_1 = r + g(r, \theta) \end{cases}$$

of $\mathcal{D}$ into $\mathbf{R^2}$ with $f, g \in C^l$ and

$$|f|_l + |g|_l \le \nu\delta$$

possesses an invariant curve of the form

$$r = c + u(\xi), \qquad \theta = \xi + v(\xi)$$

in $\mathcal{D}$ where u, v are continuously differentiable, of period 2π and satisfy

$$|u|_1 + |v|_1 < \varepsilon,$$

and c is a constant in (a, b). Moreover, the induced mapping of this curve is given by

$$\xi \to \xi + \omega,$$

where ω is incommensurable with 2π, and satisfies infinitely many conditions

$$\left| \frac{\omega}{2\pi} - \frac{p}{q} \right| \geq \gamma q^{-\tau}$$

with some positive γ, τ, for all integers $q > 0$, p. In fact, each choice of ω in the range of $\alpha(r)$ and satisfying the above inequalities give rise to such an invariant curve.

Moser's twist theorem above can be found in [21, pp. 50–54] (see also [27]). It should be pointed out that the δ does not depend on ν. It should be also noted that the period 2π can be replaced by any period T. In addition, "any area-preserving mapping" can be relaxed to "any mapping which has intersection property".

We are now in a position to prove Theorem 1.1. From Lemma 7.1 and Lemma 7.2, by Moser's twist theorem, Φ^1 has an invariant curve $\widetilde{\Gamma}$ in the annulus $(\rho, \phi) \in [c_3 A^{n\tau}, c_4 A^{n\tau}] \times \mathbf{T}^1$, $c_4 > c_3 > 0$, $A \gg 1$, $\tau > 0$. It follows that the time-one map of the original system has an invariant curve $\widetilde{\Gamma}_{A_0}$. Choosing a sequence $A_0 = A_{m0} \to \infty$ as $m \to \infty$, we have that there are countable many invariant curves $\widetilde{\Gamma}_{A_{m0}}$, clustering at ∞. Therefore any solution of the original system is bounded. This completes the proof of Theorem 1.1.

Remark 7.4. Any solutions starting from the invariant curves $\widetilde{\Gamma}_{A_{m0}}$ $(m = 1, 2, \dots)$ are quasi-periodic with frequencies $(1, \omega_m)$ in time t, where $(1, \omega_m)$ satisfies Diophantine conditions and $\omega_m > C A_{m0}^{n\tau}$. Actually, the frequencies can form a positive Lebesgue set in $\mathbf{R}$.

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