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Homoclinic solutions for subquadratic Hamiltonian systems with competition potentials

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Abstract. In this paper, we consider of the following second-order Hamiltonian system

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \qquad \forall t \in \mathbb{R},$$

where $W(t, x)$ is subquadratic at infinity. With a competition condition, we establish the existence of homoclinic solutions by using the variational methods. In our theorem, the smallest eigenvalue function $l(t)$ of $L(t)$ is not necessarily coercive or bounded from above and $W(t, x)$ is not necessarily integrable on $\mathbb{R}$ with respect to t. Our theorem generalizes many known results in the references.

Keywords: Hamiltonian systems, homoclinic solutions, subquadratic potentials, competition condition, variational methods.

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1 Introduction

In this paper, we consider the following Hamiltonian system

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \qquad \forall t \in \mathbb{R}, \tag{1.1}$$

where $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $L \in C\left(\mathbb{R}, \mathbb{R}^{N^2}\right)$ is a symmetric matrix valued function and $\nabla W(t, x)$ denotes the gradient with respect to the x variable. A nontrivial solution $u(t)$ of problem (1.1) is homoclinic if $u(t) \to 0$, $\dot{u}(t) \to 0$ as $t \to \pm\infty$ and $u(t) \not\equiv 0$.

The importance of homoclinic solutions for Hamiltonian systems in studying the dynamic behavior has been recognized. In recent years, many mathematicians used the variational methods to show the existence and multiplicity of homoclinic solutions for systems (1.1) with different growth conditions on $W(t, x)$. In this paper, we only consider the subquadratic cases. In [5], Ding assumed

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(L') letting $l(t) \equiv \inf_{|q|=1}(L(t)q,q)$, there exists $\xi > 1$ such that

$$|t|^{-\xi} l(t) \to +\infty, \quad \text{as } |t| \to +\infty.$$

By (L'), Ding showed a compact embedding theorem from $H^1(\mathbb{R})$ to $L^p(\mathbb{R})$ for $p \in (1,+\infty]$. Under some other subquadratic conditions on $W(t,x)$ with respect to x, Ding obtained the existence and multiplicity of homoclinic solutions for systems (1.1). This result has been generalized by many mathematicians. For example, in [19], Zhang introduced condition

(L'') There exists a constant $l_0 > 0$ such that $l(t) + l_0 \geq 1$ for all $t \in \mathbb{R}$ and

$$\int_{\mathbb{R}} (l(t) + l_0)^{-1}\, dt < \infty. \tag{1.2}$$

By (L''), the embedding $H^1(\mathbb{R}) \hookrightarrow L^1(\mathbb{R})$ is compact. Obviously, (L'') is weaker than (L') and both of these two conditions yield that $l^{-1}(t)$ decays fast at infinity. When $l^{-1}(t)$ has a slow decay at infinity, it is difficult for us to obtain such compact embeddings. In this case, we can consider the decaying rate of $W(t,x)$ at infinity with respect to t. Let us consider the pure power nonlinearities with weight functions, i.e. $W(t,x) = a(t)|x|^{\nu} (\nu \in (1,2))$. In [23], Zhang and Yuan assumed that $a(t)$ belongs to $L^2(\mathbb{R},\mathbb{R}^+) \cap L^{\frac{2}{2-\nu}}(\mathbb{R},\mathbb{R}^+)$ to make sure the corresponding functional is well defined and show the convergence of the (PS) sequence. This condition is weakened by Sun, Chen and J. Nieto [12] by just requiring $a \in L^{\frac{2}{2-\nu}}(\mathbb{R},\mathbb{R}^+)$. In 2014, Lv and Tang [11] obtained homoclinic solutions for systems (1.1) with more general weight functions where $a \in L^p(\mathbb{R},\mathbb{R})$ for some $p \in \left(1, \frac{2}{2-\nu}\right]$. The readers are referred to [1–3, 6–10, 13–18, 20–22] for more details.

From above papers, we know that, the decaying rates of $l^{-1}(t)$ and $a(t)$ at infinity are important for us in finding homoclinic solutions of (1.1). There is an interesting question that whether systems (1.1) possesses homoclinic solutions when $a(t)$ is unbounded or $l(t)$ is oscillating (which means $\liminf_{|t|\to\infty} l(t) < +\infty$ and $\limsup_{|t|\to\infty} l(t) = +\infty$)? Motivated by the above analysis, we are encouraged to find a twisted condition between $l(t)$ and $a(t)$ which can be stated as follows:

$(W0)$ For $b \in [1,2]$ and $\mu \in (1,2)$, there exist $\gamma \in \left(b, \frac{2b}{2+b-b\mu}\right]$ and $k \in \left[0, \frac{\gamma-b}{b\gamma}\right]$ such that $\frac{a(t)}{(l(t))^k} \in L^{\gamma}(\mathbb{R})$.

More precisely, we obtain the following theorem.

Theorem 1.1. *Suppose that $(W0)$ holds for $b = 2$ and*

$(L1)$ *one of the following statements holds:*

(i) $L \in C^2(\mathbb{R}, \mathbb{R}^{N^2})$ *and* $((L''(t) - \kappa L(t))x, x) \leq 0$ *for all* $|t| \geq \bar{r}_1$ *and* $x \in \mathbb{R}^N$*;*

(ii) $L \in C^1(\mathbb{R}, \mathbb{R}^{N^2})$ *and* $|L'(t)x| \leq \kappa |L(t)x|$ *for all* $|t| \geq \bar{r}_1$ *and* $x \in \mathbb{R}^N$

with some $\kappa > 0$ *and* $\bar{r}_1 > 0$*, where* $L'(t) = (d/dt)L(t)$ *and* $L''(t) = (d^2/dt^2)L(t)$*;*

$(L2)$ *there exists* $M_0 > 0$ *such that* $l(t) \geq M_0$ *for all* $t \in \mathbb{R}$*, where* $l(t) \equiv \inf_{|u|=1}(L(t)u,u)$*;*

$(W1)$ $W(t,0) \equiv 0$*, there exists* $a \in C(\mathbb{R}, \mathbb{R}^+)$ *such that* $|\nabla W(t,x)| \leq a(t)|x|^{\mu-1}$*;*

$(W2)$ *there exist* $\lambda \in (1,2)$*,* $\eta > 0$*,* $\zeta > 0$ *and open set* $\Omega \subset \mathbb{R}$ *such that*

$$W(t,x) \geq \eta |x|^{\lambda}, \qquad \forall (t,x) \in \Omega \times \mathbb{R}^N, |x| \leq \zeta.$$

Then system (1.1) possesses at least one nontrivial homoclinic solution.

$(L1)$ is assumed to show all the critical points of corresponding functional for systems

(1.1) are classical homoclinic solutions, which is introduced in [5]. In [11, 13, 18], the authors only considered the homoclinic solutions in sense of $u(t) \to 0$ as $|t| \to \infty$ while we consider the classical ones. To obtain the asymptotic behavior of the solutions at infinity, we can also consider the following condition

$(L3)$ there exist $\delta > 0$, $D > 0$, $q \in [1,2]$ and $r_0 > 0$ such that

$$\int_t^{t+\delta} \hat{l}^q(s)ds \le D$$

for all $|t| \ge r_0$, where $\hat{l}(t) \equiv \sup_{|u|=1}(L(t)u,u)$.

It is easy to see that $(L3)$ holds if all the eigenvalues of $L(t)$ are bounded from above. Then $(L3)$ can be seen as a generalization of the following bounded condition

$(L4)$ there exists $R > 0$ such that

$$(L(t)u,u) \le R|u|^2, \qquad \forall (t,u) \in \mathbb{R} \times \mathbb{R}^N.$$

Then we obtain the following theorem.

Theorem 1.2. *Suppose $(L2)$, $(L3)$, $(W1)$, $(W2)$ and $(W0)$ hold with $b = q$, then system (1.1) possesses at least one nontrivial homoclinic solution.*

Remark 1.3. In our theorems, condition $(W0)$ is a class of competition conditions between a and l. When $0 < \inf_{t\in\mathbb{R}} l(t) \le \sup_{t\in\mathbb{R}} l(t) < \infty$, $(W0)$ reduces to $a(t) \in L^\gamma(\mathbb{R})$, which is required in [12, 13, 18, 22]. There are examples satisfying the conditions of Theorems 1.1 and 1.2 but not the results in [2, 5, 7–14, 16–23].

Example 1.4 (Oscillating example for Theorem 1.1). Let $L(t) = l(t)Id_N$ and $W(t,x) = a(t)|x|^{\frac{8}{5}}$, where

$$l(t) = \begin{cases} \sin(\ln 2) + 1 & \text{for } |t| < 1, \\ t^{\frac{6}{7}}\left(\sin\left(\ln(t^2+1)\right) + 1\right) + 1 & \text{for } |t| \ge 1, \end{cases}$$

$$a(t) = t^{\frac{1}{20}}\left(\sin\left(\ln(t^2+1)\right) + 1\right)^{\frac{3}{10}}$$

and Id_N is the identity matrix of order N. It is easy to see that

$$\liminf_{|t|\to\infty} l(t) = 1, \quad \limsup_{|t|\to\infty} l(t) = +\infty, \quad \liminf_{|t|\to\infty} a(t) = 0 \quad \text{and} \quad \limsup_{|t|\to\infty} a(t) = +\infty.$$

Hence $l(t)$, $a(t)$ are neither coercive nor bounded from above and $l^{-1}(t)$, $(a(t))^p \notin L(\mathbb{R})$ for any $p \in (1,5]$. However, this example satisfies the conditions of Theorem 1.1 with $\gamma = 5$ and $k = \frac{3}{10}$. Here, we only need to show condition $(L1)$ is fulfilled while the other conditions can be easily checked. To check $(L1)$, we show (ii) holds, which can be verified by the following inequality

$$\left(\frac{6}{7}t^{-\frac{1}{7}}\sin\left(\ln(t^2+1)\right) + \frac{2t^{\frac{13}{7}}}{t^2+1}\cos\left(\ln(t^2+1)\right)\right)|x| \le \left(t^{\frac{6}{7}}\left(\sin\left(\ln(t^2+1)\right)+1\right)+1\right)|x|$$

for all $x \in \mathbb{R}^N$ and $|t|$ large enough.

Example 1.5 (Coercive example for Theorem 1.1)**.** There are also examples in which $l(t)$ and $a(t)$ are both coercive. Let $L(t) = \left(t^6+1\right) Id_N$ and $W(t,x) = t^{\frac{2}{5}}|x|^{\frac{3}{2}}$. If we choose $\gamma = 4$ and $k = \frac{1}{4}$, $(W0)$ is fulfilled. Moreover, other conditions of Theorem 1.1 can be easily checked. However this example does not satisfy the results in [2,5,7–14,17–23].

Example 1.6 (Oscillating example for Theorem 1.2)**.** Let

$$g(t) = \begin{cases} 2n^{\frac{8}{9}}\left(n^{\frac{8}{9}}+1\right)|t| - 2n^{\frac{17}{9}}\left(n^{\frac{8}{9}}+1\right), & n \le |t| < n + \frac{1}{2\left(n^{\frac{8}{9}}+1\right)}, \\ -2n^{\frac{8}{9}}\left(n^{\frac{8}{9}}+1\right)|t| + 2n^{\frac{17}{9}}\left(n^{\frac{8}{9}}+1\right) + 2n^{\frac{8}{9}}, & n + \frac{1}{2\left(n^{\frac{8}{9}}+1\right)} \le |t| \le n + \frac{1}{n^{\frac{8}{9}}+1}, \\ 0, & \text{otherwise} \end{cases} \tag{1.3}$$

and

$$m(t) = \begin{cases} 2n^{\frac{1}{72}}\left(n^{\frac{8}{9}}+1\right)|t| - 2n^{\frac{73}{72}}\left(n^{\frac{8}{9}}+1\right), & n \le |t| < n + \frac{1}{2\left(n^{\frac{8}{9}}+1\right)}, \\ -2n^{\frac{1}{72}}\left(n^{\frac{8}{9}}+1\right)|t| + 2n^{\frac{73}{72}}\left(n^{\frac{8}{9}}+1\right) + 2n^{\frac{1}{72}}, & n + \frac{1}{2\left(n^{\frac{8}{9}}+1\right)} \le |t| \le n + \frac{1}{n^{\frac{8}{9}}+1}, \\ 0, & \text{otherwise} \end{cases} \tag{1.4}$$

for all $n \in \mathbb{N} \cup \{0\}$. We see that $g(t), m(t) \ge 0$ and $g \notin L(\mathbb{R})$, $m \notin L(\mathbb{R})$. Let $a(t) = m(t) + e^{-|t|}$ and $L(t) = l(t) Id_N$, where $l(t) = \sqrt{g(t)+1}$. Obviously,

$$\liminf_{|t|\to\infty} l(t) = 1, \quad \limsup_{|t|\to\infty} l(t) = +\infty, \quad \liminf_{|t|\to\infty} a(t) = 0, \quad \limsup_{|t|\to\infty} a(t) = +\infty.$$

Choosing $q = 2$ and $\delta = \frac{1}{4}$, we deduce from the definitions of $\hat{l}$ and g that

$$\begin{aligned} \int_t^{t+\frac{1}{4}} \hat{l}^2(s)ds = \int_t^{t+\frac{1}{4}} l^2(s)ds &= \int_t^{t+\frac{1}{4}} (g(s)+1)ds \\ &\le \frac{1}{2}\left[\sum_{i=[|t|]-1,[|t|],[|t|]+1} \frac{i^{\frac{8}{9}}}{i^{\frac{8}{9}}+1}\right] + \frac{1}{4} \\ &\le \frac{7}{4} \end{aligned}$$

for $|t|$ is large enough. Then $(L3)$ is checked. Moreover, $l^{-1}(t)$, $(a(t))^p \notin L(\mathbb{R})$ for any $p > 1$. Here we only give the proof for $(a(t))^p \notin L(\mathbb{R})$. It follows from the definition of $a(t)$ that

$$\begin{aligned} \int_{\mathbb{R}} a^p(s)ds &\ge \sum_{n=0}^{\infty} \int_n^{n+\frac{1}{2\left(n^{\frac{8}{9}}+1\right)}} m^p(s)ds \\ &= \sum_{n=0}^{\infty} \int_n^{n+\frac{1}{2\left(n^{\frac{8}{9}}+1\right)}} \left(2n^{\frac{1}{72}}\left(n^{\frac{8}{9}}+1\right)s - 2n^{\frac{73}{72}}\left(n^{\frac{8}{9}}+1\right)\right)^p ds \\ &= \sum_{n=0}^{\infty} \frac{n^{\frac{p}{72}}}{2(p+1)\left(n^{\frac{8}{9}}+1\right)} \\ &= +\infty \end{aligned}$$

which implies $(a(t))^p \notin L(\mathbb{R})$ for all $p > 1$. Finally, we show $(W0)$ is fulfilled with $b = q = 2$. Set $W(t,x) = a(t)|x|^{\frac{3}{2}}$. Choosing $\gamma = 4$ and $k = \frac{1}{4}$, from (1.3) and (1.4), we infer that

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\frac{a(s)}{l^{\frac{1}{4}}(s)} \right)^4 ds \\
&\quad = \int_{\mathbb{R}} \frac{a^4(s)}{\sqrt{g(s)+1}} ds \\
&\quad \le \int_{\mathbb{R}} \frac{8\left(m^4(s) + e^{-4|s|}\right)}{\sqrt{g(s)+1}} ds \\
&\quad \le 8 \int_{\mathbb{R}} \frac{m^4(s)}{\sqrt{g(s)}} ds + 8 \int_{\mathbb{R}} e^{-4|s|} ds \\
&\quad \le 16 \sum_{n=0}^{\infty} n^{-\frac{63}{144}} \int_n^{n+\frac{1}{2\left(n^{\frac{8}{9}}+1\right)}} \left(2n^{\frac{1}{72}} \left(n^{\frac{8}{9}}+1\right) s - 2n^{\frac{73}{72}} \left(n^{\frac{8}{9}}+1\right) \right)^{\frac{7}{2}} ds \\
&\qquad + 16 \sum_{n=0}^{\infty} n^{-\frac{63}{144}} \int_{n+\frac{1}{2\left(n^{\frac{8}{9}}+1\right)}}^{n+\frac{1}{n^{\frac{8}{9}}+1}} \left(-2n^{\frac{1}{72}} \left(n^{\frac{8}{9}}+1\right) s + 2n^{\frac{73}{72}} \left(n^{\frac{8}{9}}+1\right) + 2n^{\frac{1}{72}} \right)^{\frac{7}{2}} ds + 4 \\
&\quad = \frac{32}{9} \sum_{n=0}^{\infty} \frac{n^{-\frac{7}{18}}}{n^{\frac{8}{9}}+1} + 4 \\
&\quad < +\infty.
\end{aligned}$$

Then all the conditions of Theorem 1.2 are satisfied. However, since a is not integrable or bounded, $l(t)$ is not bounded or coercive, our example does not satisfy the theorems in [2,5, 8,9,11–14,17,18,20–23].

2 Proof of Theorem 1.1

Set

$$E := \left\{ u \in H^1(\mathbb{R}, \mathbb{R}^N) : \int_{\mathbb{R}} (|\dot{u}(t)|^2 + (L(t)u(t), u(t)))dt < \infty \right\}$$

with

$$(u,v) = \int_{\mathbb{R}} ((\dot{u}(t), \dot{v}(t)) + (L(t)u(t), u(t)))dt.$$

By $(L2)$, the embedding $E \hookrightarrow L^p\left(\mathbb{R}, \mathbb{R}^N\right)$ is continuous for all $p \in [2, +\infty]$. Hence, for any $p \in [2, +\infty]$,

$$\|u\|_p \le C_p \|u\|, \qquad \forall u \in E \tag{2.1}$$

for some $C_p > 0$. Furthermore, let $I : E \to \mathbb{R}$ be the functional of (1.1) defined by

$$I(u) = \int_{\mathbb{R}} \left(\frac{1}{2}|\dot{u}(t)|^2 + \frac{1}{2}(L(t)u(t), u(t)) - W(t, u(t)) \right) dt. \tag{2.2}$$

First, we give the following useful estimate.

Lemma 2.1. *Let $u \in E$. For any $\theta > 0$ and $q \in [1,2]$, the following inequality holds*

$$|u(t)| \leq \theta^{\frac{1}{q^*}-1} \left(\int_t^{t+\theta} |u(s)|^q ds \right)^{\frac{1}{q}} + \theta^{\frac{1}{q^*}} \left(\int_t^{t+\theta} |\dot{u}(s)|^q \, ds \right)^{\frac{1}{q}}, \qquad \forall t \in \mathbb{R}. \tag{2.3}$$

Furthermore, if $u \in C^2(\mathbb{R}, \mathbb{R}^N)$, there holds

$$|\dot{u}(t)| \leq \theta^{\frac{1}{q^*}-1} \left(\int_t^{t+\theta} |\dot{u}(s)|^q ds \right)^{\frac{1}{q}} + \theta^{\frac{1}{q^*}} \left(\int_t^{t+\theta} |\ddot{u}(s)|^q \, ds \right)^{\frac{1}{q}}, \qquad \forall t \in \mathbb{R}, \tag{2.4}$$

where $\frac{1}{q} + \frac{1}{q^*} = 1$ ($q^* = +\infty$, if $q = 1$).

Proof. For any $t, \ \tau \in \mathbb{R}$,

$$|u(t)| \leq |u(\tau)| + \left| \int_\tau^t \dot{u}(s) ds \right|.$$

Integrating over $[t, t+\theta]$, we get

$$\begin{aligned}
\theta |u(t)| &\leq \int_t^{t+\theta} |u(\tau)| d\tau + \int_t^{t+\theta} \left| \int_\tau^t \dot{u}(s) ds \right| d\tau \\
&\leq \theta^{\frac{1}{q^*}} \left(\int_t^{t+\theta} |u(s)|^q ds \right)^{\frac{1}{q}} + \theta \int_t^{t+\theta} |\dot{u}(s)| \, ds \\
&\leq \theta^{\frac{1}{q^*}} \left(\int_t^{t+\theta} |u(s)|^q ds \right)^{\frac{1}{q}} + \theta^{\frac{1}{q^*}+1} \left(\int_t^{t+\theta} |\dot{u}(s)|^q \, ds \right)^{\frac{1}{q}},
\end{aligned}$$

which implies

$$|u(t)| \leq \theta^{\frac{1}{q^*}-1} \left(\int_t^{t+\theta} |u(s)|^q ds \right)^{\frac{1}{q}} + \theta^{\frac{1}{q^*}} \left(\int_t^{t+\theta} |\dot{u}(s)|^q \, ds \right)^{\frac{1}{q}}.$$

Then we obtain (2.3). Similarly, we can also obtain (2.4). □

Lemma 2.2. *Suppose $(L2)$, $(W0)$–$(W2)$ hold, then $I \in C^1(E, \mathbb{R})$ and*

$$\langle I'(u), v \rangle = \int_{\mathbb{R}} [(\dot{u}(t), \dot{v}(t)) + (L(t)u(t), v(t)) - (\nabla W(t, u(t)), v(t))] dt. \tag{2.5}$$

Moreover, all the critical points of I are homoclinic solutions of (1.1) if $(L1)$ holds with $b = 2$ or $(L3)$ holds with $b = q$ respectively.

Proof. First, we show that I is well defined. By $(W1)$, we infer that

$$|W(t, u(t))| = \left| \int_0^1 (\nabla W(t, su(t)), u(t)) ds \right| \leq \frac{1}{\mu} a(t) |u(t)|^\mu, \qquad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N. \tag{2.6}$$

First, we consider a general case, i.e. $\gamma \in \left(1, \frac{2}{2-\mu}\right]$ and $k \in \left[0, \frac{\gamma-1}{\gamma}\right)$. For any $\Lambda \subset \mathbb{R}$, it follows from $(W0)$ and (2.1) that

$$
\begin{aligned}
&\int_\Lambda a(t)|u(t)|^\mu dt\\
&= \int_\Lambda \frac{a(t)}{(l(t))^k}(l(t))^k|u(t)|^\mu dt\\
&\le \left(\int_\Lambda \left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{1}{\gamma}} \left(\int_\Lambda (l(t))^{\frac{k\gamma}{\gamma-1}}|u(t)|^{\frac{\mu\gamma}{\gamma-1}}dt\right)^{\frac{\gamma-1}{\gamma}}\\
&= \left(\int_\Lambda \left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{1}{\gamma}} \left(\int_\Lambda (l(t))^{\frac{k\gamma}{\gamma-1}}|u(t)|^{\frac{2k\gamma}{\gamma-1}}|u(t)|^{\frac{(\mu-2k)\gamma}{\gamma-1}}dt\right)^{\frac{\gamma-1}{\gamma}}\\
&\le \left(\int_\Lambda \left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{1}{\gamma}} \left[\left(\int_\Lambda l(t)|u(t)|^2 dt\right)^{\frac{k\gamma}{\gamma-1}} \left(\int_\Lambda |u(t)|^{\frac{(\mu-2k)\gamma}{\gamma-1-k\gamma}}dt\right)^{\frac{\gamma-1-k\gamma}{\gamma-1}}\right]^{\frac{\gamma-1}{\gamma}}\\
&\le \left(\int_\Lambda \left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{1}{\gamma}} C^{\mu-2k}_{\frac{(\mu-2k)\gamma}{\gamma-1-k\gamma}}\|u\|^\mu. \qquad (2.7)
\end{aligned}
$$

When $k=\frac{\gamma-1}{\gamma}$, we have

$$
\begin{aligned}
\int_\Lambda a(t)|u(t)|^\mu dt &= \int_\Lambda \frac{a(t)}{(l(t))^{\frac{\gamma-1}{\gamma}}}(l(t))^{\frac{\gamma-1}{\gamma}}|u(t)|^\mu dt\\
&\le \left(\int_\Lambda \left(\frac{a(t)}{(l(t))^{\frac{\gamma-1}{\gamma}}}\right)^\gamma dt\right)^{\frac{1}{\gamma}} \left(\int_\Lambda l(t)|u(t)|^{\frac{\mu\gamma}{\gamma-1}}dt\right)^{\frac{\gamma-1}{\gamma}}\\
&\le \left(\int_\Lambda \left(\frac{a(t)}{(l(t))^{\frac{\gamma-1}{\gamma}}}\right)^\gamma dt\right)^{\frac{1}{\gamma}} \left(\|u\|_\infty^{\frac{\mu\gamma}{\gamma-1}-2}\int_\Lambda l(t)|u(t)|^2 dt\right)^{\frac{\gamma-1}{\gamma}}\\
&\le \left(\int_\Lambda \left(\frac{a(t)}{(l(t))^{\frac{\gamma-1}{\gamma}}}\right)^\gamma dt\right)^{\frac{1}{\gamma}} \|u\|_\infty^{\mu-\frac{2(\gamma-1)}{\gamma}}\|u\|^{\frac{2(\gamma-1)}{\gamma}}\\
&\le \left(\int_\Lambda \left(\frac{a(t)}{(l(t))^{\frac{\gamma-1}{\gamma}}}\right)^\gamma dt\right)^{\frac{1}{\gamma}} C_\infty^{\mu-\frac{2(\gamma-1)}{\gamma}}\|u\|^\mu. \qquad (2.8)
\end{aligned}
$$

Since $\left(b,\frac{2b}{2+b-b\mu}\right]\subset\left(1,\frac{2}{2-\mu}\right]$ and $\left[0,\frac{\gamma-b}{b\gamma}\right]\subset\left[0,\frac{\gamma-1}{\gamma}\right]$ for all $b\in[1,2]$, (2.7) and (2.8) also hold when $\gamma\in\left(b,\frac{2b}{2+b-b\mu}\right]$ and $k\in\left[0,\frac{\gamma-b}{b\gamma}\right]$.

Choosing $\Lambda=\mathbb{R}$, we see I is well defined. Similar to Lemma 3.1 in [22], one shows $I\in C^1(E,\mathbb{R})$ and (2.5) holds. Finally, we show all the critical points of I are homoclinic solutions for (1.1), i.e. we need to show $u(t)\to 0$ and $\dot u(t)\to 0$ as $t\to\pm\infty$ if $u(t)$ is a critical point of I. We can easily deduce from (2.5) that $L(t)u-\nabla W(t,u)$ is the weak derivative of $\dot u$. Since $E\subset C^0\left(\mathbb{R},\mathbb{R}^N\right)$(the space of continuous functions), $W\in C^1\left(\mathbb{R}\times\mathbb{R}^N,\mathbb{R}\right)$ and $L\in C\left(\mathbb{R},\mathbb{R}^{N^2}\right)$, we know u is indeed in $C^2(\mathbb{R},\mathbb{R}^N)$. Obviously,

$$\int_t^{t+\theta}|u(s)|^q ds\le\theta^{\frac{2-q}{2}}\left(\int_t^{t+\theta}|u(s)|^2 ds\right)^{\frac{q}{2}}\to 0\quad\text{as } |t|\to+\infty \qquad (2.9)$$

and

$$\int_t^{t+\theta}|\dot u(s)|^q ds\le\theta^{\frac{2-q}{2}}\left(\int_t^{t+\theta}|\dot u(s)|^2 ds\right)^{\frac{q}{2}}\to 0\quad\text{as } |t|\to+\infty \qquad (2.10)$$

for any $\theta \in \mathbb{R}$. It follows from (2.3) that $u(t) \to 0$ as $|t| \to +\infty$. In order to prove $\dot{u}(t) \to 0$ as $|t| \to +\infty$, we show a useful estimate as follow. For any $b \in [1,2]$, it follows from $(W0)$ and (2.1) that

$$\begin{aligned}
&\int_{\mathbb{R}} |\nabla W(t,u(t))|^b dt \\
&\quad \le \int_{\mathbb{R}} a^b(t)|u(t)|^{b(\mu-1)} dt \\
&\quad = \int_{\mathbb{R}} \left(\frac{a(t)}{(l(t))^k}\right)^b (l(t))^{bk}|u(t)|^{b(\mu-1)} dt \\
&\quad \le \left(\int_{\mathbb{R}} \left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{b}{\gamma}} \left(\int_{\mathbb{R}} (l(t))^{\frac{bk\gamma}{\gamma-b}} |u(t)|^{\frac{b\gamma(\mu-1)}{\gamma-b}} dt\right)^{\frac{\gamma-b}{\gamma}} \\
&\quad \le \left(\int_{\mathbb{R}} \left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{b}{\gamma}} \left[\left(\int_{\mathbb{R}} l(t)|u(t)|^2 dt\right)^{\frac{bk\gamma}{\gamma-b}} \left(\int_{\mathbb{R}} |u(t)|^{\frac{b\gamma(\mu-1-2k)}{\gamma-b-bk\gamma}} dt\right)^{\frac{\gamma-b-bk\gamma}{\gamma-b}}\right]^{\frac{\gamma-b}{\gamma}} \\
&\quad \le \left(\int_{\mathbb{R}} \left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{b}{\gamma}} C^{b(\mu-1-2k)}_{\frac{b\gamma(\mu-1-2k)}{\gamma-b-bk\gamma}} \|u\|^{b(\mu-1)}.
\end{aligned} \tag{2.11}$$

Similarly, when $k = \frac{\gamma-b}{b\gamma}$,

$$\begin{aligned}
\int_{\mathbb{R}} |\nabla W(t,u(t))|^b dt &\le \int_{\mathbb{R}} a^b(t)|u(t)|^{b(\mu-1)} dt \\
&= \int_{\mathbb{R}} \left(\frac{a(t)}{(l(t))^{\frac{\gamma-b}{b\gamma}}}\right)^b (l(t))^{\frac{\gamma-b}{\gamma}} |u(t)|^{b(\mu-1)} dt \\
&\le \left(\int_{\mathbb{R}} \left(\frac{a(t)}{(l(t))^{\frac{\gamma-b}{b\gamma}}}\right)^\gamma dt\right)^{\frac{b}{\gamma}} \left(\int_{\mathbb{R}} l(t)|u(t)|^{\frac{b\gamma(\mu-1)}{\gamma-b}} dt\right)^{\frac{\gamma-b}{\gamma}} \\
&\le \left(\int_{\mathbb{R}} \left(\frac{a(t)}{(l(t))^{\frac{\gamma-b}{b\gamma}}}\right)^\gamma dt\right)^{\frac{b}{\gamma}} \left[\|u\|_\infty^{\frac{b\gamma(\mu-1)}{\gamma-b}-2} \int_{\mathbb{R}} l(t)|u(t)|^2 dt\right]^{\frac{\gamma-b}{\gamma}} \\
&\le \left(\int_{\mathbb{R}} \left(\frac{a(t)}{(l(t))^{\frac{\gamma-b}{b\gamma}}}\right)^\gamma dt\right)^{\frac{b}{\gamma}} C_\infty^{b(\mu-1)-\frac{2(\gamma-b)}{\gamma}} \|u\|^{b(\mu-1)}.
\end{aligned} \tag{2.12}$$

The following proof is divided into two cases.

Case 1. $(L1)$ holds with $b = 2$. Let $\mathcal{A}$ be the self-adjoint extension of $-(d^2/dt^2) + L(t)$ with $\mathfrak{D}(\mathcal{A}) \subset L^2(\mathbb{R},\mathbb{R}^N)$. Since we have $(L2)$ and (i)(or (ii)) of $(L1)$, similar to Lemma 2.3 in [5], $\mathfrak{D}(\mathcal{A})$ is continuously embedded in $W^{2,2}(\mathbb{R},\mathbb{R}^N)$. Making estimates as (2.9) and (2.10), it follows from (2.4) that $\dot{u}(t) \to 0$ as $|t| \to +\infty$ if $u \in \mathfrak{D}(\mathcal{A})$. Subsequently, we show all the critical points of I belong to $\mathfrak{D}(\mathcal{A})$. By (2.11) and (2.12) with $b = 2$, we see $\|\mathcal{A}u\|_{L^2}^2 = \int_{\mathbb{R}} |\nabla W(t,u(t))|^2 dt < \infty$. Then $u \in \mathfrak{D}(\mathcal{A})$, which shows u is a homoclinic solution for (1.1).

Case 2. $(L3)$ holds with $b = q$. Since $u \in C^2(\mathbb{R},\mathbb{R}^N)$, we deduce from $(L3)$ and (2.4) that

$$|\dot{u}(t)| \le \delta^{\frac{1}{q^*}-1} \left(\int_t^{t+\delta} |\dot{u}(s)|^q ds\right)^{\frac{1}{q}} + \delta^{\frac{1}{q^*}} \left(\int_t^{t+\delta} |\ddot{u}(s)|^q ds\right)^{\frac{1}{q}}.$$

By (2.10), we only need to consider $\int_t^{t+\delta} |\ddot{u}(s)|^q ds$. Similar to Lemma 3.1 in [22], (2.11) and (2.12), for any $\gamma \in (q, \frac{2q}{2+q-q\mu}]$ and $k \in [0, \frac{\gamma-q}{q\gamma}]$

$$\begin{aligned}
&\int_t^{t+\delta} |\ddot{u}(s)|^q ds \\
&\leq 2^{q-1} \int_t^{t+\delta} \left(|\nabla W(s,u(s))|^q + |L(s)u(s)|^q \right) ds \\
&\leq 2^{q-1} M_1 \left(\int_t^{t+\delta} \left(\frac{a(s)}{(l(s))^k} \right)^\gamma ds \right)^{\frac{q}{\gamma}} \|u\|^{q(\mu-1)} + 2^{q-1} \int_t^{t+\delta} \left| (L(s)u(s))^T L(s)u(s) \right|^{\frac{q}{2}} ds \\
&= 2^{q-1} M_1 \left(\int_t^{t+\delta} \left(\frac{a(s)}{(l(s))^k} \right)^\gamma ds \right)^{\frac{q}{\gamma}} \|u\|^{q(\mu-1)} + 2^{q-1} \int_t^{t+\delta} \left| (u(s))^T L^2(s)u(s) \right|^{\frac{q}{2}} ds \\
&= 2^{q-1} M_1 \left(\int_t^{t+\delta} \left(\frac{a(s)}{(l(s))^k} \right)^\gamma ds \right)^{\frac{q}{\gamma}} \|u\|^{q(\mu-1)} + 2^{q-1} \int_t^{t+\delta} \left| (L^2(s)u(s), u(s)) \right|^{\frac{q}{2}} ds \\
&\leq 2^{q-1} M_1 \left(\int_t^{t+\delta} \left(\frac{a(s)}{(l(s))^k} \right)^\gamma ds \right)^{\frac{q}{\gamma}} \|u\|^{q(\mu-1)} + 2^{q-1} \left[\sup_{s \geq t} |u(s)|^q \right] \int_t^{t+\delta} \hat{l}^q(s) ds \\
&\to 0 \quad \text{as } |t| \to +\infty,
\end{aligned}$$

where

$$M_1 = \begin{cases} C^{q(\mu-1-2k)}_{\frac{q\gamma(\mu-1-2k)}{\gamma-q-qk\gamma}}, & k \in \left[0, \frac{\gamma-q}{q\gamma}\right), \\ C_\infty^{q(\mu-1)-\frac{2(\gamma-q)}{\gamma}}, & k = \frac{\gamma-q}{q\gamma}. \end{cases}$$

Thus u is a homoclinic solution for (1.1). ☐

In the next lemma, we show the functional I satisfies the classical Palais–Smale ((PS) for short) condition. We say that I satisfies the (PS) condition, if any sequence $(u_i)_i$ in E such that

$$(I(u_i))_i \text{ is bounded and } I'(u_i) \to 0,$$

admits a convergent subsequence.

Lemma 2.3. *Under $(L2)$, $(W0)$ and $(W1)$, I satisfies the (PS) condition.*

Proof. Let $\{u_i\}_{i\in\mathbb{N}} \subset E$ be a sequence such that $\{I(u_i)\}_{i\in\mathbb{N}}$ is bounded and $I'(u_i) \to 0$ as $i \to +\infty$. Then there exists $B > 0$ such that $|I(u_i)| \leq B$. By (2.2), (2.7) and (2.8) with $\Lambda = \mathbb{R}$, we have

$$\|u_i\|^2 = 2I(u_i) + 2\int_{\mathbb{R}} W(t, u_i(t))\, dt \leq 2B + \frac{2M_2}{\mu} \left(\int_{\mathbb{R}} \left(\frac{a(t)}{(l(t))^k} \right)^\gamma dt \right)^{\frac{1}{\gamma}} \|u_i\|^\mu,$$

where

$$M_2 = \begin{cases} C^{\mu-2k}_{\frac{(\mu-2k)\gamma}{\gamma-1-k\gamma}}, & k \in \left[0, \frac{\gamma-1}{\gamma}\right), \\ C_\infty^{\mu-\frac{2(\gamma-1)}{\gamma}}, & k = \frac{\gamma-1}{\gamma}, \end{cases}$$

which implies $\{u_i\}_{i\in\mathbb{N}}$ is bounded in E. Hence, there exists $u_0 \in E$(up to passing to a subsequence) such that $u_i \rightharpoonup u_0$ in E and

$$\begin{aligned}
&\langle I'(u_i) - I'(u_0), u_i - u_0 \rangle \\
&\quad = \|u_i - u_0\|^2 - \int_{\mathbb{R}} (\nabla W(t, u_i(t)) - \nabla W(t, u_0(t)), u_i(t) - u_0(t))\, dt \to 0
\end{aligned} \tag{2.13}$$

as $i \to \infty$. Moreover, there exists $M_3 > 0$ such that

$$\sup_{j\in\mathbb{N}} \|u_i\|_\infty \leq M_3 \quad \text{and} \quad \|u_0\|_\infty \leq M_3. \tag{2.14}$$

For any $\varepsilon > 0$ it follows from $(W0)$ that there exists $T > 0$ such that

$$\left(\int_{|t|>T} \left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{1}{\gamma}} < \varepsilon. \tag{2.15}$$

It follows from $(W0)$ and Sobolev's compact embedding theorem in bounded domain that

$$\begin{aligned}
&\int_{|t|\leq T} (\nabla W(t, u_i(t)) - \nabla W(t, u_0(t)), u_i(t) - u_0(t))\, dt \\
&\quad \leq \int_{|t|\leq T} a(t) \left(|u_i(t)|^{\mu-1} + |u_0(t)|^{\mu-1}\right) |u_i(t) - u_0(t)|\, dt \\
&\quad \leq a_0 \left(\left(\int_{|t|\leq T} |u_i(t)|^\mu\right)^{\frac{\mu-1}{\mu}} + \left(\int_{|t|\leq T} |u_0(t)|^\mu\right)^{\frac{\mu-1}{\mu}}\right) \left(\int_{|t|\leq T} |u_i(t) - u_0(t)|^\mu\right)^{\frac{1}{\mu}} \\
&\quad \leq \varepsilon
\end{aligned} \tag{2.16}$$

for i large enough, where $a_0 = \max_{|t|\leq T} a(t)$. By $(W0)$, (2.7) and (2.8) with $\Lambda = \mathbb{R} \setminus [-T, T]$, one has

$$\begin{aligned}
&\int_{|t|>T} (\nabla W(t, u_i(t)) - \nabla W(t, u_0(t)), u_i(t) - u_0(t))\, dt \\
&\quad \leq \int_{|t|>T} |\nabla W(t, u_i(t)) - \nabla W(t, u_0(t)\| u_i(t) - u_0(t)| dt \\
&\quad \leq \int_{|t|>T} a(t) \left(|u_i(t)|^{\mu-1} + |u_0(t)|^{\mu-1}\right) (|u_i(t)| + |u_0(t)|)\, dt \\
&\quad \leq 3 \int_{|t|>T} a(t) \left(|u_i(t)|^\mu + |u_0(t)|^\mu\right) dt \\
&\quad \leq 3M_2 \left(\int_{|t|>T} \left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{1}{\gamma}} \left(\|u_i\|^\mu + \|u_0\|^\mu\right).
\end{aligned} \tag{2.17}$$

By the arbitrariness of ε, (2.15) and (2.17), we obtain

$$\int_{|t|>T} (\nabla W(t, u_i(t)) - \nabla W(t, u(t)), u_i(t) - u_0(t))\, dt \to 0 \quad \text{as } i \to +\infty. \tag{2.18}$$

Together with (2.16), we obtain

$$\int_{\mathbb{R}} (\nabla W(t, u_i(t)) - \nabla W(t, u(t)), u_i(t) - u_0(t))\, dt \to 0 \quad \text{as } i \to +\infty.$$

Consequently, we infer from (2.13) and (2.18) that $\|u_i - u_0\| \to 0$ as $i \to +\infty$. □

Proof of Theorem 1.1. By (2.2), (2.6) and (2.7) with $\Lambda = \mathbb{R}$, for any $u \in E$, we get

$$\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W(t, u(t)) dt \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{\mu} \int_{\mathbb{R}} a(t)|u(t)|^\mu dt \\
&\geq \frac{1}{2}\|u\|^2 - \frac{M_2}{\mu} \left(\int_{\mathbb{R}} \left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{1}{\gamma}} \|u\|^\mu,
\end{aligned}$$

which implies that $I(u) \to +\infty$ as $\|u\| \to +\infty$. Thus I is bounded from below and satisfies the (PS) condition. Then there exists $\bar{u}$ such that $I(\bar{u}) = c = \inf_E I(u)$. We also need to show that $\bar{u} \not\equiv 0$. Letting $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N) \setminus \{0\}$ and $s > 0$, it follows from (2.2) and $(W2)$ that

$$\begin{aligned} I(s\varphi) &= \frac{s^2}{2}\|\varphi\|^2 - \int_{\mathbb{R}} W(t, s\varphi(t))\,dt \\ &= \frac{s^2}{2}\|\varphi\|^2 - \int_{\Omega} W(t, s\varphi(t))\,dt \\ &\leq \frac{s^2}{2}\|\varphi\|^2 - \eta s^\lambda \int_{\Omega} |\varphi(t)|^\lambda\,dt, \end{aligned}$$

which implies $I(s\varphi) < 0$ when $s > 0$ small enough. Then we can deduce that $\inf_E I(u) < 0$, which implies that $\bar{u} \not\equiv 0$. □

Proof of Theorem 1.2. The only difference between Theorems 1.1 and 1.2 is the way to obtain the asymptotic behavior of the solutions for (1.1) at infinity. This has been shown in the proof of Lemma 2.2. The remaining part is similar to Theorem 1.1, we omit it here. □

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