



A remark on energy balance of the Navier–Stokes equations

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Abstract. In this short note, we provide a self-contained proof for the criterion $\nabla u \in L^{\frac{5}{2}}(0, T; L^2(\mathbb{R}^3))$ to Navier–Stokes equations energy balance, which improves some recent results on this problem.

Keywords: Navier–Stokes equations, energy equality, distributional solutions.

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1 Introduction

We are interested in the energy balance of distributional solutions to Navier–Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0, & x \in \mathbb{R}^3, t > 0 \\ \nabla \cdot u = 0, & x \in \mathbb{R}^3, t > 0 \\ u|_{t=0} = u_0(x), & x \in \mathbb{R}^3. \end{cases} \quad (1.1)$$

It is well known since the work of Leray [8] and Hopf [6], that for any $u_0 \in L^2_{\sigma}(\mathbb{R}^3)$ one can construct a global weak solutions to (1.1), namely, a function u that, for each $T > 0$, is in the class


$$u \in L^{\infty}(0, T; L^2_{\sigma}(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)) \quad (1.2)$$

and solves (1.1) in a distributional sense. Here, $L^2_{\sigma}(\mathbb{R}^3)$ is the subspace of $L^2(\mathbb{R}^3)$ of divergence-free vector functions. In addition, such a u satisfies the so-called energy inequality:

$$\|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq \|u_0\|_{L^2}^2, \quad \forall t \geq 0. \quad (1.3)$$

Much about the solutions of the Navier–Stokes equation is unknown, including uniqueness and regularity. The main barrier is the fact that the energy equality, which states that for any smooth solution u , it obeys the following energy balance:

$$\|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau = \|u_0\|_{L^2}^2, \quad \forall t \geq 0. \quad (1.4)$$

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A natural question immediately arises: does any Leray–Hopf weak (distributional) solution of the Navier–Stokes equations automatically satisfy the energy balance (1.4)? To date this question remains open, and only conditional results are available.

Energy equality is clearly a prerequisite for regularity, and can be a first step in proving conditional regularity results [2, 3, 11, 12]. Lions [9] and Ladyzhenskaya [7] proved independently that a Leray–Hopf weak solution satisfy the (global) energy equality (1.4) under the additional assumption $u \in L^4L^4$. Shinbrot [13] generalized the Lions–Ladyzhenskaya condition to

$$u \in L^r(0, T; L^s(\mathbb{R}^d)) \quad \text{with} \quad \frac{2}{r} + \frac{2}{s} \leq 1, s \geq 4. \quad (1.5)$$

Recently, Yu [14] given a new proof to the Shinbrot energy conservation criterion. In addition, Berselli and Chiodaroli in [1] prove some new energy balance criteria in terms of the gradient of the velocity. Specially, they showed that

$$\nabla u \in L^{\frac{5}{2}}(0, T; L^2(\Omega)) \quad (1.6)$$

can ensure the energy identity.

From the PDEs point of view, it is significant to study the motion of distributional (very weak) solutions of fluid equations, see Definition 1.1. In this regard, there is not any available regularity on velocity field u , apart the solution being in $L^2_{\text{loc}}(\mathbb{R}^3 \times [0, T))$. Recently, The famous mathematician Giovanni P. Galdi [4, 5] systematically studied the relation between very weak and Leray–Hopf solutions to Navier–Stokes equations, and he first proved that if distributional solution in $L^4(0, T; L^4(\mathbb{R}^3))$, and with initial data u_0 in $L^2(\mathbb{R}^3)$, then energy equality (1.4) holds true, in particular, he emphasized that the requirement (1.2) is entirely redundant. The key observation is the use of the duality argument and the above conditions to improve the regularity of the solution (i.e., $L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$).

As everyone knows, in fluid mechanics, the gradient of velocity (∇u) is an important physical quantity. Objective of this note is to prove that control the gradient of velocity, i.e., $\nabla u \in L^{\frac{5}{2}}(L^2(\mathbb{R}^3))$ along with the (necessary) condition $u_0 \in L^2_\sigma(\mathbb{R}^3)$ can ensure the energy balance. More precisely, setting

$$\mathcal{D}_T := \{\varphi \in C_0^\infty(\mathbb{R}^3 \times [0, T)) : \text{div } \varphi = 0\}.$$

Definition 1.1 (Distributional solution). Let $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, $T > 0$. The function $u \in L^2_{\text{loc}}(\mathbb{R}^3 \times [0, T))$ is a distributional solution to the Navier–Stokes equations (1.1) if

1. for any $\Phi \in \mathcal{D}_T$, we have

$$\int_0^T \int_{\mathbb{R}^3} u \cdot (\partial_t \Phi + \Delta \Phi + u \cdot \nabla \Phi) dx dt = - \int_{\mathbb{R}^3} u(x, 0) \cdot \Phi(x, 0) dx;$$

2. for any $\varphi \in C_0^\infty(\mathbb{R}^3)$, it holds that

$$\int_{\mathbb{R}^3} u \cdot \nabla \varphi dx = 0,$$

for a.e. $t \in (0, T)$.

We will show the following.

Theorem 1.2. *Suppose that $u \in L^2_{\text{loc}}(\mathbb{R}^3 \times [0, T])$ be a distributional solution in the sense of Definition 1.1 to the Navier–Stokes (1.1). If*

$$\nabla u \in L^{\frac{5}{2}}(0, T; L^2(\mathbb{R}^3)),$$

then

$$\int_{\mathbb{R}^3} |u(t, x)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u(x, \tau)|^2 dx d\tau = \int_{\mathbb{R}^3} |u_0|^2 dx$$

for any $t \in [0, T]$.

Remark 1.3. From a purely mathematical perspective, it seems that a new strategy for studying the energy balance of distributional solutions based on gradient of velocity, which may be applied to other incompressible fluid equations.

Remark 1.4. Note that this result is supercritical with respect to the Prodi–Serrin scaling since $\frac{3}{p} + \frac{2}{q} = \frac{3}{2} + \frac{4}{5} > 2$, showing that the energy balance holds even if one does not expect the full regularity of solutions to hold.

Remark 1.5. Berselli and Chiodaroli [1] obtained energy equality via $\nabla u \in L^{\frac{5}{2}}(0, T; L^2(\Omega))$, however, we want to emphasize is that the finite energy ($u \in L^\infty L^2 \cap L^2 H^1$) plays a key role in their proof.

2 Proof of Theorem 1.2

This section is devoted to proof of Theorem 1.2. For the sake of simplicity, we will proceed as if the solution is differentiable in time. The extra arguments needed to mollify in time are straightforward.

Let $\eta : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a standard mollifier, i.e. $\eta(x) = C e^{\frac{1}{|x|^2-1}}$ for $|x| < 1$ and $\eta(x) = 0$ for $|x| \geq 1$, where constant $C > 0$ selected such that $\int_{\mathbb{R}^3} \eta(x) dx = 1$. For any $\varepsilon > 0$, we define the rescaled mollifier $\eta_\varepsilon(x) = \varepsilon^{-3} \eta(\frac{x}{\varepsilon})$. For any function $f \in L^1_{\text{loc}}(\mathbb{R}^3)$, its mollified version is defined as

$$f^\varepsilon(x) = (f * \eta_\varepsilon)(x) = \int_{\mathbb{R}^3} \eta_\varepsilon(x - y) f(y) dy.$$

If $f \in W^{1,p}(\mathbb{R}^3)$, the following local approximation is well known

$$f^\varepsilon(x) \rightarrow f \quad \text{in } W^{1,p}_{\text{loc}}(\mathbb{R}^3) \quad \forall p \in [1, \infty).$$

The crucial ingredient to prove Theorem 1.2 is the following lemmas.

Lemma 2.1 ([10]). *Let ∂ be a partial derivative in one direction. Let $f, \partial f \in L^p(\mathbb{R}^+ \times \mathbb{R}^3)$, $g \in L^q(\mathbb{R}^+ \times \mathbb{R}^d)$ with $1 \leq p, q \leq \infty$, and $\frac{1}{p} + \frac{1}{q} \leq 1$. Then, we have*

$$\|\partial(fg) * \eta_\varepsilon - \partial(f(g * \eta_\varepsilon))\|_{L^r(\mathbb{R}^+ \times \mathbb{R}^3)} \leq C \|\partial f\|_{L^p(\mathbb{R}^+ \times \mathbb{R}^3)} \|g\|_{L^q(\mathbb{R}^+ \times \mathbb{R}^3)}$$

for some constant $C > 0$ independent of ε, f and g , and with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In addition,

$$\partial(fg) * \eta_\varepsilon - \partial(f(g * \eta_\varepsilon)) \rightarrow 0 \quad \text{in } L^r(\mathbb{R}^+ \times \mathbb{R}^3)$$

as $\varepsilon \rightarrow 0$, if $r < \infty$.

Lemma 2.2. Let $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ and let u be a distributional solution in the sense of Definition 1.1 to the Navier–Stokes equations (1.1) and satisfies

$$\nabla u \in L^{\frac{5}{2}}(0, T; L^2(\mathbb{R}^3)),$$

then we have

$$\sup_{t \geq 0} \|u^\varepsilon(\cdot, t)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u^\varepsilon|^2 dx d\tau \leq K, \quad \forall t \in [0, T],$$

where K is a constant depending only on $\|u_0\|_{L^2}$ and $\int_0^T \|\nabla u\|_{L^2}^{\frac{5}{2}} dt$.

Remark 2.3. Lemma 2.2 shows distributional solution u falls into the class of Leray–Hopf weak solutions provided that $\nabla u \in L^{\frac{5}{2}}(0, T; L^2(\mathbb{R}^3))$.

Proof of Lemma 2.2. Multiplying (1.1)₁ by $(u^\varepsilon)^\varepsilon$, then integrating over $(0, T) \times \mathbb{R}^3$, we infer that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |u^\varepsilon|^2 dx + \int_{\mathbb{R}^3} |\nabla u^\varepsilon|^2 dx = - \int_{\mathbb{R}^3} \operatorname{div}(u \otimes u)^\varepsilon \cdot u^\varepsilon dx. \quad (2.1)$$

Indeed, taking advantage of the interpolation inequality, Hölder’s inequality and Young’s inequality, we know that

$$\begin{aligned} \left| - \int_{\mathbb{R}^3} \operatorname{div}(u \otimes u)^\varepsilon \cdot u^\varepsilon dx \right| &\leq C \|(u \cdot \nabla u)\|_{L^{\frac{3}{2}}} \|u^\varepsilon\|_{L^3} \\ &\leq C \|u\|_{L^6} \|\nabla u\|_{L^2} \|u^\varepsilon\|_{L^3} \\ &\leq C \|\nabla u\|_{L^2}^2 \|u^\varepsilon\|_{L^3} \\ &\leq C \|\nabla u\|_{L^2}^2 \|u^\varepsilon\|_{L^2}^{\frac{1}{2}} \|u^\varepsilon\|_{L^6}^{\frac{1}{2}} \\ &\leq C \|\nabla u\|_{L^2}^{\frac{5}{2}} (\|u^\varepsilon\|_{L^2}^2 + 1). \end{aligned} \quad (2.2)$$

Then substituting estimates (2.2) into (2.1), we arrive at

$$\frac{d}{dt} \int_{\mathbb{R}^3} |u^\varepsilon|^2 dx + \int_{\mathbb{R}^3} |\nabla u^\varepsilon|^2 dx \leq C \|\nabla u\|_{L^2}^{\frac{5}{2}} (\|u^\varepsilon\|_{L^2}^2 + 1). \quad (2.3)$$

Applying Gronwall’s inequality to see that

$$\begin{aligned} \sup_{t \geq 0} \|u^\varepsilon(\cdot, t)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u^\varepsilon|^2 dx d\tau &\leq \|u_0\|_{L^2}^2 \exp C \int_0^t \|\nabla u\|_{L^2}^{\frac{5}{2}} ds \\ &\leq K, \end{aligned} \quad (2.4)$$

for all $t \in [0, T]$, where K is a constant depending only on initial data u_0 and $\int_0^T \|\nabla u\|_{L^2}^{\frac{5}{2}} dt$. Let $\varepsilon \rightarrow 0$ in (2.4), one has

$$\sup_{t \geq 0} \|u(\cdot, t)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx d\tau \leq K. \quad (2.5)$$

Then we complete the proof of Lemma 2.2. □

Proof of Theorem 1.2. With Lemma 2.1 and Lemma 2.2 in hand, we are ready to prove our main result. First, we appeal to $u \in L^\infty(L^2) \cap L^{\frac{5}{2}}(H^1)$, by interpolation inequality we show that

$$\|u\|_{L^4} \leq \|u\|_{L^2}^{\frac{1}{4}} \|u\|_{L^6}^{\frac{3}{4}},$$

By integration in $(0, T)$ one easily proves that the estimate

$$\int_0^T \|u\|_{L^4}^{\frac{10}{3}} dt \leq \int_0^T \|u\|_{L^6}^{\frac{5}{2}} \|u\|_{L^2}^{\frac{5}{6}} dt \leq \int_0^T \|\nabla u\|_{L^2}^{\frac{5}{2}} \|u\|_{L^2}^{\frac{5}{6}} dt \leq C. \quad (2.6)$$

Next, modifying the momentum equation (1.1)₁ and taking the inner-product with u^ε , thus we have

$$\int_{\mathbb{R}^3} u^\varepsilon (\partial_t u + u \cdot \nabla u - \Delta u + \nabla p)^\varepsilon dx = 0. \quad (2.7)$$

This yields

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |u^\varepsilon|^2 dx + \int_{\mathbb{R}^3} |\nabla u^\varepsilon|^2 dx = - \int_{\mathbb{R}^3} \operatorname{div}(u \otimes u)^\varepsilon \cdot u^\varepsilon dx. \quad (2.8)$$

Clearly,

$$\int_{\mathbb{R}^3} |u^\varepsilon|^2 dx - \int_{\mathbb{R}^3} |u_0^\varepsilon|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u^\varepsilon|^2 dx d\tau = -2 \int_0^t \int_{\mathbb{R}^3} \operatorname{div}(u \otimes u)^\varepsilon \cdot u^\varepsilon dx d\tau. \quad (2.9)$$

Notice that the incompressible condition $\operatorname{div} u = 0$ ensures

$$-2 \int_0^t \int_{\mathbb{R}^3} \operatorname{div}(u \otimes u^\varepsilon) \cdot u^\varepsilon dx d\tau = 0,$$

by using Hölder's equality and Lemma 2.1, one has

$$\begin{aligned} & -2 \int_0^t \int_{\mathbb{R}^3} \operatorname{div}(u \otimes u)^\varepsilon \cdot u^\varepsilon - \operatorname{div}(u \otimes u^\varepsilon) \cdot u^\varepsilon dx d\tau \\ & = 2 \int_0^t \int_{\mathbb{R}^3} [(u \otimes u)^\varepsilon - (u \otimes u^\varepsilon)] \cdot \nabla u^\varepsilon dx d\tau \\ & \leq 2 \int_0^t \int_{\mathbb{R}^3} (|(u \otimes u)^\varepsilon - u \otimes u| + |u \otimes u - u \otimes u^\varepsilon|) |\nabla u^\varepsilon| dx d\tau \\ & \leq C \|\nabla u\|_{L^{\frac{5}{2}}(0,T;L^2(\mathbb{R}^3))} \left(\|(u \otimes u)^\varepsilon - u \otimes u\|_{L^{\frac{5}{3}}(0,T;L^2(\mathbb{R}^3))} \right. \\ & \quad \left. + \|u\|_{L^{\frac{10}{3}}(L^4(\mathbb{R}^3))} \|u - u^\varepsilon\|_{L^{\frac{10}{3}}(0,T;L^4(\mathbb{R}^3))} \right). \end{aligned} \quad (2.10)$$

Thanks to (2.6) and standard properties of mollifier, we know that the right hand side of (2.10) becomes zero as $\varepsilon \rightarrow 0$, which completes the proof of this case.

Finally, letting ε go to zero in (2.9), and using the facts (2.10), what we have proved is that in the limit

$$\int_{\mathbb{R}^3} |u(t, x)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u(x, \tau)|^2 dx d\tau = \int_{\mathbb{R}^3} |u_0|^2 dx.$$

This completes the proof of Theorem 1.2. \square

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