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# The blow-up method applied to monodromic singularities

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**Abstract.** The blow-up method proves its effectiveness to characterize the integrability of the resonant saddles giving the necessary conditions to have formal integrability and the sufficiency doing the resolution of the associated recurrence differential equation using induction. In this work we apply the blow-up method to monodromic singularities in order to solve the center-focus problem. The case of nondegenerate monodromic singularities is straightforward since any real nondegenerate monodromy singularity can be embedded into a complex system with a resonant saddle. Here we apply the method to nilpotent and degenerate monodromic singularities solving the center problem when the center conditions are algebraic.

**Keywords:** monodromic singularity, blow-up, center problem, formal first integral.

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## 1 Introduction

The center-focus problem for systems of differential equations is one of the main unsolved problems in the qualitative theory of differential systems in the plane [23, 26]. For the non-degenerate monodromic singularities the center-focus problem is closely connected with integrability problem, see for instance references [37, 39]. The center-focus problem consists of providing the necessary and sufficient conditions under which a monodromic singularity has a neighborhood filled with periodic orbits. If the monodromic singularity is a non-degenerate singular point, i.e., its linear part has two purely imaginary eigenvalues, then the real differential system can be embedded in the complex plane and the singular point it transformed to a $1 : -1$ resonant saddle singular point, see [15, 16, 29, 30].

Indeed, the $1 : -1$ resonance can be generalized into a $p : -q$ resonance known as a $p : -q$ resonant singular point of a polynomial vector field in $\mathbb{C}^2$, see [19, 44].

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The characterization of the analytic integrability of several families of differential systems with a resonant saddle is studied in several works, see for instance [17–19, 28–30, 44] and references therein.

In order to find the necessary conditions of analytic integrability of a $p : -q$ resonant singular point there exist different algorithms. One of them is based on the transformation of the original system to its normal form through a series of invertible changes of variables [2]. Another algorithm propose directly the formal first integral, see [41,44]. Recently the blow-up method has been introduced to compute the necessary conditions, see [16].

Once the necessary conditions are obtained the second step is to prove their sufficiency. There is no general algorithm that works for all differential systems in order to prove the sufficiency. The sufficiency is guaranteed if, for instance, the system is Hamiltonian or time-reversible. Recall that a time-reversible system is invariant by certain symmetry. The existence of an explicit first integral well-defined in a neighborhood of the singular point guarantees also the existence of a center in a monodromic singular point. This first integral can be found through the knowledge of an integrating factor. The connections between integrating factors an analytic first integrals have been studied by different authors, see [8, 13, 31, 41] and references therein. Finally ad hoc methods to prove the sufficiency are used for some particular families, see for instance [12, 13, 19, 32, 34–36, 40, 44]. All these different algorithms to prove the sufficiency have been useless for certain differential systems. However, in [15] the blow-up method is used to prove the sufficiency doing the resolution of the associated recurrence differential equation using induction and all the open problems of previous works have been solved.

We remark that for an isolated singularity the existence of a formal first integral implies the existence of an analytic first integral, see [10,41]. Consequently, to prove the sufficiency is sufficient to prove the existence of a formal first integral. In [3] the formal integrability was studied through the existence of invariant analytic (sometimes algebraic) curves.

In this paper we use the blow-up method to approach the center-focus problem for nilpotent and degenerate monodromic singularities, also when there exists no formal integral. This method that was successfully applied for resonant saddles and nondegenerate monodromic singularities, is used here to determine necessary conditions. Also, it is also possible to prove the sufficiency when the center is formally integrable. We solve open cases and cases previously studied with very difficult techniques.

## 2 Blow-up method for monodromic non-degenerate singular points

A monodromic non-degenerate singular point at the origin of a differential system on $\mathbb{R}^2$ takes the form

$$\dot{u} = v + P(u,v), \qquad \dot{v} = -u + Q(u,v), \tag{2.1}$$

where $P(u,v)$ and $Q(u,v)$ are real analytic functions without constant and linear terms. Such singular point is a center, if and only if, the system has a first integral of the form

$$\Phi(u,v) = u^2 + v^2 + \sum_{k+l\geq 3} \phi_{kl} u^k v^l. \tag{2.2}$$

analytically defined around it, see [37,39]. Therefore, the center-focus problem reduces to the case of proving the existence of such analytic first integral. From this result straightforward emerge a method to determine the first necessary conditions to have a center, which consists in proposing a power series of the form (2.2). However the unique general method that enables us to prove the sufficiency for this first necessary conditions is to use the method developed in [15] solving the recurrence differential equation associated to the problem using induction.

The first step to apply the method is to complexify system (2.1) defining the complex variable $x = u + iv$ and system (2.1) becomes the equation $\dot{x} = ix + R(x, \bar{x})$. Considering also its complex conjugate equation we have the system

$$\dot{x} = ix + R(x, \bar{x}), \qquad \dot{\bar{x}} = -i\bar{x} + \bar{R}(x, \bar{x}).$$

If we define $y := \bar{x}$ as a new variable and $\bar{R}$ as a new function we obtain a complex system which is after the change of time $idt = dT$ written as

$$\dot{x} = x + G(x, y), \qquad \dot{y} = -y + H(x, y). \tag{2.3}$$

The power series (2.2) is now transformed into

$$\Psi(x, y) = xy + \sum_{i+j>2} \psi_{ij} x^i y^j,$$

verifying that $\mathcal{X}\Psi = \sum_{i=1} v_{2i+1}(xy)^{2i+2}$, where $\mathcal{X}$ is the vector field associated to system (2.3) and $v_{2i+1}$ are polynomials in the parameters of the system. We note that if all the polynomials $v_{2i+1}$ vanish then the power series $\Psi(x, y)$ is first integral of system (2.3). The singular point at the origin of system (2.3) is $1 : -1$ resonant saddle singular point and the values $v_{2i+1}$ are the so-called *saddle constants*, see [41, 44].

When the $1 : -1$ resonant saddle singular point at the origin is generalized into the $p : -q$ resonant saddle singular point at the origin then the differential system is of the form

$$\dot{x} = p\,x + F_1(x, y), \qquad \dot{y} = -q\,y + F_2(x, y), \tag{2.4}$$

where $F_1$ and $F_2$ are analytic functions without constant and linear terms with $p, q \in \mathbb{Z}$ and $p, q > 0$, see [14, 33, 44]. In this case a $p : -q$ resonant saddle singular point is called a resonant center, if an only if, there exists a meromorphic first integral $\Psi = x^q y^p + \sum_{i+j>p+q} \psi_{ij} x^i y^j$ around it. We recall here that if $\Psi(x, y) \in \mathbb{C}[[x, y]]$, i.e, is a formal first integral in a neighborhood of the singularity, then there also exists an analytic first integral.

The blow-up method to detect formal integrability for a resonant singular point works as follows. We perform the blow-up $(x, y) \to (x, z) = (x, y/x)$ to system (2.4) which has a resonant singular point at the origin. So that the origin is replaced by the line $x = 0$, which contains two singular points that correspond to the separatrices of the resonant point at the origin of system (2.4). These two singular points are a $(p + q) : -p$ resonant saddle and a $(p + q) : -q$ resonant saddle that we call $p_1$ and $p_2$, respectively. The method is based on the following result.

**Theorem 2.1.** *The $p : -q$ resonant singular point at the origin of system (2.4) is analytically integrable if, and only if, either $p_1$ or $p_2$ is orbitally analytically linearizable.*

The proof is based on the fact that if the $p : -q$ resonant singular has an analytic first integral $\Psi(x, y)$ then both points $p_1$ or $p_2$ have also a well-defined analytic first integral given by $\Psi(x, zx)$. The sufficiency follows from Lemma 1 of [19] using the normal orbital form of the $p : -q$ resonant system (2.4) and the first integral of such normal orbital form. From Theorem 2.1 we deduce that the necessary conditions of integrability for the $p : -q$ resonant singular point generate the same ideal that the necessary integrability conditions of the singular points $p_1$ or $p_2$.

Hence we apply the blow-up $z = y/x$ and system (2.4) is transformed into the system

$$\dot{z} = -(p+q)z + x\mathcal{F}(x,z), \qquad \dot{x} = p\,x + x^2\mathcal{G}(x,z), \tag{2.5}$$

where $\mathcal{F}(0,0) = 0$ and $x = 0$ is an invariant line of the new system. Next we propose the power series $\tilde{\mathcal{H}} = \sum_{i\geq 1}^{\infty} f_i(z)x^i$, where $f_i(z)$ are arbitrary functions of $z$ (in the case of formal integrability these functions must be polynomials). Let $\tilde{\mathcal{X}}$ be the vector field associated to system (2.5). The lower terms of equation $\tilde{\mathcal{X}}\tilde{\mathcal{H}} = 0$ give the differential equation for $f_1(z)$ given by $pf_1(z) - (p+q)zf_1'(z) = 0$ whose solution is $f_1(z) = c_1 z^{p/(p+q)}$. Taking into account that $f_1(z)$ must be a polynomial we take $c_1 = 0$ and consequently $f_1(z) = 0$. The power two of terms give the differential equation $2pf_2(z) - (p+q)zf_2'(z) = 0$ and its solution is $f_2(z) = c_2 z^{(2p)/(p+q)}$. Consequently, either $(2p)/(p+q) \in \mathbb{N}$ or we take $c_2 = 0$. Taking into account that $p, q \in \mathbb{Z}$ with $p, q > 0$ it always exists $f_{k_0}$ such that $(k_0 p)/(p+q) \in \mathbb{N}$ (or $(k_0 q)/(p+q) \in \mathbb{N}$ for saddle point $p_2$). Finally, for each power of $x$ of the equation $\tilde{\mathcal{X}}\tilde{H} = 0$ we get the differential equation

$$k\,p\,f_k(z) - (p+q)\,z\,f_k{}'(z) + g_k(z) = 0, \tag{2.6}$$

where $g_k(z)$ depends on some previous functions $f_i(z)$ for $i = k_0, \ldots, k-1$. The solution of differential equation (2.6) is given by

$$f_k(z) = c_k\, z^{\frac{kp}{p+q}} + z^{\frac{kp}{p+q}} \int \frac{z^{-1-\frac{kp}{p+q}}}{p+q}\, g_k(z)\, dz, \tag{2.7}$$

where $c_k$ is an arbitrary constant. From (2.7) it is easy to see that functions $f_k$ in (2.7) are always polynomials except when appear logarithmic terms. If the origin is not a resonant center, always exists a value $k_r$ such that for $k \geq k_r$ the functions $f_i(z)$ for $i \geq k_r$ can have logarithmic terms. In fact, the logarithmic term appears when there is a term $s^{-1}$ in the integral of (2.7). This is the case when

$$-1 - \frac{k_r p}{p+q} + m_k = -1,$$

where $m_k$ is the degree of the polynomial $g_k(s)$. So, we have $k_r = m_k(p+q)/p$. The coefficients of these logarithmic terms are the saddle constants of the original system (2.4).

Vanishing a certain number of saddle constants and checking that some of the next ones are zero we can apply the following procedure. First we apply the induction method to prove that the solution $f_k$ of recursive equation (2.6) is always a polynomial to assure that system (2.5) has a formal first integral. Second, to prove the sufficiency of the original system (2.4) we can apply the following result.

**Theorem 2.2.** *Assume that system (2.5) has a formal first integral $\tilde{\mathcal{H}}(x,z)$. If the function $\tilde{H} = \tilde{\mathcal{H}}(x, y/x)$ is well-defined at the origin of system (2.4) then this system is analytic integrable in a neighborhood of the origin.*

The idea of the method is to study the connected singular points at infinity and if they are formally integrable and the first integral can be extended up to the origin then the origin is also formally integrable. The reason of why the coordinates $(x, z = y/x)$ are better than the original coordinates $(x, y)$ is double. First because doing the blow-up we introduce $x$ as a invariant curve of the new differential system and then we can propose an expansion passing through the origin in powers of $x$ with coefficients as functions of $z$. The second is because the in the new variables $(x, z)$ the coefficients functions of $z$ are polynomials with perhaps some logarithmic terms, see [3]. This does not happens in the original variables, where the system may not have any invariant curve and if it does then the coefficient of the expansion do not have to be polynomial.

In this work we apply the same method to nilpotent and degenerate monodromic singularities in order to solve the center-focus problem. For degenerate monodromic singularities there is no general method to approach the center-focus problem. The method shows that the formal integrability of the points at infinity is intimately linked with the center problem at the origin even though the center at the origin is not formally integrable. The method determine center conditions for monodromic singularities which are algebraically solvable. In the following sections we solve several non trivial examples. The method can also be applied to systems that are not formally integrable at the monodromic singular point giving information for studying the center-focus problem.

## 3 Nilpotent monodromic singularities

In this section we consider different systems with a nilpotent singularity, and we study, using the blow-up method, the center-focus problem of such systems.

**Proposition 3.1.** *The nilpotent real cubic differential system*

$$\dot{x} = y + Ax^2y + Bxy^2 + Cy^3, \qquad \dot{y} = -x^3 + Px^2y + Kxy^2 + Ly^3. \tag{3.1}$$

*is a center if and only if $P = B + 3L = (A + K)L = 0$.*

*Proof.* In [9] was solved the center-focus problem of the nilpotent cubic system (3.1) constructing a Liapunov function and using different methods to prove the sufficiency. Indeed it is well-known that all the centers are analytically (hence formally) integrable, see [7]. Later in [22,27] the center-focus problem of such system is also solved using the fact that all the nilpotent centers are limit of non-degenerate centers. Here, we apply the blow-up method to solve it. Hence, applying the blow-up transformation

$$(x, y) \to (z, y) = (x/y, y) \tag{3.2}$$

system (3.1) becomes

$$\begin{aligned} \dot{z} &= 1 + Cy^2 + By^2z - Ly^2z + Ay^2z^2 - Ky^2z^2 - Py^2z^3 + y^2z^4, \\ \dot{y} &= y^3(L + Kz + Pz^2 - z^3), \end{aligned} \tag{3.3}$$

which has a regular point at the origin. Therefore system (3.3) is analytic integrable at the origin and the recursive differential equation do not generate logarithmic terms. Next, we propose the power series

$$\mathcal{H}(z, y) = \sum_{k=2}^{\infty} f_k(z)y^k. \tag{3.4}$$

We impose that $\dot{\mathcal{H}} = \dot{z}\partial\mathcal{H}/\partial z + \dot{y}\partial\mathcal{H}/\partial y = 0$ and equating to zero each coefficient of power of $y$ we obtain the following recursive differential equation for the functions $f_k$

$$(k-1)(L + Kz + Pz^2 - z^3)f_{k-1} + (C + Bz - Lz + Az^2 - Kz^2 - Pz^3 + z^4)f'_{k-1} + f'_{k+1} = 0. \quad (3.5)$$

Solving for the first values of $k$ we can take $f_k = 0$ for all $k$ odd and for $k$ even we find $f_2 = c_2$, where $c_2$ is an arbitrary integration constant that we can take $c_2 = 1$, then we have

$$\begin{aligned} f_4 &= \frac{1}{6}(-12Lz - 6Kz^2 - 4Pz^3 + 3z^4) + c_4, \\ f_6 &= \frac{1}{630}(P_6(z) - 60Pz^7) + c_6. \end{aligned}$$

In order to have a polynomial in the original variables $(x, y)$ we must to take $P = 0$. Then $f_8$ is a polynomial of degree 9 of the form

$$f_8 = \frac{1}{83160}\big(P_8(z) - 3696(B + 3L)z^9\big) + c_8.$$

In this case we have to take $B + 3L = 0$. Taking $B = -3L$ then $f_{10}$ is a polynomial of degree 15 given by

$$f_{10} = \frac{1}{83160}\big(P_{10}(z) - 5896800(A + K)Lz^{11}\big) + c_{10}.$$

If $(A + K)L = 0$ we have checked that some of the next $f_k$ for $k$ even are all of degree at most $k$. Now, we assume that $f_s$ have degree $s$ for $s = 2, 4, \ldots, k-1$ and solving the recursive equation (3.5) we obtain

$$f_{k+1}(z) = -\int (k-1)(L + Kz - z^3)f_{k-1} + (C - 4Lz + (A - K)z^2 + z^4)f'_{k-1} \quad (3.6)$$

where it is easy to see that the higher terms cancel, that is, if we introduce $f_{k-1}(z) = C_0 + C_1 z + \cdots + C_{k-1}z^{k-1}$ in (3.6) we get a polynomial for $f_{k+1}$ of degree at most $k+1$. Consequently, we have proven the sufficiency since we have a formal first integral at the origin that in the original variables $(x, y)$ is also formal for all the center cases. Here the blow-up method gives straightforward the necessary conditions and the sufficiency for all cases and in a unified method for all the center cases. □

**Proposition 3.2.** *Consider the nilpotent differential system*

$$\dot{x} = Ax^3 + By, \qquad \dot{y} = Cx^5 + Dx^2 y, \quad (3.7)$$

*where the unique monodromic condition is $(D - 3A)^2 + 12BC < 0$. It has a center at the origin if and only if $3A + D = 0$.*

*Proof.* The monodromic and center-focus problem of system (3.7) has been solved in [1]. Indeed, system (3.7) is a $(1, 3)$-quasihomogeneous system and consequently, $V(x, y) = Cx^6 - 3Ax^3y + Dx^3y - 3By^2$ is an inverse integrating factor of (3.7). In fact such $(p, q)$-quasihomogeneous systems of degree $r$ has a unique center condition given by

$$\int \frac{F_r(\varphi)}{G_r(\varphi)} d\varphi = 0 \quad (3.8)$$

where

$$G_r(\varphi) = p\,Q_{p+r}(\cos\varphi, \sin\varphi)\cos\varphi - q\,P_{q+r}(\cos\varphi, \sin\varphi)\sin\varphi,$$
$$F_r(\varphi) = P_{p+r}(\cos\varphi, \sin\varphi)\cos\varphi + Q_{q+r}(\cos\varphi, \sin\varphi)\sin\varphi.$$

using the weighted polar blow-up $(x,y) \to (\rho, \varphi)$ given by $x = r^p \cos\varphi$, $y = r^q \sin\varphi$. Nevertheless, the computation of condition (3.8) is very demanding and sometimes impossible, see [23,24]. However applying the blow-up method we compute the center condition in a straightforward way. We proceed in a similar way as in the previous example. After transformation (3.2) system (3.9) becomes

$$\begin{aligned}\dot{z} &= B + Ay^2z^3 - Dy^2z^3 - Cy^4z^6,\\ \dot{y} &= y^3z^2(D + Cy^2z^3).\end{aligned} \tag{3.9}$$

Since $B \neq 0$ by the monodromic condition we have that the origin of system (3.9) is also a regular point. Therefore system (3.9) has an analytic first integral around its origin. Hence, we look for a power series of the form (3.4). We compute $\dot{\mathcal{H}} = \dot{z}\partial\mathcal{H}/\partial z + \dot{y}\partial\mathcal{H}/\partial y$ for system (3.9) and equating to zero the coefficients of the same power of $y$ yields the following recurrence differential equation

$$(k-4)Cz^5 f_{k-4} + (k-2)Dz^2 f_{k-2} - Cz^6 f'_{k-4} + (Az^3 - Dz^3)f'_{k-2} + Bf'_k = 0.$$

We take $f_k = 0$ for $k$ odd and for $k$ even we can take $f_2(z) = 1$ and

$$\begin{aligned}f_4(z) &= \frac{1}{3B}(-2Dz^3) + c_4,\\ f_6(z) &= \frac{1}{9B^2}(-3BC + 3AD + D^2)z^6 + c_6,\\ f_8(z) &= \frac{1}{27B^3}(P_8(z) - 2(3A+D)(AD - BC)z^9) + c_8,\end{aligned}$$

where $P_8$ is a polynomial of at most degree 8. In order to have a polynomial in the original variables $(x,y)$ we must to take $(3A+D)(AD-BC) = 0$. So we impose $3A + D = 0$ because the other one is not compatible with the monodromic condition. In this case $f_{10}$ takes the form

$$f_{10}(z) = \frac{1}{3B^2}(24ABc_8z^3 + 36A^2c_6z^6 - 3BCc_6z^6 - 4ACc_4z^9 + 3B^2c_{10}).$$

We can take all $c_4 = c_6 = c_8 = c_{10} = 0$ and then $f_{10}(z) = 0$ and also take $f_k = 0$ for all $k \geq 10$. Next we define

$$\begin{aligned}H &= f_2\Big(\frac{x}{y}\Big)y^2 + f_4\Big(\frac{x}{y}\Big)y^4 + f_6\Big(\frac{x}{y}\Big)y^6 + f_8\Big(\frac{x}{y}\Big)y^8\\ &= y^2 + \frac{2A}{B}x^3y - \frac{C}{3B}x^6,\end{aligned}$$

which is a polynomial first integral of system (3.7) and therefore it has a center at the origin. Here the computation of the necessary condition is straightforward unlike other known methods and our method also gives directly the sufficiency. $\square$

**Proposition 3.3.** *The nilpotent differential system*

$$\dot{x} = y + x^2, \qquad \dot{y} = -x^3 + cx^4, \tag{3.10}$$

*has not any analytic first integral at the origin and it has a center at the origin if and only if $c = 0$.*

*Proof.* First we apply the blow-up transformation (3.2) and system (3.10) becomes

$$\dot{z} = 1 + yz^2 + y^2z^4 - cy^3z^5, \qquad \dot{y} = y^3z^3(-1 + cyz), \tag{3.11}$$

We propose a power series of the form (3.4) and impose that $\dot{\mathcal{H}} = \dot{z}\partial\mathcal{H}/\partial z + \dot{y}\partial\mathcal{H}/\partial y = 0$ for system (3.11) and we get the following recurrence differential equation

$$(k-3)cz^4 f_{k-3} - (k-2)z^3 f_{k-2} - cz^5 f'_{k-3} + z^4 f'_{k-2} + z^2 f'_{k-1} + f'_k = 0$$

and we can, as in previous case, take $f_2(z) = 1$, and $f_3(z) = c_3$,

$$f_4(z) = \frac{1}{2}z^4 + c_4, \qquad f_5(z) = \frac{1}{60}(45c_3z^4 - 24cz^5 - 20z^6) + c_5.$$

However, it is not possible to get a polynomial from $f_5$ in the original variables $(x, y)$. Therefore the analytic first integral at infinity cannot be extended to the origin of system (3.10). This also implies system (3.10) has not an analytic first integral at the origin. Next we propose a power series of the form

$$\mathcal{V}(z, y) = \sum_{k=1}^{\infty} v_k(z) y^k. \tag{3.12}$$

and we impose that this $\mathcal{V}$ satisfies the equation

$$\dot{z}\partial\mathcal{V}/\partial z + \dot{y}\partial\mathcal{V}/\partial y - (\partial\dot{z}/\partial z + \partial\dot{y}/\partial y)\mathcal{V} = 0, \tag{3.13}$$

which is the equation of the inverse integrating factor. As an inverse integrating factor is not coordinates free (as happens for a first integral) and it is affected by the Jacobian of the transformation when we come back to the original coordinates. In this case the recurrence differential equation is

$$6cz^4 v_{k-3} - 7z^3 v_{k-2} - 2zv_{k-1} - cz^5 v'_{k-3} + z^4 v'_{k-2} + z^2 v'_{k-1} + v'_k = 0.$$

Without loss of generality we now take $v_1 = 1$. Then $v_2 = z^2 + c_2$, and

$$\begin{aligned}
v_3(z) &= \frac{1}{2}(2c_2z^2 + z^4) + c_3, \\
v_4(z) &= \frac{1}{20}(20c_3z^2 + 15c_2z^4 - 8cz^5) + c_4, \\
v_5(z) &= 5\frac{1}{420}(420c_4z^2 + 420c_3z^4 - 252c\, c_2z^5 + 35c_2z^6 + 12cz^7) + c_5.
\end{aligned}$$

Taking into account that the inverse integrating factor for system (3.10) is obtained multiplying the power series (3.12) by the Jacobian of the transformation, we have to take $c = 0$ in a polynomial $v_5$ to ensure that $V$ is polynomial in the original variables $(x, y)$. Then

$$v_6 = \frac{1}{10080}(10080c_5z^2 + 12600c_4z^4 + 1680c_3z^6 + 525c_2z^8) + c_6.$$

Choosing $c_2 = c_3 = c_4 = c_5 = c_6 = 0$ then $v_6 = 0$ and we can choose $v_k = 0$ for all $k \geq 5$. Consequently,

$$\mathcal{V} = v_1\left(\frac{x}{y}\right)y + v_2\left(\frac{x}{y}\right)y^2 + v_3\left(\frac{x}{y}\right)y^3 + v_4\left(\frac{x}{y}\right)y^4 = y + x^2 + \frac{x^4}{2y}.$$

The inverse integrating factor of system (3.10) is obtained multiplying $\mathcal{V}$ by the Jacobian of the transformation

$$V = y\mathcal{V} = y^2 + x^2 y + \frac{x^4}{2}.$$

For a general monodromic nilpotent singularity the existence of an inverse integrating factor in a neighborhood of singularity does not guarantees the existence of a center at this singularity, but for the nilpotent monodromic singularities with leading term $(y, -x^3)$ this is true, see the result in [6, 20]. System (3.10) with $c = 0$ was studied in [11], where it was proved that there exists no analytic first integral. Consequently, here we have used the blow–up method to find an inverse integrating factor of system (3.10) which gives the condition $c = 0$ implying that system (3.10) has a center at the origin if and only if $c = 0$. □

**Proposition 3.4.** *Consider the nilpotent differential system*

$$\dot{x} = y + ax^2 + 5xy^2, \qquad \dot{y} = -2x^3 + 3xy^2 - 4y^3, \tag{3.14}$$

*where $a \in \mathbb{R}$. The first necessary condition of system (3.14) to have a center is $-98 + 47a^2 + 20a^4 = 0$. Moreover system (3.14) always has a focus at the origin.*

*Proof.* System (3.14) has a monodromic singular point at the origin if and only if $|a| < 2$, see [27]. Applying the blow-up transformation (3.2) system (3.14) takes the form

$$\dot{z} = \ 1 + 9y^2 z + ayz^2 - 3y^2 z^2 + 2y^2 z^4, \qquad \dot{y} = \ -y^3(4 - 3z + 2z^3). \tag{3.15}$$

We propose directly a power series of the form (3.12) and we impose that this $\mathcal{V}$ satisfies the equation (3.13). Recall that the transformation to the original variables $(x, y)$ will be affected by the Jacobian of the transformation. In this case the recurrence differential equation is

$$\big(3(k-3)z - (4k-11) - 2(k-1)z^3\big)v_{k-2} - 2azv_{k-1} + \big(9z - 3z^2 + 2z^4\big)v'_{k-2} + az^2 v'_{k-1} + v'_k = 0$$

and we can, as above, take $v_1(z) = 1$, and $v_2(z) = az^2 + c_2$, $v_3(z) = z + ac_2 z^2 + z^4 + c_3$

$$\begin{aligned}
v_4(z) &= 5c_2 z - \frac{3}{2}c_2 z^2 + ac_3 z^2 - 4az^3 + \frac{3}{4}az^4 + \frac{3}{2}c_2 z^4 + c_4,\\
v_5(z) &= \frac{1}{60}(P_5(z) + 60z^6 - 15a^2 z^6 + 10ac_2 z^6) + c_5,\\
v_6(z) &= \frac{1}{1680}(P_7(z) - 525az^8 + 210a^3 z^8 + 630c_2 z^8 - 140a^2 c_2 z^8) + c_6,
\end{aligned}$$

where $P_i(z)$ are determined polynomials of degree $i$. Taking into account that in the original variable the inverse integrating factor is $V = y\mathcal{V}$ the coefficient in the term with $z^8$ in $v_6$ must be zero. Then, we have $-525a + 210a^3 + 630c_2 - 140a^2 c_2 = 0$ which yields

$$c_2 = \frac{3(2a^3 - 5a)}{2(2a^2 - 9)},$$

if $2a^2 - 9 \neq 0$. Recall that if $2a^2 - 9 = 0$ this is not a monodromic case. Next, $v_7$ has the form

$$v_7(z) = \frac{P_8(z) + (14112 - 9904a^2 - 1376a^4 + 640a^6)z^9}{1680(2a^2 - 9)} + c_7,$$

where $P_8(z)$ is a determined polynomial of degree 8. The coefficient of the term with monomial $z^9$ must vanish, so,

$$16(-9+2a^2)(-98+47a^2+20a^4)=0.$$

The unique real roots of this polynomial satisfying the monodromic condition $|a|<2$ are $a=\pm 1.153741$. This last numerical value was obtained by Varin in [42] using the Bautin method after doing a generalized polar blow-up. The method developed in [42, 43] is not useful to compute the algebraic condition $-98+47a^2+20a^4=0$. Moreover, with our method we can distinguish between a center and a focus. If we compute more terms of the power series $\mathcal{V}$ the powers in $z$ that must be zero have not a common root. Therefore, the origin of system (3.14) is always a focus. The algebraic necessary center condition $-98+47a^2+20a^4=0$ was also obtained in [27] using a more involved method based in the result that all the nilpotent centers are limit of non-degenerate centers. The fact that, under monodromy the origin of (3.14) is always a focus was also derived in Proposition 26 of [21]. Here we also use that the existence of a formal inverse integrating factor defined around a nilpotent monodromic singularity with leading term $(-y,x^3)$ is a necessary and sufficient condition to have a center at the singularity, see [6, 20]. □

## 4 Degenerate monodromic singularities

In this section we consider different systems with a degenerate singularity, and using the blow–up method we study the center-focus problem. The examples proposed here show the narrow relation between the center problem and the existence of a first integral for the singular points at infinity. The necessary conditions founded by the method do not always correspond to trivial cases of centers.

**Proposition 4.1.** *Consider the differential system*

$$\dot{x}=x^2y+ax^5+y^5, \qquad \dot{y}=-xy^2-x^5+bx^4y, \tag{4.1}$$

*where $a$, $b\in\mathbb{R}$. System (4.1) has a center at the origin if and only if $5a+b=0$.*

*Proof.* In [5] it is proved that the origin of system (4.1) is always monodromic. Moreover, system (4.1) has characteristic directions because the homogeneous polynomial $xq_n(x,y)-yp_n(x,y)$, where $p_n$ and $q_n$ are the lower homogeneous terms of system (4.1), has real roots. When the singular point has characteristic directions it is not possible to apply the Bautin method in order to solve the center-focus problem, see [25].

After applying the blow-up transformation (3.2) system (4.1) takes the form

$$\begin{aligned}\dot{z}&=y^2+2z^2+ay^2z^5-by^2z^5+y^2z^6\\ \dot{y}&=-yz(1-by^2z^3+y^2z^4).\end{aligned} \tag{4.2}$$

Now, we look for a power series of the form (3.4) and we compute $\dot{\mathcal{H}}=(\partial\mathcal{H}/\partial z)\dot{z}+(\partial\mathcal{H}/\partial y)\dot{y}$ for system (4.2). We obtain the following recursive differential equation

$$(k-2)z^4(b-z)f_{k-2}-kzf_k+(1+az^5-bz^5+z^6)f'_{k-2}+2z^2f'_k=0$$

Solving for the first values of $k$ we can take $f_k = 0$ for all $k$ odd and for $k$ even we find $f_2 = 0$, $f_4 = z^2 + c_4$ where $c_4$ is an arbitrary integration constant, and

$$\begin{aligned} f_6 &= \frac{1}{6}(2 - 3az^5 - 3bz^5 + 2z^6) + c_6, \\ f_8 &= -\frac{1}{80}z^3(100a + 20b - 240bc_6 - 25a^2z^5 - 30abz^5 \\ &\quad - 5b^2z^5 + 20az^6 + 4bz^6 - 80zc_8 - 240c_6 z \log z), \end{aligned}$$

where $c_6$ and $c_8$ are arbitrary constants. Since $f_8$ must be a polynomial we have to impose $c_6 = 0$ and since it must be a polynomial in the original variables $(x, y)$ we have to impose that the terms in $z^9$ vanish, that is, $5a + b = 0$. Under this restrictions we have that

$$f_{10} = c_8 z^2 \left( \frac{2}{3} + 4az^5 + \frac{2}{3}z^6 \right) + c_{10} z^5.$$

Then taking $c_8 = c_{10} = 0$ we get $f_{10} = 0$ and we can choose $f_k = 0$ for all $k \geq 10$. Consequently

$$\mathcal{H} = f_4\Big(\frac{x}{y}\Big)y^4 + f_6\Big(\frac{x}{y}\Big)y^6 + f_8\Big(\frac{x}{y}\Big)y^8 = \frac{x^6}{3} + 2ax^5y + x^2y^2 + \frac{y^6}{3},$$

which is a polynomial first integral of system (4.1). Therefore, when $5a + b = 0$ system (4.1) has a center at the origin. It remains to see that if $5a + b \neq 0$ then system (4.1) has a focus at the origin. From [5, Theorem 2.3] is derived the geometric criteria for proving that if $5a + b \neq 0$ then system (4.1) has a focus at the origin, see Proposition 3.19 in [5]. Here our blow-up method gives straightforward the necessary condition while for applying the geometric criteria the necessary condition is needed. □

**Proposition 4.2.** *Consider the differential system*

$$\dot{x} = x^2y + ax^3 + y^5, \qquad \dot{y} = -xy^2 + bx^2y - x^3, \tag{4.3}$$

*where $a$, $b \in \mathbb{R}$. System (4.3) has a center at the origin if and only if $3a + b = 0$.*

*Proof.* The origin of system (4.3) is monodromic if and only if, $(a - b)2 - 8 < 0$, see [5]. System (4.3) has also characteristic directions. Applying the blow-up (3.2) system (4.3) takes the form

$$\begin{aligned} \dot{z} &= y^2 + 2z^2 + az^3 - bz^3 + z^4 \\ \dot{y} &= -yz(1 - bz + z^2), \end{aligned} \tag{4.4}$$

after a scaling of time. Now, we compute $\dot{\mathcal{H}} = (\partial\mathcal{H}/\partial z)\dot{z} + (\partial\mathcal{H}/\partial y)\dot{y}$ for system (4.4) where $\mathcal{H}$ is a power series of the form (3.4) and we obtain the following recursive differential equation

$$-5z(1 - bz + z^2)f_k + f'_{k-2} + (2z^2 + az^3 - bz^3 + z^4)f'_k = 0.$$

Doing the computations of the first $f_k$ we must to take $f_2 = f_3 = 0$ in order to be polynomials and

$$f_4 = c_4 e^{-\frac{2(3a+b)\arctan\left(\frac{a-b+2z}{\sqrt{8-(a-b)^2}}\right)}{\sqrt{8-(a-b)^2}}} z^2(2 + z(a - b + z)).$$

where in order to have a polynomial we have to take $3a + b = 0$ and without loss of generality $c_4 = 1$.

Next we must to take $f_5=0$ and $f_6=(2+3z^3(2+4az+z^2)^{3/2}c_6)/3$, and taking $c_6=0$ we have $f_6=2/3$. Next, $f_7=0$ and $f_8=c_8z^4(2+4az+z^2)^2$. Then taking $c_8=0$ we obtain $f_8=0$ and we can choose $f_k=0$ for all $k\geq 8$. Consequently

$$\mathcal{H}=f_4\Big(\frac{x}{y}\Big)y^4+f_6\Big(\frac{x}{y}\Big)y^6+f_8\Big(\frac{x}{y}\Big)y^8=x^4+4ax^3y+2x^2y^2+\frac{2y^6}{3},$$

which is a polynomial first integral of system (4.3). Finally to see that for $3a+b\neq 0$ we have a focus at the origin, we use also the geometric criteria developed from [5, Theorem 2.3]. In Proposition 3.16 [5] is that system (4.3) has a focus if $3a+b\neq 0$. As in the example before the blow-up method gives the necessary condition directly. □

**Proposition 4.3.** *Consider the degenerate differential system*

$$\dot{x}=cx^2y+fxy^2+dy^3,\qquad \dot{y}=\tilde{c}xy^2+fy^3+ax^5. \tag{4.5}$$

*If the origin of system (4.5) is monodromic then it is a center if, and only if, $f=0$.*

*Proof.* In [38] Medvedeva studied the stability problem of the origin of system(4.5). The first non zero focal value of system (4.5) was given in [38] through a complicate and involved method using several blow-up transformations. The monodromy problem for system (4.5) was solved in [5] where the following result was given.

**Lemma 4.4.** *The origin of system (4.5) is monodromic if and only if one of the following conditions holds:*

*a)* $d\,a<0$, $(\tilde{c}-c)(\tilde{c}-2c)>0$ *and* $d(\tilde{c}-c)<0$

*b)* $d\,a<0$, $\tilde{c}-c=0$ *and* $cd>0$.

*c)* $d\,a<0$, $\tilde{c}-2c=0$ *and* $ca>0$.

Applying the blow-up transformation (3.2) to system (4.5), the new differential system takes the form

$$\begin{aligned}\dot{z}&=d+cz^2-\tilde{c}z^2-ay^2z^6,\\ \dot{y}&=y\left(\tilde{c}z+ay^2z^5+f\right)\end{aligned}\tag{4.6}$$

with the change of time $d\tau=y^2dt$. From the monodromic condition we know that $d\neq 0$. System (4.6) has a regular point at the origin and consequently, an analytic first integral around the origin and the recursive differential equation do not generate logarithmic terms. Then the question is if this analytic first integral at infinity can be extended to the origin of the original system (4.5). In this case the recursive differential equation is

$$(k-2)az^5f_{k-2}+k(\tilde{c}z+f)f_k-az^6f'_{k-2}+(d+cz^2-\tilde{c}z^2)f'_k=0.$$

Then if $f_i=0$ for $i=1,\ldots,k-2$ we have that the value of $f_k$ is

$$f_k=c_ke^{-\frac{kf\arctan\left(\frac{\sqrt{c-\tilde{c}}\,z}{\sqrt{d}}\right)}{\sqrt{c-\tilde{c}}\sqrt{d}}}\left(d+(c-\tilde{c})z^2\right)^{-\frac{k\tilde{c}}{2(c-\tilde{c})}}.$$

In order to have a well defined function in the original variables $(x,y)$ we have to impose $f=0$. Moreover, under the monodromic condition system (4.5) has a center at the origin since it is invariant with respect to the symmetry $(x,y,t)\to(-x,y,-t)$.

To finish the proof we see that if $f \neq 0$ then system (4.5) has a focus at the origin. We apply the geometrical criteria developed in [5, Theorem 2.3]. Consider the vector field

$$\mathcal{X}_c = (cx^2y + dy^3)\frac{\partial}{\partial x} + (\tilde{c}xy^2 + ax^5)\frac{\partial}{\partial y},$$

which has a center at the origin. Let $\mathcal{X}$ the vector field associated to system (4.5). Then we compute that

$$\mathcal{X} \wedge \mathcal{X}_c = fy^2(ax^6(\tilde{c} - c)x^2y^2 - dy^4)$$

which is semi-definite under the monodromic conditions of Lemma 4.4 and by Theorem 2.3 of [5] if $f \neq 0$ system (4.5) has a focus at the origin. □

Finally we consider the differential system

$$\begin{aligned} \dot{x} &= y^3 + 2ax^3y + 2x(\alpha x^4 + \beta xy^2), \\ \dot{y} &= -x^5 - 3ax^2y^2 + 3y(\alpha x^4 + \beta xy^2), \end{aligned} \tag{4.7}$$

where $\alpha$, $\beta$, $a \in \mathbb{R}$. In [4] it was proven that system (4.7) with $\alpha\beta \neq 0$ is not orbitally reversible nor formally integrable. Moreover there are values of $(\alpha, \beta, a)$ with $a \neq 0$ and with the monodromic condition $|a| < 1/\sqrt{6}$ such that the origin of system (4.7) is a center. In fact the center condition is not algebraic in the parameters. In [23] it was also identified the center condition using the existence of an inverse integrating factor. Therefore the center problem is not algebraically solvable. As we will see, if we apply the blow-up method proposing a power series verifying the first integral equation we only find the algebraically solvable centers. So we will propose a power series satisfying the inverse integrating equation. Applying the blow-up (3.2) to system (4.7), the new differential system takes the form

$$\begin{aligned} \dot{z} &= 1 + 5ayz^3 + y^2z^6 - y^2z^5\alpha - yz^2\beta \\ \dot{y} &= -y^2z(3az + yz^4 - 3yz^3\alpha - 3\beta), \end{aligned} \tag{4.8}$$

after the scaling of time $d\tau = y^2dt$. Looking for a power series of the form (3.4) and computing the equation that satisfies a first integral we get only the center condition $\alpha = \beta = 0$ (the reader can follow the steps seeing the previous examples). Therefore the analytic first integral at infinity cannot always be extended to the origin of system (4.7). Next, we propose a power series of the form

$$\mathcal{F}(z, y) = y^{k_2} \sum_{k=0}^{\infty} v_k(z)y^k, \tag{4.9}$$

where $k_2 \in \mathbb{Q}$ and we impose that it satisfies the equation of certain inverse integrating factor $\dot{z}\partial\mathcal{F}/\partial z + \dot{y}\partial\mathcal{F}/\partial y = k_1(\partial\dot{z}/\partial z + \partial\dot{y}/\partial y)\mathcal{F}$, where $k_1 \in \mathbb{R}$. The recurrence differential equation is

$$\begin{aligned} &(-(k-2)z^5 - 3k_1z^5 - k_2z^5 + 3(k-2)z^4\alpha - 4k_1z^4\alpha + 3k_2z^4\alpha)v_{k-2} \\ &\quad + (-3(k-1)az^2 - 9ak_1z^2 - 3ak_2z^2 + 3(k-1)z\beta - 4k_1z\beta + 3k_2z\beta)v_{k-1} \\ &\quad + (z^6 - z^5\alpha)v'_{k-2} + (5az^3 - z^2\beta)v'_{k-1} + v'_k = 0 \end{aligned}$$

and we can take $v_0(z) = 1$, and

$$\begin{aligned}
v_1(z) &= 3ak_1z^3 + ak_2z^3 + 2k_1z^2\beta - \frac{3}{2}k_2z^2\beta + c_1,\\
v_2(z) &= \frac{1}{6}k_2z^6 + \frac{1}{2}a^2(9k_1^2 + 6k_1(k_2-2) + (k_2-4)k_2)z^6 - \frac{3}{5}k_2z^5\alpha\\
&\quad - \frac{3}{2}c_1z^2\beta - \frac{3}{2}c_1k_2z^2\beta + 2k_1^2z^4\beta^2 + \frac{3}{8}k_2z^4\beta^2 + \frac{9}{8}k_2^2z^4\beta^2\\
&\quad + \frac{1}{10}az^3\Big(10\,c_1(1+3k_1+k_2)\\
&\quad + (60k_1^2 + 3(7-5k_2)k_2 - k_1(28+25k_2))z^2\beta\Big)\\
&\quad + \frac{1}{10}k1(5z^6 + 8z^5\alpha + 20c_1z^2\beta - 5(1+6k_2)z^4\beta^2) + c_2.
\end{aligned}$$

We do not write here the value of $v_3(z)$ due to its length. Now, choosing the values of $k_1$, $k_2$, $c_1$, $c_2$ and $c_3$ we impose that $v_3(z) = 0$. One solution is $k_1 = 12/13$ and $k_2 = 16/13$ and $c_1 = c_2 = c_3 = 0$ which implies $v_k = 0$ for all $k \geq 3$. Consequently,

$$\mathcal{F} = y^{k_2}\Big(v_0\Big(\frac{x}{y}\Big) + v_1\Big(\frac{x}{y}\Big)y + v_2\Big(\frac{x}{y}\Big)y^2\Big) = \frac{2x^6 + 12ax^3y^2 + 3y^4}{3y^{36/13}}.$$

The inverse integrating factor for system (3.10) is obtained by multiplying $\mathcal{V} = \mathcal{F}^{\frac{13}{12}}$ by the Jacobian of the transformation and the change of time made, i.e.

$$V = y^3\mathcal{V} = y^3\mathcal{F}^{\frac{13}{12}} = \Big(y^2 + x^2y + \frac{x^4}{2}\Big)^{\frac{13}{12}}.$$

For a degenerate singular point the existence of an inverse integrating factor defined around the singular point does not guarantee the existence of a center at the singular point. In fact for system (4.7) an extra nonalgebraic condition in the parameters is needed, see [4,23].

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