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Multiplicity of solutions of Kirchhoff-type fractional Laplacian problems with critical and singular nonlinearities

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Abstract. In this article, the following Kirchhoff-type fractional Laplacian problem with singular and critical nonlinearities is studied:

$$\begin{cases} \left(a+b\|u\|^{2\mu-2}\right)(-\Delta)^s u = \lambda l(x) u^{2_s^*-1} + h(x) u^{-\gamma}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $s \in (0,1)$, $N > 2s$, $(-\Delta)^s$ is the fractional Laplace operator, $2_s^* = 2N/(N-2s)$ is the critical Sobolev exponent, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $l \in L^\infty(\Omega)$ is a non-negative function and $\max\{l(x), 0\} \not\equiv 0$, $h \in L^{\frac{2_s^*}{2_s^*+\gamma-1}}(\Omega)$ is positive almost everywhere in Ω, $\gamma \in (0,1)$, $a > 0, b > 0$, $\mu \in [1, 2_s^*/2)$ and parameter λ is a positive constant. Here we utilize a special method to recover the lack of compactness due to the appearance of the critical exponent. By imposing appropriate constraint on λ, we obtain two positive solutions to the above problem based on the Ekeland variational principle and Nehari manifold technique.

Keywords: fractional Laplacian problem, singular, critical nonlinearity, Kirchhoff-type problem.

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1 Introduction

This paper is concerned with the existence and multiplicity of positive solutions for the following Kirchhoff-type problem with singular nonlinearity and critical exponent driven by

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fractional Laplacian operator:

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)(-\Delta)^s u = \lambda l(x)u^{2_s^*-1} + h(x)u^{-\gamma}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $0 < s < 1$, $N > 2s$, $2_s^* = 2N/(N-2s)$ is the fractional critical Sobolev exponent, $M(t) = a + bt^{\mu-1}$, $a > 0, b > 0$, $\mu \in [1, 2_s^*/2)$, $l(x)$ is non-negative and $l(x) \in L^\infty(\Omega)$ satisfies $l(x) \not\equiv 0$ in Ω, $0 < \gamma < 1$ and $h \in L^{\frac{2_s^*}{2_s^*+\gamma-1}}(\Omega)$ is positive almost everywhere in Ω, parameter $\lambda > 0$ and $(-\Delta)^s$ is the fractional Laplace operator which defined up to normalization factors as

$$(-\Delta)^s \Psi(x) = 2 \lim_{\tau \to 0^+} \int_{\mathbb{R}^N \backslash B_\tau(x)} \frac{\Psi(x) - \Psi(y)}{|x-y|^{N+2s}} \mathrm{d}y, \qquad x \in \mathbb{R}^N, \tag{1.2}$$

for any $\Psi \in C_0^\infty(\mathbb{R}^N)$, where $B_\tau(x)$ is the ball with radius τ and center $x \in \mathbb{R}^N$. For more details, we can refer to [25] and references therein. The fractional elliptic problem appeared in many different practical applications and phenomena, such as resilience, phase transformation and minimal surface problems, etc. For more related introduction, see [1,2,6,20,29].

Above all, let us review the relevant progress on Kirchhoff-type equation. The Kirchhoff-type equation is a generalization of the classical D'Alembert wave equation, which was raised by Kirchhoff to describe the lateral vibration of stretched strings in [17]. The basic model for problem (1.1) can be summarized as follows:

$$\rho u_{tt} - M\left(\int_0^L u_x^2 dx\right) u_{xx} = 0,$$

where ρ, a, b, L are constants, $M(\int_0^L u_x^2 dx) := a + b(\int_0^L u_x^2 dx)^{\mu-1}$ describes the tension changes arise from changes in string length during the vibrations. Concerning the Kirchhoff term M, we consider a specific version of M,

$$M(t) = a + bt^{\mu-1}, \quad a, b > 0, \quad 1 \le \mu < 2_s^*/2. \tag{1.3}$$

Where, a represents the initial tension while b is related to the inherent properties of string (such as Young's modulus). In particular, in the case of $M(0) = 0$ while $M(t) > 0$ for all $t \in \mathbb{R}^+$, Kirchhoff-type equation is often referred as degenerate. If $M(t) \ge c > 0$ for all $t \in \mathbb{R}_0^+$ and some constant c, equation is commonly known as non-degenerate. For some advance of degenerate Kirchhoff-type problems, see for instance [3,7,32]. In addition, we refer to [8,10,13,31,33] about some existence results of non-degenerate Kirchhoff-type problems.

Next, let us present some progress of Laplacian equations involving singular terms. A general version of this type of problem can be formed as follows:

$$\begin{cases} -\Delta u = \lambda m(x) u^{-\gamma} + h(x) u^q, & \text{in}\Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{in } \partial\Omega. \end{cases} \tag{1.4}$$

In the early days, when $\lambda \equiv 1$ and $h(x) = 0$, the existence and regularity results of the solutions to problem (1.4) were studied by Boccardo et al. in [5]. The difference in results depends on the summability of m in some Lebesgue function spaces and on the value range of γ (which can be smaller, equal or larger than 1). When $0 < \gamma < 1$, in the case of $m(x) \equiv 1$ and $h(x) \equiv 0$, Crandall et al. solved problem (1.4) in [9] and learned that it has a unique weak solution. Subsequently, the multiplicity of solutions to such problems was obtained by Sun et al. in [28]. Moreover, Liu and Sun solved the Kirchhoff equation involving singular terms and Hardy potential in [23]. For equations involving the critical case, we may refer to [12, 15, 16]. To be specific, the author in [12] solved the Kirchhoff equation involving the critical exponent and obtained two different solutions. In [15], when $m(x), h(x) \equiv 1$, the authors obtained that if λ is less than a positive constant, then problem (1.4) has two positive solutions. Furthermore, Yang in [16] studied the multiplicity and asymptotic behavior of positive solutions to problem (1.4), where $0 < \gamma < 1 < q \leq (N+2)/(N-2)$. By applying variational method and sub-supersolution technique, the author learned what happens to the number and properties of solutions for the equation with different values of λ. In the setting of $\gamma = 1$, minimization theory is used by the authors in [31] to obtain a unique positive solution in the subcritical case. Regarding $\gamma > 1$, in the case of $h(x) \equiv 0$ and $\lambda \equiv 1$, the authors in [18] also got the unique solution. It is worth mentioning that Wang et al. used Ekeland's variational principle and the Nehari method to prove the existence of a unique positive solution for a Kirchhoff equation involving strong singularity in [30]. Besides, there are equations for (1.4) with Kirchhoff terms that we can refer to [19, 21, 22]. In [19], the authors obtained two different positive solutions through the variational and perturbation methods. Liao et al. in [22] studied the solutions of equation (1.4) in the weak singular case under different constraints. On the basis of [22], they solved the critical case in [21] and got the unique positive solution.

In the above context, the following class of singular Kirchhoff problem with fractional Laplace operators has been extensively studied:

$$\begin{cases} M(\int_\Omega |\nabla u(x)|^2)(-\Delta)^s u = \lambda f(x) u^{-\gamma} + g(x) u^{2_s^*-1}, & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.5}$$

in the setting of $M \equiv 1$. Mukherjee and Sreenadh in [24] studied a singular problem with critical growth and obtained two solutions, where γ can be equal to 1. In the case of $\gamma > 0$, Barrios et al. discussed the existence of solutions to the equation 1.5 in two cases: $g(x) = 0$ and $g(x) = 1$. Besides, the authors in [14] solved a variant of problem (1.5) in which λ is multiplied to the critical term. Through the variational method, they learned about the existence and multiplicity of solutions to the equation when λ takes different values. For such problems with different Kirchhoff terms, we may consult [11–13] and the references therein. Equation (1.5) was discussed in [12], where there is no weight function and the Kirchhoff term may be degenerate, the variational method and appropriate truncation theory were used to obtain two solutions. In [13], Fiscella et al. proved that equation (1.5) of the non-degenerate type has two distinct solutions by using the Nehari method. At last, the authors in [11] considered a critical degenerate Kirchhoff problem with strong singularity, and the only positive solution was obtained.

In view of the aforementioned works, in particular, according to [11–13,30], we are inspired to investigate the existence and multiplicity of solutions to problem (1.1) under appropriate assumptions. The most significant difficulty lies in the lack of compactness caused by the presence of critical term. For this, we use the method of [13] to recover compactness. Especially, we are interested in a natural problem: whether problem (1.1) can be solved in the strong singular case? We will try our best to study this situation in the future.

Here is the main result we obtain.

Theorem 1.1. *Let $s \in (0,1)$, $N > 2s$, $0 < \gamma < 1$, a be small enough and $h \in L^{\frac{2_s^*}{2_s^*+\gamma-1}}(\Omega)$ be positive a.e. in Ω. Then there exists $\Gamma_0 > 0$, when $0 < \lambda < \Gamma_0$, then problem (1.1) has at least two positive solutions with negative energies.*

Remark 1.2. Compared to the fundamental conclusion in [11], there are three main differences: (i) The range of $M(t)$ is different, in this paper we only consider non-degenerate case. (ii) We utilize a different method to recover the lack of compactness caused by the critical term. (iii) By controlling the range of λ in the weak singular case, we obtain two positive solutions.

Remark 1.3. Compared to [13], our result refines and improves the main result of [13] from the following aspects: (i) We do not need to control b to be as small as possible but we need to control $a > 0$ small enough to ensure that λ is positive in order to obtain the second positive solution. (ii) Our nonlinearities do not involve sign changing functions and we don't need to control $l(x) = \|l\|_\infty$ in $B_{\rho_0}(0)$ for some $\rho_0 > 0$.

2 Variational setting

Regarding problem (1.1), we mainly solve it in fractional Sobolev space, which is specifically defined by

$$\mathbb{M} = \left\{ \Phi \mid \Phi : \mathbb{R}^N \to \mathbb{R} \text{ is measureable}, \ \Phi|_\Omega \in L^2(\Omega), \ \frac{\Phi(x) - \Phi(y)}{|x-y|^{\frac{N+2s}{2}}} \in L^2(G) \right\}, \tag{2.1}$$

where $G = \mathbb{R}^{2N} \backslash (\Omega^c \times \Omega^c)$, with $\Omega^c = \mathbb{R}^N \backslash \Omega$. Moreover, $\mathbb{M}_0$ is defined as the linear subspace of $\mathbb{M}$, which is

$$\mathbb{M}_0 := \left\{ \Phi \in \mathbb{M} : \Phi = 0 \ \text{ a.e. in } \ \mathbb{R}^N \backslash \Omega \right\}.$$

As for the norm of two spaces, the norm of the space $\mathbb{M}$ is given as shown below:

$$\|\Phi\|_{\mathbb{M}} = \|\Phi\|_{L^2(\Omega)} + \left(\iint_G \frac{|\Phi(x) - \Phi(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2}. \tag{2.2}$$

Besides, we confirm the following norm on $\mathbb{M}_0$:

$$\|\Phi\|_{\mathbb{M}_0} := \left(\iint_G \frac{|\Phi(x) - \Phi(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2}. \tag{2.3}$$

According to Lemma 6 in [26], it is easy to know that (2.2) and (2.3) are equivalent. In addition, it is standard to verify that $(\mathbb{M}_0, \|.\|_{\mathbb{M}_0})$ is a Hilbert space and the form of scalar product in $\mathbb{M}_0$ is as follows:

$$\langle \imath, \jmath \rangle : \langle \imath, \jmath \rangle_{\mathbb{M}_0} = \iint_G \frac{(\imath(x) - \imath(y))(\jmath(x) - \jmath(y))}{|x-y|^{N+2s}} dxdy, \quad \text{for } \imath, \jmath \in \mathbb{M}_0, \tag{2.4}$$

see for example Lemma 7 in [26]. The embedding $\mathbb{M}_0 \hookrightarrow L^\eta(\Omega)$ is compact and continuous for $2 \le \eta < 2_s^*$, see [26, Lemma 8]). Then, an appropriate selection linked to the best Sobolev constant can be defined as

$$S_s = \inf_{\Phi \in \mathbb{M}_0 \backslash \{0\}} S_s(\Phi) = \frac{\iint_G \frac{|\Phi(x)-\Phi(y)|^2}{|x-y|^{N+2s}} dxdy}{\left(\int_\Omega |\Phi(x)|^{2_s^*} dx \right)^{2/2_s^*}}. \tag{2.5}$$

In what follows, for the sake of simplicity of notations, we shall denote $\|\cdot\|_{\mathbb{M}_0}$ and $\|\cdot\|_{L^\eta(\Omega)}$ by $\|\cdot\|$ and $\|\cdot\|_\eta$ for any $\eta \in [2,\infty]$.

In the process of obtaining multiple solutions, we will use Nehari manifold method and fibering maps. Before this, let us first introduce the definition of weak solutions to problem (1.1).

Definition 2.1. $u \in \mathbb{M}_0$ is a weak solution of problem (1.1) if for all $\ell \in \mathbb{M}_0$ the following weak formulation is satisfied:

$$a\langle u,\ell\rangle + b\|u\|^{2\mu-2}\langle u,\ell\rangle - \lambda\int_\Omega l(x)|u|^{2_s^*-1}\ell dx - \int_\Omega h(x)|u|^{-\gamma}\ell dx = 0.$$

The energy functional associated to problem (1.1): $\mathcal{I}:\mathbb{M}_0 \to \mathbb{R}$ is defined as

$$\mathcal{I}(u) = \frac{a}{2}\|u\|^2 + \frac{b}{2\mu}\|u\|^{2\mu} - \frac{\lambda}{2_s^*}\int_\Omega l(x)|u|^{2_s^*}dx - \frac{1}{1-\gamma}\int_\Omega h(x)|u|^{1-\gamma}dx. \tag{2.6}$$

3 Fibering maps analysis

For any $u \in \mathbb{M}_0$, we first introduce the fibering map: $\phi_u(t):(0,\infty)\to\mathbb{R}$, defined as

$$\phi_u(t) = \mathcal{I}(tu) = \frac{a}{2}t^2\|u\|^2 + \frac{b}{2\mu}t^{2\mu}\|u\|^{2\mu} - \lambda\frac{t^{2_s^*}}{2_s^*}\int_\Omega l(x)|u|^{2_s^*}dx - \frac{t^{1-\gamma}}{1-\gamma}\int_\Omega h(x)|u|^{1-\gamma}dx.$$

Through simple calculation, we get

$$\phi_u'(t) = at\|u\|^2 + bt^{2\mu-1}\|u\|^{2\mu} - \lambda t^{2_s^*-1}\int_\Omega l(x)|u|^{2_s^*}dx - t^{-\gamma}\int_\Omega h(x)|u|^{1-\gamma}dx,$$

where in particular

$$\phi_u'(1) = a\|u\|^2 + b\|u\|^{2\mu} - \lambda\int_\Omega l(x)|u|^{2_s^*}dx - \int_\Omega h(x)|u|^{1-\gamma}dx. \tag{3.1}$$

From this, we may define the constrained set as

$$\mathbb{X} = \left\{u \in \mathbb{M}_0 : a\|u\|^2 + b\|u\|^{2\mu} - \lambda\int_\Omega l(x)|u|^{2_s^*}dx - \int_\Omega h(x)|u|^{1-\gamma}dx = 0\right\}. \tag{3.2}$$

Furthermore,

$$\phi_u''(t) = a\|u\|^2 + (2\mu-1)bt^{2\mu-2}\|u\|^{2\mu} - \lambda(2_s^*-1)t^{2_s^*-2}\int_\Omega l(x)|u|^{2_s^*}dx + \gamma t^{-\gamma-1}\int_\Omega h(x)|u|^{1-\gamma}dx.$$

Apparently,

$$\phi_u''(1) = a\|u\|^2 + (2\mu-1)b\|u\|^{2\mu} - \lambda(2_s^*-1)\int_\Omega l(x)|u|^{2_s^*}dx + \gamma\int_\Omega h(x)|u|^{1-\gamma}dx. \tag{3.3}$$

As a matter of fact, the two weak solutions we want are in $\mathbb{X}$. In order to better explore the existence of solutions, $\mathbb{X}$ can be further decomposed into $\mathbb{X}^+$, $\mathbb{X}^-$ and $\mathbb{X}^0$:

$$\mathbb{X}^+ = \left\{u\in\mathbb{X} : a(1+\gamma)\|u\|^2 + b(2\mu-1+\gamma)\|u\|^{2\mu} - \lambda(2_s^*-1+\gamma)\int_\Omega l(x)|u|^{2_s^*}dx > 0\right\}, \tag{3.4}$$

$$\mathbb{X}^- = \left\{u\in\mathbb{X} : a(1+\gamma)\|u\|^2 + b(2\mu-1+\gamma)\|u\|^{2\mu} - \lambda(2_s^*-1+\gamma)\int_\Omega l(x)|u|^{2_s^*}dx < 0\right\}, \tag{3.5}$$

$$\mathbb{X}^0 = \left\{u\in\mathbb{X} : a(1+\gamma)\|u\|^2 + b(2\mu-1+\gamma)\|u\|^{2\mu} - \lambda(2_s^*-1+\gamma)\int_\Omega l(x)|u|^{2_s^*}dx = 0\right\}. \tag{3.6}$$

4 Technical lemmas

In this section, we shall present several relevant lemmas in this section, which will be helpful for the proof of Theorem 1.1.

Lemma 4.1. *When* $0 < \lambda < \Gamma_1$ *hold, where*

$$\Gamma_1 = \left(\frac{1+\gamma}{2_s^*-2}\right)\left(\frac{a(2_s^*-2)}{2_s^*+\gamma-1}\right)^{\frac{2_s^*+\gamma-1}{1+\gamma}} S_s^{\frac{2_s^*-1+\gamma}{1+\gamma}} \|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}}^{\frac{2-2_s^*}{1+\gamma}} \|l\|_\infty^{-1},$$

there exist unique $t_0 = t_0(u) > 0$, $t_- = t_-(u) > 0$, $t_+ = t_+(u) > 0$, *with* $t_- < t_0 < t_+$, *such that* $t_+u \in \mathbb{X}^+$, $t_-u \in \mathbb{X}^-$.

Proof. For any $u \in \mathbb{M}_0$, we may write $\psi_u(t)$ in the form

$$\psi_u(t) = at^{2-2_s^*}\|u\|^2 + bt^{2\mu-2_s^*}\|u\|^{2\mu} - t^{1-\gamma-2_s^*}\int_\Omega h(x)|u|^{1-\gamma}dx, \qquad t > 0. \tag{4.1}$$

It is noticeable that if $\psi_u(t) = \lambda\int_\Omega l(x)|u|^{2_s^*}dx$, that is

$$at^{2-2_s^*}\|u\|^2 + bt^{2\mu-2_s^*}\|u\|^{2\mu} - t^{1-\gamma-2_s^*}\int_\Omega h(x)|u|^{1-\gamma}dx = \lambda\int_\Omega l(x)|u|^{2_s^*}dx, \tag{4.2}$$

multiplying $t^{2_s^*}$ on the both sides of the equation, one has

$$a\|tu\|^2 + b\|tu\|^{2\mu} - \int_\Omega h(x)|tu|^{1-\gamma}dx = \lambda\int_\Omega l(x)|tu|^{2_s^*}dx, \tag{4.3}$$

then we can deduce that $tu \in \mathbb{X}$.

We can easily infer from (4.1) that $\lim_{t\to 0^+}\psi_u(t) = -\infty$ and $\lim_{t\to\infty}\psi_u(t) = 0$. Furthermore, one step derivative calculation can get

$$\begin{aligned}\psi_u'(t) &= a(2-2_s^*)t^{1-2_s^*}\|u\|^2 + b(2\mu-2_s^*)t^{2\mu-1-2_s^*}\|u\|^{2\mu} \\ &\quad + (2_s^*+\gamma-1)t^{-\gamma-2_s^*}\int_\Omega h(x)|u|^{1-\gamma}dx.\end{aligned} \tag{4.4}$$

Based on the fact that $1 < 2\mu < 2_s^*$ and $0 < \gamma < 1$, one can obtain that $\lim_{t\to 0^+}\psi_u'(t) > 0$ and $\lim_{t\to\infty}\psi_u'(t) < 0$.

Rewrite $\psi_u'(t) = t^{2\mu-1-2_s^*}g_u(t)$, where

$$g_u(t) = a(2-2_s^*)t^{2-2\mu}\|u\|^2 + b(2\mu-2_s^*)\|u\|^{2\mu} + (2_s^*+\gamma-1)t^{1-\gamma-2\mu}\int_\Omega h(x)|u|^{1-\gamma}dx.$$

If

$$g_u'(t) = a(2-2_s^*)(2-2\mu)t^{1-2\mu}\|u\|^2 - (1-\gamma-2_s^*)(1-\gamma-2\mu)t^{-\gamma-2\mu}\int_\Omega h(x)|u|^{1-\gamma}dx = 0,$$

then it could be seen that there exists a unique

$$t_1 = \left(\frac{(2_s^*-1+\gamma)(2\mu-1+\gamma)\int_\Omega h(x)|u|^{-\gamma}dx}{a(2_s^*-2)(2\mu-2)\|u\|^2}\right)^{\frac{1}{1+\gamma}} > 0$$

such that $g_u'(t_1) = 0$. Similarly, since $1 < 2\mu < 2_s^*$ and $0 < \gamma < 1$, we have $\lim_{t\to 0^+}g_u(t) = +\infty$ and $\lim_{t\to+\infty}g_u(t) = b(2\mu-2_s^*)\|u\|^{2\mu} < 0$. Also, $\lim_{t\to 0^+}g_u'(t) < 0$ and $\lim_{t\to+\infty}g_u'(t) > 0$.

Subsequently, we infer that there is only one $t_0 > 0$ that satisfies $g_u(t_0) = 0$. Actually, it follows from $\psi'_u(t) = t^{2\mu-1-2_s^*} g_u(t)$ that t_0 is a unique critical point of $\psi_u(t)$, which is the global maximum point. In another word, this means that when $0 < t < t_0$, $\psi_u(t)$ is increasing. $\psi_u(t)$ is decreasing in the range greater than t_0 and $\psi'_u(t_0) = 0$. We define

$$\psi_u(t_0) = \max_{t>0} \psi_u(t) = \max_{t>0}(b\|u\|^{2\mu} t^{2\mu-2_s^*} + \varphi_u(t)) \geq \max_{t>0} \varphi_u(t), \tag{4.5}$$

where

$$\varphi_u(t) = at^{2-2_s^*}\|u\|^2 - t^{1-\gamma-2_s^*} \int_\Omega h(x)|u|^{1-\gamma} dx.$$

With respect to $\varphi_u(t)$, there holds

$$\varphi'_u(t) = a(2-2_s^*)t^{1-2_s^*}\|u\|^2 - (1-\gamma-2_s^*)t^{-\gamma-2_s^*} \int_\Omega h(x)|u|^{1-\gamma} dx,$$

we observe that $\lim_{t\to 0^+} \varphi'_u(t) > 0$ and $\lim_{t\to+\infty} \varphi'_u(t) < 0$,

$$\max_{t>0} \varphi_u(t) = \left(\frac{1+\gamma}{2_s^*-2}\right)\left(\frac{2_s^*-2}{2_s^*+\gamma-1}\right)^{\frac{2_s^*+\gamma-1}{1+\gamma}} \frac{(a\|u\|^2)^{\frac{2_s^*+\gamma-1}{1+\gamma}}}{\left(\int_\Omega h(x)|u|^{1-\gamma} dx\right)^{\frac{2_s^*-2}{1+\gamma}}}. \tag{4.6}$$

Hence by (4.5) and (4.6), we obtain

$$\begin{aligned}
&\psi_u(t_0) - \lambda \int_\Omega l(x)|u|^{2_s^*} dx \\
&\geq \left(\frac{1+\gamma}{2_s^*-2}\right)\left(\frac{2_s^*-2}{2_s^*+\gamma-1}\right)^{\frac{2_s^*+\gamma-1}{1+\gamma}} \frac{(a\|u\|^2)^{\frac{2_s^*+\gamma-1}{1+\gamma}}}{\left(\int_\Omega h(x)|u|^{1-\gamma} dx\right)^{\frac{2_s^*-2}{1+\gamma}}} - \lambda \int_\Omega l(x)|u|^{2_s^*} dx \\
&\geq \left(\frac{1+\gamma}{2_s^*-2}\right)\left(\frac{a(2_s^*-2)}{2_s^*+\gamma-1}\right)^{\frac{2_s^*+\gamma-1}{1+\gamma}} \frac{\|u\|^{\frac{22_s^*+2\gamma-2}{1+\gamma}}}{\left[\left(\int_\Omega h(x)^{\frac{2_s^*}{2_s^*-1+\gamma}}\right)^{\frac{2_s^*-1+\gamma}{2_s^*}} \left(\int_\Omega |u|^{2_s^*} dx\right)^{\frac{1-\gamma}{2_s^*}}\right]^{\frac{2_s^*-2}{1+\gamma}}} \\
&\quad - \lambda\|l\|_\infty S_s^{-\frac{2_s^*}{2}} \|u\|^{2_s^*} \\
&\geq \|u\|^{2_s^*} \left(\frac{1+\gamma}{2_s^*-2}\right)\left(\frac{a(2_s^*-2)}{2_s^*+\gamma-1}\right)^{\frac{2_s^*+\gamma-1}{1+\gamma}} S_s^{\frac{(1-\gamma)(2_s^*-2)}{2(1+\gamma)}} \|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}}^{\frac{2-2_s^*}{1+\gamma}} - \lambda\|l\|_\infty S_s^{-\frac{2_s^*}{2}} \|u\|^{2_s^*} \\
&> 0,
\end{aligned} \tag{4.7}$$

for all $0 < \lambda < \left(\frac{1+\gamma}{2_s^*-2}\right)\left(\frac{a(2_s^*-2)}{2_s^*+\gamma-1}\right)^{\frac{2_s^*+\gamma-1}{1+\gamma}} S_s^{\frac{2_s^*-1+\gamma}{1+\gamma}} \|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}}^{\frac{2-2_s^*}{1+\gamma}} \|l\|_\infty^{-1} = \Gamma_1$. From (4.7), we can observe that there are unique $t_+ = t_+(u) < t_0$ and $t_- = t_-(u) > t_0$ satisfying

$$\psi_u(t_+) = \lambda \int_\Omega l(x)|u|^{2_s^*} dx = \psi_u(t_-).$$

Similar to (4.2) and (4.3), we confirm that $t_+u \in \mathbb{X}$ and $t_-u \in \mathbb{X}$. Since $\psi'_u(t_+) > 0$ and $\psi'_u(t_-) < 0$, we can get $t_+u \in \mathbb{X}^+$ and $t_-u \in \mathbb{X}^-$. Specifically,

$$\psi'_u(t_+) = a\|u\|^2(2-2_s^*)t_+^{1-2_s^*} + b\|u\|^{2\mu}(2\mu-2_s^*)t_+^{2\mu-1-2_s^*} - (1-\gamma-2_s^*)t_+^{-\gamma-2_s^*} \int_\Omega h(x)|u|^{1-\gamma} dx > 0$$

multiplying $t_+^{2_s^*+1}$ on the both sides of the inequation, one has

$$a\|t_+u\|^2(2-2_s^*)+b\|t_+u\|^{2\mu}(2\mu-2_s^*)-(1-\gamma-2_s^*)\int_\Omega h(x)|t_+u|^{1-\gamma}dx>0. \tag{4.8}$$

As to the definition of $\mathbb{X}^+$, the prerequisite is $u\in\mathbb{X}$,

$$\begin{aligned}\phi_t''(1)&=a\|u\|^2+(2\mu-1)b\|u\|^{2\mu}-\lambda(2_s^*-1)\int_\Omega l(x)|u|^{2_s^*}dx+\gamma\int_\Omega h(x)|u|^{1-\gamma}dx\\&=a\|u\|^2+(2\mu-1)b\|u\|^{2\mu}-(2_s^*-1)\big(a\|u\|^2+b\|u\|^{2\mu}\\&\quad-\int_\Omega h(x)|u|^{1-\gamma}dx\big)+\gamma\int_\Omega h(x)|u|^{1-\gamma}dx\\&=a\|u\|^2(2-2_s^*)+b(2\mu-2_s^*)\|u\|^{2\mu}-(1-\gamma-2_s^*)\int_\Omega h(x)|u|^{1-\gamma}dx.\end{aligned}$$

Therefore, another expression of $\mathbb{X}^+$ can be written as

$$\mathbb{X}^+=\left\{u\in\mathbb{X}:a\|u\|^2(2-2_s^*)+b(2\mu-2_s^*)\|u\|^{2\mu}-(1-\gamma-2_s^*)\int_\Omega h(x)|u|^{1-\gamma}dx>0\right\}. \tag{4.9}$$

Similarly,

$$\mathbb{X}^-=\left\{u\in\mathbb{X}:a\|u\|^2(2-2_s^*)+b(2\mu-2_s^*)\|u\|^{2\mu}-(1-\gamma-2_s^*)\int_\Omega h(x)|u|^{1-\gamma}dx<0\right\}, \tag{4.10}$$

$$\mathbb{X}^0=\left\{u\in\mathbb{X}:a\|u\|^2(2-2_s^*)+b(2\mu-2_s^*)\|u\|^{2\mu}-(1-\gamma-2_s^*)\int_\Omega h(x)|u|^{1-\gamma}dx=0\right\}. \tag{4.11}$$

Because (4.8) is established, we know that $t_+u\in\mathbb{X}^+$. At the same time, $t_-u\in\mathbb{X}^-$ can be obtained using the same method. □

Lemma 4.2. *There is $\Gamma_2>0$ satisfies $\mathbb{X}^0=\{0\}$ for all $0<\lambda<\Gamma_2$, where*

$$\Gamma_2=\frac{2\left[(1+\gamma)(2\mu-1+\gamma)ab\right]^{\frac{1}{2}}}{(2_s^*+\gamma-1)\|l\|_\infty}\frac{S_s^{\frac{(\mu+1)(2_s^*+\gamma-1)}{2(\mu+\gamma)}}}{\left[\frac{(2_s^*+\gamma-1)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}}}{2\left[(2_s^*-2)(2_s^*-2\mu)ab\right]^{\frac{1}{2}}}\right]^{\frac{2_s^*-\mu-1}{\mu+\gamma}}}.$$

Proof. We can prove it in two cases.

Case 1: $u\in\mathbb{X}\setminus\{0\}$ and $\int_\Omega l(x)|u|^{2_s^*}dx=0$.

According to the definition of $\mathbb{X}$, it follows that (3.2) that

$$a\|u\|^2+b\|u\|^{2\mu}-\int_\Omega h(x)|u|^{1-\gamma}dx=0.$$

On account of $0<\gamma<1$, we extrapolate that

$$\begin{aligned}\phi_u''(1)&=a\|u\|^2+b(2\mu-1)\|u\|^{2\mu}+\gamma\int_\Omega h(x)|u|^{1-\gamma}dx\\&=a\|u\|^2+b(2\mu-1)\|u\|^{2\mu}+\gamma(a\|u\|^2+b\|u\|^{2\mu})\\&=a(1+\gamma)\|u\|^2+(2\mu-1+\gamma)b\|u\|^{2\mu}>0.\end{aligned} \tag{4.12}$$

From this, we can learn that $u\notin\mathbb{X}^0$.

Case 2: $u \in \mathbb{X}\setminus\{0\}$ and $\int_\Omega l(x)|u|^{2_s^*}dx \neq 0$.
We may paradoxically assume there exists $u \in \mathbb{X}^0$ and $u \not\equiv 0$. On the basis of (3.2) and (3.3), we obtain

$$a(1+\gamma)\|u\|^2 + b(2\mu-1+\gamma)\|u\|^{2\mu} - (2_s^*-1+\gamma)\lambda\int_\Omega l(x)|u|^{2_s^*}dx = 0 \tag{4.13}$$

and

$$a(2-2_s^*)\|u\|^2 + b(2\mu-2_s^*)\|u\|^{2\mu} - (1-\gamma-2_s^*)\int_\Omega h(x)|u|^{1-\gamma}dx = 0. \tag{4.14}$$

Inspired by (4.13), we may define $H : \mathbb{X} \to \mathbb{R}$ as

$$H(u) = \frac{a(1+\gamma)\|u\|^2 + b(2\mu-1+\gamma)\|u\|^{2\mu}}{(2_s^*+\gamma-1)\lambda} - \int_\Omega l(x)|u|^{2_s^*}dx.$$

Obviously, if $u \in \mathbb{X}^0$, then $H(u) = 0$. Using (2.5) and the basic inequality $(\varrho+\kappa) \geq 2(\varrho\kappa)^{\frac{1}{2}}$, for any $\varrho, \kappa \geq 0$, we conclude that

$$\begin{aligned} H(u) &\geq \frac{2\left[(1+\gamma)(2\mu-1+\gamma)ab\right]^{\frac{1}{2}}}{(2_s^*+\gamma-1)\lambda}\|u\|^{\mu+1} - \|l\|_\infty S_s^{-\frac{2_s^*}{2}}\|u\|^{2_s^*} \\ &\geq \|u\|^{2_s^*}\left(\frac{2\left[(1+\gamma)(2\mu-1+\gamma)ab\right]^{\frac{1}{2}}}{(2_s^*+\gamma-1)\lambda}\frac{1}{\|u\|^{2_s^*-\mu-1}} - \|l\|_\infty S_s^{-\frac{2_s^*}{2}}\right). \end{aligned}$$

Besides, by (2.5), (4.14) and the Hölder inequality, we know

$$\begin{aligned} 2\left[(2_s^*-2\mu)(2_s^*-2)ab\right]^{\frac{1}{2}}\|u\|^{\mu+1} &\leq (2_s^*+\gamma-1)\left(\int_\Omega h(x)^{\frac{2_s^*}{2_s^*-1+\gamma}}dx\right)^{\frac{2_s^*-1+\gamma}{2_s^*}}\left(\int_\Omega |u|^{2_s^*}dx\right)^{\frac{1-\gamma}{2_s^*}} \\ &\leq (2_s^*+\gamma-1)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}}\|u\|^{1-\gamma}. \end{aligned}$$

Therefore,

$$\|u\| \leq \left[\frac{(2_s^*+\gamma-1)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}}}{2\left[(2_s^*-2\mu)(2_s^*-2)ab\right]^{\frac{1}{2}}}\right]^{\frac{1}{\mu+\gamma}}.$$

We control

$$\frac{2\left[(1+\gamma)(2\mu-1+\gamma)ab\right]^{\frac{1}{2}}}{(2_s^*+\gamma-1)\lambda}\frac{1}{\|u\|^{2_s^*-\mu-1}} - \|l\|_\infty S_s^{-\frac{2_s^*}{2}} > 0,$$

which leads to the following conclusion

$$\begin{aligned} \lambda &< \frac{2\left[(1+\gamma)(2\mu-1+\gamma)ab\right]^{\frac{1}{2}}}{(2_s^*+\gamma-1)\|l\|_\infty}\frac{S_s^{\frac{2_s^*}{2}}}{\left[\frac{(2_s^*+\gamma-1)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}}}{2\left[(2_s^*-2)(2_s^*-2\mu)ab\right]^{\frac{1}{2}}}\right]^{\frac{2_s^*-\mu-1}{\mu+\gamma}}} \\ &= \frac{2\left[(1+\gamma)(2\mu-1+\gamma)ab\right]^{\frac{1}{2}}}{(2_s^*+\gamma-1)\|l\|_\infty}\frac{S_s^{\frac{(\mu+1)(2_s^*+\gamma-1)}{2(\mu+\gamma)}}}{\left[\frac{(2_s^*+\gamma-1)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}}}{2\left[(2_s^*-2)(2_s^*-2\mu)ab\right]^{\frac{1}{2}}}\right]^{\frac{2_s^*-\mu-1}{\mu+\gamma}}} = \Gamma_2. \end{aligned}$$

Since $2_s^* > 2\mu$, $H(u) > 0$ for all $u \in \mathbb{X}^0 \setminus \{0\}$ can be confirmed. This causes the desired contradiction. □

Lemma 4.3. *$\mathcal{I}$, in addition to being coercive, is bounded from below on $\mathbb{X}$.*

Proof. For all $u \in \mathbb{X}$, we can deduce that

$$\begin{aligned}
\mathcal{I}(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{2\mu}\|u\|^{2\mu} - \frac{1}{1-\gamma}\int_\Omega h(x)|u|^{1-\gamma}dx - \frac{1}{2_s^*}\left(a\|u\|^2 + b\|u\|^{2\mu} - \int_\Omega h(x)|u|^{1-\gamma}dx\right) \\
&= \left(\frac{1}{2} - \frac{1}{2_s^*}\right) a\|u\|^2 + \left(\frac{1}{2\mu} - \frac{1}{2_s^*}\right) b\|u\|^{2\mu} - \left(\frac{1}{1-\gamma} - \frac{1}{2_s^*}\right)\int_\Omega h(x)|u|^{1-\gamma}dx \\
&\geq \left(\frac{1}{2} - \frac{1}{2_s^*}\right) a\|u\|^2 - \left(\frac{1}{1-\gamma} - \frac{1}{2_s^*}\right)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}}\|u\|^{1-\gamma}
\end{aligned}$$

from the condition that $2\mu < 2_s^*$ and (2.5). Based on the fact of $1-\gamma < 2$, it can be determined that $\mathcal{I}$ is coercive. In addition, we may define

$$G_a(q) = \left(\frac{1}{2} - \frac{1}{2_s^*}\right) aq^2 - \left(\frac{1}{1-\gamma} - \frac{1}{2_s^*}\right)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}} q^{1-\gamma},$$

then

$$\begin{aligned}
G_a'(q) &= \frac{2_s^*-2}{2_s^*}aq - \frac{2_s^*-1+\gamma}{2_s^*}\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}} q^{-\gamma}, \\
G_a''(q) &= \frac{2_s^*-2}{2_s^*}a + \frac{2_s^*-1+\gamma}{2_s^*}\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}} \gamma q^{-\gamma-1}.
\end{aligned}$$

We can obtain a unique stationary point q_{min}, where

$$q_{min} = \left(\frac{(2_s^*-1+\gamma)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}}}{(2_s^*-2)a}\right)^{\frac{1}{1+\gamma}},$$

and

$$G_a''(q_{min}) = \frac{a(2_s^*-2)(1+\gamma)}{2_s^*} > 0.$$

Then $G_a(q)$ attains its minimum at q_{min}. Accordingly,

$$\begin{aligned}
\mathcal{I}(u) \geq{}& \frac{\left((2_s^*-2)a\right)^{\frac{\gamma-1}{\gamma+1}}}{22_s^*}\left((2_s^*-1+\gamma)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}}\right)^{\frac{2}{1+\gamma}} \\
&- \frac{1}{2_s^*(1-\gamma)}\left((2_s^*-2)a\right)^{\frac{\gamma-1}{\gamma+1}}\left((2_s^*-1+\gamma)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}}\right)^{\frac{2}{1+\gamma}} \\
={}& \frac{\gamma+1}{22_s^*(\gamma-1)}\left((2_s^*-2)a\right)^{\frac{\gamma-1}{\gamma+1}}\left((2_s^*-1+\gamma)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}}\right)^{\frac{2}{1+\gamma}} > -C_0
\end{aligned}$$

for some constant $C_0 > 0$. This proof is completed. □

Lemma 4.4. *Let $\lambda \in (0, \Gamma_2)$, assume that $\gamma \in (0,1)$, then $\|u\| > \rho$ for all $u \in \mathbb{X}^-$, where*

$$\rho = \left(\frac{2\sqrt{(1+\gamma)(2\mu-1+\gamma)ab}}{(2_s^*+\gamma-1)\lambda\|l\|_\infty S_s^{-\frac{2_s^*}{2}}}\right)^{\frac{1}{2_s^*-\mu-1}}.$$

Proof. If $u \in \mathbb{X}^- \subset \mathbb{X}$, from (3.3), then we are sure that

$$a\|u\|^2 + b\|u\|^{2\mu} - \lambda \int_\Omega l(x)|u|^{2_s^*}dx - \int_\Omega h(x)|u|^{1-\gamma}dx = 0$$

and

$$a\|u\|^2 + (2\mu-1)b\|u\|^{2\mu} - (2_s^*-1)\lambda \int_\Omega l(x)|u|^{2_s^*}dx + \gamma \int_\Omega h(x)|u|^{1-\gamma}dx < 0,$$

which yields

$$\begin{aligned} &a(1+\gamma)\|u\|^2 + (2\mu-1+\gamma)b\|u\|^{2\mu} \\ &\qquad < (2_s^*+\gamma-1)\lambda \int_\Omega l(x)|u|^{2_s^*}dx \le (2_s^*+\gamma-1)\lambda\|l\|_\infty S_s^{-\frac{2_s^*}{2}}\|u\|^{2_s^*}. \end{aligned}$$

Hence, we infer that $\rho < \|u\|$. □

Lemma 4.5. *Assume that $u_n \to u$ in $\mathbb{M}_0$, then*

$$\lim_{n\to+\infty} \int_\Omega l(x)|u_n|^{2_s^*}dx = \int_\Omega l(x)|u|^{2_s^*}dx, \tag{4.15}$$

and

$$\lim_{n\to+\infty} \int_\Omega h(x)|u_n|^{1-\gamma}dx = \int_\Omega h(x)|u|^{1-\gamma}dx. \tag{4.16}$$

Proof. Let $\{u_n\} \subset \mathbb{M}_0$ and $u_n \to u$ in $\mathbb{M}_0$. Due to $l \in L^\infty(\Omega)$ and $u_n \to u$, we deduce that there must be $C_1 > 0$ and $C_2 > 0$ satisfying $\|u_n\| \le C_1$ and $|l(x)| \le C_2$ a.e. in Ω. Set $k_n(x) = l(x)^{\frac{1}{2_s^*}}u_n$, $k(x) = l(x)^{\frac{1}{2_s^*}}u$, then

$$\left(\int_\Omega |k_n(x)|^{2_s^*}dx\right)^{\frac{1}{2_s^*}} = \left(\int_\Omega l(x)|u_n|^{2_s^*}dx\right)^{\frac{1}{2_s^*}} \le C_2{}^{\frac{1}{2_s^*}}\left(\int_\Omega |u_n|^{2_s^*}dx\right)^{\frac{1}{2_s^*}} \le C_2{}^{\frac{1}{2_s^*}}S_s^{-\frac{1}{2}}C_1. \tag{4.17}$$

It can be clearly determined from this that

$$\begin{cases} \{k_n\} \text{ is bounded in } L^{2_s^*}(\Omega), \\ k_n \to k \text{ a.e. in } \Omega. \end{cases}$$

Moreover,

$$\begin{aligned} \int_\Omega |k_n(x)-k(x)|^{2_s^*}dx = \int_\Omega l(x)|u_n-u|^{2_s^*}dx \le C_2\int_\Omega |u_n-u|^{2_s^*}dx \\ \le C_2\|u_n-u\|_{2_s^*}^{2_s^*} \to 0, \end{aligned} \tag{4.18}$$

for n large enough. All prerequisites have been met, and the Brézis–Lieb lemma can be used to obtain

$$\lim_{n\to+\infty}\left(\int_\Omega |k_n(x)|^{2_s^*}dx\right)^{\frac{1}{2_s^*}} = \left(\int_\Omega |k(x)|^{2_s^*}dx\right)^{2_s^*} + \lim_{n\to+\infty}\left(\int_\Omega |k_n(x)-k(x)|^{2_s^*}dx\right)^{\frac{1}{2_s^*}}.$$

On account of (4.18), we obtain (4.15). Using the same method, we can prove that (4.16) is valid. □

Lemma 4.6. *For all $0 < \lambda < \Gamma_2$, $\mathbb{X}^+ \cup \mathbb{X}^0$ and $\mathbb{X}^-$ are closed sets in G_0-topology.*

Proof. We prove this lemma in two parts. Let us first prove that $\mathbb{X}^+ \cup \mathbb{X}^0$ is a closed set.

Part 1: Suppose $\{u_n\} \subset \mathbb{X}^+ \cup \mathbb{X}^0$ and $u_n \to u_0$ in $\mathbb{M}_0$, we need to prove that $u_0 \in \mathbb{X}^+ \cup \mathbb{X}^0$. Since $\{u_n\} \subset \mathbb{X}^+ \cup \mathbb{X}^0$, we get

$$a(1+\gamma)\|u_n\|^2 + b(2\mu - 1 + \gamma)\|u_n\|^{2\mu} - (2_s^* + \gamma - 1)\lambda \int_\Omega l(x)|u_n|^{2_s^*} dx \geq 0.$$

Since $|\|u_n\| - \|u_0\|| \leq \|u_n - u_0\| \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \|u_n\|^2 = \|u_0\|^2, \qquad \lim_{n\to\infty} \|u_n\|^{2\mu} = \|u_0\|^{2\mu}.$$

Then, letting $n \to \infty$, it follows from Lemma 4.5 that

$$a(1+\gamma)\|u_0\|^2 + b(2\mu - 1 + \gamma)\|u_0\|^{2\mu} - (2_s^* + \gamma - 1)\lambda \int_\Omega l(x)|u_0|^{2_s^*} dx \geq 0.$$

Therefore, $\mathbb{X}^+ \cup \mathbb{X}^0$ is a closed set.

Part 2: Suppose that $\{u_n\} \subset \mathbb{X}^-$ such that $u_n \to u_0$ in $\mathbb{M}_0$. We infer that $u_0 \in \overline{\mathbb{X}^-} = \mathbb{X}^- \cup \{0\}$. By using Lemma 4.4, we have

$$\|u_0\| = \lim_{n\to\infty} \|u_n\| \geq \rho > 0. \tag{4.19}$$

Therefore $u_0 \neq 0$, which implies $u_0 \in \mathbb{X}^-$. This proof in completed. □

Lemma 4.7. *Let $u \in \mathbb{X}^+$(respectively $\mathbb{X}^-$) with $u \geq 0$, $0 < \gamma < 1$ and $h \in L^{\frac{2_s^*}{2_s^*+\gamma-1}}(\Omega)$. Subsequently, there exist $\varepsilon > 0$ and the continuous function $\varsigma : B_\varepsilon(0) \to \mathbb{R}^+$ satisfying*

$$\varsigma(z) > 0, \qquad \varsigma(0) = 1, \qquad \varsigma(z)(u+z) \in \mathbb{X}^{\pm}$$

for any $z \in B_\varepsilon(0)$, where $B_\varepsilon(0) = \{z \in \mathbb{M}_0 : \|z\| < \varepsilon\}$.

Proof. With regard to any $u \in \mathbb{X}^+ \subset \mathbb{X}$, define $Q : \mathbb{M}_0 \times \mathbb{R}^+ \to \mathbb{R}$ as follows

$$\begin{aligned} Q(z,\omega) = {} & \omega^{1+\gamma} a\|u+z\|^2 + \omega^{2\mu-1+\gamma} b\|u+z\|^{2\mu} \\ & - \omega^{2_s^*-1+\gamma}\lambda \int_\Omega l(x)|u+z|^{2_s^*} dx - \int_\Omega h(x)|u+z|^{1-\gamma} dx. \end{aligned}$$

Differentiating the above equation, we determine that

$$\begin{aligned} \frac{\partial Q}{\partial \omega} = {} & a(1+\gamma)\omega^\gamma \|u+z\|^2 + b(2\mu-1+\gamma)\omega^{2\mu-2+\gamma}\|u+z\|^{2\mu} \\ & - (2_s^* - 1 + \gamma)\lambda\omega^{2_s^*-2+\gamma} \int_\Omega l(x)|u+z|^{2_s^*} dx. \end{aligned}$$

Due to $u \in \mathbb{X}^+ \subset \mathbb{X}$, it is clear that

$$Q(0,1) = a\|u\|^2 + b\|u\|^{2\mu} - \int_\Omega h(x)|u|^{1-\gamma} dx - \lambda \int_\Omega l(x)|u|^{2_s^*} dx = 0 \tag{4.20}$$

and

$$\frac{\partial Q}{\partial \omega}(0,1) = a(1+\gamma)\|u\|^2 + b(2\mu-1+\gamma)\|u\|^{2\mu} - (2_s^* - 1 + \gamma)\lambda \int_\Omega l(x)|u|^{2_s^*} dx > 0. \tag{4.21}$$

The implicit function theorem applies to Q at the point $(0,1)$. A continuous function $\omega = \varsigma(z) > 0$ can be got, it can be seen that $\varsigma(0) = 1$ from (4.20). There is $\varepsilon^* > 0$. So $Q(z, \varsigma(z)) = 0$ for any $z \in \mathbb{M}_0$ with $\|z\| < \varepsilon^*$.

$$\begin{aligned} Q(z,w) &= Q(z,\varsigma(z)) \\ &= \varsigma^{1+\gamma}(z)a\|u+z\|^2 + \varsigma^{2\mu-1+\gamma}(z)b\|u+z\|^{2\mu} \\ &\quad -\varsigma^{2_s^*-1+\gamma}(z)\lambda\int_\Omega l(x)|u+z|^{2_s^*}dx - \int_\Omega h(x)|u+z|^{1-\gamma}dx \\ &= [a\|\varsigma(z)(u+z)\|^2 + b\|\varsigma(z)(u+z)\|^{2\mu} \\ &\quad -\lambda\int_\Omega l(x)|\varsigma(z)(u+z)|^{2_s^*}dx - \int_\Omega h(x)|\varsigma(z)(u+z)|^{1-\gamma}dx]/\varsigma^{1-\gamma}(z) \\ &= 0, \end{aligned} \tag{4.22}$$

that is $\varsigma(z)(u+z) \in \mathbb{X}$ for any $z \in \mathbb{M}_0$ with $\|z\| < \varepsilon^*$.

$$\begin{aligned} &\frac{\partial Q}{\partial \omega}(z,\varsigma(z)) \\ &= \frac{a(\gamma+1)\|\varsigma(z)(u+z)\|^2 + b(2\mu-1+\gamma)\|\varsigma(z)(u+z)\|^{2\mu} - (2_s^*-1+\gamma)\lambda\int_\Omega l(x)|\varsigma(z)(u+z)|^{2_s^*}dx}{\varsigma^{2-\gamma}(z)}. \end{aligned}$$

Taking sufficiently small $\varepsilon > 0$ so that $\varepsilon < \varepsilon^*$, we determine that

$$\varsigma(z)(u+z) \in \mathbb{X}^+, \qquad \forall z \in \mathbb{M}_0,\ \|z\| < \varepsilon.$$

As for $u \in \mathbb{X}^-$, we can proceed similarly to arrive at the same conclusion. □

5 Proof of Theorem 1.1

At present, let us show that problem (1.1) has a positive solution on each of $\mathbb{X}^+$ and $\mathbb{X}^-$, respectively. From Lemma 4.1, when $0 < \lambda < \Gamma_1$, one has $\mathbb{X}^\pm \neq \emptyset$. We complete this proof in two steps.

Step 1: We analyze problem (1.1) on $\mathbb{X}^+ \cup \mathbb{X}^0$.

According to Lemma 4.6, for $0 < \lambda < \Gamma_2$, we know $\mathbb{X}^+ \cup \mathbb{X}^0$ must be a closed set in $\mathbb{M}_0$. In the light of Lemma 4.3, $\mathcal{I}$ can be determined to be coercive and bounded below, $c^+ = \inf_{u\in\mathbb{X}^+\cup\mathbb{X}^0}\mathcal{I}$ can be clearly defined. Then, this minimization problem can be handled by Ekeland's variational principle. Then, a sequence $\{u_k\} \subset \mathbb{X}^+ \cup \mathbb{X}^0$ exists and satisfies the following properties:

$$(i)\ \mathcal{I}(u_k) < \inf_{u\in\mathbb{X}^+\cup\mathbb{X}^0}\mathcal{I}(u) + \frac{1}{k}, \qquad (ii)\ \mathcal{I}(u_k) \leq \mathcal{I}(u) + \frac{1}{k}\|u_k - u\|, \qquad \forall u \in \mathbb{X}^+ \cup \mathbb{X}^0. \tag{5.1}$$

By means of $\mathcal{I}(u) = \mathcal{I}(|u|)$, we know that $u_k(x) \geq 0$ almost everywhere in Ω. Significantly, $\{u_k\}$ must be bounded in $\mathbb{M}_0$, going to a subsequence if necessary, let us represent the subsequence in terms of $\{u_n\}$. There exists u_0 satisfies

$$\begin{aligned} u_n &\rightharpoonup u_0 \quad \text{in } \mathbb{M}_0, \\ u_n &\to u_0 \quad \text{a.e. in } \Omega, \\ u_n &\rightharpoonup u_0 \quad \text{in } L^{2_s^*}, \\ u_n &\to u_0 \quad \text{in } L^r(\Omega) \text{ for } 2 \leq r < 2_s^*, \end{aligned} \tag{5.2}$$

as $n \to \infty$. For all $u \in \mathbb{X}^+$, it follows from (3.4) and $2\mu < 2_s^*$ that

$$\begin{aligned}\mathcal{I} &= \frac{a}{2}\|u\|^2 + \frac{b}{2\mu}\|u\|^{2\mu} - \frac{1}{2_s^*}\lambda\int_\Omega l(x)|u|^{2_s^*}dx - \frac{1}{1-\gamma}\int_\Omega h(x)|u|^{1-\gamma}dx \\ &= -\frac{a(1+\gamma)}{2(1-\gamma)}\|u\|^2 - \frac{2\mu+\gamma-1}{2\mu(1-\gamma)}b\|u\|^{2\mu} + \frac{\gamma-1+2_s^*}{2_s^*(1-\gamma)}\lambda\int_\Omega l(x)|u|^{2_s^*}dx \\ &< -\frac{a(1+\gamma)}{2(1-\gamma)}\|u\|^2 - \frac{2\mu+\gamma-1}{2\mu(1-\gamma)}b\|u\|^{2\mu} + \frac{a(1+\gamma)}{2_s^*(1-\gamma)}\|u\|^2 + \frac{2\mu+\gamma-1}{2_s^*(1-\gamma)}b\|u\|^{2\mu} \\ &< -\frac{a(1+\gamma)}{(1-\gamma)}\frac{(2_s^*-2)}{22_s^*}\|u\|^2 \\ &< 0.\end{aligned} \tag{5.3}$$

So we are sure that $\inf_{u\in\mathbb{X}^+}\mathcal{I}(u) < 0$. Thus, $c^+ = \inf_{u\in\mathbb{X}^+}\mathcal{I}(u) < 0$, which in particular implies we might as well consider a subsequence $\{u_n\} \subset \mathbb{X}^+$. As to this fact, in terms of Lemma 4.7 with $u = u_n$, a series of functions ς_n satisfying $\varsigma_n(0) = 1$ can be obtained. Meanwhile, for $\varphi \in \mathbb{M}_0$ with $\varphi \geq 0$ and $\wp > 0$ sufficiently small, the fact that $\varsigma_n(\wp\varphi)(u_n + \wp\varphi) \in \mathbb{X}^+$ holds can be established. With these basic facts in mind, it is easy to know

$$a\|u_n\|^2 + b\|u_n\|^{2\mu} - \lambda\int_\Omega l(x)|u_n|^{2_s^*}dx - \int_\Omega h(x)|u_n|^{1-\gamma}dx = 0 \tag{5.4}$$

and

$$\begin{aligned}&a\varsigma_n^2(\wp\varphi)\|u_n+\wp\varphi\|^2 + b\varsigma_n^{2\mu}(\wp\varphi)\|u_n+\wp\varphi\|^{2\mu} \\ &\quad - \varsigma_n^{2_s^*}(\wp\varphi)\lambda\int_\Omega l(x)|u_n+\wp\varphi|^{2_s^*}dx - \varsigma_n^{1-\gamma}(\wp\varphi)\int_\Omega h(x)|u_n+\wp\varphi|^{1-\gamma}dx = 0.\end{aligned} \tag{5.5}$$

It should be noted that $\varsigma_n'(0)$ is treated by us as the derivative of ς_n at zero, and its specific representation is as follows:

$$\varsigma_n'(0) = (\varsigma_n', \varphi) := \lim_{\wp\to 0}\frac{\varsigma_n(\wp\varphi)-1}{\wp} \in [-\infty, +\infty],$$

for all $\varphi \in \mathbb{M}_0$. Now let us prove when $\lambda < \Gamma_1$, $\{u_n\} \subset \mathbb{X}^\pm$ satisfies (5.1). Then, (ς_n', φ) must be uniformly bounded for all $\varphi \in \mathbb{M}_0$ with $\varphi \geq 0$. In particular, here we just consider $\{u_n\} \subset \mathbb{X}^+$, the case on $\mathbb{X}^-$ can be proved in the same way.

It follows from (5.4) and (5.5) that

$$\begin{aligned}0 &= a[(\varsigma_n^2(\wp\varphi)-1)\|u_n+\wp\varphi\|^2 + \|u_n+\wp\varphi\|^2 - \|u_n\|^2] \\ &\quad + b[(\varsigma_n^{2\mu}(\wp\varphi)-1)\|u_n+\wp\varphi\|^{2\mu} + \|u_n+\wp\varphi\|^{2\mu} - \|u_n\|^{2\mu}] \\ &\quad - (\varsigma_n^{2_s^*}(\wp\varphi)-1)\lambda\int_\Omega l(x)|u_n+\wp\varphi|^{2_s^*}dx - \lambda\int_\Omega l(x)(|u_n+\wp\varphi|^{2_s^*} - |u_n|^{2_s^*})dx \\ &\quad - (\varsigma_n^{1-\gamma}(\wp\varphi)-1)\int_\Omega h(x)|u_n+\wp\varphi|^{1-\gamma} - \int_\Omega h(x)(|u_n+\wp\varphi|^{1-\gamma} - |u_n|^{1-\gamma})dx \\ &\leq a(\varsigma_n^2(\wp\varphi)-1)\|u_n+\wp\varphi\|^2 + a(\|u_n+\wp\varphi\|^2 - \|u_n\|^2) \\ &\quad + b(\varsigma_n^{2\mu}(\wp\varphi)-1)\|u_n+\wp\varphi\|^{2\mu} + b(\|u_n+\wp\varphi\|^{2\mu} - \|u_n\|^{2\mu}) \\ &\quad - (\varsigma_n^{2_s^*}(\wp\varphi)-1)\lambda\int_\Omega l(x)|u_n+\wp\varphi|^{2_s^*}dx - \lambda\int_\Omega l(x)(|u_n+\wp\varphi|^{2_s^*} - |u_n|^{2_s^*})dx \\ &\quad - (\varsigma_n^{1-\gamma}(\wp\varphi)-1)\int_\Omega h(x)|u_k+\wp\varphi|^{1-\gamma}.\end{aligned}$$

Afterwards, dividing the above inequation by $\wp > 0$, we get

$$\begin{aligned} &\frac{\varsigma_n(\wp\varphi)-1}{\wp}\Big[a(\varsigma_n(\wp\varphi)+1)\|u_n+\wp\varphi\|^2 + b\frac{\varsigma_n^{2\mu}(\wp\varphi)-1}{\varsigma_n(\wp\varphi)-1}\|u_n+\wp\varphi\|^{2\mu} \\ &-\frac{\varsigma_n^{2_s^*}(\wp\varphi)-1}{\varsigma_n(\wp\varphi)-1}\lambda\int_\Omega l(x)|u_n+\wp\varphi|^{2_s^*}dx - \frac{\varsigma_n^{1-\gamma}(\wp\varphi)-1}{\varsigma_n(\wp\varphi)-1}\int_\Omega h(x)|u_n+\wp\varphi|^{1-\gamma}dx\Big] \\ &+a\frac{\|u_n+\wp\varphi\|^2-\|u_n\|^2}{\wp} + b\frac{\|u_n+\wp\varphi\|^{2\mu}-\|u_n\|^{2\mu}}{\wp} - \lambda\int_\Omega l(x)\frac{|u_n+\wp\varphi|^{2_s^*}-|u_n|^{2_s^*}}{\wp}dx \geq 0. \end{aligned}$$

Letting $\wp \to 0$, we extrapolate that

$$\begin{aligned} &(\varsigma_n', \varphi)\Big[2a\|u_n\|^2 + 2\mu b\|u_n\|^{2\mu} - 2_s^*\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx - (1-\gamma)\int_\Omega h(x)|u_n|^{1-\gamma}dx\Big] \\ &+2a\langle u_n,\varphi\rangle + 2\mu b\|u_n\|^{2\mu-2}\langle u_n,\varphi\rangle - 2_s^*\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx \geq 0. \end{aligned} \tag{5.6}$$

According to $\{u_n\} \in \mathbb{X}$, using (5.4) in (5.6), we have

$$\begin{aligned} &(\varsigma_n', \varphi)\Big[a(1+\gamma)\|u_n\|^2 + (2\mu-1+\gamma)b\|u_n\|^{2\mu} - (2_s^*-1+\gamma)\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx\Big] \\ &+2a\langle u_n,\varphi\rangle + 2\mu b\|u_n\|^{2\mu-2}\langle u_n,\varphi\rangle - 2_s^*\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx \geq 0, \end{aligned} \tag{5.7}$$

that is

$$(\varsigma_n', \varphi) \geq \frac{-(2a\langle u_n,\varphi\rangle + 2\mu b\|u_n\|^{2\mu-2}\langle u_n,\varphi\rangle - 2_s^*\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx)}{a(1+\gamma)\|u_n\|^2 + (2\mu-1+\gamma)b\|u_n\|^{2\mu} - (2_s^*-1+\gamma)\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx}.$$

Since $\{u_n\}$ is bounded in $\mathbb{M}_0$, the above inequality means that (ς_n', φ) is bounded from below uniformly for any $\varphi \in \mathbb{M}_0$ with $\varphi \geq 0$, that is $(\varsigma_n', \varphi) \neq -\infty$.

Now we have to prove that (ς_n', φ) is bounded from above. By (5.1)-(ii), we have

$$\frac{\|u_n - \varsigma_n(\wp\varphi)(u_n+\wp\varphi)\|}{n} \geq \mathcal{I}(u_n) - \mathcal{I}[\varsigma_n(\wp\varphi)(u_n+\wp\varphi)] \tag{5.8}$$

and

$$\begin{aligned} \frac{\|u_n - \varsigma_n(\wp\varphi)(u_n+\wp\varphi)\|}{n} &= \frac{\|(1-\varsigma_n(\wp\varphi))u_n - \varsigma_n(\wp\varphi)\wp\varphi\|}{n} \\ &\leq \frac{\|(1-\varsigma_n(\wp\varphi))u_n\|}{n} + \frac{\|-\varsigma_n(\wp\varphi)\wp\varphi\|}{n} \\ &\leq |\varsigma_n(\wp\varphi)-1|\frac{\|u_n\|}{n} + \wp\zeta_n(\wp\varphi)\frac{\|\varphi\|}{n}, \end{aligned} \tag{5.9}$$

which implies

$$\begin{aligned} &\Big|\varsigma_n(\wp\varphi)-1\Big|\frac{\|u_n\|}{n} + \wp\varsigma_n(\wp\varphi)\frac{\|\varphi\|}{n} \geq \mathcal{I}(u_n) - \mathcal{I}[\varsigma_n(\wp\varphi)(u_n+\wp\varphi)] \\ &= \frac{a(1+\gamma)}{2(1-\gamma)}\Big[(\varsigma_n^2(\wp\varphi)-1)\|u_n+\wp\varphi\|^2 + (\|u_n+\wp\varphi\|^2 - \|u_n\|^2)\Big] \\ &\quad+\frac{b(2\mu-1+\gamma)}{2\mu(1-\gamma)}\Big[(\varsigma_n^{2\mu}(\wp\varphi)-1)\|u_n+\wp\varphi\|^{2\mu} + (\|u_n+\wp\varphi\|^{2\mu} - \|u_n\|^{2\mu})\Big] \\ &\quad-\frac{2_s^*-1+\gamma}{2_s^*(1-\gamma)}\lambda\Big[(\varsigma_n^{2_s^*}(\wp\varphi)-1)\int_\Omega l(x)|u_n+\wp\varphi|^{2_s^*} + \int_\Omega l(x)|u_n+\wp\varphi|^{2_s^*} - l(x)|u_n|^{2_s^*}dx\Big]. \end{aligned} \tag{5.10}$$

Dividing (5.10) by $\wp > 0$, and letting $\wp \to 0$, we deduce

$$\begin{aligned}
&\left|(\varsigma_n', \varphi)\right| \frac{\|u_n\|}{n} + \lim_{\wp \to 0} \varsigma_n(\wp\varphi) \frac{\|\varphi\|}{n} \\
&\geq a\frac{1+\gamma}{1-\gamma}\left[(\varsigma_n', \varphi)\|u_n\|^2 + \langle u_n, \varphi\rangle\right] \\
&\quad + b\frac{2\mu-1+\gamma}{1-\gamma}\left[(\varsigma_n', \varphi)\|u_n\|^{2\mu} + \|u_n\|^{2\mu-2}\langle u_n, \varphi\rangle\right] \\
&\quad - \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\left[(\varsigma_n', \varphi)\int_\Omega l(x)|u_n|^{2_s^*}dx + \int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx\right].
\end{aligned}$$

So

$$\begin{aligned}
&\left|(\varsigma_n', \varphi)\right| \frac{\|u_n\|}{n} + \frac{\|\varphi\|}{n} \\
&\geq (\varsigma_n', \varphi)\left[a\frac{1+\gamma}{1-\gamma}\|u_n\|^2 + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu} - \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx\right] \\
&\quad + a\frac{1+\gamma}{1-\gamma}\langle u_n, \varphi\rangle + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu-2}\langle u_n, \varphi\rangle \\
&\quad - \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx.
\end{aligned}$$

If $(\varsigma_n', \varphi) \geq 0$, then

$$\begin{aligned}
(\varsigma_n', \varphi) &\leq \frac{\frac{\|\varphi\|}{n} - \left(a\frac{1+\gamma}{1-\gamma}\langle u_n, \varphi\rangle + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu-2}\langle u_n, \varphi\rangle - \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx\right)}{\left[a\frac{1+\gamma}{1-\gamma}\|u_n\|^2 + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu} + \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx\right] - \frac{\|u_n\|}{n}} \\
&\leq \frac{\frac{\|\varphi\|}{n} + \left|a\frac{1+\gamma}{1-\gamma}\langle u_n, \varphi\rangle + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu-2}\langle u_n, \varphi\rangle - \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx\right|}{\left[a\frac{1+\gamma}{1-\gamma}\|u_n\|^2 + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu} + \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx\right] - \frac{\|u_n\|}{n}}.
\end{aligned} \tag{5.11}$$

If $(\varsigma_n', \varphi) < 0$, that is

$$\begin{aligned}
(\varsigma_n', \varphi) &\leq \frac{\frac{\|\varphi\|}{n} - \left(a\frac{1+\gamma}{1-\gamma}\langle u_n, \varphi\rangle + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu-2}\langle u_n, \varphi\rangle - \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx\right)}{\left[a\frac{1+\gamma}{1-\gamma}\|u_n\|^2 + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu} + \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx\right] + \frac{\|u_n\|}{n}} \\
&\leq \frac{\frac{\|\varphi\|}{n} + \left|a\frac{1+\gamma}{1-\gamma}\langle u_n, \varphi\rangle + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu-2}\langle u_n, \varphi\rangle - \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx\right|}{\left[a\frac{1+\gamma}{1-\gamma}\|u_n\|^2 + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu} + \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx\right] + \frac{\|u_n\|}{n}}.
\end{aligned} \tag{5.12}$$

Combining (5.11) and (5.12), we deduce that

$$(\varsigma_n', \varphi) \leq \frac{\frac{\|\varphi\|}{n} + \left|a\frac{1+\gamma}{1-\gamma}\langle u_n, \varphi\rangle + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu-2}\langle u_n, \varphi\rangle - \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx\right|}{\left[a\frac{1+\gamma}{1-\gamma}\|u_n\|^2 + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu} + \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx\right] - \frac{\|u_n\|}{n}}.$$

By the boundedness of $\{u_n\}$, the above inequality can already explain $(\varsigma_n', \varphi) \neq +\infty$. In summary, there is a positive constant C_3 such that $|(\varsigma_n', \varphi)| \leq C_3$.

At present, again we use (5.8) and (5.9) in combination and divide by $\wp > 0$.

$$
\begin{aligned}
&\Big|\frac{\varsigma_n(\wp\varphi)-1}{\wp}\Big|\frac{\|u_n\|}{n}+\varsigma_n(\wp\varphi)\frac{\|\varphi\|}{n}\\
&\geq \frac{\mathcal{I}(u_n)-\mathcal{I}[\varsigma_n(\wp\varphi)(u_n+\wp\varphi)]}{\wp}\\
&= -\frac{\varsigma_n(\wp\varphi)-1}{\wp}\Bigg[\frac{a(\varsigma_n(\wp\varphi)+1)}{2}\|u_n+\wp\varphi\|^2+\frac{b(\varsigma_n^{2\mu}(\wp\varphi)-1)}{2\mu(\varsigma_n(\wp\varphi)-1)}\|u_n+\wp\varphi\|^{2\mu}\\
&\quad-\frac{\varsigma_n^{2_s^*}(\wp\varphi)-1}{2_s^*(\varsigma_n(\wp\varphi)-1)}\lambda\int_\Omega l(x)|u_n+\wp\varphi|^{2_s^*}dx-\frac{\varsigma_n^{1-\gamma}(\wp\varphi)-1}{(1-\gamma)(\varsigma_n(\wp\varphi)-1)}\int_\Omega h(x)|u_k+\wp\varphi|^{1-\gamma}dx\Bigg]\\
&\quad-\Bigg[\frac{a(\|u_n+\wp\varphi\|^2-\|u_n\|^2)}{2\wp}+\frac{b(\|u_n+\wp\varphi\|^{2\mu}-\|u_n\|^{2\mu})}{2\mu\wp}\\
&\quad-\lambda\int_\Omega\frac{l(x)|u_n+\wp\varphi|^{2_s^*}-l(x)|u_n|^{2_s^*}}{2_s^*\wp}-\int_\Omega\frac{h(x)|u_n+\wp\varphi|^{1-\gamma}-h(x)|u_n|^{1-\gamma}}{(1-\gamma)\wp}dx\Bigg]. \qquad (5.13)
\end{aligned}
$$

Based on the above inequality, now we let $\wp \to 0$. With the help of Fatou's lemma, we infer

$$
\begin{aligned}
&|(\varsigma_n',\varphi)|\frac{\|u_n\|}{n}+\lim_{\wp\to 0}\varsigma_n(\tau\varphi)\frac{\|\varphi\|}{n}=|(\varsigma_n',\varphi)|\frac{\|u_n\|}{n}+\frac{\|\varphi\|}{n}\\
&\geq -(\varsigma_n,\varphi)\Big[a\|u_n\|^2+b\|u_n\|^{2\mu}-\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx-\int_\Omega h(x)|u_n|^{1-\gamma}dx\Big]\\
&\quad-\Big[a\langle u_n,\varphi\rangle+b\|u_n\|^{2\mu-2}\langle u_n,\varphi\rangle-\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx\\
&\quad-\int_\Omega\liminf_{\wp\to 0}\frac{h(x)|u_n+\wp\varphi|^{1-\gamma}-h(x)|u_n|^{1-\gamma}}{(1-\gamma)\wp}dx\Big]\\
&=-\Big[a\langle u_n,\varphi\rangle+b\|u_n\|^{2\mu-2}\langle u_n,\varphi\rangle-\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx-\int_\Omega h(x)|u_n|^{-\gamma}\varphi dx\Big],
\end{aligned}
$$

owing to $u_n \in \mathbb{X}$ and $|(\varsigma_n',\varphi)| \leq C_3$ uniformly for large n. Consequently,

$$
\begin{aligned}
&(a+b\|u_n\|^{2\mu-2})\langle u_n,\varphi\rangle-\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx+\frac{|(\varsigma_n',\varphi)\|u_n\|+\|\varphi\||}{n}\\
&\geq\int_\Omega h(x)|u_n|^{-\gamma}\varphi dx, \qquad (5.14)
\end{aligned}
$$

which implies that as $n \to \infty$

$$
a\langle u_n,\varphi\rangle+b\|u_n\|^{2\mu-2}\langle u_n,\varphi\rangle-\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx-\int_\Omega h(x)|u_n|^{-\gamma}\varphi dx\geq o_n(1), \qquad (5.15)
$$

for any $\varphi \in \mathbb{M}_0$ with $\varphi \geq 0$.

After that, our purpose is to prove that (5.15) applicable to any arbitrary $\ell \in \mathbb{M}_0$. We set $\psi_\varepsilon = u_n + \varepsilon\ell$ with $\varepsilon > 0$ and $\ell \in \mathbb{M}_0$. Denoting $\Omega_\varepsilon = \{x \in \mathbb{R}^N : \psi_\varepsilon(x) \leq 0\}$. Afterwards, letting $\varphi = \psi_\varepsilon^+$ in (5.15), we confirm

$$\begin{aligned}
o_n(1) &\le (a+b\|u_n\|^{2\mu-2})\langle u_n, \psi_\varepsilon^+\rangle - \lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\psi_\varepsilon^+ dx - \int_\Omega h(x)|u_n|^{-\gamma}\psi_\varepsilon^+ dx \\
&= (a+b\|u_n\|^{2\mu-2})\langle u_n, \psi_\varepsilon + \psi_\varepsilon^-\rangle - \lambda\int_\Omega l(x)|u_n|^{2_s^*-1}(\psi_\varepsilon + \psi_\varepsilon^-)dx \\
&\quad - \int_\Omega h(x)|u_n|^{-\gamma}(\psi_\varepsilon + \psi_\varepsilon^-)dx \\
&= (a+b\|u_n\|^{2\mu-2})\langle u_n, u_n + \varepsilon\ell\rangle + (a+b\|u_n\|^{2\mu-2})\langle u_n, \psi_\varepsilon^-\rangle \\
&\quad - \Big(\int_\Omega - \int_{\Omega_\varepsilon}\Big)\big[\lambda l(x)|u_n|^{2_s^*-1}(u_n+\varepsilon\ell) + h(x)|u_n|^{-\gamma}(u_n+\varepsilon\ell)\big]dx \\
&= \Big[a\|u_n\|^2 + b\|u_n\|^{2\mu} - \lambda\int_\Omega l(x)|u_n|^{2_s^*}dx - \int_\Omega h(x)|u_n|^{1-\gamma}dx\Big] \\
&\quad + \varepsilon\Big[(a+b\|u_n\|^{2\mu-2})\langle u_n, \ell\rangle - \lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\ell dx - \int_\Omega h(x)|u_n|^{-\gamma}\ell dx\Big] \\
&\quad + (a+b\|u_n\|^{2\mu-2})\langle u_n, \psi_\varepsilon^-\rangle + \int_{\Omega_\varepsilon}[\lambda l(x)|u_n|^{2_s^*-1}(u_n+\varepsilon\ell) + h(x)|u_n|^{-\gamma}(u_n+\varepsilon\ell)]dx \\
&\quad = \Big[a\|u_n\|^2 + b\|u_n\|^{2\mu} - \lambda\int_\Omega l(x)|u_n|^{2_s^*}dx - \int_\Omega h(x)|u_n|^{1-\gamma}dx\Big] \\
&\quad + \varepsilon\Big[(a+b\|u_n\|^{2\mu-2})\langle u_n, \ell\rangle - \lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\ell dx - \int_\Omega h(x)|u_n|^{-\gamma}\ell dx\Big] \\
&\quad + (a+b\|u_n\|^{2\mu-2})\langle u_n, \psi_\varepsilon^-\rangle + \lambda\int_{\Omega_\epsilon} l(x)|u_n|^{2_s^*-1}(u_n+\varepsilon\ell)dx \\
&\quad + \int_{\Omega_\epsilon} h(x)|u_n|^{-\gamma}(u_n+\varepsilon\ell)dx.
\end{aligned}$$

Note that $u_n \in \mathbb{X}$ and $u_n + \varepsilon\ell \le 0$ in Ω_ε, thus

$$\int_{\Omega_\varepsilon} h(x)|u_n|^{-\gamma}(u_n+\varepsilon\ell)dx < 0.$$

Considering these facts, we deduce that

$$\begin{aligned}
o_n(1) &\le \varepsilon\Big[(a+b\|u_n\|^{2\mu-2})\langle u_n, \ell\rangle - \lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\ell dx - \int_\Omega h(x)|u_n|^{-\gamma}\ell dx\Big] \\
&\quad + (a+b\|u_n\|^{2\mu-2})\langle u_n, \psi_\varepsilon^-\rangle + \lambda\int_{\Omega_\varepsilon} l(x)|u_n|^{2_s^*-1}(u_n+\varepsilon\ell)dx \\
&\quad + \int_{\Omega_\varepsilon} h(x)|u_n|^{-\gamma}(u_n+\varepsilon\ell)dx \\
&\le \varepsilon\Big[(a+b\|u_n\|^{2\mu-2})\langle u_n, \ell\rangle - \lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\ell dx - \int_\Omega h(x)|u_n|^{-\gamma}\ell dx\Big] \\
&\quad + (a+b\|u_n\|^{2\mu-2})\langle u_n, \psi_\varepsilon^-\rangle + \lambda\int_{\Omega_\varepsilon} l(x)|u_n|^{2_s^*-1}(u_n+\varepsilon\ell)dx.
\end{aligned} \tag{5.16}$$

Then, denote

$$\Im_\varepsilon(x,y) = \frac{(u_n(x)-u_n(y))(\psi_\varepsilon^-(x)-\psi_\varepsilon^-(y))}{|x-y|^{N+2s}}$$

and

$$\Im(x,y) = \frac{(u_n(x)-u_n(y))(\ell(x)-\ell(y))}{|x-y|^{N+2s}}.$$

The definition of scalar products and the symmetry of the fraction kernel can be used here. Therefore, we may write

$$\begin{aligned}\langle u_n, \psi_\varepsilon^- \rangle &= \iint_G \frac{(u_n(x)-u_n(y))(\psi_\varepsilon^-(x)-\psi_\varepsilon^-(y))}{|x-y|^{N+2s}}dxdy \\ &= \iint_{(\mathbb{R}^N\times\mathbb{R}^N)\setminus(\Omega^c\times\Omega^c)} \Im_\varepsilon(x,y)dxdy \\ &= \left(\iint_{\Omega\times\Omega} + \iint_{\Omega\times(\mathbb{R}^N\setminus\Omega)} + \iint_{(\mathbb{R}^N\setminus\Omega)\times\Omega}\right)\Im_\varepsilon(x,y)dxdy \\ &= \iint_{\Omega\times\Omega}\Im_\varepsilon(x,y)dxdy + 2\iint_{\Omega\times(\mathbb{R}^N\setminus\Omega)}\Im_\varepsilon(x,y)dxdy.\end{aligned}$$

It is worth noting that $\psi_\varepsilon^- = 0$ in the case of ψ_ε is not in Ω_ε. From this, we

$$\begin{aligned}&\iint_{\Omega\times\Omega}\Im_\varepsilon(x,y)dxdy + 2\iint_{\Omega\times(\mathbb{R}^N\setminus\Omega)}\Im_\varepsilon(x,y)dxdy \\ &\quad = \left(\iint_{\Omega_\varepsilon\times\Omega_\varepsilon} +2\iint_{\Omega_\varepsilon\times(\Omega\setminus\Omega_\varepsilon)} +2\iint_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus\Omega)}\right)\Im_\varepsilon(x,y)dxdy \\ &\quad = \iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\Im_\varepsilon(x,y)dxdy + 2\iint_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus\Omega_\varepsilon)}\Im_\varepsilon(x,y)dxdy.\end{aligned}$$

Next,

$$\begin{aligned}&\iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{(u_n(x)-u_n(y))(\psi_\varepsilon^-(x)-\psi_\varepsilon^-(y))}{|x-y|^{N+2s}}dxdy \\ &= \iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{(u_n(x)-u_n(y))(\psi_\varepsilon^+(x)-\psi_\varepsilon^+(y))}{|x-y|^{N+2s}}dxdy \\ &\quad - \iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{(u_n(x)-u_n(y))(\psi_\varepsilon(x)-\psi_\varepsilon(y))}{|x-y|^{N+2s}}dxdy \\ &= -\iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{(u_n(x)-u_n(y))^2}{|x-y|^{N+2s}}dxdy - \varepsilon\iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{(u_n(x)-u_n(y))(\ell(x)-\ell(y))}{|x-y|^{N+2s}}dxdy \\ &\leq -\varepsilon\iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\Im(x,y)dxdy.\end{aligned}$$

In the same way,

$$2\iint_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus\Omega_\varepsilon)}\Im_\varepsilon(x,y)dxdy \leq -2\varepsilon\iint_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus\Omega_\varepsilon)}\Im(x,y)dxdy.$$

In combination with the above, we can obtain

$$\langle u_n, \psi_\varepsilon^-\rangle \leq -\varepsilon\left(\iint_{\Omega_\varepsilon\times\Omega_\varepsilon} +2\iint_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus\Omega_\varepsilon)}\right)\Im(x,y)dxdy \leq 2\varepsilon\iint_{\Omega_\varepsilon\times\mathbb{R}^N}|\Im(x,y)|dxdy. \tag{5.17}$$

Apparently, $\Im(x,y)\in L^1(\mathbb{R}^N\times\mathbb{R}^N)$. Besides, for any $\sigma>0$, there exists R_σ sufficiently large.

From the basic definition of Ω_ε, we infer $\Omega_\varepsilon \subset \operatorname{supp}\ell$.

Since

$$\iint_{\Omega_\varepsilon\times\mathbb{R}^N}|\Im(x,y)|dxdy = \iint_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus B_{R_\sigma})}|\Im(x,y)|dxdy + \iint_{\Omega_\varepsilon\times B_{R_\sigma}}|\Im(x,y)|dxdy,$$

for the first item, we may obtain

$$\iint_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus B_{R_\sigma})}|\Im(x,y)|dxdy < \iint_{(\operatorname{supp}\ell)\times(\mathbb{R}^N\setminus B_{R_\sigma})}|\Im(x,y)|dxdy \quad < \frac{\sigma}{2}.$$

Also, we know that $|\Omega_\varepsilon \times B_{R_\sigma}| \to 0$ as $\varepsilon \to 0^+$. The absolute continuity of the integral can be used, so there exists δ_σ and ε_σ such that for any $\varepsilon \in (0, \varepsilon_\sigma]$,

$$|\Omega_\varepsilon \times B_{R_\sigma}| < \delta_\sigma, \text{and} \iint_{\Omega_\varepsilon \times B_{R_\sigma}} |\Im(x,y)| dxdy < \frac{\sigma}{2}.$$

Accordingly, for all $\varepsilon \in (0, \varepsilon_\sigma]$,

$$\iint_{\Omega_\varepsilon \times \mathbb{R}^N} |\Im(x,y)| dxdy < \sigma,$$

$$\lim_{\varepsilon \to 0^+} \iint_{\Omega_\varepsilon \times \mathbb{R}^N} |\Im(x,y)| dxdy = 0.$$

Thus, according to (5.17), we get

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \langle u_n, \psi_\varepsilon^- \rangle = 0. \tag{5.18}$$

With respect to $\int_{\Omega_\varepsilon} l(x)|u_n|^{2_s^*-1}(u_n + \varepsilon\ell)dx$, since $\{u_n + \varepsilon\ell \le 0\} \to 0$ as $\varepsilon \to 0$, we obtain

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} l(x)|u_n|^{2_s^*-1}(u_n + \varepsilon\phi)dx = 0. \tag{5.19}$$

Finally, dividing by ε and letting $\varepsilon \to 0$ in (5.16), for n large enough, we get from (5.18) and (5.19) that

$$a\langle u_n, \ell\rangle + b||u_n||^{2\mu-2}\langle u_n, \ell\rangle - \lambda \int_\Omega l(x)|u_n|^{2_s^*-1}\ell dx - \int_\Omega h(x)|u_n|^{-\gamma}\ell dx \ge o_n(1). \tag{5.20}$$

Replace ℓ in (5.20) with $-\ell$, and the inequality is also true. Thus it can be seen that

$$a\langle u_n, \ell\rangle + b\|u_n\|^{2\mu-2}\langle u_n, \ell\rangle - \lambda \int_\Omega l(x)|u_n|^{2_s^*-1}\ell dx - \int_\Omega h(x)|u_n|^{-\gamma}\ell dx = o_n(1) \tag{5.21}$$

as $n \to \infty$.

Step 2: We analyze problem (1.1) on $\mathbb{X}^-$.

We have learned that $\mathcal{I}$ is bounded from below on $\mathbb{X}^-$ and coercive from Lemma 4.3. And it turns out that $\mathbb{X}^-$ is a closed set. Ekeland's variational principle can also be used. We may define $c^- = \inf_{u\in\mathbb{X}^-} \mathcal{I}$ accordingly. There exists a $\xi \in \mathbb{X}^-$, we deduce that

$$a(1+\gamma)\|\xi\|^2 + b(2\mu - 1 + \gamma)\|\xi\|^{2\mu} - \lambda(2_s^* - 1 + \gamma) \int_\Omega l(x)|\xi|^{2_s^*} dx < 0.$$

From (3.5) it follows that

$$b\|\xi\|^{2\mu} < \lambda \frac{2_s^* - 1 + \gamma}{2\mu - 1 + \gamma} \int_\Omega l(x)|\xi|^{2_s^*} dx - a\frac{1+\gamma}{2\mu - 1 + \gamma}\|\xi\|^2.$$

$$\begin{aligned}
\mathcal{I}(\xi) &= \frac{2_s^* - 2}{22_s^*} a\|\xi\|^2 + \frac{2_s^* - 2\mu}{2\mu 2_s^*} b\|\xi\|^{2\mu} - \frac{2_s^* + \gamma - 1}{2_s^*(1-\gamma)} \int_\Omega h(x)|\xi|^{1-\gamma} dx \\
&< \frac{2_s^* - 2}{22_s^*} a\|\xi\|^2 + \frac{2_s^* - 2\mu}{2\mu 2_s^*} \left[\lambda \frac{2_s^* - 1 + \gamma}{2\mu - 1 + \gamma} \int_\Omega l(x)|\xi|^{2_s^*} dx - a\frac{1+\gamma}{2\mu - 1 + \gamma}\|\xi\|^2\right] \\
&\quad - \frac{2_s^* + \gamma - 1}{2_s^*(1-\gamma)} \int_\Omega h(x)|\xi|^{1-\gamma} dx \\
&< \frac{\mu}{2\mu - 1 + \gamma} a\|\xi\|^2 + \lambda\left(\frac{2_s^* - 2\mu}{2\mu 2_s^*}\right)\left(\frac{2_s^* - 1 + \gamma}{2\mu - 1 + \gamma}\right) \int_\Omega l(x)|\xi|^{2_s^*} dx \\
&\quad - \frac{2_s^* + \gamma - 1}{2_s^*(1-\gamma)} \int_\Omega h(x)|\xi|^{1-\gamma} dx.
\end{aligned}$$

Let

$$\frac{\mu}{2\mu-1+\gamma}a\|\xi\|^2+\lambda\left(\frac{2_s^*-2\mu}{2\mu 2_s^*}\right)\left(\frac{2_s^*-1+\gamma}{2\mu-1+\gamma}\right)\int_\Omega l(x)|\xi|^{2_s^*}dx \\ -\frac{2_s^*+\gamma-1}{2_s^*(1-\gamma)}\int_\Omega h(x)|\xi|^{1-\gamma}dx<0. \quad (5.22)$$

Then (5.22) implies that

$$0<\lambda<\left(\frac{2_s^*+\gamma-1}{2_s^*(1-\gamma)}\int_\Omega h(x)|\xi|^{1-\gamma}dx-\frac{\mu}{2\mu-1+\gamma}a\|\xi\|^2\right)\frac{2\mu 2_s^*(2\mu-1+\gamma)}{(2_s^*-2\mu)(2_s^*-1+\gamma)\int_\Omega l(x)|\xi|^{2_s^*}dx} \\ =\Gamma_*.$$

In order to guarantee that λ is positive, we have to make a sufficiently small. Hence, when $0<\lambda<\Gamma_*$, $\mathcal{I}(\xi)<0$. Furthermore, we know that $c^-<0$.

Based on what is explained above, there exists a sequence $\{v_k\}\subset\mathbb{X}^-$ satisfies the following properties

$$(i)\ \mathcal{I}(v_k)<c^-+\frac{1}{k}, \qquad (ii)\ \mathcal{I}(v_k)\leq\mathcal{I}(v)+\frac{1}{k}\|v_k-v\|, \qquad \forall v\in\mathbb{X}^-. \quad (5.23)$$

Similarly, let us assume that $v_k(x)\geq 0$ for all $x\in\Omega$. Because $\mathbb{X}^-$ does not contain $\{0\}$. So $v_k(x)>0$ for all $x\in\Omega$. Apparently, $\{v_k\}$ is bounded in $\mathbb{M}_0$, we use $\{v_n\}$ to represent its subsequence, so there will be $v_0>0$ such that

$$\begin{aligned} &v_n\rightharpoonup v_0 && \text{in } \mathbb{M}_0,\\ &v_n\rightharpoonup v_0 && \text{in } L^{2_s^*},\\ &v_n\to v_0 && \text{a.e. in } \Omega,\\ &v_n\to v_0 && \text{in } L^\eta(\Omega) \text{ for } 2\leq\eta<2_s^*, \end{aligned} \quad (5.24)$$

owing to $\mathbb{X}^-$ is a closed set. Applying Lemma 4.7 with $u=v_n$, $\varphi\in\mathbb{M}_0$, $\varphi\geq 0$ and $\wp>0$ small enough. A series of continuous functions satisfying $\varsigma_n(0)=1$ and $\varsigma_n(\wp\varphi)(v_n+\wp\varphi)\in\mathbb{X}^-$ can certainly be obtained. The proof procedure in **Step 1** can be used again to obtain

$$a\langle v_n,\ell\rangle+b||v_n||^{2\mu-2}\langle v_n,\ell\rangle-\lambda\int_\Omega l(x)|v_n|^{2_s^*-1}\ell dx-\int_\Omega h(x)|v_n|^{-\gamma}\ell dx\geq o_n(1) \quad (5.25)$$

as $n\to\infty$.

Lemma 5.1. *For $0<\lambda<\Gamma_1$, let $\{u_k\}\subset\mathbb{X}^+$ in **Step 1** and $\{v_k\}\subset\mathbb{X}^-$ in **Step 2** respectively satisfying (5.1) and (5.23) and simultaneously satisfying $\mathcal{I}\to c<C_\lambda$ as $k\to\infty$, where*

$$C_\lambda=\frac{s}{N}S_s^{\frac{N}{2s}}\left(\frac{a^{\frac{2_s^*}{2}}}{\|l\|_\infty}\right)^{\frac{2}{2_s^*-2}}\lambda^{\frac{2}{2-2_s^*}}-\frac{2\mu-1+\gamma}{2_s^*(1-\gamma)2\mu}\frac{\left((2_s^*+\gamma-1)\|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}}S_s^{\frac{\gamma-1}{2}}\right)^{\frac{2\mu}{2\mu-1+\gamma}}}{[b(2_s^*-2\mu)]^{\frac{1-\gamma}{2\mu-1+\gamma}}}.$$

Then, both $\{u_k\}$ and $\{v_k\}$ have strongly convergent subsequences in $\mathbb{M}_0$.

Proof. Let us just think about $\{u_k\}\subset\mathbb{X}^+$, the case $\{v_k\}\subset\mathbb{X}^-$ can be obtained in the same way. Note that $\{u_k\}$ is bounded in $\mathbb{M}_0$ and $u_k\geq 0$. Furthermore, there is a subsequence $\{u_n\}$

that satisfies

$$\begin{aligned} u_n &\rightharpoonup u_0 \quad \text{in } \mathbb{M}_0, \\ u_n &\rightharpoonup u_0 \quad \text{in } L^{2_s^*}, \\ u_n &\to u_0 \quad \text{a.e. in } \ \Omega, \\ u_n &\le \hbar \qquad \text{a.e. in } \ \Omega, \\ u_n &\to u_0 \quad \text{in } L^r(\Omega) \text{ for } 2 \le r < 2_s^*, \\ \|u_n\| &\to \zeta, \end{aligned} \tag{5.26}$$

as $n \to \infty$, with $\hbar \in L^r(\Omega)$ for a fixed $r \in [1, 2_s^*)$ and $u_0 \ge 0$. If $\zeta = 0$, that is $\|u_n - 0\| \to 0$ as $n \to 0$, which implies $u_n \to 0$ in $\mathbb{M}_0$ as $n \to \infty$. The situation of $\zeta > 0$ will be considered below.

According to (5.26), we get

$$\begin{aligned} 2\langle u_n, u_0\rangle &= 2\langle u_0, u_0\rangle + o(1) \\ &= \langle u_n, u_n\rangle - \langle u_n, u_n\rangle + 2\langle u_0, u_0\rangle + o(1), \end{aligned}$$

as $n \to \infty$, which implies

$$\|u_n\|^2 = \|u_n - u_0\|^2 + \|u_0\|^2 + o(1) \tag{5.27}$$

as $n \to \infty$. Applying the Brézis–Lieb lemma and the process in Lemma 4.5, we have

$$\int_\Omega l(x)|u_n|^{2_s^*} dx = \int_\Omega l(x)|u_n - u_0|^{2_s^*} dx + \int_\Omega l(x)|u_0|^{2_s^*} dx + o(1) \tag{5.28}$$

as $n \to \infty$. We infer from (5.21), (5.27) and (5.28) that, as $n \to \infty$

$$\begin{aligned} o(1) &= (a + b\|u_n\|^{2\mu-2})\langle u_n, u_n - u_0\rangle - \lambda \int_\Omega l(x)|u_n|^{2_s^*-1}(u_n - u_0)dx \\ &\quad - \int_\Omega h(x)|u_n|^{-\gamma}(u_n - u_0)dx \\ &= (a + b\zeta^{2\mu-2})(\zeta^2 - \|u_0\|^2) - \lambda \int_\Omega l(x)|u_n|^{2_s^*} dx \\ &\quad + \lambda \int_\Omega l(x)|u_0|^{2_s^*} dx - \int_\Omega h(x)|u_n|^{-\gamma}(u_n - u_0)dx + o(1) \\ &= (a + b\zeta^{2\mu-2})\|u_n - u_0\|^2 - \lambda \int_\Omega l(x)|u_n - u_0|^{2_s^*} dx - \int_\Omega h(x)|u_n|^{-\gamma}(u_n - u_0)dx + o(1). \end{aligned}$$

Consequently,

$$\begin{aligned} &(a + b\zeta^{2\mu-2}) \lim_{n\to\infty} \|u_n - u_0\|^2 \\ &\quad = \lim_{n\to\infty} \lambda \int_\Omega l(x)|u_n - u_0|^{2_s^*} dx + \lim_{n\to\infty} \int_\Omega h(x)|u_n|^{1-\gamma}dx - h(x)|u_n|^{-\gamma}u_0 dx. \end{aligned} \tag{5.29}$$

By (5.26), we have $u_n^{1-\gamma} \le \hbar^{1-\gamma}$ a.e. in Ω. Then, the dominated convergence theorem can be used, that is

$$\lim_{n\to\infty} \int_\Omega h(x)|u_n|^{1-\gamma}dx = \int_\Omega h(x)|u_0|^{1-\gamma}dx. \tag{5.30}$$

In the light of (5.20), we know that $h(x)u_n^{-\gamma}u_0 dx \in L^1(\Omega)$. Then Fatou's lemma yields

$$\liminf_{n\to\infty} \int_\Omega h(x)u_n^{-\gamma}u_0 dx \ge \int_\Omega h(x)u_0^{1-\gamma}dx. \tag{5.31}$$

For convenience, we define

$$\aleph^{2_s^*} = \lim_{n\to\infty} \int_\Omega l(x)|u_n - u_0|^{2_s^*} dx. \tag{5.32}$$

Combining (5.29), (5.30) and (5.31), we get

$$(a + b\zeta^{2\mu-2}) \lim_{n\to\infty} \|u_n - u_0\|^2 \le \lambda \aleph^{2_s^*}, \tag{5.33}$$

which means that $0 \le \aleph$. If $\aleph = 0$, we can immediately infer that $u_n \to u_0$ in $\mathbb{M}_0$ owing to $\zeta > 0$. Let us paradoxically say that $\aleph > 0$ to complete this proof. From (5.32), as $n \to \infty$, we obtain

$$\aleph^{2_s^*} \le \|l\|_\infty S_s^{-\frac{2_s^*}{2}} \lim_{n\to\infty} \|u_n - u_0\|^{2_s^*},$$

which implies that

$$\|l\|_\infty^{-\frac{2}{2_s^*}} \aleph^2 S_s \le \lim_{n\to\infty} \|u_n - u_0\|^2. \tag{5.34}$$

Combining (5.33) and (5.34), we have

$$\aleph^{2_s^*-2} \ge (a + b\zeta^{2\mu-2})\|l\|_\infty^{-\frac{2}{2_s^*}} S_s \lambda^{-1}. \tag{5.35}$$

As $n \to \infty$, it is easy to get

$$\aleph^{2_s^*} \ge (a + b\zeta^{2\mu-2})(\zeta^2 - \|u_0\|^2)\lambda^{-1} \tag{5.36}$$

from (5.33). Using (5.34) and (5.35) in (5.36), we have

$$\begin{aligned}
(\aleph^{2_s^*})^{\frac{2_s^*-2}{2}} &\ge (a + b\zeta^{2\mu-2})^{\frac{2_s^*-2}{2}} (\zeta^2 - \|u_0\|^2)^{\frac{2_s^*-2}{2}} \lambda^{\frac{2-2_s^*}{2}} \\
&= (a + b\zeta^{2\mu-2})^{\frac{2_s^*-2}{2}} (\lim_{n\to\infty} \|u_n - u_0\|^2)^{\frac{2_s^*-2}{2}} \lambda^{\frac{2-2_s^*}{2}} \\
&\ge (a + b\zeta^{2\mu-2})^{\frac{2_s^*-2}{2}} (\|l\|_\infty^{-\frac{2}{2_s^*}} S_s)^{\frac{2_s^*-2}{2}} \aleph^{2_s^*-2} \lambda^{\frac{2-2_s^*}{2}} \\
&\ge (a + b\zeta^{2\mu-2})^{\frac{2_s^*}{2}} S_s^{\frac{2_s^*}{2}} \|l\|_\infty^{-1} \lambda^{-\frac{2_s^*}{2}}.
\end{aligned} \tag{5.37}$$

At the same time, according to (5.34) and (5.35), we get

$$\begin{aligned}
(\zeta^2 - \|u_0\|^2)^{\frac{2_s^*-2}{2}} &\ge \|l\|_\infty^{-\frac{2_s^*-2}{2_s^*}} \aleph^{2_s^*-2} S_s^{\frac{2_s^*-2}{2}} \\
&\ge (a + b\zeta^{2\mu-2})\|l\|_\infty^{-1} S_s^{\frac{2_s^*}{2}} \lambda^{-1}
\end{aligned} \tag{5.38}$$

as $n \to \infty$. Consequently, we have

$$(\zeta^2)^{\frac{2_s^*-2}{2}} \ge (\zeta^2 - \|u_0\|^2)^{\frac{2_s^*-2}{2}} \ge (a + b\zeta^{2\mu-2})\|l\|_\infty^{-1} S_s^{\frac{2_s^*}{2}} \lambda^{-1}, \tag{5.39}$$

that is

$$\zeta^2 \ge S_s^{\frac{N}{2s}} \|l\|_\infty^{-\frac{2}{2_s^*-2}} (a + b\zeta^{2\mu-2})^{\frac{2}{2_s^*-2}} \lambda^{\frac{2}{2-2_s^*}} \ge S_s^{\frac{N}{2s}} \|l\|_\infty^{-\frac{2}{2_s^*-2}} a^{\frac{2}{2_s^*-2}} \lambda^{\frac{2}{2-2_s^*}}. \tag{5.40}$$

We define

$$F(u_n, \phi) = (a + b\|u_n\|^{2\mu-2})\langle u_n, \phi\rangle - \lambda \int_\Omega l(x)|u_n|^{2_s^*-1}\phi dx - \int_\Omega h(x)|u_n|^{-\gamma}\phi dx \tag{5.41}$$

for any $\phi \in \mathbb{M}_0$. Subsequently,

$$
\begin{aligned}
&\mathcal{I}(u_n) - \frac{1}{2_s^*}F(u_n, u_n) \\
&= \left(\frac{1}{2} - \frac{1}{2_s^*}\right) a\|u_n\|^2 + \left(\frac{1}{2\mu} - \frac{1}{2_s^*}\right) b\|u_n\|^{2\mu} - \left(\frac{1}{1-\gamma} - \frac{1}{2_s^*}\right) \int_\Omega h(x)|u_n|^{1-\gamma}dx \\
&\geq \left(\frac{1}{2} - \frac{1}{2_s^*}\right) a\|u_n\|^2 + \left(\frac{1}{2\mu} - \frac{1}{2_s^*}\right) b\|u_n\|^{2\mu} \\
&\quad - \left(\frac{1}{1-\gamma} - \frac{1}{2_s^*}\right) S_s^{\frac{\gamma-1}{2}} \|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}} \|u_n\|^{1-\gamma}.
\end{aligned} \tag{5.42}
$$

Define

$$
P(t) = \left(\frac{1}{2\mu} - \frac{1}{2_s^*}\right) bt^{2\mu} - \left(\frac{1}{1-\gamma} - \frac{1}{2_s^*}\right) S_s^{\frac{\gamma-1}{2}} \|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}} t^{1-\gamma},
$$

$$
P'(t) = \left(\frac{1}{2\mu} - \frac{1}{2_s^*}\right) 2\mu bt^{2\mu-1} - \left(\frac{1}{1-\gamma} - \frac{1}{2_s^*}\right) S_s^{\frac{\gamma-1}{2}} \|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}} (1-\gamma)t^{-\gamma}.
$$

When $p'(t) = 0$, we can get

$$
t = \left[\frac{2_s^* - 1 + \gamma}{b(2_s^* - 2\mu)}\right]^{\frac{1}{2\mu-1+\gamma}} S_s^{\frac{\gamma-1}{2(2\mu-1+\gamma)}} \|h\|^{\frac{1}{2\mu-1+\gamma}}_{\frac{2_s^*}{2_s^*+\gamma-1}}.
$$

Hence we have

$$
\begin{aligned}
P(t) &\geq \frac{2_s^* - 2\mu}{2\mu 2_s^*} b \left[\frac{2_s^* - 1 + \gamma}{b(2_s^* - 2\mu)}\right]^{\frac{2\mu}{2\mu-1+\gamma}} S_s^{\frac{(\gamma-1)\mu}{2\mu-1+\gamma}} \|h\|^{\frac{2\mu}{2\mu-1+\gamma}}_{\frac{2_s^*}{2_s^*+\gamma-1}} \\
&\quad - \frac{2_s^* - 1 + \gamma}{(1-\gamma)2_s^*} S_s^{\frac{\gamma-1}{2}} \|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}} \left[\frac{2_s^* - 1 + \gamma}{b(2_s^* - 2\mu)}\right]^{\frac{1-\gamma}{2\mu-1+\gamma}} S_s^{\frac{-\gamma^2+2\gamma-1}{2(2\mu-1+\gamma)}} \|h\|^{\frac{1-\gamma}{2\mu-1+\gamma}}_{\frac{2_s^*}{2_s^*+\gamma-1}} \\
&= -\frac{2\mu - 1 + \gamma}{2_s^*(1-\gamma)2\mu} \frac{\left((2_s^* + \gamma - 1)\|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}} S_s^{\frac{\gamma-1}{2}}\right)^{\frac{2\mu}{2\mu-1+\gamma}}}{[b(2_s^* - 2\mu)]^{\frac{1-\gamma}{2\mu-1+\gamma}}}.
\end{aligned} \tag{5.43}
$$

Then, from (5.42), we have

$$
\begin{aligned}
\mathcal{I}(u_n) - \frac{1}{2_s^*}F(u_n, u_n) &\geq \left(\frac{1}{2} - \frac{1}{2_s^*}\right) a\|u_n\|^2 \\
&\quad - \frac{2\mu - 1 + \gamma}{2_s^*(1-\gamma)2\mu} \frac{\left((2_s^* + \gamma - 1)\|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}} S_s^{\frac{\gamma-1}{2}}\right)^{\frac{2\mu}{2\mu-1+\gamma}}}{[b(2_s^* - 2\mu)]^{\frac{1-\gamma}{2\mu-1+\gamma}}}.
\end{aligned} \tag{5.44}
$$

Letting $n \to \infty$, we get

$$
c \geq \frac{s}{N} S_s^{\frac{N}{2s}} \left(\frac{a^{\frac{2_s^*}{2}}}{\|l\|_\infty}\right)^{\frac{2}{2_s^*-2}} \lambda^{\frac{2}{2-2_s^*}} - \frac{2\mu - 1 + \gamma}{2_s^*(1-\gamma)2\mu} \frac{\left((2_s^* + \gamma - 1)\|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}} S_s^{\frac{\gamma-1}{2}}\right)^{\frac{2\mu}{2\mu-1+\gamma}}}{[b(2_s^* - 2\mu)]^{\frac{1-\gamma}{2\mu-1+\gamma}}},
$$

which contradicts the assumption $c < C_\lambda$. This proof is completed. □

Let us fix $\lambda < \Gamma_0 = \min\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_*\}$, with Γ_1 and Γ_2 given respectively in Lemma 4.1 and Lemma 4.2, and

$$\Gamma_3 = \frac{a^{\frac{2_s^*}{2}}}{\|l\|_\infty}\left(\frac{s}{N}S_s^{\frac{N}{2s}}\frac{2_s^*(1-\gamma)2\mu}{2\mu-1+\gamma}\right)^{\frac{2_s^*-2}{2}}\left[\frac{[b(2_s^*-2\mu)]^{\frac{1-\gamma}{2\mu-1+\gamma}}}{(2_s^*-1+\gamma)\|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}}S_s^{\frac{\gamma-1}{2}}}\right]^{\frac{2_s^*-2}{2}},$$

which shows that $C_\lambda > 0$. From (5.3) and (5.22), we know that $c^+ < 0 < C_\lambda$ and $c^- < 0 < C_\lambda$. Applying the Lemma 5.1, the minimization sequence $\{u_n\}$ will satisfy $u_n \to u_0$ in $\mathbb{M}_0$, and the minimization sequence $\{v_n\}$ will satisfy $v_n \to v_0$ in $\mathbb{M}_0$. According to (5.20) and (5.25), we can separately obtain

$$a\langle u_0, \ell\rangle + b\|u_0\|^{2\mu-2}\langle u_0, \ell\rangle - \lambda\int_\Omega l(x)|u_0|^{2_s^*-1}\ell dx - \int_\Omega h(x)|u_0|^{-\gamma}\ell dx \geq 0,$$

and

$$a\langle v_0, \ell\rangle + b||v_0||^{2\mu-2}\langle v_0, \ell\rangle - \lambda\int_\Omega l(x)|v_0|^{2_s^*-1}\ell dx - \int_\Omega h(x)|v_0|^{-\gamma}\ell dx \geq 0$$

for any $\ell \in \mathbb{M}_0$. From the two inequalities above, we know $h(x)|u_0|^{-\gamma}\ell$ and $h(x)|v_0|^{-\gamma}\ell$ are integrable, which imply that $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$ in Ω, then the strong maximum principle (see Proposition 2.2.8 in [27]) yields that $u_0 > 0$ and $v_0 > 0$ in Ω. According to the arbitrariness of ℓ, we know that (5.20) fits any $\ell \in \mathbb{M}_0$. It follows that

$$a\langle u_0, \ell\rangle + b\|u_0\|^{2\mu-2}\langle u_0, \ell\rangle - \lambda\int_\Omega l(x)|u_0|^{2_s^*-1}\ell dx - \int_\Omega h(x)|u_0|^{-\gamma}\ell dx = 0,$$

and

$$a\langle v_0, \ell\rangle + b||v_0||^{2\mu-2}\langle v_0, \ell\rangle - \lambda\int_\Omega l(x)|v_0|^{2_s^*-1}\ell dx - \int_\Omega h(x)|v_0|^{-\gamma}\ell dx = 0$$

as $n \to \infty$. This indicates that problem (1.1) has a positive solution on both $\mathbb{X}^+$ and $\mathbb{X}^-$, respectively.

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