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Existence of positive solutions for generalized quasilinear elliptic equations with Sobolev critical growth

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Abstract. In this paper, we are devoted to establishing that the existence of positive solutions for a class of generalized quasilinear elliptic equations in $\mathbb{R}^N$ with Sobolev critical growth, which have appeared from plasma physics, as well as high-power ultrashort laser in matter. To begin, by changing the variable, quasilinear equations are transformed into semilinear equations. The positive solutions to semilinear equations are then presented using the Mountain Pass Theorem for locally Lipschitz functionals and the Concentration-Compactness Principle. Finally, an inverse translation reveals the presence of positive solutions to the original quasilinear equations.

Keywords: variational methods, Sobolev critical growth, locally Lipschitz functional.

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1 Introduction

In this paper, we aim at studying the existence of positive solutions for the following generalized quasilinear elliptic equations

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(x,u), \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $g \in C^1(\mathbb{R}, \mathbb{R}^+)$ is an even function and $g'(t) \geq 0$ for all $t \geq 0$ and $g(0) = 1$, the potential $V \in C(\mathbb{R}^N, \mathbb{R})$, h is a measurable function defined on $\mathbb{R}^N \times \mathbb{R}$. These equations are closely related to the existence of standing wave solutions for the following quasilinear Schrödinger equations

$$i\partial_t z = -\Delta z + E(x)z - \sigma(x, |z|^2)z - \Delta[l(|z|^2)]l'(|z|^2)z, \tag{1.2}$$

where $z : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$, $E : \mathbb{R}^N \to \mathbb{R}$ is a potential function and $\sigma : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $l : \mathbb{R} \to \mathbb{R}$ are suitable functions. Notice that equation (1.2) can be reduced to elliptic equations with the

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following formal structure (see [15]) by setting $z(x,t) = \exp(-iFt)u(x)$, where $F \in \mathbb{R}$ and u is a real function,

$$-\Delta u + V(x)u - \Delta[l(u^2)]l'(u^2)u = h(x,u), \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Furthermore, we take $g^2(u) = 1 + \frac{(l'(u^2))^2}{2}$, then (1.3) turns into (1.1) (see [16,28]).

To the best of our knowledge, the quasilinear equation (1.1) have been utilized to simulate a range of physical phenomena corresponding to various types of $g(u)$ in several fields of mathematical physics. For instance, the case $g^2(u) = 1 + 2u^2$, that is, $l(u) = u$ in (1.3), we get

$$-\Delta u + V(x)u - u\Delta(u^2) = h(x,u), \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

which simulates the time evolution of the condensate wave function in a superfluid film (see [20]). For equation (1.1), if we take $g^2(u) = 1 + \frac{u^2}{2(1+u^2)}$, that is, $l(u) = \sqrt{1+u}$, we get

$$-\Delta u + V(x)u - [\Delta\sqrt{1+u^2}]\frac{u}{2\sqrt{1+u^2}}u = h(x,u), \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

Equation (1.5) is often used as a model of the self-channeling of a high-power ultrashort laser in matter (see [8,12,25]). For more physical backgrounds about equation (1.1), readers can refer to [7,19,23,24] and the references within. So, the study for general quasilinear elliptic equation (1.1) is meaningful and important.

In [21], the quasilinear equation (1.4) was firstly transformed to a semilinear one by using a change of variable. Then, they chose an Orlicz space as the working space and obtained the existence of positive solutions to equation (1.4) by using the Mountain Pass theorem. Afterwards, the same change of variable was applied in [14,30,31], but the usual Sobolev space framework was used as the working space. For example, Silva and Vieira in [30] obtained the existence of positive solutions of equation (1.4) in an asymptotically periodic condition with critical growth. In [28], Shen and Wang used a new change of variable developed by (1.4) to show the existence of positive solutions of equation (1.1) when $h(x,u)$ was superlinear and subcritical. Following that, by applying the same modification in variable as in [28] and the classical Mountain Pass Theorem, Deng et al. in [16] proved the existence of a positive solution for equation (1.1) where nonlinearity was critical growth, and Shi and Chen in [29] proved the existence of positive solutions of equation (1.1) when nonlinearity was periodic or asymptotically periodic cases with critical growth. On the other hand, Candela et al. in [9] considered the more general quasilinear ellptic equation:

$$-\operatorname{div}(A(x,u)|\nabla u|^{p-2}\nabla u) + \frac{1}{p}A_u(x,u)|\nabla u|^p + V(x)|u|^{p-2}u = h(x,u), \quad \text{in } \mathbb{R}^N,$$

with $p > 1$ and $A : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ such that $A_u(x,u) = \frac{\partial A}{\partial u}(x,u)$. When $A(x,u) = g^2(u)$ and $p = 2$, the above equation turns to (1.1) with $N = 3$. They employed an entirely new approach to deal with (1.1) because the arguments of change of variable frequently need $g(u)$ to meet certain particular assumptions, and the features of $g(u)$ directly affect the hypotheses on the nonlinear term $h(x,u)$. By using the Mountain Pass Theorem with the weak Cerami Palais Smale condition, they established the existence of week-bounded solutions under certain appropriate hypotheses on $V(x)$ and $h(x,u)$, which are independent of $g(u)$.

It is worth pointing out that the continuity of nonlinearity is always required in these aforementioned papers. It seems that there are no results concerning equation (1.1) with discontinuous nonlinearities. Actually, many free boundary problems and obstacle problems arising from physics can be described with nonlinear partial differential equations with discontinuous nonlinearities. For more problems with discontinuous nonlinearities, readers can refer to [1,3,6,10,22,26,35] and their references. Hence, our goal is to discuss the existence of positive solutions for problem (1.1) with discontinuous nonlinearities. In this paper, we consider equation (1.1) with $h(x,t) = \kappa f(x,t) + g(t)|G(t)|^{2^*-2}G(t)$, where $\kappa > 0$, $G(t) = \int_0^t g(s)ds$, $2^* = \frac{2N}{N-2}$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a discontinuous function. We rewrite equation (1.1) as follows:

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = \kappa f(x,u) + g(u)|G(u)|^{2^*-2}G(u), \quad \text{in } \mathbb{R}^N. \tag{1.6}$$

The hypotheses on the function V and f are the following:

(V_1) There exist a function $V_p(x) \in C(\mathbb{R}^N, \mathbb{R})$, $\mathbb{Z}^N$-periodic with respect to variable x, satisfying

$$V_p(x) \geq V_0, \qquad \forall x \in \mathbb{R}^N,$$

and a function $W(x) \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap C(\mathbb{R}^N, \mathbb{R})$ with $W(x) \geq 0$ such that

$$V(x) = V_p(x) - W(x) \geq W_0, \qquad \forall x \in \mathbb{R}^N,$$

where V_0, W_0 are positive constants and the inequality $W(x) \geq 0$ is strict on a subset of positive measure in $\mathbb{R}^N$.

(f_1) $f(x,t)$ is a measurable function defined on $\mathbb{R}^N \times \mathbb{R}$ and the functions

$$\underline{f}(x,t) := \lim_{\delta \downarrow 0} \text{ess inf}\{f(x,s); |t-s| < \delta\}$$

and

$$\overline{f}(x,t) := \lim_{\delta \downarrow 0} \text{ess sup}\{f(x,s); |t-s| < \delta\},$$

are N-mensurable (see [11]).

(f_2) $f(x,t) \equiv 0$ if $t \leq 0$ and $\limsup_{t \to 0^+} \frac{f(x,t)}{t} = 0$, uniformly in $x \in \mathbb{R}^N$.

(f_3) There are $C > 0$ and $q \in (2, 2^*)$ such that

$$|f(x,t)| \leq C(1 + g(t)|G(t)|^{q-1}), \qquad \forall (x,t) \in \mathbb{R}^N \times [0, \infty).$$

(f_4) There exists $\theta \in (2, 2^*)$ such that

$$0 \leq \theta g(t)F(x,t) \leq G(t) \min\{f(x,t), \varrho\}, \qquad \forall \varrho \in \partial_t F(x,t) \quad \text{and} \quad \forall (x,t) \in \mathbb{R}^N \times [0,\infty),$$

where $F(x,t) = \int_0^t f(x,s)ds$ and $\partial_t F(x,t) := [\underline{f}(x,t), \overline{f}(x,t)]$ denotes the generalized gradient of $F(x,t)$ with respect to variable t (see [4]).

Here, we provide a nonlinearity f that satisfies the assumptions above as following: fixed $T > 0$, let us consider the function

$$f(x,t) = \begin{cases} 0, & t \in (-\infty, 0], \\ g(t)(G(t))^{q-2}\left[G(t) - \arctan(G(t))\right], & t \in (0, T], \\ g(t)(G(t))^{q-2}\left[G(t) - \mu \arctan(G(t))\right], & t \in (T, +\infty), \end{cases}$$

where $0 < \mu < 1$.

The asymptotic periodicity of f at infinity is given by the following condition:

(f_5) There exists a function $f_p(x,t) \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $\mathbb{Z}^N$-periodic with respect to variable x, such that $f_p(x,t) \equiv 0$ if $t \le 0$ and $\frac{f_p(x,t)}{g(t)G(t)}$ is nondecreasing for all $t > 0$.

(f_6) There exists $\nu \in (2, 2^*)$ such that

$$0 \le \nu g(t) F_p(x,t) \le G(t) f_p(x,t), \qquad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where $F_p(x,t) = \int_0^t f_p(x,s)ds$.

(f_7) $F(x,t) \ge F_p(x,t) = \int_0^t f_p(x,s)ds$, $\forall (x,t) \in \mathbb{R}^N \times [0,\infty)$ and

$$|f(x,t) - f_p(x,t)| \le \pi(x) g(t) |G(t)|^{q-1}, \qquad \forall (x,t) \in \mathbb{R}^N \times [0,\infty),$$

$$|\varrho - f_p(x,t)| \le \pi(x) g(t) |G(t)|^{q-1}, \qquad \forall (x,t) \in \mathbb{R}^N \times [0,\infty), \text{ and } \varrho \in \partial_t F(x,t),$$

where $\pi(x) > 0$ for all $x \in \mathbb{R}^N$, $\pi(x) \in L^\infty(\mathbb{R}^N)$, and $\pi(x) \to 0$ as $|x| \to \infty$.

Next, we provide a suitable example of function $f(x,t)$ that satisfies the assumptions (f_1)–(f_7). Fixed $T > 0$, let

$$f(x,t) = \begin{cases} 0, & t \in (-\infty, 0], \\ f_p(x,t), & t \in (0,T], \\ f_p(x,t) + \mu \exp(-|x|) g(t) (G(t))^{q-2} \arctan(G(t)), & t \in (T, +\infty), \end{cases}$$

where $0 < \mu < 1$ and

$$f_p(x,t) = \begin{cases} 0, & t \in (-\infty, 0], \\ g(t)(G(t))^{q-2} \left[G(t) - \arctan(G(t)) \right], & t \in (0, +\infty). \end{cases}$$

Since $f(x,t)$ is discontinuous, inspired by [11] and [27], we give the definition of solutions for (1.6). We say a function u is a solution for the multivalued problem (1.6) if it satisfies

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u - g(u)|G(u)|^{2^*-2}G(u) \in \kappa \hat{f}(x,u), \quad \text{a.e. in } \mathbb{R}^N, \tag{1.7}$$

where $\hat{f}$ is the multivalued function

$$\hat{f}(x,t) = [\underline{f}(x,t), \overline{f}(x,t)].$$

Below, we describe the main results of this paper.

Theorem 1.1 (The non periodic case)**.** *Assume that (V_1) and (f_1)–(f_7) are satisfied. Then, there exists $\kappa^* > 0$ such that the problem (1.6) possesses a positive solution for all $\kappa \ge \kappa^*$.*

When f is $\mathbb{Z}^N$ periodic and $V = V_p$ given by (V_1), problem (1.6) can turns to the following periodic problem:

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V_p(x)u = \kappa f(x,u) + g(u)|G(u)|^{2^*-2}G(u), \quad \text{in } \mathbb{R}^N. \tag{1.8}$$

For periodic problem (1.8), we may state:

Theorem 1.2 (The periodic case)**.** *Assume that (f_1)–(f_4) are satisfied and f is $\mathbb{Z}^N$ periodic. Then, there exists $\kappa^* > 0$ such that the problem (1.8) possesses a positive solution for all $\kappa \ge \kappa^*$.*

Remark 1.3. As is known to all, the discontinuity of nonlinearity causes a lack of functional differentiability. In this paper, as f is discontinuous, the modified energy functional associated with (1.6) is only locally Lipschitz continuous. The classical variational methods cannot be directly utilized for nonsmooth functionals. For smooth functionals, it is essential that the energy functional can be studied on the Nehari manifold and that the mountain pass level is equal to the minimum of the energy functional on the Nehari manifold. However, these results are not valid for nonsmooth cases. Hence, the proof of the existence of solutions for equation (1.6) is more difficult.

Remark 1.4. Similar equations have been considered in [9]. However, our assumptions on nonliearities are critical growth and discontinuous.

Below we give a sketch of the proofs of our main results:

1) Firstly, we make a change of variable to reduce the quasilinear problem (1.6) to a semilinear problem (2.1) which can be well defined in $H^1(\mathbb{R}^N)$ and satisfies the geometric hypotheses of the Mountain Pass Theorem. Hence, we get a $(PS)_c$ sequence associated with the minimax level. And by using standard arguments, we show that the weak convergence limit of $(PS)_c$ sequence is a solution of the problem (2.1).

2) Furthermore, for adopting the similar technical scheme due to [30], we assume this solution is trivial. Thereby, we get a nontrivial critical point of the functional associated with the periodic case, and use the nontrivial critical point to construct a special path to prove that the maximum of the functional associated with (2.1) is strictly less than the one of the functional associated with the periodic case, which is a contradiction.

3) Hence, we obtain the existence of nontrivial solutions of the problem (2.1). Finally, by Lemma 2.2, Theorem 1.1 is proved.

The outline of the article is as follows: in Section 2, we introduce the variational setting associated with problem (1.6) and some basic knowledge of the critical point theory of locally Lipschitz continuous functionals. In Section 3, we prove the geometric structure of the Mountain Pass Theorem and some preliminary lemmas. In Section 4 and Section 5, we prove Theorem 1.1 and Theorem 1.2, respectively.

Throughout this paper, we make use of the following notations:

- M, C, C_ε denote positive constants, which may vary from line to line.
- $L^p(\mathbb{R}^N)$ denotes the Lebesgue space with the norm $\|\cdot\|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{\frac{1}{p}}$ for $1 \le p \le \infty$.
- The dual space of a Banach space X will be denoted by X^*.
- The strong (respectively, weak) convergence is denoted by $\to$ (respectively, $\rightharpoonup$).
- $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$.
- Denote the function space $D^{1,2}(\mathbb{R}^N) := \{v \in L^{2^*}(\mathbb{R}^N) : |\nabla v| \in L^2(\mathbb{R}^N)\}$. Here, S is the best constant that verifies
$$S\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{\frac{2}{2^*}} \le \int_{\mathbb{R}^N} |\nabla u|^2, \quad \text{for all } u \in D^{1,2}(\mathbb{R}^N).$$
- Denote the function space $H^1(\mathbb{R}^N) = \{v \in L^2(\mathbb{R}^N) : |\nabla v| \in L^2(\mathbb{R}^N)\}$ with the usual norm
$$\|v\|_{H^1}^2 = \int_{\mathbb{R}^N} \left(|\nabla v|^2 + |v|^2\right).$$

2 Variational setting and preliminary knowledge

From the variational point of view, we note that we may not directly apply the variational method to deal with the problem (1.6), since the functional associated with (1.6) may not be well defined in general $H^1(\mathbb{R}^N)$. The first difficulty associated with (1.6) is to find an appropriate function space where the functional responding to (1.6) is well defined. In order to overcome this difficulty, we we change the variables $u = G^{-1}(v)$, where G is defined as

$$v = G(u) = \int_0^u g(t)dt$$

by Shen and Wang in [28].

Now, we present some important properties about the functions g, G and G^{-1}, which proofs can be found in [16].

Lemma 2.1. *The functions $g(s)$ and $G(s) = \int_0^s g(t)dt$ enjoy the following properties.*

(i) *$G(s)$ and $G^{-1}(t)$ are odd and strictly increasing.*

(ii) *For all $s \geq 0$, $t \geq 0$,*

$$G(s) \leq g(s)s, \qquad G^{-1}(t) \leq \frac{t}{g(0)} = t.$$

(iii) *For all $t \geq 0$, $\frac{G^{-1}(t)}{t}$ is nonincreasing and*

$$\lim_{t\to 0} \frac{G^{-1}(t)}{t} = \frac{1}{g(0)} = 1, \qquad \lim_{t\to\infty} \frac{G^{-1}(t)}{t} = \begin{cases} \dfrac{1}{g(\infty)}, & \text{if } g \text{ is bounded,} \\ o(1), & \text{if } g \text{ is unbounded.} \end{cases}$$

(iv) *Denote $T(t) = \frac{G^{-1}(t)}{g(G^{-1}(t))}$, then $t^2T'(t) \leq T(t)t$, $\forall t \in \mathbb{R}$.*

After the change of variable $u = G^{-1}(v)$, the problem(1.6) can be rewritten as follows:

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = \frac{f(x, G^{-1}(v))}{g(G^{-1}(v))} + |v|^{2^*-2}v, \quad \text{in } \mathbb{R}^N. \tag{2.1}$$

As a consequence of Lemma 2.1, the functional associated with (2.1) is well defined in $H^1(\mathbb{R}^N)$.

Lemma 2.2. *From Lemma 2.1, direct calculations demonstrate that $u = G^{-1}(v)$ shall be a solution of the equation (1.6) when v is a solution of the problem (2.1). That is to say, $v \in H^1(\mathbb{R}^N)$ satisfies*

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} - |v|^{2^*-2}v \in \frac{\hat{f}(x, G^{-1}(v))}{g(G^{-1}(v))} \quad \text{a.e. in } \mathbb{R}^N, \tag{2.2}$$

where

$$\frac{\hat{f}(x, G^{-1}(v))}{g(G^{-1}(v))} = \left[\frac{\underline{f}(x, G^{-1}(v))}{g(G^{-1}(v))}, \frac{\overline{f}(x, G^{-1}(v))}{g(G^{-1}(v))}\right].$$

From the above commentaries, in order to find solutions to equation (1.6), it suffices to study the existence of solutions to equation (2.1). The second difficulty associated with (2.1) is that the classical critical theory for smooth functionals cannot be directly applied to (2.1) since the function $f(x, G^{-1}(t))$ is discontinuous. To study nonsmooth problems like (2.1), we will apply the critical point theory of locally Lipschitz continuous functionals developed by Clarke [13]. For the convenience of the readers, here we provide some relevant knowledge of the critical point theory of locally Lipschitz continuous functionals.

Let X be a real Banach space and $I : X \to \mathbb{R}$.

Definition 2.3 ([27]). If given $v \in X$ there exists an open neighborhood $U := U_v \subset X$ and some constant $C_U > 0$ such that

$$|I(v_1) - I(v_2)| \leq C_U \|v_1 - v_2\|_X, \qquad v_i \in U, \ \ i = 1, 2.$$

We call that I is locally Lipschitz continuous ($I \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ for short).

Definition 2.4 ([27]). The generalized directional derivative of $I \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ at v in the direction of $\tilde{v} \in X$ is defined by

$$I^{\circ}(v; \tilde{v}) = \limsup_{u \to 0 \ t \downarrow 0} \frac{I(v + u + t\tilde{v}) - I(v + u)}{t}.$$

The definition 2.4 implies that $I^{\circ}(v; .)$ is continuous, convex and its subdifferential at $w \in X$ is given by

$$\partial I^{\circ}(v; w) = \{\mu \in X^*; I^{\circ}(v; u) \geq I^{\circ}(v; w) + \langle \mu, u - w \rangle, \quad \forall u \in X\},$$

where X^* is the dual of X and $\langle ., . \rangle$ is the duality paring between X^* and X.

Definition 2.5. ([27]) The general gradient of $I \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ at v is the set

$$\partial I(v) = \{\mu \in X^*; I^{\circ}(v; u) \geq \langle \mu, u \rangle, \quad \forall u \in X\}.$$

Since $I^{\circ}(v; 0) = 0$, $\partial I(v)$ is the subdifferential of $I^{\circ}(v; 0)$. Moreover, $\partial I(v) \subset X^*$ is convex, nonempty and weak* compact. If I is C^1 functional, $\partial I(v) = \{I'(v)\}$. We denote by $\lambda_I(u)$ the following real number

$$\lambda_I(v) := \min\{\|\mu\|_{X^*};\ \mu \in \partial I(v)\}.$$

Definition 2.6 ([27]). An element $v \in X$ is a critical point of I if $0 \in \partial I(v)$ or equivalently, when $\lambda_I(v) = 0$.

Lemma 2.7. *If $I_1 \in C^1(X, \mathbb{R})$ and $I_2 \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$, then*

$$\partial(I_1 + I_2)(v) = \{I_1'(v)\} + \partial I_2(v), \qquad \forall v \in X.$$

Lemma 2.8 ([13] and [34]). *Let Y be a Banach space and $j : Y \to X$ be a continuously differentiable function. Then $I \circ j$ is locally Lipschitz and*

$$\partial(I \circ j)(v) \subset \partial I(j(v)) \circ j'(v), \qquad \forall v \in Y.$$

Lemma 2.9 ([13]). *Let $I \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ with $I(0) = 0$ and X be a Banach space. Suppose there are constants α, $\rho > 0$ and function $e \in X$, such that* *(i)* $I(v) \geq \alpha$, *for all* $v \in X$ *with* $\|v\|_X = \rho$,

(ii) $I(e) < 0$ *and* $\|e\|_X > r$.

Let

$$c = \inf_{\gamma \in \Gamma_I} \max_{t \in [0,1]} I(\gamma(t)),$$

where

$$\Gamma_I = \{\gamma \in C([0,1], X) : \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

Then, $c \geq \alpha$ *and there exists a sequence* $\{v_n\} \subset X$ *satisfying* $I(v_n) \to c$ *and* $\lambda_I(v_n) \to 0$. *The sequence* $\{v_n\}$ *is called a* $(PS)_c$ *sequence for* I.

3 Some preliminary lemmas

Hypotheses (f_1)–(f_3) imply that, for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$\begin{aligned} |f(x,t)| &\leq \varepsilon|t| + C_\varepsilon g(t)|G(t)|^{q-1}, \qquad \forall t \in \mathbb{R},\ \forall x \in \mathbb{R}^N, \\ |F(x,t)| &\leq \frac{\varepsilon}{2}|t|^2 + \frac{C_\varepsilon}{q}|G(t)|^q, \qquad \forall t \in \mathbb{R},\ \forall x \in \mathbb{R}^N. \end{aligned} \tag{3.1}$$

From the second inequality of (3.1) and Lemma 2.1-(ii), we can prove

$$\Psi(v) = \int_{\mathbb{R}^N} F(x, G^{-1}(v)) \leq \int_{\mathbb{R}^N} \left(\frac{\varepsilon}{2}|v|^2 + \frac{C_\varepsilon}{q}|v|^q \right) \leq C(\|v\|_2 + \|v\|_q), \tag{3.2}$$

so functional Ψ is well defined in $H^1(\mathbb{R}^N)$. However, in order to apply variational methods for locally Lipschitz functionals, it is preferable to deal with the functional Ψ in a more appropriate space, that is $\Psi : L^\Phi(\mathbb{R}^N) \to \mathbb{R}$, for $\Phi(t) = |t|^2 + |t|^q$, where $L^\Phi(\mathbb{R}^N)$ denotes the Orlicz space associated with the N-function Φ. In this paper, we are working in $\mathbb{R}^N$ and the conditions on f yield

$$|F(x, G^{-1}(t))| \leq C(|t|^2 + |t|^q), \qquad \forall t \in \mathbb{R}, \tag{3.3}$$

then Ψ is not well defined in $L^p(\mathbb{R}^N)$. The above estimate involving the function F suggests that the best space to work is the Orlicz space $L^\Phi(\mathbb{R}^N)$. In bounded domains, the Orlicz space $L^\Phi(\mathbb{R}^N)$ is not necessary. In this case, (3.2) implies that the functional Ψ is well defined in $L^p(\Omega)$. Since $2 < q < 2^*$, we obtain that the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^\Phi(\mathbb{R}^N)$ is continuous and Φ satisfies Δ_2 condition which ensures that $L^\Phi(\mathbb{R}^N)$ and $L^{\tilde{\Phi}}(\mathbb{R}^N)$ are reflexive spaces ($\tilde{\Phi}$ is the conjugate function of Φ (see [17])). Hence, given $\varsigma \in (L^\Phi(\mathbb{R}^N))^*$, we get

$$\varsigma(v) = \int_{\mathbb{R}^N} u_\varsigma v, \qquad \forall v \in L^\Phi(\mathbb{R}^N),$$

for some $u_\varsigma \in L^{\tilde{\Phi}}(\mathbb{R}^N)$. Essentially, by the definition of Φ and (f_1)–(f_3) the conditions below occur:

$$\begin{aligned} |\zeta| &\leq \varepsilon|t| + C_\varepsilon|t|^{q-1} \leq C\Phi'(|t|), \qquad \forall \zeta \in \partial_t F(x, G^{-1}(t)), \\ |F(x, G^{-1}(t))| &\leq \frac{\varepsilon}{2}|t|^2 + \frac{C_\varepsilon}{q}|t|^{q-1} \leq C\Phi(t), \end{aligned} \tag{3.4}$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$. Here, $\partial_t F(x, G^{-1}(t))$ denotes the generalized gradient of $F(x, G^{-1}(t))$ with respect to variable t. The above information involving Ψ and Φ is crucial in the below.

The next two lemmas establish important properties of the functional Ψ given in (3.2).

Lemma 3.1 ([2, Theorem 4.1] and [5, Theorem 4.2])**.** *Assume* (3.4)*. Then, the functional* $\Psi : L^\Phi(\mathbb{R}^N) \to \mathbb{R}$ *given by*

$$\Psi(v) = \int_{\mathbb{R}^N} F(x, G^{-1}(v)), \qquad v \in L^\Phi(\mathbb{R}^N),$$

is well defined and $\Psi \in \mathrm{Lip}_{\mathrm{loc}}(L^\Phi(\mathbb{R}^N), \mathbb{R})$*. Furthermore,*

$$\partial\Psi(v) \subset \partial_t F(x, G^{-1}(v)), \qquad \forall\, v \in L^\Phi(\mathbb{R}^N),$$

in the sense that for every $\varrho^* \in \partial\Psi(v) \subset (L^\Phi(\mathbb{R}^N))^* \cong L^{\tilde{\Phi}}(\mathbb{R}^N)$ *there exists* $\varrho \in L^{\tilde{\Phi}}(\mathbb{R}^N)$ *such that*

$$\varrho(x) \in \partial_t F(x, G^{-1}(v(x))) = \left[\frac{\underline{f}(x, G^{-1}(v(x)))}{g(G^{-1}(v(x)))}, \frac{\overline{f}(x, G^{-1}(v(x)))}{g(G^{-1}(v(x)))} \right] \quad \textit{a.e. in } \mathbb{R}^N$$

and

$$\langle \varrho^*, v \rangle = \int_{\mathbb{R}^N} \varrho v, \qquad \forall\, v \in L^\Phi(\mathbb{R}^N).$$

As a similar consequence of Proposition 2.3 in [5], we obtain the following lemma and the proof will be omitted. More details can be found in [5].

Lemma 3.2. *Assume* (3.4)*. If* $\{v_n\} \subset H^1(\mathbb{R}^N)$ *satisfies* $v_n \rightharpoonup v$ *in* $H^1(\mathbb{R}^N)$ *and* $\varrho_n \in \partial\Psi(v_n)$ *satisfies* $\varrho_n \overset{*}{\rightharpoonup} \varrho$ *in* $(H^1(\mathbb{R}^N))^*$*, then* $\varrho \in \partial\Psi(v)$*.*

We consider $H^1(\mathbb{R}^N)$ endowed with the following norm

$$\|v\|^2 = \int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x)|v|^2 \right).$$

Under the assumption (V_1), the norm $\|\cdot\|$ is equivalent to the standard norm $\|\cdot\|_{H^1}$.

In order to get the positive solutions, we consider the functional corresponding to (2.1) given by $J(v) = Q(v) - \kappa\Psi(v)$, $v \in H^1(\mathbb{R}^N)$, where

$$Q(v) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x)|G^{-1}(v)|^2 \right) - \frac{1}{2^*} \int_{\mathbb{R}^N} (v^+)^{2^*}$$

and

$$\Psi(v) = \int_{\mathbb{R}^N} F(x, G^{-1}(v)).$$

By standard arguments, we get the functional $Q \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and

$$\langle Q'(v), \varphi \rangle = \int_{\mathbb{R}^N} \left(\nabla u \nabla \varphi + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \varphi \right) - \int_{\mathbb{R}^N} (v^+)^{2^*-1} \varphi,$$

for all $v, \varphi \in H^1(\mathbb{R}^N)$. Then, by Lemma 3.1, $J \in \mathrm{Lip}_{\mathrm{loc}}(H^1(\mathbb{R}^N), \mathbb{R})$ and

$$\partial J(v) = \{Q'(v)\} - \kappa \partial\Psi(v). \tag{3.5}$$

Lemma 3.3. *Assume that* (V_1) *and* (f_1)–(f_3) *are satisfied. Then there exist* $\rho, \alpha > 0$*, such that*

$$J(v) \geq \alpha, \qquad \forall v \in H^1(\mathbb{R}^N) \textit{ with } \|v\| = \rho.$$

Proof. Since $\lim_{|t|\to 0}\frac{G^{-1}(t)}{t}=1$, by (3.1), for any $\varepsilon>0$, there is C_ε such that

$$|F(x,G^{-1}(t))|\le \varepsilon|t|^2+C_\varepsilon|t|^q,\ \forall t\in\mathbb{R}.$$

From Lemma 2.1, we have

$$\lim_{s\to+\infty}\frac{G^{-1}(t)}{t}=\begin{cases}\dfrac{1}{g(\infty)}, & \text{if } g \text{ is bounded},\\ o(1), & \text{if } g \text{ is unbounded}.\end{cases}$$

If g is bounded, in view of that $\frac{G^{-1}(t)}{t}$ is nonincreasing, we get

$$\begin{aligned} J(v)&=\frac12\int_{\mathbb{R}^N}\left(|\nabla v|^2+V(x)|G^{-1}(v)|^2\right)-\kappa\int_{\mathbb{R}^N}F(x,G^{-1}(v))-\frac{1}{2^*}\int_{\mathbb{R}^N}(v^+)^{2^*}\\ &\ge\frac12\int_{\mathbb{R}^N}\left(|\nabla v|^2+W_0|G^{-1}(v)|^2\right)-\kappa\int_{\mathbb{R}^N}(\varepsilon|v|^2+C_\varepsilon|v|^q)-\frac{1}{2^*}\int_{\mathbb{R}^N}(v^+)^{2^*}\\ &\ge\frac12\int_{\mathbb{R}^N}|\nabla v|^2+\left(\frac{W_0}{2g(\infty)}-\kappa\varepsilon\right)\int_{\mathbb{R}^N}|v|^2-\kappa C_\varepsilon\int_{\mathbb{R}^N}|v|^q-\frac{1}{2^*}\int_{\mathbb{R}^N}|v|^{2^*}. \end{aligned}\tag{3.6}$$

If g is unbounded, we set $\Upsilon(t):=-\frac12 W_0|G^{-1}(t)|^2+\kappa F(x,G^{-1}(t))$, then

$$\lim_{t\to0}\frac{\Upsilon(t)}{t^2}=-\frac{W_0}{2}<0,\qquad \lim_{t\to+\infty}\frac{\Upsilon(t)}{t^{2^*}}=0.$$

Therefore,

$$\begin{aligned} J(v)&=\frac12\int_{\mathbb{R}^N}\left(|\nabla v|^2+V(x)|G^{-1}(v)|^2\right)-\kappa\int_{\mathbb{R}^N}F(x,G^{-1}(v))-\frac{1}{2^*}\int_{\mathbb{R}^N}(v^+)^{2^*}\\ &\ge\frac12\int_{\mathbb{R}^N}|\nabla v|^2-\int_{\mathbb{R}^N}\Upsilon(v)-\frac{1}{2^*}\int_{\mathbb{R}^N}(v^+)^{2^*}\\ &\ge\frac12\int_{\mathbb{R}^N}|\nabla v|^2+\left(\frac{W_0}{2}-\varepsilon\right)\int_{\mathbb{R}^N}|v|^2-C_\varepsilon\int_{\mathbb{R}^N}|v|^{2^*}-\frac{1}{2^*}\int_{\mathbb{R}^N}|v|^{2^*}. \end{aligned}\tag{3.7}$$

By (3.6), (3.7) and Sobolev's inequality, we get

$$J(v)\ge C\|v\|^2-C\|v\|^q-C\|v\|^{2^*}.$$

Since $2<q<2^*$, taking $\rho>0$ sufficiently small, we conclude that there exists $\alpha>0$ such that

$$J(v)\ge\alpha,\qquad \forall v\in H^1(\mathbb{R}^N)\text{ with }\|v\|=\rho.$$

This proof is completed. □

Lemma 3.4. *Suppose that (V_1) and (f_4) are satisfied. Then, for all $\kappa>0$, there exists function $e\in H^1(\mathbb{R}^N)$ such that $J(e)\le 0$ and $\|e\|>\rho$.*

Proof. Fixing $\phi\in H^1(\mathbb{R}^N)$ with $\phi\ge0$ and $\phi\not\equiv0$, by Lemma 2.1-(i), (ii), we get

$$\begin{aligned} J(t\phi)&=\frac12\int_{\mathbb{R}^N}\left(|t\nabla\phi|^2+V(x)|G^{-1}(t\phi)|^2\right)-\kappa\int_{\mathbb{R}^N}F(x,G^{-1}(t\phi))-\frac{1}{2^*}\int_{\mathbb{R}^N}(t\phi)^{2^*}\\ &\le\frac{t^2}{2}\|\phi\|^2-\frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N}\phi^{2^*}. \end{aligned}$$

Then, we can choose some t_0 large enough such that $\|t_0\phi\|>\rho$ and $J(t_0\phi)<0$. The lemma is completed when $e=t_0\phi$. □

Note that $J(0) = 0$ and by Lemma 3.3 and Lemma 3.4, (i) and (ii) of Lemma 2.9 are satisfied. Thereby, we may define

$$c_\kappa = \inf_{\gamma \in \Gamma_J} \max_{t\in[0,1]} J(\gamma(t)),$$

where

$$\Gamma_J = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, J(\gamma(1)) < 0\}.$$

By Lemma 2.9, there exists a sequence $\{v_n\} \subset H^1(\mathbb{R}^N)$ satisfying $J(v_n) \to c_\kappa$ and $\lambda_J(v_n) \to 0$. Namely, the sequence $\{v_n\}$ is a $(PS)_{c_\kappa}$ sequence for functional J.

Lemma 3.5. *Assume that (V_1), (f_1) and (f_4) hold. Then any $(PS)_{c_\kappa}$ sequence for J is bounded in $H^1(\mathbb{R}^N)$.*

Proof. Let $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a $(PS)_{c_\kappa}$ sequence for J, that is,

$$J(v_n) \to c_\kappa \quad \text{and} \quad \lambda_J(v_n) \to 0, \quad \text{as } n \to \infty.$$

Then, there exists $w_n \in \partial J(v_n) \subset (H^1(\mathbb{R}^N))^*$ such that

$$\|w_n\|_* = \lambda_J(v_n) = o_n(1)$$

and

$$w_n = Q'(v_n) - \varrho_n,$$

where $\|w_n\|_* := \|w_n\|_{(H^1(\mathbb{R}^N))^*}$ and $\varrho_n \in \partial\Psi(v_n) \subset L^{\tilde{\Phi}}(\mathbb{R}^N)$.

Therefore, we obtain that

$$\begin{aligned} c + 1 + \|v_n\| &\geq J(v_n) - \frac{1}{\theta}\langle w_n, v_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} |\nabla v_n|^2 + \int_{\mathbb{R}^N} V(x)G^{-1}(v_n)\left(\frac{1}{2}G^{-1}(v_n) - \frac{1}{\theta}\frac{v_n}{g(G^{-1}(v_n))}\right) \\ &\quad - \kappa\int_{\mathbb{R}^N}\left(F(x, G^{-1}(v_n)) - \frac{1}{\theta}\varrho_n v_n\right) - \left(\frac{1}{2^*} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} |v_n^+|^{2^*}. \end{aligned}$$

From (f_4) and $\varrho_n(x) \in \left[\frac{\underline{f}(x,G^{-1}(v_n(x)))}{g(G^{-1}(v_n(x)))}, \frac{\overline{f}(x,G^{-1}(v_n(x)))}{g(G^{-1}(v_n(x)))}\right]$ a.e. in $\mathbb{R}^N$, we have

$$\frac{1}{\theta}\varrho_n v_n \geq F(x, G^{-1}(v_n)) \geq 0 \quad \text{a.e. in } \mathbb{R}^N.$$

Hence, by Lemma 2.1-(ii), we get

$$c + 1 + \|v_n\| \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2\right). \tag{3.8}$$

For all $t \geq 1$, by (f_4), we can verify that there exists some $C > 0$ such that $CF(x,t) \geq (G(t))^\theta \geq (G(t))^2$. Then

$$\begin{aligned} \int_{\{|G^{-1}(v_n)|>1\}} V(x)v_n^2 &\leq \kappa C\int_{\{|G^{-1}(v_n)|>1\}} F(x, G^{-1}(v_n)) \\ &\leq \kappa C\int_{\mathbb{R}^N} F(x, G^{-1}(v_n)) + \frac{C}{2^*}\int_{\mathbb{R}^N} |v_n^+|^{2^*} \\ &= C\left[\frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2\right) - J(v_n)\right] \\ &= C\left[\frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2\right) - c_\kappa + o_n(1)\right]. \end{aligned} \tag{3.9}$$

For $\{|G^{-1}(v_n)| \leq 1\}$, by Lemma 2.1-(ii) and $g'(t) \geq 0$ for all $t \geq 0$, we have

$$\begin{aligned}\frac{1}{g^2(1)}\int_{\{|G^{-1}(v_n)|\leq 1\}} V(x)v_n^2 &\leq \int_{\{|G^{-1}(v_n)|\leq 1\}} V(x)|G^{-1}(v_n)|^2 \\ &\leq \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2.\end{aligned} \tag{3.10}$$

By (3.8)–(3.10), we deduce that

$$\begin{aligned}\|v_n\|^2 &= \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)|v_n|^2\right) \\ &\leq C\int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2\right) + C \\ &\leq C\|v_n\| + C,\end{aligned}$$

which implies that the sequence $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. □

Next, the following lemma shows the behavior of c_κ associated with the parameter κ.

Lemma 3.6. *Suppose that (V_1) and (f_4) are satisfied, then $\lim_{\kappa\to+\infty} c_\kappa = 0$.*

Proof. Since $J(v)$ is nonsmooth functional, unlike the method used to prove Lemma 3.1 in [29], we will not use the Nehari manifold. For ϕ given by Lemma 3.4, it follows that there is $t_\kappa > 0$ satisfying

$$J(t_\kappa\phi) = \max_{t\geq 0} J(t\phi) \geq \alpha > 0.$$

Then, we have

$$\frac{t_\kappa^2}{2}\int_{\mathbb{R}^N} |\nabla\phi|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(t_\kappa\phi)|^2 \geq \kappa\int_{\mathbb{R}^N} F(x, G^{-1}(t_\kappa\phi)) + \frac{t_\kappa^{2^*}}{2^*}\int_{\mathbb{R}^N} \phi^{2^*}.$$

By (f_4), we get

$$\frac{t_\kappa^2}{2}\left(\int_{\mathbb{R}^N} |\nabla\phi|^2 + \int_{\mathbb{R}^N} V(x)|\phi|^2\right) \geq \frac{t_\kappa^{2^*}}{2^*}\int_{\mathbb{R}^N} \phi^{2^*},$$

which implies that t_κ is bounded.

Next, we will prove that $t_\kappa \to 0$ as $\kappa \to +\infty$. Suppose, by contradiction, that there exists a sequence $\kappa_n \to +\infty$ and a constant $\bar{t} > 0$ such that $t_{\kappa_n} \to \bar{t}$ as $n \to \infty$. The boundedness of t_{κ_n} implies that there is $M > 0$ such that

$$\frac{t_{\kappa_n}^2}{2}\int_{\mathbb{R}^N} |\nabla\phi|^2 + \int_{\mathbb{R}^N} V(x)|G^{-1}(t_{\kappa_n}\phi)|^2 \leq M.$$

Hence,

$$\kappa_n\int_{\mathbb{R}^N} F(x, G^{-1}(t_{\kappa_n}\phi)) + \frac{t_{\kappa_n}^{2^*}}{2^*}\int_{\mathbb{R}^N} \phi^{2^*} \leq M.$$

If $\bar{t} > 0$, we have that

$$\lim_{n\to\infty}\left[\kappa_n\int_{\mathbb{R}^N} F(x, G^{-1}(t_{\kappa_n}\phi)) + \frac{t_{\kappa_n}^{2^*}}{2^*}\int_{\mathbb{R}^N} \phi^{2^*}\right] = +\infty$$

which is absurd. Thus, we have $t_\kappa \to 0$ as $\kappa \to +\infty$.

Observe that

$$J(t_\kappa\phi) \leq \frac{t_\kappa^2}{2}\int_{\mathbb{R}^N} |\nabla\phi|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(t_\kappa\phi)|^2 \leq \frac{t_\kappa^2}{2}\|\phi\|^2.$$

Due to $t_\kappa \to 0$ as $\kappa \to +\infty$, we get $c_\kappa \leq J(t_\kappa\phi) \to 0$ as $\kappa \to +\infty$, which finishes the proof. □

Lemma 3.7. *Suppose that (V_1) and (f_1)–(f_3) are satisfied. Let $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a $(PS)_{c_\kappa}$ sequence for J with $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$. Then there is $\kappa^* > 0$. When $\kappa > \kappa^*$, there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ and $\delta > 0$ such that*

$$\limsup_{n\to\infty} \int_{B_1(y_n)} |v_n|^2 \geq \delta > 0.$$

Proof. By Lemma 3.5, there exists a constant $\kappa^* > 0$ satisfying

$$c_\kappa < \frac{1}{N} S^{\frac{N}{2}},$$

for all $\kappa > \kappa^*$. Suppose, by contradiction, that $\{v_n\}$ is vanishing. Then, from Lions compactness lemma [33], we deduce that $v_n \to 0$ in $L^r(\mathbb{R}^N)$ for all $2 < r < 2^*$. From $|G^{-1}(v_n)| \leq |v_n|$, we get $G^{-1}(v_n) \to 0$ in $L^r(\mathbb{R}^N)$ for all $2 < r < 2^*$. Since $\{v_n\}$ is a $(PS)_{c_\kappa}$ sequence for J, there exists $w_n \in \partial J(v_n)$ with $\|w_n\|_* = \lambda_J(v_n) = o_n(1)$ and $w_n = Q'(v_n) - \varrho_n$, where $\varrho_n \in \partial\Psi(v_n)$. By (3.1) and Lemma 3.1, we have

$$\int_{\mathbb{R}^N} F(x, G^{-1}(v_n)) \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \varrho_n v_n \to 0, \quad \text{as } n \to \infty. \tag{3.11}$$

Therefore, by (3.11), we have

$$\begin{aligned} c_\kappa + o_n(1) &= J(v_n) - \frac{1}{2}\langle w_n, v_n\rangle \\ &= \frac{1}{2}\int_{\mathbb{R}^N} V(x)\left[|G^{-1}(v_n)|^2 - \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n\right] + \frac{1}{N}\int_{\mathbb{R}^N} |v_n^+|^{2^*}. \end{aligned} \tag{3.12}$$

We claim that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)\left[|G^{-1}(v_n)|^2 - \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n\right] = 0. \tag{3.13}$$

For proving (3.13), we only verify that

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)\left[|G^{-1}(v_n)|^2 - |v_n|^2\right] &= 0, \\ \lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)\left[|v_n|^2 - \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n\right] &= 0. \end{aligned} \tag{3.14}$$

For $\delta > 0$ to be chosen later, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} V(x)\left[|G^{-1}(v_n)|^2 - |v_n|^2\right] \\ &\quad = \int_{\{|v_n|>\delta\}} V(x)\left[|G^{-1}(v_n)|^2 - |v_n|^2\right] + \int_{\{|v_n|\leq\delta\}} V(x)\left[|G^{-1}(v_n)|^2 - |v_n|^2\right]. \end{aligned}$$

By Lemma 2.1-(ii) and (V_1), we get

$$\begin{aligned} \int_{\{|v_n|>\delta\}} V(x)\left[|G^{-1}(v_n)|^2 - |v_n|^2\right] &\leq 2\|V\|_\infty \int_{\{|v_n|>\delta\}} |v_n|^2 \\ &\leq \frac{2\|V\|_\infty}{\delta^{r-2}} \int_{\mathbb{R}^N} |v_n|^r = o_n(1), \end{aligned} \tag{3.15}$$

where $2 < r < 2^*$.

On the other hand, given $\varepsilon > 0$, by Lemma 2.1-(iii), we choose $\delta > 0$ so that

$$\left|\left(\frac{G^{-1}(s)}{s}\right)^2 - 1\right| < \varepsilon, \quad \text{if } |s| \leq \delta.$$

Then, we have

$$\begin{aligned}&\limsup_{n\to\infty}\int_{\{0<|v_n|\le\delta\}}V(x)|v_n|^2\left|\left(\frac{G^{-1}(v_n)}{v_n}\right)^2-1\right|\\&\le\|V\|_\infty\limsup_{n\to\infty}\int_{\{0<|v_n|\le\delta\}}|v_n|^2\left|\left(\frac{G^{-1}(v_n)}{v_n}\right)^2-1\right|\\&\le\varepsilon\limsup_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^2.\end{aligned}$$

Hence

$$\begin{aligned}&\limsup_{n\to\infty}\int_{\mathbb{R}^N}V(x)\left[|G^{-1}(v_n)|^2-|v_n|^2\right]\\&\quad\le\limsup_{n\to\infty}\int_{\{|v_n|>\delta\}}V(x)\left[|G^{-1}(v_n)|^2-|v_n|^2\right]+\varepsilon\limsup_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^2.\end{aligned}$$

As $\varepsilon>0$ is arbitrary and $\{v_n\}\subset H^1(\mathbb{R}^N)$ is bounded, using (3.15), we have the first limit in (3.14).

By Lemma 2.1-(ii), (iii) and the fact that

$$(G^{-1})'(s)=\frac{1}{g(G^{-1}(s))}\to 1\quad\text{as } s\to 0,$$

the verification of the second limit in (3.14) is similar to the first one. Therefore, our claim (3.13) is true.

Then, by (3.12) and (3.13), we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n^+|^{2^*}=Nc_\kappa. \tag{3.16}$$

From the fact $\langle w_n,v_n\rangle=o_n(1)$ and the second limit in (3.11), we get

$$\int_{\mathbb{R}^N}|\nabla v_n|^2+\int_{\mathbb{R}^N}V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n-\int_{\mathbb{R}^N}|v_n^+|^{2^*}=o_n(1). \tag{3.17}$$

From the definition of $G^{-1}(s)$ and (V_1), the second integral in (3.17) is nonnegative. Then, we have

$$\int_{\mathbb{R}^N}|\nabla v_n|^2\le\int_{\mathbb{R}^N}|v_n^+|^{2^*}+o_n(1). \tag{3.18}$$

By the definition of S, (3.16) and (3.18), it follows that

$$\int_{\mathbb{R}^N}|v_n^+|^{2^*}\le\int_{\mathbb{R}^N}|v_n|^{2^*}\le S^{-\frac{2^*}{2}}\left(\int_{\mathbb{R}^N}|\nabla v_n|^2\right)^{\frac{2^*}{2}}\le S^{-\frac{2^*}{2}}\left(\int_{\mathbb{R}^N}|v_n^+|^{2^*}+o_n(1)\right)^{\frac{2^*}{2}}.$$

Taking $n\to\infty$ in the above inequality, in view of (3.16), we get

$$Nc_\kappa\le S^{-\frac{2^*}{2}}(Nc_\kappa)^{\frac{2^*}{2}},$$

that is,

$$c_\kappa\ge\frac{1}{N}S^{\frac{N}{2}},$$

which is a contradiction. Hence $\{v_n\}$ is non-vanishing. This concludes the proof. □

4 Proof of Theorem 1.1

In the following, we will prove that there exists $v \in H^1(\mathbb{R}^N)$ is a positive solution of problem (1.6). With this aim in mind, we need to show that there is $v \in H^1(\mathbb{R}^N)$ and $v > 0$ such that

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} - v^{2^*-1} \in \left[\frac{\underline{f}(x, G^{-1}(v))}{g(G^{-1}(v))}, \frac{\overline{f}(x, G^{-1}(v))}{g(G^{-1}(v))}\right] \quad \text{a.e. in } \mathbb{R}^N.$$

By Lemma 3.3 and Lemma 3.4, the functional J satisfies all hypotheses of Lemma 2.9. Then, by Lemma 2.9 and Lemma 3.5 , there exists a bounded sequence $\{v_n\} \subset H^1(\mathbb{R}^N)$ satisfying

$$J(v_n) \to c_\kappa \geq \alpha > 0 \quad \text{and} \quad \lambda_J(v_n) \to 0, \quad \text{as } n \to \infty,$$

where

$$c_\kappa = \inf_{\gamma \in \Gamma_J} \max_{t \in [0,1]} J(\gamma(t)),$$

and

$$\Gamma_J = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, J(\gamma(1)) < 0\}.$$

Therefore, there exists $w_n \in \partial J(v_n)$ such that $\|w_n\|_* = \lambda_J(v_n)$, $w_n = Q'(v_n) - \varrho_n$ where $\varrho_n \in \partial\Psi(v_n)$. For all $\psi \in H^1(\mathbb{R}^N)$,

$$\langle w_n, \psi\rangle = \langle Q'(v_n), \psi\rangle - \langle \varrho_n, \psi\rangle, \qquad \forall n \in \mathbb{N}.$$

Since $H^1(\mathbb{R}^N)$ is reflexive, taking a subsequence if necessary, there exists $v \in H^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$. Thus, we obtain

$$\langle \varrho_n, \psi\rangle = \langle Q'(v_n), \psi\rangle - \langle w_n, \psi\rangle \to \langle Q'(v), \psi\rangle, \quad \text{as } n \to \infty,$$

that is, $\varrho_n \overset{*}{\rightharpoonup} Q'(v)$ in $(H^1(\mathbb{R}^N))^*$. By Lemma 3.2, we get $Q'(v) \in \partial\Psi(v)$. Then, there exits $\varrho \in \partial\Psi(v)$ such that $Q'(v) = \varrho$ and

$$\int_{\mathbb{R}^N} \left(\nabla v \nabla \psi + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\psi\right) - \int_{\mathbb{R}^N} (v^+)^{2^*-1}\psi = \int_{\mathbb{R}^N} \varrho\psi, \qquad \forall \psi \in H^1(\mathbb{R}^N),$$

where

$$\varrho(x) \in \left[\frac{\underline{f}(x, G^{-1}(v(x)))}{g(G^{-1}(v(x)))}, \frac{\overline{f}(x, G^{-1}(v(x)))}{g(G^{-1}(v(x)))}\right] \quad \text{a.e. in } \mathbb{R}^N.$$

Taking $\psi = v^- := \min\{v, 0\}$, we obtain

$$\int_{\mathbb{R}^N} \left(|\nabla v^-|^2 + V(x)\frac{G^{-1}(v^-)}{g(G^{-1}(v^-))}v^-\right) \leq 0,$$

which implies $v^- \equiv 0$. Thus, we get $v = v^+ \geq 0$ satisfying

$$\begin{cases} -\Delta v + V(x)\dfrac{G^{-1}(v)}{g(G^{-1}(v))} = \varrho + v^{2^*-1} & \text{in } \mathbb{R}^N, \\ v \in H^1(\mathbb{R}^N). \end{cases}$$

Furthermore, since $\varrho \in L^{\tilde{\Psi}}(\mathbb{R}^N) \subset L^{\frac{q}{q-1}}_{\text{loc}}(\mathbb{R}^N)$, the elliptic regularity theory gives that $v \in W^{2,\frac{2^*}{2^*-1}}_{\text{loc}}(\mathbb{R}^N)$ and v satisfies

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = \varrho + v^{2^*-1} \quad \text{a.e. in } \mathbb{R}^N,$$

that is,

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} - v^{2^*-1} \in \left[\frac{\underline{f}(x, G^{-1}(v))}{g(G^{-1}(v))}, \frac{\overline{f}(x, G^{-1}(v))}{g(G^{-1}(v))}\right] \quad \text{a.e. in } \mathbb{R}^N.$$

Finally, in order to prove Theorem 1.1, it suffices to verify that v is nontrivial. Suppose, by contradiction, that v is trivial. Then, we claim that in this case $\{v_n\}$ is also a $(PS)_{c_\kappa}$ sequence for J_p defined by

$$J_p(v) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 + V_p(x)|G^{-1}(v)|^2\right) - \frac{1}{2^*}\int_{\mathbb{R}^N}(v^+)^{2^*} - \int_{\mathbb{R}^N} F_p(x, G^{-1}(v)),$$

for $v \in H^1(\mathbb{R}^N)$ and J_p possesses a nontrivial critical point. It is well known that $J_p \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$, with

$$\langle J_p'(v), \varphi\rangle = \int_{\mathbb{R}^N}\left(\nabla u \nabla\varphi + V_p(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\varphi\right) - \int_{\mathbb{R}^N}(v^+)^{2^*-1}\varphi - \int_{\mathbb{R}^N}\frac{f_p(x, G^{-1}(v))}{g(G^{-1}(v))}\varphi,$$

for all $\varphi \in H^1(\mathbb{R}^N)$.

Lemma 4.1. *If $\{v_n\}$ is given by the above, then*

$$\varrho_n - \Psi_p'(v_n) \to 0 \quad \textit{and} \quad \Psi(v_n) - \Psi_p(v_n) \to 0,$$

where

$$\Psi(v) = \int_{\mathbb{R}^N} F(x, G^{-1}(v)) \quad \textit{and} \quad \Psi_p(v) = \int_{\mathbb{R}^N} F_p(x, G^{-1}(v)).$$

Proof. For any $\varphi \in H^1(\mathbb{R}^N)$ with $\|\varphi\| \leq 1$, by (f_7), we obtain

$$\begin{aligned}\left|\langle \varrho_n - \Psi_p'(v_n), \varphi\rangle\right| &\leq \int_{\mathbb{R}^N}\left|\varrho_n - \frac{f_p(x, G^{-1}(v_n))}{g(G^{-1}(v_n))}\right||\varphi| \\ &\leq \int_{\mathbb{R}^N}\pi(x)|v_n|^{q-1}|\varphi| \\ &\leq \left(\int_{\mathbb{R}^N}|\pi(x)|^{\frac{q}{q-1}}|v_n|^q\right)^{\frac{q-1}{q}}\|\varphi\|_q \\ &\leq C\left(\int_{\mathbb{R}^N}|\pi(x)|^{\frac{q}{q-1}}|v_n|^q\right)^{\frac{q-1}{q}}.\end{aligned}$$

Since $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, there is $M > 0$ with $\|v_n\|_q \leq M$ for all $n \in \mathbb{N}$. Using the fact that $\pi(x) \to 0$ as $|x| \to +\infty$, given $\varepsilon > 0$ there is $R_\varepsilon > 0$ such that $|\pi(x)| \leq \varepsilon$ for $|x| > R_\varepsilon$. Since $H^1(B_{R_\varepsilon}(0)) \hookrightarrow L^q(B_{R_\varepsilon}(0))$ is compact, we have $v_n \to 0$ in $L^q(B_{R_\varepsilon}(0))$. Thus, there is $n_0 \in \mathbb{N}$ satisfying $\|v_n\|_{L^q(B_{R_\varepsilon}(0))} \leq \varepsilon$, for all $n \geq n_0$, and so,

$$\begin{aligned}\int_{\mathbb{R}^N}|\pi(x)|^{\frac{q}{q-1}}|v_n|^q &= \int_{B_{R_\varepsilon}(0)}|\pi(x)|^{\frac{q}{q-1}}|v_n|^q + \int_{B^c_{R_\varepsilon}(0)}|\pi(x)|^{\frac{q}{q-1}}|v_n|^q \\ &\leq \|\pi(x)\|_\infty^{\frac{q}{q-1}}\int_{B_{R_\varepsilon}(0)}|v_n|^q + \varepsilon^{\frac{q}{q-1}}\int_{\mathbb{R}^N}|v_n|^q \\ &\leq \varepsilon^q\|\pi(x)\|_\infty^{\frac{q}{q-1}} + \varepsilon^{\frac{q}{q-1}}M^q.\end{aligned}$$

As ε is arbitrary,

$$\varrho_n - \Psi_p'(v_n) \to 0 \quad \text{in } (H^1(\mathbb{R}^N))^*.$$

A similar argument guarantees that

$$\Psi(v_n) - \Psi_p(v_n) \to 0 \quad \text{in } \mathbb{R}. \qquad \square$$

Since $W(x) \in L^{\frac{N}{2}}(\mathbb{R}^N)$ and $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, we can conclude

$$\int_{\mathbb{R}^N} W(x)|G^{-1}(v_n)|^2 \le \int_{\mathbb{R}^N} W(x)|v_n|^2 \to 0. \tag{4.1}$$

From Lemma 4.1 and (4.1), we deduce

$$\begin{aligned} |J(v_n) - J_p(v_n)| &= \left| \frac{1}{2}\int_{\mathbb{R}^N} W(x)|G^{-1}(v_n)|^2 + \kappa \int_{\mathbb{R}^N} F(x, G^{-1}(v_n)) - F_p(x, G^{-1}(v_n)) \right| \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} W(x)|G^{-1}(v_n)|^2 + \kappa \left| \int_{\mathbb{R}^N} F(x, G^{-1}(v_n)) - F_p(x, G^{-1}(v_n) \right| \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} W(x)|v_n|^2 + \kappa \left|\Psi(v_n) - \Psi_p(v_n)\right| \\ &= o_n(1), \end{aligned} \tag{4.2}$$

which shows that $J_p(v_n) \to c_\kappa$ as $n \to \infty$.

On the other hand, note that $w_n = Q'(v_n) - \varrho_n$ and $\|w_n\|_* = \lambda_J(v_n) = o_n(1)$, where $\varrho_n \in \partial\Psi(v_n)$. From Lemma 4.1 and (4.1), for $\varphi \in H^1(\mathbb{R}^N)$ with $\|\varphi\| \le 1$, we obtain

$$\begin{aligned} &\left| \langle w_n, \varphi\rangle - \langle J_p'(v_n), \varphi\rangle \right| \\ &= \left| \int_{\mathbb{R}^N} W(x) \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\varphi + \kappa \int_{\mathbb{R}^N} \left(\varrho_n \varphi - \frac{f_p(x, G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi \right) \right| \\ &\le \int_{\mathbb{R}^N} W(x) \left| \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} \right| |\varphi| + \kappa \left| \int_{\mathbb{R}^N} \left(\varrho_n \varphi - \frac{f_p(x, G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi \right) \right| \\ &\le \left(\int_{\mathbb{R}^N} W(x)|v_n|^2 \right)^{\frac{1}{2}} \|w\|_{\frac{N}{2}} \|\varphi\|_{2^*}^{\frac{1}{2}} + \kappa \left| \langle \varrho_n - \Psi_p'(v_n), \varphi\rangle \right| \\ &= o_n(1), \end{aligned} \tag{4.3}$$

which shows that $J_p'(v_n) \to 0$, as $n \to \infty$. Thus, by (4.2) and (4.3), $\{v_n\}$ is a $(PS)_{c_\kappa}$ sequence for J_p.

As we suppose that v is trivial, by Lemma 3.7, there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ and $\delta > 0$ such that

$$\limsup_{n\to\infty} \int_{B_1(y_n)} |v_n|^2 \ge \delta > 0.$$

So, we can find a sequence $\{z_n\} \subset \mathbb{Z}^N$ such that $|z_n - y_n| < \sqrt{N}$ for all $n \in \mathbb{N}$ and

$$\limsup_{n\to\infty} \int_{B_{1+\sqrt{N}}(z_n)} |v_n|^2 \ge \limsup_{n\to\infty} \int_{B_1(y_n)} |v_n|^2 \ge \delta > 0. \tag{4.4}$$

Since $v_n \to v$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for all $s \in [1, 2^*)$ and $v = 0$, we may suppose that $|z_n| \to \infty$ up to a subsequence. Denote $\hat{v}_n(x) = v_n(x + z_n)$. Since $\{v_n\}$ is a $(PS)_{c_\kappa}$ sequence for J_p, in view of the periodicities of V_p and f_p, $\{\hat{v}_n\}$ is also a $(PS)_{c_\kappa}$ sequence for J_p. As $\{v_n\}$ is bounded in

$H^1(\mathbb{R}^N)$, it follows that $\{\hat{v}_n\}$ is also bounded in $H^1(\mathbb{R}^N)$. Without loss of generality, we may suppose that

$$\begin{cases} \hat{v}_n \rightharpoonup \hat{v} & \text{in } H^1(\mathbb{R}^N), \\ \hat{v}_n \to \hat{v} & \text{in } L^r_{\text{loc}}(\mathbb{R}^N),\ \forall\, r \in [1, 2^*), \\ \hat{v}_n \to \hat{v} & \text{a.e. in } \mathbb{R}^N, \end{cases}$$

then $J_p'(\hat{v}) = 0$. By (4.4), going to a subsequence if necessary, there exists $n_1 \in \mathbb{N}$ such that

$$\int_{B_{1+\sqrt{N}}(z_n)} |v_n|^2 \geq \frac{\delta}{2} > 0, \qquad \forall\, n \geq n_1.$$

Since $\hat{v}_n(x) = v_n(x + z_n)$ and $\hat{v}_n \to \hat{v}$ in $L^2_{\text{loc}}(\mathbb{R}^N)$, we get

$$\int_{B_{1+\sqrt{N}}(0)} |\hat{v}|^2 = \lim_{n\to\infty} \int_{B_{1+\sqrt{N}}(0)} |\hat{v}_n|^2 = \lim_{n\to\infty} \int_{B_{1+\sqrt{N}}(z_n)} |v_n|^2 \geq \frac{\delta}{2} > 0,$$

which shows $\hat{v} \not\equiv 0$. Besides,

$$0 = \langle J_p'(\hat{v}), \hat{v}^- \rangle = \int_{\mathbb{R}^N} \left(|\nabla \hat{v}^-|^2 + V(x) \frac{G^{-1}(\hat{v}^-)}{g(G^{-1}(\hat{v}^-))} \hat{v}^- \right),$$

which implies $\hat{v} = \hat{v}^+ \geq 0$. Thus, by Fatou's Lemma and (f_6), we have

$$\begin{aligned} c_\kappa &= \limsup_{n\to\infty} \left[J_p(\hat{v}_n) - \frac{1}{2} \langle J_p'(\hat{v}_n), \hat{v}_n \rangle \right] \\ &= \frac{1}{2} \limsup_{n\to\infty} \int_{\mathbb{R}^N} V_p(x) \left[|G^{-1}(\hat{v}_n)|^2 - \frac{G^{-1}(\hat{v}_n)}{g(G^{-1}(\hat{v}_n))} \hat{v}_n \right] + \frac{1}{N} \limsup_{n\to\infty} \int_{\mathbb{R}^N} |\hat{v}_n^+|^{2^*} \\ &\quad - \kappa \limsup_{n\to\infty} \int_{\mathbb{R}^N} \left[F_p(x, G^{-1}(\hat{v}_n)) - \frac{1}{2} \frac{f_p(x, G^{-1}(\hat{v}_n))}{g(G^{-1}(\hat{v}_n))} \hat{v}_n \right] \\ &\geq \frac{1}{2} \int_{\mathbb{R}^N} V_p(x) \left[|G^{-1}(\hat{v})|^2 - \frac{G^{-1}(\hat{v})}{g(G^{-1}(\hat{v}))} \hat{v} \right] + \frac{1}{N} \int_{\mathbb{R}^N} |\hat{v}^+|^{2^*} \\ &\quad - \kappa \int_{\mathbb{R}^N} \left[F_p(x, G^{-1}(\hat{v})) - \frac{1}{2} \frac{f_p(x, G^{-1}(\hat{v}))}{g(G^{-1}(\hat{v}))} \hat{v} \right] \\ &= J_p(\hat{v}) - \frac{1}{2} \langle J_p'(\hat{v}), \hat{v} \rangle = J_p(\hat{v}). \end{aligned}$$

Thus, $\hat{v}$ is a nontrivial critical point of J_p and $J_p(\hat{v}) \leq c_\kappa$.

Claim 4.2. $\hat{v} > 0$ in $\mathbb{R}^N$.

For proving the result, we adapt the same ideas used in [30]. Since $\hat{v}$ is a critical point of J_p (namely $J_p'(\hat{v}) = 0$), $\hat{v}$ is a weak solution of the following equation

$$-\Delta \hat{v} = \zeta, \quad \text{a.e. in } \mathbb{R}^N, \tag{4.5}$$

where

$$\zeta(x, \hat{v}) = \hat{v}^{2^*-1} + \kappa \frac{f_p(x, G^{-1}(\hat{v}))}{g(G^{-1}(\hat{v}))} - V_p(x) \frac{G^{-1}(\hat{v})}{g(G^{-1}(\hat{v}))}.$$

From the conditions (V), (f_7), (3.1) and the Lemma 2.1-(ii), we get

$$\begin{aligned} &\left| \hat{v}^{2^*-1} + \kappa \frac{f_p(x, G^{-1}(\hat{v}))}{g(G^{-1}(\hat{v}))} - V_p(x) \frac{G^{-1}(\hat{v})}{g(G^{-1}(\hat{v}))} \right| \\ &\quad \leq |\hat{v}|^{2^*-1} + \kappa \left(\varepsilon |\hat{v}| + C_\varepsilon |\hat{v}|^{q-1} + \pi(x) |\hat{v}|^{q-1} \right) + V_p(x) |\hat{v}| \\ &\quad \leq C \left(|\hat{v}|^{2^*-1} + |\hat{v}|^{q-1} + |\hat{v}| \right) \\ &\quad \leq C \left(|\hat{v}|^{2^*-1} + 1 \right). \end{aligned} \tag{4.6}$$

Using a result concluded by Brézis–Kato (see [32]), it yields that $\zeta(x, \hat{v}) \in L^r(B_R(0))$ for every $r \in [1, +\infty)$, with $R > 0$ arbitrary. By standard elliptic regularity theory, we get that $\hat{v} \in W^{2,r}(B_R(0))$. So, there exits some $\sigma \in (0, 1)$ such that $\hat{v} \in C^{1,\sigma}_{\text{loc}}(\mathbb{R}^N)$.

Arguing by contradiction, we assume that there exits $x_0 \in \mathbb{R}^N$ such that $\hat{v}(x_0) = 0$. Meanwhile, we have

$$\begin{aligned} -\Delta \hat{v}(x) + b(x) \hat{v}(x) = V_p(x) &\left(\frac{\hat{v}(x)}{g(G^{-1}(\hat{v}(x)))} - \frac{G^{-1}(\hat{v}(x))}{g(G^{-1}(\hat{v}(x)))} \right) \\ &+ \hat{v}^{2^*-1}(x) + \kappa \frac{f_p(x, G^{-1}(\hat{v}(x)))}{g(G^{-1}(\hat{v}(x)))}, \end{aligned} \tag{4.7}$$

where $b(x) := \frac{V_p(x)}{g(G^{-1}(\hat{v}(x)))} \geq 0$, for $x \in \mathbb{R}^N$. Combining Lemma 2.1-(i) and (ii), we get $\frac{\hat{v}(x)}{g(G^{-1}(\hat{v}(x)))} - \frac{G^{-1}(\hat{v}(x))}{g(G^{-1}(\hat{v}(x)))} \geq 0$. By the hypotheses of $V_p(x)$, we know $-\Delta \hat{v}(x) + b(x)\hat{v}(x) \geq 0$. In view of (V_1), $b(x)$ is continuous in $\mathbb{R}^N$. Thus, applying the Maximum Principle for the weak solution (see [18]) on an arbitrary ball centered in x_0, we get that $\hat{v} \equiv 0$. This is a contradiction.

Claim 4.3. There exists a curve $\gamma(t) : [0, 1] \to H^1(\mathbb{R}^N)$ such that

$$\begin{cases} \gamma(0) = 0, \ J_p(\gamma(1)) < 0, \ \hat{v} \in \gamma([0, 1]), \\ \gamma(t)(x) > 0, \ \forall x \in \mathbb{R}^N, \ t \in (0, 1], \\ \max_{t \in [0,1]} J_p(\gamma(t)) = J_p(\hat{v}). \end{cases} \tag{4.8}$$

Defining the function $\tilde{\gamma}(t)(x) = t\hat{v}(x)$ for $t \geq 0$, we have

$$\begin{aligned} J_p(\tilde{\gamma}(t)) = J_p(t\hat{v}) &= \frac{1}{2} \int_{\mathbb{R}^N} \left(|t \nabla \hat{v}|^2 + V_p(x) |G^{-1}(t\hat{v})|^2 \right) - \kappa \int_{\mathbb{R}^N} F_p(x, G^{-1}(t\hat{v})) - \frac{1}{2^*} \int_{\mathbb{R}^N} |t\hat{v}|^{2^*} \\ &\leq \frac{t^2}{2} \int_{\mathbb{R}^N} \left(|\nabla \hat{v}|^2 + V_p(x) |\hat{v}|^2 \right) - \frac{t^{2^*}}{2^*} \int_{\mathbb{R}^N} |\hat{v}|^{2^*}. \end{aligned}$$

Therefore, we may choose a sufficiently large constant $L > 1$ such that $J_p(\tilde{\gamma}(L)) < 0$ with $\tilde{\gamma}(t)(x) > 0$, for all $(x, t) \in \mathbb{R}^N \times (0, L]$. Furthermore, since $\hat{v}$ is a critical point of J_p, set $\varsigma(t) = J_p(t\hat{v})$ and we may write

$$\begin{aligned} \varsigma'(t) &= t \int_{\mathbb{R}^N} |\nabla \hat{v}|^2 + \int_{\mathbb{R}^N} V_p(x) \frac{G^{-1}(t\hat{v})}{g(G^{-1}(t\hat{v}))} \hat{v} - \kappa \int_{\mathbb{R}^N} \frac{f_p(x, G^{-1}(t\hat{v}))}{g(G^{-1}(t\hat{v}))} \hat{v} - t^{2^*-1} \int_{\mathbb{R}^N} |\hat{v}|^{2^*} \\ &= t \left\{ \int_{\mathbb{R}^N} |\nabla \hat{v}|^2 + \int_{\mathbb{R}^N} \left[V_p(x) \frac{G^{-1}(t\hat{v})}{g(G^{-1}(t\hat{v}))t\hat{v}} - \kappa \frac{f_p(x, G^{-1}(t\hat{v}))}{g(G^{-1}(t\hat{v}))t\hat{v}} - (t\hat{v})^{2^*-2} \right] \hat{v}^2 \right\}. \end{aligned}$$

As a direct consequence of Lemma 2.1-(iv) and (f_5), fixed $x \in \mathbb{R}^N$, the function $\eta : (0, +\infty) \to \mathbb{R}$ defined by

$$\eta(t) = V_p(x)\frac{G^{-1}(t)}{g(G^{-1}(t))t} - \kappa\frac{f_p(x, G^{-1}(t))}{g(G^{-1}(t))t} - t^{2^*-2}$$

is decreasing.

Since $\hat{v}$ is a critical point of J_p, we have $\varsigma'(1) = 0$. Moreover, $\varsigma(t) > 0$ for $0 < t < 1$ and $\varsigma(t) < 0$ for $t > 1$. Hence,

$$J_p(\hat{v}) = \varsigma(1) = \max_{t\geq 0}\varsigma(t) = \max_{t\geq 0} J_p(t\hat{v}) = \max_{t\in[0,L]} J_p(t\hat{v}) = \max_{t\in[0,L]} J_p(\tilde{\gamma}(t)).$$

Let $\gamma(t) = \tilde{\gamma}(tL)$. We can check the curve $\gamma(t)$ satisfies (4.8). From $J(\phi) \leq J_p(\phi)$ for all $\phi \in H^1(\mathbb{R}^N)$, we get $\gamma \in \Gamma_J$.

Due to the fact that $\gamma \in \Gamma_J$ satisfies (4.8) and the inequality $W(x) \geq 0$ is strict on a subset of positive measure in $\mathbb{R}^N$, we deduce that

$$c_\lambda \leq \max_{t\in[0,1]} J(\gamma(t)) := J(\gamma(\bar{t})) < J_p(\gamma(\bar{t})) \leq \max_{t\in[0,1]} J_p(\gamma(t)) = J_p(\hat{v}) \leq c_\lambda,$$

which is absurd.

Thus, we conclude that v is a nontrivial solution to problem (2.1). An argument similar to Claim 4.2 shows $v > 0$ in $\mathbb{R}^N$. By Lemma 2.2, problem (1.6) possesses a positive solution $u = G^{-1}(v)$.

5 Proof of Theorem 1.2

The following section gives the proof of Theorem 1.2. First, note that the lemmas in Section 3 are not dependent on the periodicity of function f, but only on its growth, meaning all of them are also valid here. As f satisfies (f_1)–(f_4), by Lemmas 3.3, 3.4 and 3.5, there is a bounded $(PS)_{c_\kappa}$ sequence for J, denoted by $\{v_n\} \subset H^1(\mathbb{R}^N)$, that is,

$$J(v_n) \to c_\kappa \geq \alpha > 0 \quad \text{and} \quad \lambda_J(v_n) \to 0, \quad \text{as } n \to \infty.$$

Consider $w_n \in \partial J(v_n)$ such that $\|w_n\|_* = \lambda_J(v_n) = o_n(1)$ and $w_n = Q'(v_n) - \varrho_n$, where $\varrho_n \in \partial\Psi(v_n)$. Without loss of generality, we may suppose that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$. If v is nontrivial, then Theorem 1.2 is proved. Indeed, repeating the analogous arguments as in the initial steps of the proof of Theorem 1.1, we can instantly obtain that $v = v^+ \geq 0$ and satisfies

$$-\Delta v + V_p(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = \varrho + v^{2^*-1} \quad \text{a.e. } \mathbb{R}^N,$$

that is,

$$-\Delta v + V_p(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} - v^{2^*-1} \in \left[\frac{\underline{f}(x, G^{-1}(v))}{g(G^{-1}(v))}, \frac{\overline{f}(x, G^{-1}(v))}{g(G^{-1}(v))}\right] \quad \text{a.e. in } \mathbb{R}^N.$$

By the argument similar to the one used in Claim 4.2, we can show $v > 0$. Then, $u = G^{-1}(v)$ will be a positive solution of problem (1.8).

Hence, in order to prove Theorem 1.2, it suffices to assume that $v = 0$.

In view of Lemma 3.6, it follows that there exists κ^* such that $c_\kappa < \frac{1}{N}S^{\frac{N}{2}}$ for all $\kappa > \kappa^*$. Furthermore, by Lemma (3.8), there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ and $\delta > 0$ such that

$$\limsup_{n\to\infty} \int_{B_1(y_n)} |v_n|^2 \geq \delta > 0, \qquad \text{for all } n \in \mathbb{N}. \tag{5.1}$$

Since $v_n \to v$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ and $v = 0$, we may suppose that $|y_n| \to \infty$ up to a subsequence. As in the proof of Theorem 1.1, without loss of generality, we can suppose that $\{y_n\} \subset \mathbb{Z}^N$. Defining $\tilde{v}_n(x) = v_n(x + y_n)$, we get $\|\tilde{v}_n\| = \|v_n\|$. Then, taking a subsequence if necessary, there exists $\tilde{v} \in H^1(\mathbb{R}^N)$ such that

$$\begin{cases} \tilde{v}_n \rightharpoonup \tilde{v} & \text{in } H^1(\mathbb{R}^N), \\ \tilde{v}_n \to \tilde{v} & \text{in } L^r_{\text{loc}}(\mathbb{R}^N),\ \forall\, r \in [1, 2^*), \\ \tilde{v}_n \to \tilde{v} & \text{a.e. in } \mathbb{R}^N. \end{cases}$$

The fact that

$$\int_{B_1(0)} |\tilde{v}|^2 = \lim_{n\to\infty} \int_{B_1(0)} |\tilde{v}_n|^2 = \lim_{n\to\infty} \int_{B_1(y_n)} |v_n|^2,$$

and (5.1) imply that $\tilde{v} \neq 0$.

Now, we claim $\tilde{v}$ is a nontrivial solution of periodic problem.

First, we note that $\varrho_n \in \partial\Psi(v_n)$. By the definition of $\partial\Psi(v_n)$,

$$\Psi^\circ(v_n, \psi) \geq \langle \varrho_n, \psi \rangle, \qquad \forall \psi \in L^\Phi(\mathbb{R}^N).$$

Since $H^1(\mathbb{R}^N) \hookrightarrow L^\Phi(\mathbb{R}^N)$ is continuous, a simple change variable implies

$$\begin{aligned} \Psi^\circ(v_n; \psi(\cdot - y_n)) &\geq \langle \varrho_n, \psi(\cdot - y_n) \rangle \\ &= \int_{\mathbb{R}^N} \varrho_n \psi(\cdot - y_n) \\ &= \int_{\mathbb{R}^N} \varrho_n(\cdot + y_n)\psi \\ &= \langle \tilde{\varrho}_n, \psi \rangle, \end{aligned} \tag{5.2}$$

where $\tilde{\varrho}_n = \varrho_n(\cdot + y_n)$. Meanwhile, we can easily verify

$$\Psi(v_n + h + t\psi(\cdot - y_n)) = \Psi(\tilde{v}_n + h(\cdot + y_n) + t\psi) \quad \text{and} \quad \Psi(v_n + h) = \Psi(\tilde{v}_n + h(\cdot + y_n)),$$

where $h \in H^1(\mathbb{R}^N)$ and $t \in \mathbb{R}$. Thus, directly calculations demonstrate

$$\Psi^\circ(v_n + \psi(\cdot - y_n)) = \Psi^\circ(\tilde{v}_n + \psi). \tag{5.3}$$

By (5.2) and (5.3), we get

$$\Psi^\circ(\tilde{v}_n + \psi) \geq \langle \tilde{\varrho}_n, \psi \rangle, \qquad \forall \psi \in H^1(\mathbb{R}^N),$$

which shows $\tilde{\varrho}_n \in \partial\Psi(\tilde{v}_n)$. Furthermore, for all $\psi \in H^1(\mathbb{R}^N)$, we have

$$\begin{aligned} \langle w_n, \psi(\cdot - y_n) \rangle &= \langle Q'(v_n), \psi(\cdot - y_n) \rangle - \langle \varrho_n, \psi(\cdot - y_n) \rangle \\ &= \langle Q'(\tilde{v}_n), \psi \rangle - \langle \tilde{\varrho}_n, \psi \rangle. \end{aligned}$$

Setting $\langle w_n, \psi(\cdot - y_n) \rangle = \langle \tilde{w}_n, \psi \rangle$, we assert

$$\tilde{w}_n = Q'(\tilde{v}_n) - \tilde{\varrho}_n. \tag{5.4}$$

Claim 5.1. $\tilde{w}_n \in \partial J(\tilde{v}_n)$.

Similarly, by change of variables, we get

$$J^\circ(v_n; \psi(\cdot - y_n)) = J^\circ(\tilde{v}_n; \psi). \tag{5.5}$$

And as $w_n \in \partial J(v_n)$, then

$$J^\circ(v_n; \psi(\cdot - y_n)) \geq \langle w_n, \psi(\cdot - y_n)\rangle = \langle \tilde{w}_n, \psi\rangle, \qquad \forall \psi \in H^1(\mathbb{R}^N). \tag{5.6}$$

Combining (5.5) and (5.6), we have

$$J^\circ(\tilde{v}_n; \psi) \geq \langle \tilde{w}_n, \psi\rangle, \qquad \forall \psi \in H^1(\mathbb{R}^N),$$

which shows $\tilde{w}_n \in \partial J(\tilde{v}_n)$.

Moreover, by definition of $\tilde{w}_n$, we get

$$\|\tilde{w}_n\|_* = \sup_{\psi \in H^1(\mathbb{R}^N)} \frac{|\langle \tilde{w}_n, \psi\rangle|}{\|\psi\|} \leq \|w_n\|_*, \qquad \forall n \in \mathbb{N}.$$

Therefore,

$$0 \leq \|\tilde{w}_n\|_* \leq \|w_n\|_* = \lambda_J(v_n) = o_n(1). \tag{5.7}$$

By (5.4), we get

$$\langle \tilde{w}_n, \psi\rangle = \langle Q'(\tilde{v}_n), \psi\rangle - \langle \tilde{\varrho}_n, \psi\rangle, \qquad \forall \psi \in H^1(\mathbb{R}^N). \tag{5.8}$$

From (5.7) and (5.8), we obtain

$$\langle \tilde{\varrho}_n, \psi\rangle = \langle Q'(\tilde{v}_n), \psi\rangle - \langle \tilde{w}_n, \psi\rangle \to \langle Q'(\tilde{v}), \psi\rangle, \quad \text{as } n \to \infty,$$

that is, $\tilde{\varrho}_n \overset{*}{\rightharpoonup} Q'(\tilde{v})$ in $(H^1(\mathbb{R}^N))^*$.

This limit together with Lemma 3.2 shows that $Q'(\tilde{v}) \in \partial\Psi(v)$. Thereby, $Q'(\tilde{v}) = \tilde{\varrho} \in \partial\Psi(\tilde{v})$, and so,

$$\int_{\mathbb{R}^N} \left(\nabla\tilde{v}\nabla\psi + V_p(x)\frac{G^{-1}(\tilde{v})}{g(G^{-1}(\tilde{v}))}\psi\right) - \int_{\mathbb{R}^N} (\tilde{v}^+)^{2^*-1}\psi = \int_{\mathbb{R}^N} \tilde{\varrho}\psi, \qquad \forall\, \psi \in H^1(\mathbb{R}^N),$$

where

$$\tilde{\varrho}(x) \in \left[\frac{\underline{f}(x, G^{-1}(\tilde{v}(x)))}{g(G^{-1}(\tilde{v}(x)))}, \frac{\overline{f}(x, G^{-1}(\tilde{v}(x)))}{g(G^{-1}(\tilde{v}(x)))}\right] \quad \text{a.e. in } \mathbb{R}^N.$$

Repeating the analogous steps of the proof of Theorem 1.1, $\tilde{v} = \tilde{v}^+ \geq 0$ and satisfies

$$-\Delta\tilde{v} + V_p(x)\frac{G^{-1}(\tilde{v})}{g(G^{-1}(\tilde{v}))} = \tilde{\varrho} + \tilde{v}^{2^*-1} \quad \text{a.e. } \mathbb{R}^N,$$

that is,

$$-\Delta\tilde{v} + V_p(x)\frac{G^{-1}(\tilde{v})}{g(G^{-1}(\tilde{v}))} - \tilde{v}^{2^*-1} \in \left[\frac{\underline{f}(x, G^{-1}(\tilde{v}))}{g(G^{-1}(\tilde{v}))}, \frac{\overline{f}(x, G^{-1}(\tilde{v}))}{g(G^{-1}(\tilde{v}))}\right] \quad \text{a.e. in } \mathbb{R}^N.$$

Similarly, we also have $\tilde{v} > 0$ in $\mathbb{R}^N$ by the analogous argument used in Claim 4.2. Since (1.8) is only the periodic case for (1.6), Lemma 2.2 is also valid. Hence, we can see that $u = G^{-1}(\tilde{v})$ will be a positive solution of problem (1.8).

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