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Existence of solution for a generalized Schrödinger–Poisson system via bifurcation theory

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Abstract. In this paper, we study a generalized Schrödinger–Poisson system in a bounded domain of $\mathbb{R}^3$ and involving an asymptotically linear nonlinearity. We prove the existence of positive solutions using bifurcation theory.

Keywords: bifurcation theory, positive solutions, Schrödinger–Poisson system, topological degree.

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1 Introduction

This paper is concerned with the existence of solutions for the problem

$$\begin{cases} -\Delta u + \phi(x)u = \lambda f(u) \text{ in } \Omega, \\ -\Delta \phi(x) = g(u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ \phi > 0 \text{ in } \Omega, \\ u(x) = \phi(x) = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{P}$$

where $0 < \lambda$ is a parameter, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $f \in C^1([0,\infty),\mathbb{R})$ and $g \in C(\mathbb{R},[0,\infty))$.

When the function $g(t) = t^2$, this system represents the well known Schrödinger–Poisson (or Schrödinger–Maxwell) equations, that have been widely studied in the recent past. This equation appears in the mean field approach for the Hartree–Fock model and as a nonlinear Schrödinger equation that takes into account the electrostatic field generated by the wave, see [7,10,14,15].

Recently, many authors have studied the existence, non-existence and multiplicity of solutions of the problem

$$\begin{cases} -\Delta u + \lambda\phi(x)u = z(u) \text{ in } \Omega, \\ -\Delta \phi(x) = u^2 \text{ in } \Omega, \\ u(x) = \phi(x) = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

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where z is a superlinear function; see for example [1,3,8,9,11,13] and the references therein. To prove their results they used the reduction argument and then employed variational methods. It is worth pointing out that in the proof of Theorem 2.1 of [13] the authors used the Leray–Schauder degree to prove the existence of a positive solution when the parameter λ is small enough. Also, in the references of the papers mentioned above the reader will find many works dealing with Schrödinger–Poisson systems where $\Omega = \mathbb{R}^3$.

Motivated by the papers above and Ambrosetti and Hess [4], we are interested in studying system (P) when f is asymptotically linear and g satisfies some suitable assumptions. Specifically, we introduce the following assumptions:

(F_1) $f \in C^1([0,\infty),\mathbb{R})$, $f(0) = 0$ and $m_0 = \lim\limits_{t\to 0^+} \dfrac{f(t)}{t} > 0$ (namely $m_0 = f'_+(0)$);

(F_2) There exist $m_\infty > 0$, a function h and a constant C such that

$$f(t) = m_\infty t + h(t), \text{ where } h \in C^{0,1}(\mathbb{R}^+,\mathbb{R}) \text{ and } |h(t)| \le C, \forall t \in \mathbb{R}^+ (\mathbb{R}^+ = [0,\infty));$$

(G_1) $g(t) = t^{2p}$, where $0 < p < 2$;

(G_2) $g \in C(\mathbb{R},(0,\infty))$ and there exist the limit $\lim\limits_{t\to\infty} g(t) = g(\infty)$ and a constant $c > 0$ such that $0 < g(t) < c$ for all $t \in \mathbb{R}$.

Some examples of functions satisfying the above assumptions are as follows.

Example 1.1.

(a) The function $f(t) = t - t^{10}, t \ge 0$, satisfies (F_1).

(b) The function $f(t) = t - arctg(t^2), t \ge 0$, satisfies (F_1) and (F_2).

(c) The function $f(t) = t, t \ge 0$, satisfies (F_1) and (F_2).

(e) The function $g(t) = \frac{t^2}{1+t^2} + 1, t \in \mathbb{R}$, satisfies (G_2).

(g) The function $g(t) = \frac{|t|}{1+t^2} + 1, t \in \mathbb{R}$, satisfies (G_2).

As we can see, the function f is allowed to change sign. Before stating our main results, we need some definitions and notations. First, we introduce the Banach space

$$X = C(\overline{\Omega},\mathbb{R})$$

endowed with the norm $\|u\| = \sup_{x\in\overline{\Omega}} |u(x)|$ for $u \in X$.

We say that $(\lambda, u, \phi_u) \in \mathbb{R} \times [(H_0^1(\Omega) \times H_0^1(\Omega)) \cap (X \times X)]$ is a solution of (P) if $u > 0$ in Ω, $\phi_u > 0$ in Ω and

$$\int_\Omega \nabla u \nabla \varphi dx + \int_\Omega \phi_u(x) u \varphi dx = \lambda \int_\Omega f(u)\varphi dx, \tag{1.1}$$

$$\int_\Omega \nabla \phi_u \nabla \psi dx = \int_\Omega g(u)\psi dx, \tag{1.2}$$

for all $(\varphi,\psi) \in H_0^1(\Omega) \times H_0^1(\Omega)$. When $u > 0$ in Ω, (u,ϕ_u) is a positive solution. Moreover, we say that (λ, u, ϕ_u) is a weak solution of (P) if $(u,\phi_u) \in H_0^1(\Omega) \times H_0^1(\Omega)$ and it satisfies (1.1)–(1.2). It turns out that weak solutions are solutions provided f has subcritical growth (see Lemma 2.4).

A bifurcation point for (P) is a number $\lambda^* \in \mathbb{R}$ such that there exists a sequence $(\lambda_n, u_n, \phi_{u_n})$ $\in \mathbb{R} \times [(H_0^1(\Omega) \times H_0^1(\Omega)) \cap (X \times X)]$ satisfying the following properties:

(*i*) $\lambda_n \longrightarrow \lambda^*$;

(*ii*) $(\lambda_n, u_n, \phi_{u_n})$ is a solution of (P) with $u_n \neq 0$ and $\|u_n\| \longrightarrow 0$.

We say that $\lambda^* \in \mathbb{R}$ is a bifurcation point from infinity of (P) if there exists a sequence $(\lambda_n, u_n, \phi_{u_n}) \in \mathbb{R} \times [(H_0^1(\Omega) \times H_0^1(\Omega)) \cap (X \times X)]$ satisfying the following properties:

(*i*) $\lambda_n \longrightarrow \lambda^*$;

(*ii*) $(\lambda_n, u_n, \phi_{u_n})$ is a solution of (P) and $\|u_n\| \longrightarrow +\infty$.

It is well known that under the assumption (G_2) there exists a unique solution $\phi_\infty \in H_0^1(\Omega) \cap X$ of the problem

$$\begin{cases} -\Delta u = g(\infty) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Also, there exists a unique solution $\phi_0 \in H_0^1(\Omega) \cap X$ of the problem

$$\begin{cases} -\Delta u = g(0) \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\phi_0 > 0$ in Ω if $g(0) > 0$ holds.

Let us denote by $\lambda_1[\phi_\infty]$ and φ_∞ the first eigenvalue and the positive eigenfunction normalized by $\|\varphi_\infty\| = 1$, respectively, of the eigenvalue problem

$$-\Delta u + \phi_\infty(x) u = \lambda u \text{ in } \Omega,$$

$$u = 0 \text{ on } \partial\Omega.$$

Similarly, let us denote by $\lambda_1[\phi_0]$ and φ_0 the first eigenvalue and the positive eigenfunction normalized by $\|\varphi_0\| = 1$, respectively, of the eigenvalue problem

$$-\Delta u + \phi_0(x) u = \lambda u \text{ in } \Omega,$$

$$u = 0 \text{ on } \partial\Omega.$$

We observe that if $g(0) = 0$ then $\lambda_1[\phi_0]$ and φ_0 are the first eigenvalue and the positive eigenfunction, respectively, of $(-\Delta, H_0^1(\Omega))$.

Now we are ready to state our main results.

Theorem 1.2. *Suppose that* (F_1) *and* (G_1) *hold. Then* $\lambda_0 = \lambda_1[\phi_0]/m_0$ *is the unique bifurcation point of* (P). *In addition, the continuum* Σ_0 *emanating from* $(\lambda_0, 0)$ *is unbounded. The same conclusion holds under the assumptions* (F_1) *and* (G_2).

Theorem 1.3. *Assume that* (F_2) *and* (G_2) *hold. Then* $\lambda_\infty = \lambda_1[\phi_\infty]/m_\infty$ *is the unique bifurcation point from infinity of* (P). *Moreover, there exists a subset* Σ_∞ *in* $\mathbb{R} \times X$ *of solutions of* (P) *such that* $\tilde{\Sigma}_\infty = \{(\lambda, z) : (\lambda, z/\|z\|^2) \in \Sigma_\infty\} \cup \{(\lambda_\infty, 0)\}$ *is connected and unbounded.*

After a bibliography review, we did not find any paper involving bifurcation theory and problems involving a generalized Schrödinger–Poisson system in a bounded domain as in the problem (P). Inspired by this fact, in the present paper we show that it is possible to apply the Leray–Schauder degree theory and the global bifurcation result due to Rabinowitz [12] to study the existence of solution for (P). To carry out this program, we first use the reduction argument (see [2]), which says that (P) is equivalent to a nonlocal problem (see problem (S)). After, we follow the same methodology as Ambrosetti and Hess [4]. However as we are working with a nonlocal problem it is necessary to do a careful study on some estimates and convergences involving the nonlocal term $\phi_u u$. Also, the calculation of Leray–Schauder degree of some maps involving the nonlocal term $\phi_u u$ must be justified (see Lemma 3.2). The reader is invited to verify that when $g(0) \neq 0$ the bifurcation points of Theorems 1.2 and 1.3 are different from those found in [4]. Moreover, under additional assumptions on f and g we will show that the bifurcation point found in our work is supercritical (the nontrivial solutions branch off on the right of λ_∞), while under the same assumption on f, the bifurcation point found in [4] is subcritical (the branching is on the left of bifurcation point).

Finally, we would like to point out that our results are new even in the case where $g(t) = t^2$ (that is, $p = 1$ in (G_1)), which is the case considered in the papers mentioned above and which allows us to apply variational methods. Indeed, in the papers mentioned above they did not study the existence of bifurcation points for problems of type (P). Also, they did not consider asymptotically linear nonlinearities as in our work. Thus, our work is the first to deal with the existence of bifurcation points and the continuum emanating from these points for Problem (P) with asymptotically linear nonlinearities even in the case when $p = 1$.

The paper is organized as follows. Section 2 is devoted to some preliminaries. In Section 3, we prove Theorem 1.2. In Section 4, we prove Theorem 1.3. In section 5 we will show a result of multiplicity of solutions under additional assumptions on f and g.

Notation. Throughout this paper, we make use of the following notations:

- $L^p(\Omega)$, for $1 \leq p \leq \infty$, denotes the Lebesgue space with usual norm denoted by $|u|_p$.

- $H_0^1(\Omega)$ denotes the Sobolev space endowed with inner product

$$(u, v)_H = \int_\Omega \nabla u \nabla v, \ \forall u, v \in H_0^1(\Omega).$$

  The norm associated with this inner product will be denoted by $\| \ \|_H$.

- $W^{2,k}(\Omega)$ denotes the Sobolev space with norm $\|u\|_{W^{2,k}} = \left(\sum_{|\alpha| \leq 2} \|D^\alpha u\|_k^k\right)^{1/k}$.

- If u is a measurable function, we denote by u^- the negative part of u, which is given by $u^- = \max\{-u, 0\}$.

- The function $d(x, \partial\Omega)$ denotes the distance from a point $x \in \overline{\Omega}$ to the boundary $\partial\Omega$, where $\overline{\Omega} = \Omega \cup \partial\Omega$ is the closure of $\Omega \subset \mathbb{R}^N$.

- $\deg(I - \Psi, \mathcal{W}, 0)$ denotes the Leray–Schauder degree of $I - \Psi$ in $\mathcal{W}$ with respect to 0, where $\mathcal{W} \subset X$ is a bounded open set and $\Psi : \overline{\mathcal{W}} \longrightarrow X$ is a compact operator.

- $B_r(0) \subset X$ denotes the ball centered at $0 \in X$ with radius $r > 0$.

- $c, c_1, c_2, \ldots$ and $C, C_1, C_2, \ldots$ are possibly different positive constants which may change from line to line.

2 Preliminary results

Throughout this paper, unless it is explicitly stated, we will assume that (G_1) or (G_2) holds. In this section we will establish some results that we will need in the next sections.

For all $u \in L^{3/p}(\Omega)$ there exists a unique $\phi_u \in H_0^1(\Omega)$ which solves

$$-\Delta\phi = g(u(x)) \ \text{ in } \Omega,$$

and there holds

$$\phi_u(x) = \int_\Omega \frac{g(u(y))}{|x-y|} dy.$$

By L^p-theory one has $\phi_u \in W^{2,3/p}(\Omega)$, $0 < p < 2$, and so $\phi_u \in X$ (because $6/p > 3$). Since $g(u) \geq 0$, then by the maximum principle $\phi_u \geq 0$. Moreover, if $u \neq 0$ then $\phi_u > 0$ in Ω. Also, we have the following estimates.

Lemma 2.1. *For every $u \in L^{3/p}(\Omega)$ there holds*

$$\|\phi_u\| \leq C_2 |g(u)|_{3/p},$$

for some constant $C_2 > 0$ independent of u. In particular, if $u \in X$, then

$$\|\phi_u\| \leq C\|g(u)\|, \tag{2.1}$$

for some constant $C > 0$ independent of u.

Proof. By L^p-theory one has $\phi_u \in W^{2,3/p}(\Omega)$ and

$$\|\phi_u\|_{W^{2,3/p}} \leq C_1 |g(u)|_{3/p},$$

for some constant $C_1 > 0$, which depends only on Ω and p.

Combining this inequality with the embedding of $W^{2,3/p}(\Omega)$ into X we get

$$\|\phi_u\| \leq C_2 |g(u)|_{3/p},$$

for some constant $C_2 > 0$, which depends only on Ω and p.

If in addition $u \in X$, then the inequality $|g(u)|_{3/p} \leq |\Omega|^{p/3}\|g(u)\|$ is valid, and therefore

$$\|\phi_u\| \leq C\|g(u)\|,$$

where $C = C_2(\Omega)|\Omega|^{p/3}$. This completes the proof of the lemma. □

We recall that a map $J : X \to X$ is bounded if it maps bounded sets onto bounded sets.

In order to apply Bifurcation Theory we will need the following lemma.

Lemma 2.2. *The map $\mathcal{J} : X \longrightarrow X$ defined by setting $\mathcal{J}(u) = \phi_u$ is continuous and bounded.*

Proof. Let $\{u_n\} \subset X$ be a sequence such that $u_n \to u$ in X. As $\phi_{u_n} - \phi_u \in H_0^1(\Omega)$ satisfies

$$-\Delta(\phi_{u_n} - \phi_u) = g(u_n) - g(u) \ in\ \Omega,$$

by elliptic regularity it follows that

$$\|\phi_{u_n} - \phi_u\| \leq C\|g(u_n) - g(u)\|,$$

for some constant $C > 0$ independent of u_n and u. Since $u_n \to u$ in X implies $g(u_n) \to g(u)$ in X, from the last inequality one deduces that $\phi_{u_n} \to \phi_u$ in X. This proves that the map $\mathcal{J}$ is continuous in X.

Finally, the boundedness of $\mathcal{J}$ follows from (2.1), and the proof is completed. □

Our next result establishes the positivity of weak solutions to a variational inequality.

Lemma 2.3. *Let $\phi \in X$ and suppose that $u \in H_0^1(\Omega)$ satisfies*

$$\begin{cases} -\Delta u + \phi(x)u \geq 0 \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega. \end{cases}$$

Then either $u \equiv 0$, or there exists $\epsilon > 0$ such that $u(x) \geq \epsilon d(x, \partial\Omega)$ in Ω.

Proof. Let $k = \|\phi\|$ and assume that $u \not\equiv 0$. In this case, we get

$$-\Delta u + ku \geq -\Delta u + \phi(x)u \geq 0 \text{ in } \Omega,$$

namely, u satisfies

$$\begin{cases} -\Delta u + ku \geq 0 \text{ in } \Omega, \\ u \gneqq 0 \text{ in } \Omega. \end{cases}$$

This allows us to apply Theorem 3 of Brezis–Nirenberg [6] to deduce that $u(x) \geq \epsilon d(x, \partial\Omega)$ in Ω, for some $\epsilon > 0$. This completes the proof. □

Now, we consider the nonlocal problem

$$\begin{cases} -\Delta u + \phi_u(x)u = z(u) \text{ in } \Omega, \\ u(x) = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{Q}$$

under the following assumption on $z \in C(\mathbb{R}, \mathbb{R})$:

(H) $|z(t)| \leq c_1 + c_2|t|^q$, where $c_1, c_2 > 0$ are constants and $0 < q < 2^* - 1$.

Lemma 2.4. *Suppose that (H) holds. Then every $u \in H_0^1(\Omega)$ which is a weak solution of (Q) belongs to X.*

Proof. Indeed, $u \in H_0^1(\Omega)$ is a weak solution of the problem

$$-\Delta u = h(x, u) \text{ in } \Omega,$$

where $h(x,t) = z(t) - \phi_u(x)t$. From Lemma 2.2 and (H) one infers that

$$|h(x,t)| \leq c_3 + c_4|t|^q,$$

for all $x \in \Omega, t \in \mathbb{R}$ and some constants $c_3, c_4 > 0$. Thus, a standard bootstrap argument implies that $u \in X$. This completes the proof. □

3 Global bifurcation

The main goal of this section is to prove Theorem 1.2. To do this we need some definitions and auxiliary lemmas.

It is well known that Problem (P) is equivalent to the nonlocal problem

$$\begin{cases} -\Delta u + \phi_u(x)u = \lambda f(u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u(x) = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{S}$$

We extend the function f to a continuous function $\tilde{f}$ defined on $\mathbb{R}$ in such a way that $\tilde{f}(t) = f(0)$ for all $t < 0$. Then, we can consider the nonlocal problem

$$\begin{cases} -\Delta u + \phi_u(x)u = \lambda \tilde{f}(u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u(x) = 0 \text{ on } \partial\Omega. \end{cases} \tag{$\tilde{S}$}$$

Now we prove the following result.

Lemma 3.1. *Assume that either (F_1) or (F_2) is satisfied. Then Problems (S) and $(\tilde{S})$ are equivalent.*

Proof. It is clear that if u is a solution of (S) then it is also a solution of $(\tilde{S})$. Now, we assume that u is a solution of $(\tilde{S})$. Taking u^- as test function in $(\tilde{S})$ we get

$$-\|u^-\|_H^2 - \int_\Omega \phi_u(x)(u^-)^2 = \int_\Omega \lambda f(0)u^-,$$

which implies $\|u^-\|_H = 0$, that is, $u \geq 0$ in Ω. Thus $\tilde{f}(u) = f(u)$ in Ω, and if either (F_1) or (F_2) is satisfied then

$$|\tilde{f}(t)| = |f(t)| \leq c_1|t|, \ \forall t \in [0, \|u\| + 1),$$

and for some constant $c_1 > 0$. Therefore, u satisfies

$$\begin{cases} -\Delta u + (\phi_u(x) + \lambda c_1)u \geq 0 \text{ in } \Omega, \\ u \gneqq 0 \text{ in } \Omega, \\ u(x) = 0 \ \text{ on } \partial\Omega, \end{cases}$$

and from Lemma 2.3 one infers that $u > 0$ in Ω. This completes the proof. $\square$

Due to Lemma 3.1, the proof of Theorems 1.2 and 1.3 is reduced to proving the existence of the bifurcation points of Problem $(\tilde{S})$. To study Problem $(\tilde{S})$ we will transform it into a functional equation. From now on we will denote by K the Green operator of $-\Delta$ on $H_0^1(\Omega)$. It is well known that K is compact as a map from X in itself. From Lemma 2.2 it follows that the map $F_\lambda : X \to X$ given by

$$F_\lambda(u) = \lambda \tilde{f}(u) - \phi_u u$$

is continuous and bounded. As a consequence, the map $T : \mathbb{R} \times X \to X$ defined by $T(\lambda, u) = K(F_\lambda(u))$ is compact and Problem $(\tilde{S})$ is equivalent to the functional equation

$$\Phi(\lambda, u) = 0,$$

where $\Phi(\lambda, u) = u - T(\lambda, u)$ for $(\lambda, u) \in \mathbb{R} \times X$.

The first property of the map Φ that we highlight is the following.

Lemma 3.2. *For every $\mu \in [0, 1]$ the function $u \equiv 0$ is the unique solution of the problem*

$$\begin{cases} -\Delta u + \mu\phi_u(x)u = 0 \textit{ in } \Omega, \\ u \in H_0^1(\Omega) \cap X. \end{cases} \tag{A}$$

In particular,

$$\deg(\Phi(0, \cdot), B_r(0), 0) = 1,$$

for all $r > 0$.

Proof. If u satisfies (A) then,

$$\|u\|_H^2 + \mu \int_\Omega \phi_u u^2 = 0,$$

and this implies that $\|u\|_H = 0$, namely that $u \equiv 0$.

Thus the homotopy $H(\mu, u) = u - \mu T(0, u), (\mu, u) \in [0, 1] \times X$, is admissible on the ball $B_r(0)$, for all $r > 0$. Using the homotopy invariance, it follows that

$$\deg(H(1, \cdot), B_r(0), 0) = \deg(I, B_r(0), 0) = 1,$$

and since $\Phi(0, \cdot) = H(1, \cdot)$, we get $\deg(\Phi(0, \cdot), B_r(0), 0) = 1$. □

Now, let us give the precise definition of bifurcation point of the functional equation $\Phi(\lambda, u) = 0$.

Definition 3.3. We say that λ_* is a bifurcation point of $\Phi(\lambda, u) = 0$ if there exists a sequence $(\lambda_n, u_n) \in \mathbb{R} \times X$, with $u_n \neq 0$, such that $\lambda_n \longrightarrow \lambda_*$, $\|u_n\| \longrightarrow 0$ and $\Phi(\lambda_n, u_n) = 0$.

It turns out that the bifurcation points of $\Phi(\lambda, u) = 0$ are the bifurcation points of ($\tilde{S}$) (and therefore are also the bifurcation points of (P)).

Denoting by

$$\Sigma_\Phi = \{(\lambda, u) \in \mathbb{R} \times X : \Phi(\lambda, u) = 0, \ u \neq 0\},$$

and taking the closure $\overline{\Sigma}_\Phi$ of Σ_Φ, we see that λ_* is a bifurcation point of $\Phi(\lambda, u) = 0$ if and only if $(\lambda_*, 0) \in \overline{\Sigma}_\Phi$.

For each $\lambda \in \mathbb{R}$ fixed, the index of $\Phi_\lambda = \Phi(\lambda, \cdot)$ relative to 0, denoted by $i(\Phi_\lambda, 0)$, is defined by

$$i(\Phi_\lambda, 0) = \lim_{\epsilon \to 0} \deg(\Phi_\lambda, B_\epsilon(0), 0).$$

To prove Theorem 1.2 we have to prove the change of index of $\Phi(\lambda, \cdot)$ as λ crosses $\lambda = \lambda_0$. The proof is based on the following lemmas.

Lemma 3.4. *Let $\Lambda \subset \mathbb{R}^+$ be a compact interval with $\lambda_0 \notin \Lambda$. Then there exists $\epsilon > 0$ satisfying*

$$\Phi(\lambda, u) \neq 0, \qquad \forall \lambda \in \Lambda, \quad \forall 0 < \|u\| \leq \epsilon.$$

Proof. We argue by contradiction assuming that there exists a sequence $(\lambda_n, u_n) \in \Lambda \times X$ satisfying

$$\lambda_n \longrightarrow \lambda \neq \lambda_0, \qquad \|u_n\| \longrightarrow 0,$$
$$\Phi(\lambda_n, u_n) = 0, \qquad u_n > 0.$$

Now, we divide the equation $u_n = K(F_{\lambda_n}(u_n))$ by $\|u_n\|$ to get

$$v_n = K\left(\frac{F_{\lambda_n}(u_n)}{\|u_n\|}\right), \quad \text{where } v_n = \frac{u_n}{\|u_n\|}.$$

We claim that the sequence $\{\frac{F_{\lambda_n}(u_n)}{\|u_n\|}\}$ is bounded in $\Lambda \times X$. To prove this claim, let $\delta > 0$ such that $|f(t)| \leq (m_0 + 1)|t|$ for all $0 < t < \delta$ (the existence of δ is guaranteed by (F_1)). Since $\|u_n\| \longrightarrow 0$ there exists $n_0 \in \mathbb{N}$ such that $\|u_n\| < \delta$ for all $n > n_0$. From this and (2.1) we deduce that

$$\|F_{\lambda_n}(u_n)\| \leq C(\|u_n\| + \|g(u_n)\| \|u_n\|),$$

for all $n > n_0$ and for some constant $C > 0$ independent of n.

Therefore,

$$\|\frac{F_{\lambda_n}(u_n)}{\|u_n\|}\| \leq C(1+\|g(u_n)\|) \leq C(1+\max_{t\in[-\delta,\delta]}|g(t)|),$$

for $n > n_0$, which implies that the sequence $\{\frac{F_{\lambda_n}(u_n)}{\|u_n\|}\}$ is bounded in $\Lambda \times X$.

Since K is compact, from $v_n = K(\frac{F_{\lambda_n}(u_n)}{\|u_n\|})$ we deduce that, up to a subsequence, v_n strongly converges to some $v \in X$ with $\|v\| = 1$. Then, by Lemmas 2.1 and 2.2 and (F_1) one infers

$$\frac{F_{\lambda_n}(u_n)}{\|u_n\|} \longrightarrow (\lambda m_0 - \phi_0)v \text{ in } X,$$

and therefore

$$v = K((\lambda m_0 - \phi_0)v).$$

But this says that v is a solution of the problem

$$\begin{cases} -\Delta v + \phi_0(x)v = \lambda m_0 v \text{ in } \Omega, \\ v \geq 0 \ in\ \Omega, \end{cases}$$

and from Lemma 2.3 one infers that $v > 0$ in Ω. As a consequence v is an eigenfunction of norm one associated to λ.

Using φ_0 as a test function in this eigenvalue problem we obtain

$$\lambda_1[\phi_0]\int_\Omega v\varphi_0 = \int_\Omega \nabla v \nabla \varphi_0 dx + \int_\Omega \phi_0 v \varphi_0 dx = \lambda m_0 \int_\Omega v\varphi_0,$$

and we conclude that $\lambda_1[\phi_0] = \lambda m_0$, which is a contradiction and the proof is finished. □

As a consequence of the proof of Lemma 3.4 we obtain the following corollary.

Corollary 3.5. *The unique possible bifurcation point of solutions is $\lambda = \lambda_0$.*

Lemma 3.6. *If $\lambda < \lambda_0$ then $i(\Phi_\lambda, 0) = 1$.*

Proof. Fix any $\lambda < \lambda_0$ and take $\Lambda = [0, \lambda]$. For $t \in [0,1]$, the parameter $t\lambda$ belongs to Λ and from Lemma 3.4 it follows that $\Phi(t\lambda, u) \neq 0$ for all $0 < \|u\| \leq \epsilon$, where $\epsilon > 0$ is given by Lemma 3.4. Consider the homotopy $H(t,u) = \Phi(t\lambda, u)$. Using the homotopy invariance, we get

$$\deg(H(1,\cdot), B_\epsilon(0), 0) = \deg(H(0,\cdot), B_\epsilon(0), 0),$$

namely

$$i(\Phi_\lambda, 0) = \deg(\Phi_\lambda, B_\epsilon(0), 0) = \deg(\Phi_0, B_\epsilon(0), 0) = 1,$$

where we have used Lemma 3.2 in the last equality. This completes the proof. □

Lemma 3.7. *For every $\lambda > \lambda_0$ there exists $\delta > 0$ such that*

$$\Phi(\lambda, u) \neq \tau\varphi_1, \quad \forall 0 < \|u\| \leq \delta, \quad \forall \tau \geq 0.$$

Proof. We fix $\lambda > \lambda_0$ and we assume, by contradiction, that there exist sequences $u_n \in X$ and $\tau_n \geq 0$ satisfying $u_n > 0$ in Ω, $\|u_n\| \longrightarrow 0$ and

$$\Phi(\lambda, u_n) = \tau_n \varphi_1,$$

or, equivalently,

$$u_n = K(F_\lambda(u_n)) + \tau_n \varphi_1.$$

Dividing this equation by $\|u_n\|$ one finds

$$v_n = K\left(\frac{F_\lambda(u_n)}{\|u_n\|}\right) + \varphi_1 \frac{\tau_n}{\|u_n\|}, \quad \text{where } v_n = \frac{u_n}{\|u_n\|}.$$

Arguing as in the proof of Lemma 3.4, we see that the sequence $\{\frac{F_\lambda(u_n)}{\|u_n\|}\}$ is bounded in X. Thus, using the compactness of K, we deduce that, up to a subsequence, $K(\frac{F_\lambda(u_n)}{\|u_n\|})$ is convergent and hence $\tau_n/\|u_n\|$ is bounded. Passing again to a subsequence, if necessary, we can assume that $\tau_n/\|u_n\| \longrightarrow \tau \geq 0$ and $u_n/\|u_n\| \longrightarrow v$ with $v \in X$ and $\|v\| = 1$. Arguing as we have done in the proof of Lemma 3.4, it is easy to see that v satisfies

$$\begin{cases} -\Delta v + \phi_0 v = \lambda m_0 v + \tau \lambda_1 \varphi_1 \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega, \\ \|v\| = 1. \end{cases}$$

Then, using φ_0 as a test function in this problem we obtain

$$\lambda_1[\phi_0] \int_\Omega v\varphi_0 = \lambda m_0 \int_\Omega v\varphi_0 + \int_\Omega \tau\lambda_1\varphi_1\varphi_0 \geq \lambda m_0 \int_\Omega v\varphi_0,$$

which implies that $\lambda_0 \geq \lambda$, a contradiction. The proof is finished. □

Lemma 3.8. *If $\lambda > \lambda_0$ then $i(\Phi_\lambda, 0) = 0$.*

Proof. If $\lambda > \lambda_0$ then, from Lemma 3.7, we derive that

$$\deg(\Phi_\lambda, B_\delta(0), 0) = \deg(\Phi_\lambda - \tau\varphi_1, B_\delta(0), 0), \quad \forall \tau > 0,$$

where $\delta > 0$ is given by Lemma 3.7.

But, again using Lemma 3.7, the problem

$$\begin{cases} -\Delta w + \phi_w(x) w = \lambda \tilde{f}(w) + \tau\lambda_1\varphi_1 \text{ in } \Omega, \\ w = 0 \text{ in } \partial\Omega, \end{cases}$$

has no nontrivial solution satisfying $0 < \|u\| \leq \delta$. Since, $w = 0$ is not a solution provided that $\tau > 0$, we deduce that

$$i(\Phi_\lambda, 0) = \deg(\Phi_\lambda, B_\delta(0), 0) = \deg(\Phi_\lambda - \tau\varphi_1, B_\delta(0), 0) = 0, \quad \forall \lambda > \lambda_0.$$

This completes the proof. □

Now, we are ready to prove Theorem 1.2.

Proof. (*of Theorem 1.2*) Assume that λ_0 is no bifurcation point. Then there exists $\epsilon > 0$ such that

$$\Phi_\lambda(u) \neq 0, \text{ for all } \lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon] \text{ and } 0 < \|u\| \leq \epsilon.$$

Thus, if we take

$$\lambda_0 - \epsilon < \tilde{\lambda} < \lambda_0 < \hat{\lambda} < \lambda_0 + \epsilon$$

one has

$$\deg(\Phi_{\tilde{\lambda}}, B_\epsilon(0), 0) = \deg(\Phi_{\hat{\lambda}}, B_\epsilon(0), 0),$$

and therefore,

$$i(\Phi_{\tilde{\lambda}}, 0) = i(\Phi_{\hat{\lambda}}, 0),$$

which contradicts Lemmas 3.6 and 3.8. Moreover, from Corollary 3.5 λ_0 is the unique bifurcation point for (P).

As a consequence, one can repeat the arguments carried out in the proof of the Global Bifurcation Theorem due to Rabinowitz [12] to show the existence of Σ_0. This completes the proof. □

4 Bifurcation from infinity

In this section we are going to prove Theorem 1.3. Hereafter we will assume that (F_2) and (G_2) hold. We start with the following definition.

Definition 4.1. We say that λ_∞ is a bifurcation point from infinity of $\Phi(\lambda, u) = 0$ if there exists a sequence $(\lambda_n, u_n) \in \mathbb{R} \times X$ satisfying

$$\lambda_n \longrightarrow \lambda_\infty, \quad \|u_n\| \longrightarrow +\infty, \quad \Phi(\lambda_n, u_n) = 0.$$

It turns out that the bifurcation points from infinity of $\Phi(\lambda, u) = 0$ are the bifurcation points from infinity of ($\tilde{S}$) (and therefore are also the bifurcation points from infinity of (P)).

Following [4], if we make the Kelvin transform

$$z = \frac{u}{\|u\|^2}, \quad \text{with } u \neq 0,$$

we derive that

$$\Phi(\lambda, u) = 0, \;\; u \neq 0 \Leftrightarrow z - \|z\|^2 T\left(\lambda, \frac{z}{\|z\|^2}\right) = 0, \quad z \neq 0.$$

Thus we are led to define the map

$$\tilde{\Phi}(\lambda, z) = \begin{cases} z - \|z\|^2 T(\lambda, \frac{z}{\|z\|^2}), \text{ if } z \neq 0, \\ 0, \text{ if } z = 0. \end{cases}$$

Moreover, using Lemma 2.1 we find that

$$\|z\|^2 \|\phi_{z/\|z\|^2} \frac{z}{\|z\|^2}\| \leq C\|z\|,$$

for all $z \neq 0$ and some constant $C > 0$ independent of z. As a consequence we obtain

$$\lim_{z \to 0} \|z\|^2 \phi_{z/\|z\|^2} \frac{z}{\|z\|^2} = 0.$$

From this limit and assumption on f it readily follows that $\tilde{\Phi}$ is continuous. In particular, $\tilde{\Phi}$ is a compact perturbation of the identity and λ_∞ is a bifurcation point from infinity for $\Phi(\lambda, u) = 0$ if and only if λ_∞ is a bifurcation point for $\tilde{\Phi}(\lambda, z) = 0$. Moreover, arguing as in the proof of Lemma 3.2, we immediately deduce the following property:

$$\deg(\tilde{\Phi}(0, \cdot), B_\epsilon(0), 0) = 1, \text{ for all } \epsilon > 0.$$

The proof of Theorem 1.3 is based on the following lemmas.

Lemma 4.2. *Let $\Lambda \subset [0, \lambda_\infty)$ be any compact interval. Then*

(a) *there exists $r > 0$ such that $\Phi_\lambda(u) \neq 0$, for all $\lambda \in \Lambda$ and $\|u\| \geq r$,*

(b) *λ_∞ is the only possible bifurcation from infinity for $\Phi(\lambda, u) = 0$,*

(c) *$i(\tilde{\Phi}_\lambda, 0) = 1$ for all $\lambda < \lambda_\infty$.*

Proof. (a) We argue by contradiction assuming that there exists a sequence $(\lambda_n, u_n) \in \Lambda \times X$ satisfying

$$\lambda_n \longrightarrow \lambda \neq \lambda_\infty, \quad \|u_n\| \longrightarrow \infty,$$
$$\Phi(\lambda_n, u_n) = 0, \quad u_n > 0.$$

Setting $v_n = \|u_n\|^{-1} u_n$, we find

$$v_n = K\left(\lambda_n \frac{f(u_n)}{\|u_n\|} - \phi_{u_n} v_n\right).$$

By Lemma 2.1 we infer that there exists a constant $C > 0$ such that

$$\left\|\lambda_n \frac{f(u_n)}{\|u_n\|} - \phi_{u_n} v_n\right\| \leq \left[\left\|\lambda_n \left(m_\infty v_n + \frac{h(u_n)}{\|u_n\|}\right)\right\| + \|\phi_{u_n} v_n\|\right] \leq C, \text{ for all } n \in \mathbb{N}.$$

Since K is compact, from $v_n = K\big(\lambda_n \frac{f(u_n)}{\|u_n\|} - \phi_{u_n} v_n\big)$ we deduce that, up to a subsequence, v_n strongly converges to some $v \in X$ with $\|v\| = 1$. Note also that v_n converges weakly to v in $H_0^1(\Omega)$ and $v \geq 0$ in Ω. Moreover, there holds

$$\int_\Omega \nabla v_n \nabla \varphi dx + \int_\Omega \phi_{u_n} v_n \varphi dx = \int_\Omega \lambda_n \frac{f(u_n)}{\|u_n\|} \varphi dx, \quad \varphi \in H_0^1(\Omega). \tag{4.1}$$

On the other hand, the boundedness of g and the L^p-theory imply that, up to a subsequence, ϕ_{u_n} converges weakly in $H_0^1(\Omega)$ and strongly in X, to some $\phi \in H_0^1(\Omega) \cap X$. Thus, by the Lebesgue dominated convergence theorem we yield

$$\int_\Omega \nabla v \nabla \varphi dx + \int_\Omega \phi v \varphi dx = \int_\Omega \lambda m_\infty v \varphi dx, \quad \varphi \in H_0^1(\Omega), \tag{4.2}$$

which together with Lemma 2.3 implies that $v > 0$ in Ω. As a consequence we get that $u_n(x) = \|u_n\| v_n(x) \longrightarrow \infty$ for all $x \in \Omega$, and applying the Lebesgue dominated convergence theorem we found that

$$\int_\Omega \nabla \phi \nabla \varphi dx = \int_\Omega g(\infty) \varphi dx, \quad \varphi \in H_0^1(\Omega),$$

namely $\phi = \phi_\infty$.

Finally, using φ_∞ as a test function in (4.2) we obtain

$$\lambda_1[\phi_\infty] \int_\Omega v \varphi_\infty dx = \lambda m_\infty \int_\Omega v \varphi_\infty dx,$$

and we conclude that $\lambda_\infty = \lambda$, which is a contradiction. This contradiction proves (a).

Statement (b) follows immediately from (a). Regarding (c), fix any $\lambda < \lambda_\infty$ and take $\Lambda = [0, \lambda]$. For $t \in [0, 1]$, the parameter $t\lambda$ belongs to Λ and from (a) it follows that $u \neq T(t\lambda, u)$

for all $\|u\| \geq r$. This implies that $\tilde{\Phi}(t\lambda, z) \neq 0$ for all $0 < \|z\| \leq 1/r$. Consider the homotopy $H(t,z) = \tilde{\Phi}(t\lambda, z)$. Using the homotopy invariance, we get

$$\deg(\tilde{\Phi}_\lambda, B_{1/r}(0), 0) = \deg(\tilde{\Phi}_0, B_{1/r}(0), 0),$$

namely

$$i(\tilde{\Phi}_\lambda, 0) = \deg(\tilde{\Phi}_0, B_{1/r}(0), 0) = 1,$$

proving (c).

□

Lemma 4.3. *Let $\lambda > \lambda_\infty$. Then*

(a) *there exists $\epsilon > 0$ such that $\Phi_\lambda(u) \neq \tau\varphi_1$, for all $\tau \geq 0$ and $\|u\| \geq \epsilon$,*

(b) *$i(\tilde{\Phi}_\lambda, 0) = 0$ for all $\lambda > \lambda_\infty$.*

Proof. (a) We fix $\lambda > \lambda_\infty$ and we assume, by contradiction, that there exist $\tau_n \geq 0$ and $\|u_n\| \to \infty$ such that $\Phi_\lambda(u_n) = \tau_n\varphi_1$, namely

$$u_n - \tau_n\varphi_1 = K(\lambda f(u_n) - \phi_{u_n}u_n).$$

Setting $v_n = \|u_n\|^{-1}u_n$, we get

$$v_n - \tau_n\|u_n\|^{-1}\varphi_1 = K\left(\lambda\frac{f(u_n)}{\|u_n\|} - \phi_{u_n}v_n\right),$$

and arguing as in Lemma 4.2, one readily shows that the sequence $\left\{\frac{f(u_n)}{\|u_n\|} - \phi_{u_n}v_n\right\}$ is bounded in X. Thus, using the compactness of K, we deduce that, up to a subsequence,

$$K\left(\lambda\frac{f(u_n)}{\|u_n\|} - \phi_{u_n}v_n\right)$$

is convergent and hence $\tau_n/\|u_n\|$ is bounded. Passing again to a subsequence, if necessary, we can assume that $\tau_n/\|u_n\| \longrightarrow \tau \geq 0$ and $u_n/\|u_n\| \longrightarrow v$ with $v \in X$ and $\|v\| = 1$. Arguing as we have done in the proof of Lemma 4.2, we can deduce that $u_n(x) \to \infty$ for all $x \in \Omega$ and that v satisfies

$$\begin{cases} -\Delta v + \phi_\infty v = \lambda m_\infty v + \tau\lambda_1\varphi_1 \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega, \\ \|v\| = 1. \end{cases}$$

Therefore, using φ_∞ as a test function in this problem we obtain

$$\lambda_1[\phi_\infty]\int_\Omega v\varphi_\infty \geq \lambda m_\infty \int_\Omega v\varphi_\infty,$$

and we conclude that $\lambda_\infty \geq \lambda$, which is a contradiction. This proves (a).

(b) Take $\tau = t\|u\|^2$, with $t \in [0,1]$. By (a) it follows that $\Phi_\lambda(u) \neq t\|u\|^2\varphi_1$ for all $\|u\| \geq \epsilon$. This implies

$$\tilde{\Phi}_\lambda(z) \neq t\varphi_1, \quad \forall 0 < \|z\| \leq \frac{1}{\epsilon}, \quad \forall t \in [0,1]. \tag{4.3}$$

Using the homotopy $H(t,z) = \tilde{\Phi}_\lambda(z) - t\varphi_1$ on the ball $B_{1/\epsilon}(0)$ we find

$$i(\tilde{\Phi}_\lambda, 0) = \deg(\tilde{\Phi}_\lambda, B_{1/\epsilon}(0), 0) = \deg(\tilde{\Phi}_\lambda - \varphi_1, B_{1/\epsilon}(0), 0).$$

The latter degree is zero because (4.3), with $t = 1$, implies that $\tilde{\Phi}_\lambda(z) = \varphi_1$ has no solution on $B_{1/\epsilon}(0)$. This proves (b). □

Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Arguing as in the proof of Theorem 1.2, the Lemmas 4.2 and 4.3 ensure that λ_∞ is the unique bifurcation point for the equation $\tilde{\Phi}(\lambda, z) = 0$, and that from $(\lambda_\infty, 0)$ emanates an unbounded continuum of solutions $\tilde{\Sigma}_\infty = \{(\lambda, z) : \tilde{\Phi}(\lambda, z) = 0\}$ in $\mathbb{R} \times X$. Moreover, $(\lambda, z) \in \tilde{\Sigma}_\infty, z \neq 0$, if and only if $(\lambda, z/\|z\|^2) \in \Sigma_\Phi = \{(\lambda, u) : \Phi(\lambda, u) = 0,\ u \neq 0\}$. We define $\Sigma_\infty = \{(\lambda, z/\|z\|^2) : (\lambda, z) \in \tilde{\Sigma}_\infty, z \neq 0\}$. Therefore, $\Sigma_\infty \subset \Sigma_\Phi$ and

$$\tilde{\Sigma}_\infty = \{(\lambda, z) : (\lambda, z/\|z\|^2) \in \Sigma_\infty\} \cup \{(\lambda_\infty, 0)\}$$

is connected and unbounded. This completes the proof. □

Remark 4.4. The reader can ask why we consider only assumption (G_2) in Theorem 1.3. To answer this question, we recall that in the proof of Lemmas 4.2 and 4.3 the boundedness of the sequence $\{\phi_{u_n} v_n\}$ plays a fundamental role. However, under the assumption (G_1), we have the inequality $\|\phi_{u_n}\| \leq C\|u_n\|^{2q}$, which does not ensure the boundedness of the sequence $\{\phi_{u_n} v_n\}$ as $\|u_n\| \longrightarrow \infty$.

5 Multiplicity of solutions

Throughout this section we will use the same notation as in the previous sections. In this section we will apply Theorems 1.2 and 1.3 to show a result of multiplicity of solutions for (P) under additional assumptions on f and g. Specifically, we introduce the following assumptions:

(F_3) $2^{-1}m_\infty t \leq f(t) \leq m_\infty t$ for all $t \geq 0$ and $f'_+(0) = m_\infty$;

(G_3) $g(\infty) = \lim\limits_{t\to\infty} g(t) = \lim\limits_{t\to 0} g(t)$ and $g(\infty) \leq g(t)$ for all $t \in \mathbb{R}$.

Assume that (G_3) is valid. For every $u \in H_0^1(\Omega)$ we have

$$-\Delta\phi_u = g(u) \geq g(\infty) = -\Delta\phi_\infty \text{ in } \Omega,$$

which implies

$$\phi_u \geq \phi_\infty \text{ in } \Omega. \tag{5.1}$$

Moreover, if we define

$$\tilde{g} = \sup_{t\geq 0} g(t) \quad \text{and} \quad -\Delta\phi_{\tilde{g}} = \tilde{g}, \phi_{\tilde{g}} \in H_0^1(\Omega),$$

we can show that $\phi_u \leq \phi_{\tilde{g}}$ in Ω (using the same argument as above) for all $u \in H_0^1(\Omega)$.

Let us denote by $\lambda_1[\phi_{\tilde{g}}]$ and $\varphi_{\tilde{g}}$ the first eigenvalue and the positive eigenfunction normalized by $\|\varphi_{\tilde{g}}\| = 1$, respectively, of the eigenvalue problem

$$-\Delta u + \phi_{\tilde{g}}(x)u = \lambda u \text{ in } \Omega,$$

$$u = 0 \text{ on } \partial\Omega.$$

Let us point out that under the assumptions (F_1)–(F_3) and (G_2)–(G_3) one has $\lambda_0 = \lambda_\infty$. Now we have the following lemma.

Lemma 5.1. *Suppose that* (F_3) *and* (G_3) *hold. If Problem* (S) *has a solution, then* $\lambda_0 \le \lambda \le \lambda_{\tilde{g}}$, *where* $\lambda_{\tilde{g}} = 2\lambda_1[\phi_{\tilde{g}}]/m_\infty$.

Proof. Indeed, if u is a solution of (S), then

$$\begin{aligned}
\lambda_1[\phi_\infty]\int_\Omega u\varphi_\infty dx &= \int_\Omega \nabla u \nabla \varphi_\infty dx + \int_\Omega \phi_\infty u \varphi_\infty dx \\
&\le \int_\Omega \nabla u \nabla \varphi_\infty dx + \int_\Omega \phi_u u \varphi_\infty dx \quad \text{(by (5.1))} \\
&= \lambda \int_\Omega f(u)\varphi_\infty dx \\
&\le \lambda m_\infty \int_\Omega u\varphi_\infty dx \quad \text{(by } (F_3)).
\end{aligned}$$

This now implies $\lambda \ge \lambda_0$.

Similarly,

$$\begin{aligned}
2^{-1}m_\infty \lambda \int_\Omega u\varphi_{\tilde{g}} dx &\le \int_\Omega \lambda f(u)\varphi_{\tilde{g}} dx \\
&= \int_\Omega \nabla u \nabla \varphi_{\tilde{g}} dx + \int_\Omega \phi_u u \varphi_{\tilde{g}} dx \\
&\le \int_\Omega \nabla u \nabla \varphi_{\tilde{g}} dx + \int_\Omega \phi_{\tilde{g}} u \varphi_{\tilde{g}} dx \\
&= \lambda_1[\phi_{\tilde{g}}]\int_\Omega u\varphi_{\tilde{g}} dx,
\end{aligned}$$

whence we infer that $\lambda_{\tilde{g}} \ge \lambda$. This proves the lemma. □

The main result of this section is the following theorem.

Theorem 5.2. *Assume that* (F_1)–(F_3) *and* (G_2)–(G_3) *hold. Then*

(a) $\Sigma_0 = \Sigma_\infty \cup \{(\lambda_\infty, 0)\}$,

(b) *there exists* $\epsilon > 0$ *such that Problem* (P) *has at least two solutions for* $\lambda_0 < \lambda < \lambda_0 + \epsilon$.

Proof. (a) First of all, let us remark that since $\tilde{\Sigma}_\infty$ is connected and $\tilde{\Sigma}_\infty \cap (\mathbb{R} \times \{0\}) = \{(\lambda_\infty, 0)\}$ then $\tilde{\Sigma}_\infty - \{(\lambda_\infty, 0)\}$ is connected. Now, the map $W : \tilde{\Sigma}_\infty - \{(\lambda_\infty, 0)\} \longrightarrow \Sigma_\Phi$ given by

$$W(\lambda, z) = (\lambda, z/\|z\|^2)$$

is continuous. Thus $W(\tilde{\Sigma} - \{(\lambda_*, 0)\}) = \Sigma_\infty$ is a connected subset of Σ_Φ. Using Lemma 5.1 and that λ_∞ is the unique bifurcation point of $\Phi(\lambda, u) = 0$ and the unique bifurcation point from infinity of $\Phi(\lambda, u) = 0$ we can see that $\overline{\Sigma_\infty} = \Sigma_\infty \cup \{(\lambda_\infty, 0)\}$ (which is a connected subset of Σ_Φ too).

Finally, we will show that $\Sigma_0 = \Sigma_\infty \cup \{(\lambda_\infty, 0)\}$. Clearly, $\Sigma_\infty \cup \{(\lambda_\infty, 0)\} \subset \Sigma_0$. We assume now that $(\lambda, u) \in \Sigma_0 - \{(\lambda_\infty, 0)\}$, namely, $u \ne 0$ and

$$u - T(\lambda, u) = 0.$$

Let us write $(\lambda, u) = (\lambda, z/\|z\|^2)$, where $z = u/\|u\|^2$. Thus, the last equality above can be rewritten as

$$\frac{z}{\|z\|^2} - T(\lambda, z/\|z\|^2) = 0,$$

which implies $(\lambda, u) = (\lambda, z/\|z\|^2) \in \Sigma_\infty$.

Then one finds:

$$\Sigma_0 - \{(\lambda_\infty, 0)\} \subset \Sigma_\infty$$

and as $\Sigma_\infty \subset \Sigma_0$ we conclude that $\Sigma_0 = \Sigma_\infty \cup \{(\lambda_\infty, 0)\}$. This proves (a).

(b) Let $u_\lambda \in \Sigma_0$ and $v_\lambda \in \Sigma_\infty$ be the solutions of (P) obtained in Theorems 1.2 and 1.3, respectively. By using the fact that $\|u_\lambda\| \to 0$ and $\|v_\lambda\| \to \infty$ as $\lambda \to \lambda_0$ and Lemma 5.1, we deduce that there exists $\epsilon > 0$ such that

$$\|u_\lambda\| < 1 < \|v_\lambda\| \quad \text{for } \lambda_0 < \lambda < \lambda_0 + \epsilon.$$

This allows us to conclude that $u_\lambda \neq v_\lambda$, and therefore u_λ and v_λ are two distinct solutions of (P) for $\lambda_0 < \lambda < \lambda_0 + \epsilon$. □

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