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# Global bifurcation curves of nonlocal elliptic equations with oscillatory nonlinear term

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**Abstract.** We study the one-dimensional nonlocal elliptic equation of Kirchhoff type with oscillatory nonlinear term. We establish the precise asymptotic formulas for the bifurcation curves $\lambda(\alpha)$ as $\alpha \to \infty$ and $\alpha \to 0$, where $\alpha := \|u_\lambda\|_\infty$ and $u_\lambda$ is the solution associated with $\lambda$. We show that the second term of $\lambda(\alpha)$ is oscillatory as $\alpha \to \infty$.

**Keywords:** nonlocal elliptic equations, oscillatory bifurcation curves, asymptotic formulas.

**2020 Mathematics Subject Classification:** 34C23, 34F10.

## 1 Introduction

We consider the following one-dimensional nonlocal elliptic equation

$$\begin{cases} -(b\|u'\|_2^2+1)u''(x) = \lambda(u(x)^p + u(x)\sin^2 u(x)), \quad x \in I := (0,1), \\ u(x) > 0, \quad x \in I, \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $p > 1, b \geq 0$ are given constants, $\lambda > 0$ is a bifurcation parameter and $\|\cdot\|_2$ denotes the usual $L^2$-norm.

The purpose of this paper is to establish the asymptotic formulas for bifurcation curves $\lambda = \lambda(\alpha)$ of (1.1) as $\alpha \to \infty$ to understand well how the oscillatory term gives effect to the bifurcation curves. Here $\alpha := \|u_\lambda\|_\infty$ and $u_\lambda$ is a solution of (1.1) associated with $\lambda > 0$. When we consider the case where $b = 0$, we use the following notation to avoid the confusion:

$$\begin{cases} -v''(x) = \mu(v(x)^p + v(x)\sin^2 v(x)), \quad x \in I, \\ v(x) > 0, \quad x \in I, \\ v(0) = v(1) = 0, \end{cases} \tag{1.2}$$

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where $\mu > 0$ is the bifurcation parameter. A solution pair of (1.2) is usually represented as $(\mu, v_\mu) \in \mathbb{R}_+ \times C^2(\bar{I})$, where $v_\mu$ is a solution of (1.2) associated with $\mu$. In this paper, we adopt the explicit expression of the solution pair of (1.2), which was introduced in [12, Theorem 2.1]. That is, the solution pair $(\mu, v_\mu) \in \mathbb{R}_+ \times C^2(\bar{I})$ of (1.2) is parametrized by using a new parameter $\alpha > 0$. More precisely, let $\alpha > 0$ be an arbitrary given constant. Then by using the time map argument, we are able to obtain the unique solution pair $(\mu, v_\mu) \in \mathbb{R}_+ \times C^2(\bar{I})$ of (1.2) satisfying $\alpha = \|v_\mu\|_\infty$. Besides, $\mu$ is parametrized by $\alpha$, namely, $\mu = \mu(\alpha)$ and it is a continuous function of $\alpha$. The important point is that the solution pair $(\mu, v_\mu)$ satisfying $\alpha = \|v_\mu\|_\infty$ is parametrized by the supremum norm $\alpha = \|v_\mu\|_\infty$ such as $(\mu(\alpha), v_{\mu(\alpha)})$. For simplicity, we write $v_\alpha := v_{\mu(\alpha)}$ in what follows.

Equation (1.1) is the nonlocal elliptic problem of Kirchhoff type motivated by the problem in [7]:

$$\begin{cases} -A\left(\int_0^1 (u'(x))^q dx\right) u''(x) = \lambda f(u(x)), \quad x \in I, \\ u(0) = u'(1) = 0, \end{cases} \tag{1.3}$$

where $A = A(y)$, which is called Kirchhoff function (cf. [10, 15]), is a continuous function of $y \geq 0$. Nonlocal problems have been investigated by many authors and there are quite many manuscripts which treated the problems with the backgrounds in physics, biology, engineering and so on. We refer to [1–4, 6–9, 11, 13, 14], and the references therein. One of the main interests there are existence, nonexistence and the number of positive and nodal solutions. However, there seems to be a few works which considered (1.3) from a view-point of bifurcation problems. We refer to [16–21] and the references therein. As far as the author knows, there are no works which treat the nonlinear oscillatory eigenvalue problem such as (1.2). Therefore, there seems no works which treat nonlocal bifurcation problems with oscillatory nonlinear term, so our results here seem to be novel. Our approach are mainly the time-map method and the complicated calculation of definite integrals.

The relationship between $\lambda(\alpha)$ and $\mu(\alpha)$ is as follows. Let $\alpha > 0$ be an arbitrary given constant. Assume that there exists a solution pair $(\lambda(\alpha), u_\alpha) \in \mathbb{R} \times C^2(\bar{I})$ with $\|u_\alpha\|_\infty = \alpha$. Then we have

$$-u''_\alpha(x) = \frac{\lambda(\alpha)}{b\|u'_\alpha\|_2^2 + 1}(u_\alpha(x)^p + u_\alpha(x)\sin^2 u_\alpha(x)). \tag{1.4}$$

We note that $\|u_\alpha\|_\infty = \alpha$. Then we find that $u_\alpha = v_\alpha$ and $\frac{\lambda(\alpha)}{b\|u'_\alpha\|_2^2+1} = \mu(\alpha)$, since the solution pair $(\mu(\alpha), v_\alpha) \in \mathbb{R}_+ \times C^2(\bar{I})$ of (1.2) with $\|v_\alpha\|_\infty = \alpha$ is unique (cf. [12]). This implies

$$\lambda(\alpha) = (b\|v'_\alpha\|^2 + 1)\mu(\alpha). \tag{1.5}$$

Therefore, to obtain $\lambda(\alpha)$, we need to obtain both $\mu(\alpha)$ and $\|v'_\alpha\|_2$.

Now we state our results. We first consider the case $p > 2$.

**Theorem 1.1.** *Consider* (1.2)*. Let* $p > 2$*. Then as* $\alpha \to \infty$*,*

$$\mu(\alpha) = 2(p+1)\alpha^{1-p}\left\{C_{0,p} + \left(C_1 + \frac{1}{2}C_{11}\right)\alpha^{1-p} + \frac{1}{2}(C_{12} + C_{21})\alpha^{-p} + \frac{1}{2}C_{22}\alpha^{-(p+1)} + (C_2 + C_3)\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\right\}^2, \tag{1.6}$$

*where*

$$
\begin{aligned}
C_{0,p} &:= \int_0^1 \frac{1}{\sqrt{1-s^{p+1}}}ds, \\
C_1 &:= -\frac{p+1}{8}\int_0^1 \frac{1-s^2}{(1-s^{p+1})^{3/2}}ds, \\
C_{11} &:= \frac{2}{p+1}\int_0^{\pi/2} \cos(2\alpha \sin^{2/(p+1}\theta)\sin^{(3-p)/(p+1)} d\theta, \\
C_{12} &:= \frac{p-1}{2(p+1)}\int_0^{\pi/2} (\sin 2\alpha - \sin(2\alpha\sin^{2/(p+1)}\theta))\sin^{(1-p)/(p+1)}\theta d\theta \\
&\quad + \frac{p+1}{4}\int_0^1 \frac{1-s}{(1-s^{p+1})^{3/2}}\sin(2\alpha s)ds, \\
C_{21} &:= -\frac{1}{p+1}\int_0^{\pi/2} \sin(2\alpha\sin^{2/(p+1)}\theta)\sin^{(3-p)/(p+1)}\theta d\theta \\
C_{22} &:= \frac{4(p-1)}{p+1}\int_0^{\pi/2} (\cos 2\alpha - \cos(2\alpha\sin^{2/(p+1)}\theta))\sin^{(1-p)/(p+1)}\theta d\theta, \\
C_2 &:= \frac{3(p+1)^2}{128}\int_0^1 \frac{(1-s^2)^2}{(1-s^{p+1})^{5/2}}ds, \\
C_3 &:= -\frac{3}{32}(p+1)^2\int_0^1 \left(\int_0^s \frac{1-y^2}{(1-y^{p+1})^{5/2}}dy\right)\cos(2\alpha s)ds.
\end{aligned} \tag{1.7}
$$

**Theorem 1.2.** *Consider (1.2). Let $p > 2$ and $\alpha \gg 1$. Then the following asymptotic formula for $\|v_\alpha'\|_2^2$ holds.*

$$
\|v_\alpha'\|_2^2 = 4\alpha^2\{G_0 + G_1\alpha^{1-p} + G_2\alpha^{-p} + G_3\alpha^{-(p+1)} + G_4\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\},
$$

*where*

$$
\begin{aligned}
G_0 &:= C_{0,p}E_{0,p}, \\
G_1 &:= C_{0,p}E_1 + \left(C_1 + \frac{1}{2}C_{11}\right)E_{0,p}, \\
G_2 &:= \frac{1}{2}\left(C_{12} + C_{12}\right)E_{0,p} + C_{0,p}E_2, \\
G_3 &:= \frac{1}{2}C_{22}E_{0,p} + C_{0,p}E_3, \\
G_4 &:= (C_2 + C_3)E_{0,p} + C_{0,p}E_4 + \left(C_1 + \frac{1}{2}C_{11}\right)E_1, \\
E_{0,p} &:= \int_0^1 \sqrt{1-s^{p+1}}ds, \\
E_1 &:= \frac{p+1}{8}\int_0^1 \frac{1-s^4}{\sqrt{1-s^{p+1}}}ds, \\
E_2 &:= -\frac{1}{4}\int_0^{\pi/2} \{\sin 2\alpha - \sin^{2/(p+1)}\theta\sin(2\alpha\sin^{2/(p+1)}\theta)\}\sin^{(1-p)/(p+1)}\theta d\theta, \\
E_3 &:= -\frac{1}{8}\int_0^1 \{\cos 2\alpha - \cos(2\alpha\sin^{2/(p+1)}\theta)\}\sin^{(1-p)/(p+1)} d\theta, \\
E_4 &:= -\frac{(p+1)^2}{128}\int_0^1 \frac{(1-s^2)^2}{(1-s^{p+1})^{3/2}}ds, \\
E_5 &:= \frac{2}{p+1}\int_0^1 \frac{1-s^{p+1}}{\sqrt{1-s^4}}ds.
\end{aligned}
$$

**Remark 1.3.** We should note that the order of the lower terms of $\mu(\alpha)$ in (1.6) changes according to $p$. Indeed, if we expand the bracket of the r.h.s. of (1.6), then the terms with

$$C_{0,p}^2, \alpha^{1-p}, \alpha^p, \alpha^{-(p+1)}, \alpha^{2(1-p)}, \alpha^{1-2p}$$

appear. Then for $\alpha \gg 1$, clearly, the first term is $C_{0,p}^2$ and the second is $\alpha^{1-p}$. Besides, we have

$$\begin{cases} \alpha^{2(1-p)} \gg \alpha^{-p} \gg \alpha^{1-2p} \gg \alpha^{-(p+1)} & (1 < p < 2), \\ \alpha^{-p} \sim \alpha^{2(1-p)} \gg \alpha^{-(p+1)} \sim \alpha^{1-2p} & (p = 2), \\ \alpha^{-p} \gg \alpha^{2(1-p)} \gg \alpha^{-(p+1)} \gg \alpha^{1-2p} & (2 < p < 3), \\ \alpha^{-p} \gg \alpha^{-(p+1)} \sim \alpha^{2(1-p)} \gg \alpha^{1-2p} & (p = 3), \\ \alpha^{-p} \gg \alpha^{-(p+1)} \gg \alpha^{2(1-p)} \gg \alpha^{1-2p} & (p > 3). \end{cases} \tag{1.8}$$

Therefore, if $p > 2$, then the third term in the bracket of the r.h.s. of (1.6) is $\alpha^{-p}$. However, if $1 < p < 2$, then the third term is $\alpha^{2(1-p)}$. Moreover, if $p$ is very close to 1, then $1 - p \doteqdot 0$. Therefore, we have the sequence of the lower term, which are greater than $\alpha^{-p}$ in (1.6). In principle, it is possible to calculate them precisely. However, since the calculation is long and tedious, we do not carry out here.

**Theorem 1.4.** *Consider* (1.2).

*(i) Let* $1 < p < 2$. *Then as* $\alpha \to \infty$,

$$\mu(\alpha) = 2(p+1)\alpha^{1-p}\left\{C_{0,p} + \left(C_1 + \frac{1}{2}C_{11}\right)\alpha^{1-p} + (C_2 + C_3)\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\right\}^2.$$

*(ii) Let* $p = 2$. *Then as* $\alpha \to \infty$,

$$\mu(\alpha) = 6\alpha^{-1}\left\{C_{0,p} + \left(C_1 + \frac{1}{2}C_{11}\right)\alpha^{-1} + \left(\frac{1}{2}C_{12} + \frac{1}{2}C_{21} + C_2 + C_3\right)\alpha^{-2} + o(\alpha^{2(1-p)})\right\}^2.$$

**Theorem 1.5.** *Consider* (1.2).

*(i) Let* $1 < p < 2$. *Then as* $\alpha \to \infty$,

$$\|v_\alpha'\|_2^2 = 4\alpha^2\{G_0 + G_1\alpha^{1-p} + G_4\alpha^{2(1-p)} + G_2\alpha^{-p} + o(\alpha^{2(1-p)})\}.$$

*(ii) Let* $p = 2$. *Then as* $\alpha \to \infty$,

$$\|v_\alpha'\|_2^2 = 4\alpha^2\{G_0 + G_1\alpha^{-1} + (G_2 + G_4)\alpha^{-2} + o(\alpha^{-2})\}.$$

Theorems 1.4 and 1.5 are obtained directly from Theorems 1.1 and 1.2. So we omit the proofs.

We now consider (1.1).

**Theorem 1.6.** *Consider* (1.1) *with* $b > 0$.

*(i) Let* $p > 2$ *and* $\alpha \gg 1$. *Then the following asymptotic formula for* $\lambda(\alpha)$ *holds.*

$$\begin{aligned}\lambda(\alpha) = 2(p+1)\alpha^{1-p}&\left\{C_{0,p} + \left(C_1 + \frac{1}{2}C_{11}\right)\alpha^{1-p}\right.\\ &\left.+\frac{1}{2}(C_{12} + C_{21})\alpha^{-p} + \frac{1}{2}C_{22}\alpha^{-(p+1)} + C_2\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\right\}^2\\ &\times\left\{4b\alpha^2\{G_0 + G_1\alpha^{1-p} + G_2\alpha^{-p} + G_3\alpha^{-(p+1)} + G_4\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\} + 1\right\}.\end{aligned}$$

*(ii) Let $p = 2$. Then as $\alpha \to \infty$,*

$$\lambda(\alpha) = 6\alpha^{-1}\left\{C_{0,p} + \left(C_1 + \frac{1}{2}C_{11}\right)\alpha^{-1} + \left(\frac{1}{2}C_{12} + \frac{1}{2}C_{21} + C_2 + C_3\right)\alpha^{-2} + o(\alpha^{2(1-p)})\right\}^2 \times \left\{4b\alpha^2\{G_0 + G_1\alpha^{-1} + (G_2 + G_4)\alpha^{-2} + o(\alpha^{-2})\} + 1\right\}.$$

*(iii) Let $1 < p < 2$ Then as $\alpha \to \infty$,*

$$\lambda(\alpha) = 2(p+1)\alpha^{1-p}\left\{C_{0,p} + \left(C_1 + \frac{1}{2}C_{11}\right)\alpha^{1-p} + C_2\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\right\}^2 \times \left\{4b\alpha^2\{G_0 + G_1\alpha^{1-p} + G_4\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\} + 1\right\}.$$

We see from Theorem 1.6 that, roughly speaking, the asymptotic behaviors of $\lambda(\alpha)$ as $\alpha \to \infty$ are:

$$\lambda(\alpha) \sim \alpha^{3-p}.$$

We obtain Theorem 1.6 immediately by (1.5), Theorems 1.1, 1.2, 1.4 and 1.5. So we omit the proof.

Now we establish the asymptotic formulas for $\mu(\alpha)$ as $\alpha \to 0$ to understand the entire structure of $\mu(\alpha)$. We put

$$H_2 := -\frac{2}{p+1}\int_0^1 \frac{1-s^{p+1}}{(1-s^4)^{3/2}}ds,$$
$$H_n := -2^{2n-2}(-1)^n\left\{\frac{1}{(2n-1)!}\int_0^1 \frac{1-s^{2n-1}}{(1-s^4)^{3/2}}ds - \frac{1}{(2n)!}\int_0^1 \frac{1-s^{2n}}{(1-s^4)^{3/2}}ds\right\}$$

for $n \geq 3$. Furthermore, let

$$L_1 := -\frac{p+1}{8}\int_0^1 \frac{1-s^4}{(1-s^{p+1})^{3/2}}ds, \tag{1.9}$$

$$L_2 := -\frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-s^{p+1}}}K(s)ds, \tag{1.10}$$

$$K(s) := -2^3(p+1)\left\{\frac{1}{5!}\frac{1-s^5}{1-s^{p+1}} - \frac{1}{6!}\frac{1-s^6}{1-s^{p+1}} + O(\alpha^{7-p})\right\}. \tag{1.11}$$

**Theorem 1.7.** *Consider (1.2).*

*(i) Let $1 < p < 3$. Then as $\alpha \to 0$,*

$$\mu(\alpha) = 2(p+1)\alpha^{1-p}\left\{C_{0,p} + L_1\alpha^{3-p} + L_2\alpha^{5-p} + O(\alpha^{7-p})\right\}^2. \tag{1.12}$$

*(ii) Let $p = 3$. Then as $\alpha \to 0$,*

$$\mu(\alpha) = 4\alpha^{-2}\left\{C_{0,3} + \frac{1}{2}H_3\alpha^2 + O(\alpha^4)\right\}^2. \tag{1.13}$$

*(iii) Let $3 < p \leq 5$. Then as $\alpha \to 0$,*

$$\mu(\alpha) = 8\alpha^{-2}\left\{C_{0,3} + H_2\alpha^{p-3} + H_3\alpha^2 + O(\alpha^4)\right\}^2. \tag{1.14}$$

*(iv) Assume that $p > 5$. Then as $\alpha \to 0$,*

$$\mu(\alpha) = 8\alpha^{-2} \left\{C_{0,3} + H_3\alpha^2 + o(\alpha^2)\right\}^2. \tag{1.15}$$

Finally, we establish the asymptotic formulas for $\lambda(\alpha)$ as $\alpha \to 0$.

**Theorem 1.8.** *Consider (1.1).*

*(i) Let $1 < p < 3$. Then as $\alpha \to 0$,*

$$\begin{aligned}\lambda(\alpha) = 2(p+1)\alpha^{1-p} &\left\{C_{0,p} + L_1\alpha^{3-p} + L_2\alpha^{5-p} + O(\alpha^{7-p})\right\}^2 \\ &\times \left\{4b\alpha^2 \left\{E_{0,p}C_{0,p} + (E_{0,p}L_1 + C_{0,p}E_1)\alpha^{3-p} + o(\alpha^{3-p})\right\} + 1\right\}.\end{aligned}$$

*(ii) Let $p = 3$. Then as $\alpha \to 0$,*

$$\lambda(\alpha) = 4\alpha^{-2}(1 + 4bE_{0,3}C_{0,3}\alpha^2 + o(\alpha^2)) \left\{C_{0,3} + \frac{1}{2}H_3\alpha^2 + O(\alpha^4)\right\}^2.$$

*(iii) Let $3 < p \le 5$. Then as $\alpha \to 0$,*

$$\begin{aligned}\lambda(\alpha) = 8\alpha^{-2} &\left\{C_{0,3} + H_2\alpha^{p-3} + H_3\alpha^2 + O(\alpha^4)\right\}^2 \\ &\times \left[4b\alpha^2 \left\{C_{0,3} + H_2\alpha^{p-3} + H_3\alpha^2 + O(\alpha^4)\right\} \left\{E_{0,3} + E_5\alpha^{p-3}(1 + o(1))\right\} + 1\right].\end{aligned}$$

*(iv) Let $p > 5$. Then as $\alpha \to 0$,*

$$\begin{aligned}\lambda(\alpha) = 8\alpha^{-2} &\left\{C_{0,3} + H_3\alpha^2 + o(\alpha^2)\right\}^2 \\ &\times \left[4b\alpha^2 \left\{C_{0,3} + H_3\alpha^2 + o(\alpha^2)\right\} \left\{E_{0,3} + E_5\alpha^{p-3}(1 + o(1))\right\} + 1\right].\end{aligned}$$

By Theorem 1.8, we see that as $\alpha \to 0$,

$$\lambda(\alpha) \sim \begin{cases} \alpha^{1-p} & (1 < p \le 3), \\ \alpha^{-2} & (p > 3). \end{cases}$$

By Theorems 1.1, 1.4, 1.6 and 1.7, we obtain the qualitative shapes of $\mu(\alpha)$ and $\lambda(\alpha)$.

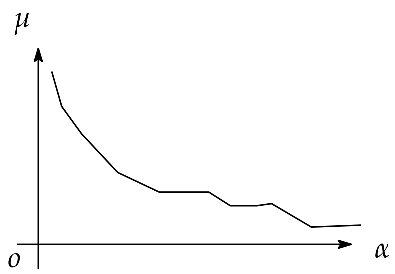


Figure 1.1: The graph of $\mu(\alpha)$

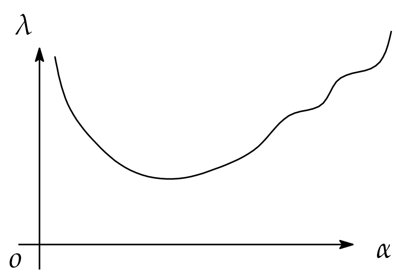


Figure 1.2: The graph of $\lambda(\alpha)$ $(1 < p < 3)$

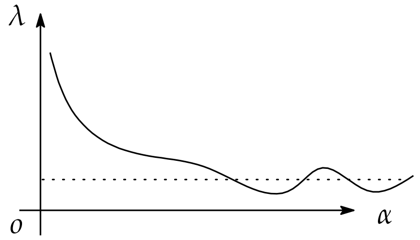


Figure 1.3: The graph of $\lambda(\alpha)$ $(p = 3)$

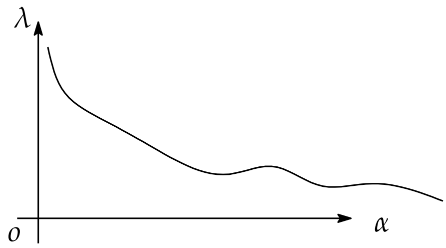


Figure 1.4: The graph of $\lambda(\alpha)$ $(p > 3)$

## 2 Proofs of Theorems 1.1 and 1.2

In this section, let $p > 2$ and we consider (1.2). In what follows, $C$ denotes various positive constants independent of $\alpha \gg 1$. By [5], we know that if $v_\alpha$ is a solution of (1.2), then $v_\alpha$ satisfies

$$v_\alpha(x) = v_\alpha(1-x), \quad 0 \le x \le \frac{1}{2}, \tag{2.1}$$

$$\alpha := \|v_\alpha\|_\infty = v_\alpha\left(\frac{1}{2}\right), \tag{2.2}$$

$$v_\alpha'(x) > 0, \qquad 0 \le x < \frac{1}{2}. \tag{2.3}$$

We put

$$f(\theta) := \theta^p + \theta \sin^2\theta, \tag{2.4}$$

$$F(\theta) := \int_0^\theta f(y)dy = \frac{1}{p+1}\theta^{p+1} + \frac{1}{4}\theta^2 - \frac{1}{4}\theta\sin 2\theta - \frac{1}{8}\cos 2\theta + \frac{1}{8}. \tag{2.5}$$

Let $\alpha > 0$ be an arbitrary given constant. We write $\mu = \mu(\alpha)$ and $v_\alpha := v_{\mu(\alpha)}$ in what follows. By (1.2), for $x \in \bar{I}$, we have

$$\{v_\alpha''(x) + \mu f(v_\alpha(x))\}v_\alpha'(x) = 0.$$

By this and (2.2), for $x \in \bar{I}$, we have

$$\frac{1}{2}v_\alpha'(x)^2 + \mu F(v_\alpha(x)) = \text{constant} = \mu F\left(v_\alpha\left(\frac{1}{2}\right)\right) = \mu F(\alpha).$$

By this and (2.3), for $0 \le x \le 1/2$, we have

$$\begin{aligned} v_\alpha'(x) &= \sqrt{2\mu(F(\alpha) - F(v_\alpha(x)))} \qquad (2.6)\\ &= \sqrt{\frac{2\mu}{p+1}}\sqrt{(\alpha^{p+1} - v_\alpha(x)^{p+1}) + \frac{p+1}{4}(\alpha^2 - v_\alpha(x)^2) - A_\alpha(v_\alpha(x)) - B_\alpha(v_\alpha(x))}, \end{aligned}$$

where

$$A_\alpha(v_\alpha(x)) := \frac{p+1}{4}(\alpha\sin 2\alpha - v_\alpha(x)\sin(2v_\alpha(x))), \tag{2.7}$$

$$B_\alpha(v_\alpha(x)) := \frac{p+1}{8}(\cos 2\alpha - \cos(2v_\alpha(x))). \tag{2.8}$$

Note that $A_\alpha(v_\alpha(x)) \ll \alpha^2, B_\alpha(v_\alpha(x)) \ll \alpha^2$. By this and putting $v_\alpha(x) = \alpha s$, we have

$$\begin{aligned}
\frac{1}{2} &= \int_0^{1/2} 1dx \\
&= \sqrt{\frac{p+1}{2\mu}} \int_0^{1/2} \frac{v_\alpha'(x)dx}{\sqrt{(\alpha^{p+1} - v_\alpha(x)^p) + \frac{p+1}{4}(\alpha^2 - v_\alpha(x)^2) - A_\alpha(v_\alpha(x)) - B_\alpha(v_\alpha(x))}} \\
&= \sqrt{\frac{p+1}{2\mu}} \alpha^{(1-p)/2} \int_0^1 \frac{ds}{\sqrt{(1 - s^{p+1}) + \frac{p+1}{4}\alpha^{1-p}(1-s^2) - \frac{1}{\alpha^{p+1}} A_\alpha(\alpha s) - \frac{1}{\alpha^{p+1}} B_\alpha(\alpha s)}} \\
&= \sqrt{\frac{p+1}{2\mu}} \alpha^{(1-p)/2} \int_0^1 \frac{1}{\sqrt{1 - s^{p+1}}} \frac{ds}{\sqrt{1 + \frac{p+1}{4}\alpha^{1-p}\frac{1-s^2}{1-s^{p+1}} - \frac{1}{\alpha^{p+1}}\frac{A_\alpha(\alpha s)}{1-s^{p+1}} - \frac{1}{\alpha^{p+1}}\frac{B_\alpha(\alpha s)}{1-s^{p+1}}}}.
\end{aligned}$$

This along with Taylor expansion implies that

$$\begin{aligned}
\sqrt{\mu} &= \sqrt{2(p+1)}\alpha^{(1-p)/2} \\
&\quad \times \int_0^1 \frac{1}{\sqrt{1 - s^{p+1}}} \left\{ 1 - \frac{p+1}{8}\alpha^{1-p}\frac{1-s^2}{1-s^{p+1}} + \frac{1}{2}\frac{1}{\alpha^{p+1}}\frac{A_\alpha(\alpha s)}{1-s^{p+1}} + \frac{1}{2}\frac{1}{\alpha^{p+1}}\frac{B_\alpha(\alpha s)}{1-s^{p+1}} \right. \\
&\quad \left. + \frac{3}{8}\left(\frac{p+1}{4}\alpha^{1-p}\frac{1-s^2}{1-s^{p+1}}\right)^2 - \frac{3}{16}(p+1)\alpha^{-2p}\frac{1-s^2}{(1-s^{p+1})^2}A_\alpha(\alpha s) + o(\alpha^{2(1-p)}) \right\} ds \\
&= \sqrt{2(p+1)}\alpha^{(1-p)/2}\left[C_{0,p} + C_1\alpha^{1-p} + I + II + C_2\alpha^{2(1-p)} + III + o(\alpha^{2(1-p)})\right],
\end{aligned} \tag{2.9}$$

where

$$I = \frac{1}{2}\alpha^{-(p+1)} I_1 := \frac{1}{2}\alpha^{-(p+1)} \int_0^1 \frac{A_\alpha(\alpha s)}{(1-s^{p+1})^{3/2}} ds, \tag{2.10}$$

$$II = \frac{1}{2}\alpha^{-(p+1)} II_1 := \frac{1}{2}\alpha^{-(p+1)} \int_0^1 \frac{B_\alpha(\alpha s)}{(1-s^{p+1})^{3/2}} ds, \tag{2.11}$$

$$III = -\frac{3}{16}(p+1)\alpha^{-2p} \int_0^1 \frac{1-s^2}{(1-s^{p+1})^{5/2}} A_\alpha(\alpha s) ds. \tag{2.12}$$

**Lemma 2.1.** *Let $\alpha \gg 1$. Then*

$$I_1 = \int_0^1 \frac{A_\alpha(\alpha s)}{(1-s^{p+1})^{3/2}} ds = C_{11}\alpha^2 + C_{12}\alpha, \tag{2.13}$$

$$II_1 = \int_0^1 \frac{B_\alpha(\alpha s)}{(1-s^{p+1})^{3/2}} ds = C_{21}\alpha + C_{22}. \tag{2.14}$$

*Proof.* We first note that the definite integrals $C_{11}$, $C_{12}$, $C_{21}$, $C_{22}$ exist, since we have $-1 < (1-p)/(p+1) < (3-p)/(p+1)$. We first prove (2.13). We put $s := \sin^{2/(p+1)}\theta$.

Then by integration by parts, we have

$$
\begin{aligned}
I_1 &= \frac{p+1}{4}\alpha\int_0^1 \frac{1}{\sqrt{1-s^{p+1}}}\frac{\sin 2\alpha-\sin(2\alpha s)}{1-s^{p+1}}ds \qquad (2.15)\\
&\quad+\frac{p+1}{4}\alpha\int_0^1\frac{(1-s)}{(1-s^{p+1})^{3/2}}\sin(2\alpha s)ds\\
&=\frac{1}{2}\alpha\int_0^{\pi/2}\frac{1}{\cos^2\theta}\left[\left\{\sin 2\alpha-\sin(2\alpha\sin^{2/(p+1)}\theta)\right\}\sin^{(1-p)/(p+1)}\theta\right]d\theta\\
&\quad+\frac{p+1}{4}\alpha\int_0^1\frac{(1-s)}{(1-s^{p+1})^{3/2}}\sin(2\alpha s)ds\\
&=\frac{1}{2}\alpha\int_0^{\pi/2}(\tan\theta)'\left[\left\{\sin 2\alpha-\sin(2\alpha\sin^{2/(p+1)}\theta)\right\}\sin^{(1-p)/(p+1)}\theta\right]d\theta\\
&\quad+\frac{p+1}{4}\alpha\int_0^1\frac{(1-s)}{(1-s^{p+1})^{3/2}}\sin(2\alpha s)ds\\
&=\frac{1}{2}\alpha\left[\tan\theta\left[\left\{\sin 2\alpha-\sin(2\alpha\sin^{2/(p+1)}\theta\right\}\sin^{(1-p)/(p+1)}\theta\right]\right]_0^{\pi/2} \qquad (*)\\
&\quad-\frac{1}{2}\alpha\int_0^{\pi/2}\frac{\sin\theta}{\cos\theta}\left\{-\frac{4}{p+1}\alpha\cos(2\alpha\sin^{2/(p+1}\theta)\sin^{(2-2p)/(p+1)}\theta\cos\theta\right.\\
&\qquad\qquad\left.-\frac{p-1}{p+1}(\sin 2\alpha-\sin(2\alpha\sin^{2/(p+1)}\theta))\sin^{-2p/(p+1)}\theta\cos\theta\right\}d\theta\\
&\quad+\frac{p+1}{4}\alpha\int_0^1\frac{(1-s)}{(1-s^{p+1})^{3/2}}\sin(2\alpha s)ds\\
&=\frac{2}{p+1}\alpha^2\int_0^{\pi/2}\cos(2\alpha\sin^{2/(p+1}\theta)\sin^{(3-p)/(p+1)}d\theta\\
&\quad+\frac{p-1}{2(p+1)}\alpha\int_0^{\pi/2}(\sin 2\alpha-\sin(2\alpha\sin^{2/(p+1)}\theta))\sin^{(1-p)/(p+1)}\theta d\theta\\
&\quad+\frac{p+1}{4}\alpha\int_0^1\frac{(1-s)}{(1-s^{p+1})^{3/2}}\sin(2\alpha s)ds\\
&=: C_{11}\alpha^2+C_{12}\alpha.
\end{aligned}
$$

We remark that by l'Hôpital's rule and direct calculation, we easily obtain that $(*)$ in (2.15) and $(**)$ in (2.16) below are equal to 0. Next, we put $s := \sin^{2/(p+1)}\theta$. Then by integration by parts, we have

$$
\begin{aligned}
II_1 &= \frac{1}{4}\int_0^{\pi/2}\frac{1}{\cos^2\theta}\left\{\cos 2\alpha-\cos(2\alpha\sin^{2/(p+1)}\theta)\right\}\sin^{(1-p)/(p+1)}\theta d\theta \qquad (2.16)\\
&=\frac{1}{4}\int_0^{\pi/2}(\tan\theta)'\left\{\cos 2\alpha-\cos(2\alpha\sin^{2/(p+1)}\theta)\right\}\sin^{(1-p)/(p+1)}\theta d\theta\\
&=\frac{1}{4}\left[\tan\theta\left\{\cos 2\alpha-\cos(2\alpha\sin^{2/(p+1)}\theta)\right\}\sin^{(1-p)/(p+1)}\theta\right]_0^{\pi/2} \qquad (**)\\
&\quad-\frac{1}{p+1}\alpha\int_0^{\pi/2}\sin(2\alpha\sin^{2/(p+1)}\theta)\sin^{(3-p)/(p+1)}\theta d\theta\\
&\quad+\frac{4(p-1)}{p+1}\int_0^{\pi/2}(\cos 2\alpha-\cos(2\alpha\sin^{2/(p+1)}\theta))\sin^{(1-p)/(p+1)}\theta d\theta\\
&=: C_{21}\alpha+C_{22}.
\end{aligned}
$$

Thus the proof is complete. □

**Lemma 2.2.** *Let $\alpha \gg 1$. Then*

$$III = C_3\alpha^{2(1-p)} + o(\alpha^{2(1-p)}). \tag{2.17}$$

*Proof.* by (2.7) and (2.12), we have

$$\begin{aligned} III &= -\frac{3}{64}(p+1)^2\alpha^{-2p}\int_0^1 \frac{1-s^2}{(1-s^{p+1})^{5/2}}\{\alpha\sin 2\alpha - \alpha s\sin(2\alpha s)\}\,ds \\ &= -\frac{3}{64}(p+1)^2\alpha^{-2p+1}\int_0^1 \frac{1-s^2}{(1-s^{p+1})^{5/2}}\{\sin 2\alpha - \sin(2\alpha s)\}\,ds \\ &\quad -\frac{3}{64}(p+1)^2\alpha^{-2p+1}\int_0^1 \frac{(1-s^2)(1-s)}{(1-s^{p+1})^{5/2}}\sin(2\alpha s)ds. \\ &=: -\frac{3}{64}(p+1)^2\alpha^{-2p+1}III_1 + O(\alpha^{-2p+1}). \end{aligned} \tag{2.18}$$

We show that $III_1 \sim \alpha$. We note that $(1-y^2)/(1-y^{p+1})^{5/2} \le (1-y^2)^{-3/2}$ for $0 \le y \le 1$. By this and integration by parts, we have

$$\begin{aligned} III_1 &= \lim_{\epsilon\to 0}\int_0^{1-\epsilon}\frac{d}{ds}\left(\int_0^s \frac{1-y^2}{(1-y^{p+1})^{5/2}}dy\right)\{\sin 2\alpha - \sin(2\alpha s)\}\,ds \\ &= \lim_{\epsilon\to 0}\left[\left(\int_0^s \frac{1-y^2}{(1-y^{p+1})^{5/2}}dy\right)\{\sin 2\alpha - \sin(2\alpha s)\}\right]_0^{1-\epsilon} \\ &\quad + 2\alpha\lim_{\epsilon\to 0}\int_0^{1-\epsilon}\left(\int_0^s \frac{1-y^2}{(1-y^{p+1})^{5/2}}dy\right)\cos(2\alpha s)ds \\ &= 2\alpha(1+o(1))\int_0^1\left(\int_0^s \frac{1-y^2}{(1-y^{p+1})^{5/2}}dy\right)\cos(2\alpha s)ds. \end{aligned}$$

By this and (2.18), we have (2.17). Thus the proof is complete. □

*Proof of Theorem 1.1.* By (2.9) and Lemma 2.1, for $\alpha \gg 1$, we obtain

$$\begin{aligned} \sqrt{\mu} = \sqrt{2(p+1)}\alpha^{(1-p)/2}\Big\{&C_{0,p} + (C_1 + \frac{1}{2}C_{11})\alpha^{1-p} \\ &+\frac{1}{2}(C_{12}+C_{21})\alpha^{-p} + \frac{1}{2}C_{22}\alpha^{-(p+1)} + (C_2+C_3)\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\Big\}. \end{aligned} \tag{2.19}$$

By this, we obtain Theorem 1.1. Thus the proof is complete. □

We next prove Theorem 1.2.

**Lemma 2.3.** *Let $v_\alpha$ be the solution of (1.2) associated with $\mu > 0$ such that $\|v_\alpha\|_\infty = \alpha > 0$. Then for $\alpha \gg 1$*

$$\|v_\alpha'\|_2^2 = 4\alpha^2\{G_0 + G_1\alpha^{1-p} + G_2\alpha^{-p} + G_3\alpha^{-(p+1)} + G_4\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\}. \tag{2.20}$$

*Proof.* By (2.6), putting $v_\alpha(x) = \alpha s$ and Taylor expansion, we obtain

$$\begin{aligned}
\|v_\alpha'\|_2^2 &= 2\int_0^{1/2} v_\alpha'(x) v_\alpha'(x) dx \qquad (2.21)\\
&= 2\sqrt{\frac{2\mu}{p+1}} \\
&\quad \times \int_0^{1/2} \sqrt{(\alpha^{p+1} - v_\alpha(x)^p) + \frac{p+1}{4}(\alpha^2 - v_\alpha(x)^2) - A_\alpha(v_\alpha(x)) - B_\alpha(v_\alpha(x))} v_\alpha'(x) dx \\
&= 2\sqrt{\frac{2\mu}{p+1}} \alpha^{(p+3)/2} \int_0^1 \sqrt{1 - s^{p+1}} \\
&\quad \times \sqrt{1 + \frac{p+1}{4}\alpha^{1-p}\frac{1-s^2}{1-s^{p+1}} - \frac{1}{\alpha^{p+1}}\frac{A_\alpha(\alpha s)}{1-s^{p+1}} - \frac{1}{\alpha^{p+1}}\frac{B_\alpha(\alpha s)}{1-s^{p+1}}} ds \\
&= 2\sqrt{\frac{2\mu}{p+1}} \alpha^{(p+3)/2} \int_0^1 \sqrt{1 - s^{p+1}} \left\{1 + \frac{p+1}{8}\alpha^{1-p}\frac{1-s^2}{1-s^{p+1}} - \frac{1}{2\alpha^{p+1}}\frac{A_\alpha(\alpha s)}{1-s^{p+1}}\right. \\
&\quad - \frac{1}{2\alpha^{p+1}}\frac{B_\alpha(\alpha s)}{1-s^{p+1}} - \frac{(p+1)^2}{128}\alpha^{2(1-p)}\left(\frac{1-s^2}{1-s^{p+1}}\right)^2 \\
&\quad \left. + \frac{1}{64}(p+1)^2\alpha^{-2p}\frac{1-s^2}{1-s^{p+1}}(\alpha\sin 2\alpha - \alpha s \sin(2\alpha s)) + o(\alpha^{2(1-p)})\right\} ds.
\end{aligned}$$

By putting $s = \sin^{2/(p+1)}\theta$, we have

$$\begin{aligned}
\int_0^1 \frac{A_\alpha(\alpha s)}{\sqrt{1-s^{p+1}}} ds &= \frac{p+1}{4}\alpha \int_0^1 \frac{\sin 2\alpha - s\sin(2\alpha s)}{\sqrt{1-s^{p+1}}} ds \qquad (2.22)\\
&= \frac{1}{2}\alpha \int_0^{\pi/2} \{\sin 2\alpha - \sin^{2/(p+1)}\theta \sin(2\alpha \sin^{2/(p+1)}\theta)\} \sin^{(1-p)/(p+1)}\theta d\theta, \\
\int_0^1 \frac{B_\alpha(\alpha s)}{\sqrt{1-s^{p+1}}} ds &= \frac{p+1}{8} \int_0^1 \frac{\cos 2\alpha - \cos(2\alpha s)}{\sqrt{1-s^{p+1}}} ds \qquad (2.23)\\
&= \frac{1}{4} \int_0^1 \{\cos 2\alpha - \cos(2\alpha \sin^{2/(p+1)}\theta)\} \sin^{(1-p)/(p+1)} d\theta.
\end{aligned}$$

By (2.21)–(2.23), we have

$$\|v_\alpha'\|_2^2 = 2\sqrt{\frac{2\mu}{p+1}}\alpha^{(p+3)/2}\left\{E_{0,p} + E_1\alpha^{1-p} + E_2\alpha^{-p} + E_3\alpha^{-(p+1)} + E_4\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\right\}. \qquad (2.24)$$

By this, (2.19)–(2.24), we have

$$\begin{aligned}
\|v_\alpha'\|_2^2 = 4\alpha^2 &\left\{C_{0,p} + \left(C_1 + \frac{1}{2}C_{11}\right)\alpha^{1-p} + \frac{1}{2}(C_{12} + C_{21})\alpha^{-p}\right. \\
&\quad \left. + \frac{1}{2}C_{22}\alpha^{-(p+1)} + (C_2 + C_3)\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\right\} \\
&\times \left\{E_{0,p} + E_1\alpha^{1-p} + E_2\alpha^{-p} + E_3\alpha^{-(p+1)} + E_4\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\right\} \\
= 4\alpha^2 &\{G_0 + G_1\alpha^{1-p} + G_2\alpha^{-p} + G_3\alpha^{-(p+1)} + G_4\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\}.
\end{aligned}$$

This implies (2.20). Thus the proof is complete. □

## 3 Proof of Theorem 1.7

In this section, let $0 < \alpha \ll 1$. We put $w_\alpha := v_\alpha/\alpha$. By (2.5) and Taylor expansion, we have

$$\begin{aligned} F(\alpha) &= \frac{1}{p+1}\alpha^{p+1} + \frac{1}{4}\alpha^2 - \frac{1}{4}\alpha\left\{2\alpha - \frac{1}{3!}(2\alpha)^3 + \sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{(2n-1)!}(2\alpha)^{2n-1}\right\} \\ &\quad - \frac{1}{8}\left\{1 - \frac{1}{2!}(2\alpha)^2 + \frac{1}{4!}(2\alpha)^4 + \sum_{n=3}^{\infty}\frac{(-1)^n}{(2n)!}(2\alpha)^{2n}\right\} + \frac{1}{8}, \\ F(v_\alpha) &= \frac{1}{p+1}\alpha^{p+1}w_\alpha^{p+1} + \frac{1}{4}\alpha^2 w_\alpha(x)^2 \\ &\quad - \frac{1}{4}\alpha w_\alpha\left\{2\alpha w_\alpha - \frac{1}{3!}(2\alpha w_\alpha)^3 + \sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{(2n-1)!}(2\alpha w_\alpha)^{2n-1}\right\} \\ &\quad - \frac{1}{8}\left\{1 - \frac{1}{2!}(2\alpha w_\alpha)^2 + \frac{1}{4!}(2\alpha w_\alpha)^4 + \sum_{n=3}^{\infty}\frac{(-1)^n}{(2n)!}(2\alpha w_\alpha)^{2n}\right\} + \frac{1}{8}. \end{aligned}$$

By the same argument as that to obtain (2.6), for $0 \le x \le 1$, we have

$$\begin{aligned} \frac{1}{2}\alpha^2 w_\alpha'(x)^2 = \mu\Bigg\{ & \frac{1}{p+1}\alpha^{p+1}(1 - w_\alpha(x)^{p+1}) + \frac{1}{4}\alpha^4(1 - w_\alpha(x)^4) \\ & + \frac{1}{4}\alpha\sum_{n=3}^{\infty}\frac{(-1)^n}{(2n-1)!}2^{2n-1}\alpha^{2n-1}(1 - w_\alpha(x)^{2n-1}) \\ & - \frac{1}{8}\sum_{n=3}^{\infty}\frac{(-1)^n}{(2n)!}2^{2n}\alpha^{2n}(1 - w_\alpha(x)^{2n})\Bigg\}. \end{aligned}$$

We put

$$\begin{aligned} H_\alpha(w_\alpha) &:= \frac{1}{4}\sum_{n=3}^{\infty}\frac{(-1)^n}{(2n-1)!}2^{2n-1}\alpha^{2n}(1 - w_\alpha(x)^{2n-1}) \\ &= \sum_{n=3}^{\infty}\frac{(-1)^n}{(2n-1)!}2^{2n-3}\alpha^{2n}(1 - w_\alpha(x)^{2n-1}), \\ J_\alpha(w_\alpha) &= -\frac{1}{8}\sum_{n=3}^{\infty}\frac{(-1)^n}{(2n)!}2^{2n}\alpha^{2n}(1 - w_\alpha(x)^{2n}) \\ &= -\sum_{n=3}^{\infty}\frac{(-1)^n}{(2n)!}2^{2n-3}\alpha^{2n}(1 - w_\alpha(x)^{2n}). \end{aligned}$$

We put

$$\begin{aligned} M_\alpha(w_\alpha) &:= H_\alpha(w_\alpha) + J_\alpha(w_\alpha(x)) \\ &= \sum_{n=3}^{\infty}(-1)^n 2^{2n-3}\left\{\frac{1}{(2n-1)!}(1 - w_\alpha(x)^{2n-1}) - \frac{1}{(2n)!}(1 - w_\alpha(x)^{2n})\right\}\alpha^{2n}. \end{aligned}$$

By this and (2.3), for $0 \le x \le 1/2$, we have

$$w_\alpha'(x) = \sqrt{2\mu}\alpha^{-1}\sqrt{\frac{1}{p+1}\alpha^{p+1}(1 - w_\alpha(x)^{p+1}) + \frac{1}{4}\alpha^4(1 - w_\alpha(x)^4) + M_\alpha(w_\alpha)}. \qquad (3.1)$$

(i) Let $1 < p < 3$. Then by (3.1), we have

$$w'_\alpha(x) = \sqrt{2\mu\alpha^{-2}}\sqrt{\frac{\alpha^{p+1}}{p+1}}\sqrt{1-w_\alpha(x)^{p+1}}\sqrt{1+\frac{p+1}{4}\alpha^{3-p}\frac{1-w_\alpha(x)^4}{1-w_\alpha(x)^{p+1}}+K(w_\alpha)\alpha^{5-p}}, \quad (3.2)$$

where

$$K(w_\alpha(x)) := -2^3(p+1)\left\{\frac{1}{5!}\frac{1-w_\alpha(x)^5}{1-w_\alpha(x)^{p+1}} - \frac{1}{6!}\frac{1-w_\alpha(x)^6}{1-w_\alpha(x)^{p+1}}\right\}.$$

By (3.2) and Taylor expansion, we have

$$\begin{aligned}\sqrt{\frac{\mu}{2(p+1)}}\alpha^{(p-1)/2} &= \int_0^{1/2}\frac{w'_\alpha(x)}{\sqrt{1-w_\alpha(x)^{p+1}}\sqrt{1+\frac{p+1}{4}\alpha^{3-p}\frac{1-w_\alpha(x)^4}{1-w_\alpha(x)^{p+1}}+K(w_\alpha(x))\alpha^{5-p}}}dx \\ &= \int_0^1 \frac{1}{\sqrt{1-s^{p+1}}}\left\{1-\frac{p+1}{8}\alpha^{3-p}\frac{1-s^4}{1-s^{p+1}}-\frac{1}{2}K(s)\alpha^{5-p}+O(\alpha^{5-p})\right\}ds.\end{aligned}$$

This implies from (1.7), (1.9) and (1.10) that

$$\sqrt{\mu} = \sqrt{2(p+1)}\alpha^{-(p-1)/2}\left\{C_{0,p}+L_1\alpha^{3-p}+L_2\alpha^{5-p}+O(\alpha^{7-p})\right\}. \quad (3.3)$$

This implies (1.12).

(ii) Let $p = 3$. Then by (3.1), we have

$$\begin{aligned}w'_\alpha(x) &= \sqrt{2\mu}\alpha^{-1}\sqrt{\frac{1}{2}\alpha^4(1-w_\alpha(x)^4)+M_\alpha(w_\alpha(x))} \\ &= \sqrt{\mu}\alpha\sqrt{1-w_\alpha(x)^4}\sqrt{1+2\alpha^{-4}\frac{M_\alpha(w_\alpha(x))}{1-w_\alpha(x)^4}}.\end{aligned}$$

This along with Taylor expansion implies that

$$\begin{aligned}\frac{1}{2}\sqrt{\mu} &= \alpha^{-1}\int_0^{1/2}\frac{w'_\alpha(x)}{\sqrt{1-w_\alpha(x)^4}\sqrt{1+2\alpha^{-4}\frac{M_\alpha(w_\alpha(x))}{1-w_\alpha(x)^4}}}dx \\ &= \alpha^{-1}\int_0^1\frac{1}{\sqrt{1-s^4}}\left\{1-\alpha^{-4}\frac{M_\alpha(s)}{1-s^4}+O(\alpha^4)\right\}ds.\end{aligned}$$

By this, we obtain

$$\begin{aligned}\sqrt{\mu} &= 2\alpha^{-1}\int_0^1\frac{1}{\sqrt{1-s^4}}\left\{1+8\alpha^2\left(\frac{1}{5!}\frac{1-s^5}{1-s^4}-\frac{1}{6!}\frac{1-s^6}{1-s^4}\right)+O(\alpha^4)\right\}ds \\ &= 2\alpha^{-1}\left\{C_{0,3}+\frac{1}{2}H_3\alpha^2+O(\alpha^4)\right\}.\end{aligned}$$

This implies (1.13).

(iii) Let $3 < p \le 5$. Then by (3.1), we have

$$\frac{1}{2}\sqrt{2\mu} = 2\alpha\int_0^{1/2}\frac{w'_\alpha(x)}{\alpha^2\sqrt{1-w_\alpha(x)^4}\sqrt{1+\frac{4}{p+1}\alpha^{p-3}\frac{1-w_\alpha(x)^{p+1}}{1-w_\alpha(x)^4}+Q_\alpha(w_\alpha(x))}}dx,$$

where

$$Q_\alpha(w_\alpha) := 4\alpha^{-4}\frac{M_\alpha(w_\alpha)}{1-w_\alpha(x)^4}.$$

By this and Taylor expansion, we have

$$\begin{aligned}\sqrt{\frac{\mu}{2}} &= 2\alpha^{-1}\int_0^1 \frac{1}{\sqrt{1-s^4}} \\ &\quad \times \left\{1-\frac{2}{p+1}\alpha^{p-3}\frac{1-s^{p+1}}{1-s^4}+\alpha^2\frac{2^4}{5!}\frac{1-s^5}{1-s^4}-\alpha^2\frac{2^4}{6!}\frac{1-s^6}{1-s^4}+O(\alpha^4)\right\}ds \\ &= 2\alpha^{-1}\left\{C_{0,3}+H_2\alpha^{p-3}+H_3\alpha^2+O(\alpha^4)\right\}.\end{aligned} \tag{3.4}$$

This implies (1.14).

(iv) Assume that $p > 5$. Then by (3.4), we have

$$\sqrt{\frac{\mu}{2}} = 2\alpha^{-1}\left\{C_{0,3}+H_3\alpha^2+o(\alpha^2)\right\}. \tag{3.5}$$

This implies (1.15). Thus the proof of Theorem 1.7 is complete. □

## 4 Proof of Theorem 1.8

In this section, we assume that $0 < \alpha \ll 1$. By Taylor expansion, we have

$$\begin{aligned}v_\alpha(x)\sin^2 v_\alpha(x) &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n-1}}{(2n)!}v_\alpha(x)^{2n+1} \\ &= v_\alpha(x)^3-\frac{1}{3}v_\alpha(x)^5+\frac{2}{45}v_\alpha(x)^7+O(v_\alpha(x)^9).\end{aligned} \tag{4.1}$$

(i) Let $1 < p < 3$. Then by (2.6), (4.1), Taylor expansion and putting $v_\alpha = \theta = \alpha s$, we have

$$\begin{aligned}\|v_\alpha'\|_2^2 &= 2\int_0^{1/2} v_\alpha'(x)v_\alpha'(x)dx \\ &= 2\sqrt{2\mu}\int_0^{1/2}\sqrt{\frac{1}{p+1}(\alpha^{p+1}-v_\alpha(x)^{p+1})+\frac{1}{4}(\alpha^4-v_\alpha(x)^4)(1+o(1))}\,v_\alpha'(x)dx \\ &= 2\sqrt{2\mu}\int_0^{\alpha}\sqrt{\frac{1}{p+1}(\alpha^{p+1}-\theta^{p+1})+\frac{1}{4}(\alpha^4-\theta^4)(1+o(1))}\,d\theta \\ &= 2\sqrt{\frac{2\mu}{p+1}}\alpha^{(p+3)/2}\int_0^1\sqrt{1-s^{p+1}}\sqrt{1+\frac{p+1}{4}\alpha^{3-p}\frac{1-s^4}{1-s^{p+1}}(1+o(1))}\,ds \\ &= 2\sqrt{\frac{2\mu}{p+1}}\alpha^{(p+3)/2}\int_0^1\sqrt{1-s^{p+1}}\left\{1+\frac{p+1}{8}\alpha^{3-p}\frac{1-s^4}{1-s^{p+1}}(1+o(1))\right\}ds \\ &= 2\sqrt{\frac{2}{p+1}}\sqrt{\mu}\alpha^{(p+3)/2}\left\{E_{0,p}+E_1\alpha^{3-p}+o(\alpha^{3-p})\right\}.\end{aligned} \tag{4.2}$$

By this and (3.3), we have

$$\begin{aligned}\|v_\alpha'\|_2^2 &= 2\sqrt{\frac{2}{p+1}}\alpha^{(p+3)/2}\left\{E_{0,p}+E_1\alpha^{3-p}+o(\alpha^{3-p})\right\} \\ &\quad \times\sqrt{2(p+1)}\alpha^{-(p-1)/2}\left\{C_{0,p}+L_1\alpha^{3-p}+L_2\alpha^{5-p}+O(\alpha^{7-p})\right\} \\ &= 4\alpha^2\left\{E_{0,p}C_{0,p}+(E_{0,p}L_1+C_{0,p}E_1)\alpha^{3-p}+o(\alpha^{3-p})\right\}.\end{aligned}$$

By this, (1.5) and Theorem 1.7 (i), we have

$$\begin{aligned}\lambda(\alpha) &= 2(p+1)\alpha^{1-p}\left\{C_{0,p}+L_1\alpha^{3-p}+L_2\alpha^{5-p}+O(\alpha^{7-p})\right\}^2 \\ &\quad \times\left\{4b\alpha^2\left\{E_{0,p}C_{0,p}+(E_{0,p}L_1+C_{0,p}E_1)\alpha^{3-p}+o(\alpha^{3-p})\right\}+1\right\}.\end{aligned}$$

(ii) Let $p=3$. Then by (4.2) and putting $s=v_\alpha(x)/\alpha$, we have

$$\begin{aligned}\|v_\alpha'\|_2^2 &= 2\sqrt{\mu}(1+o(1))\int_0^{1/2}\sqrt{\alpha^4-v_\alpha(x)^4}v_\alpha'(x)dx \\ &= 2\sqrt{\mu}(1+o(1))\alpha^3\int_0^1\sqrt{1-s^4}ds \\ &= 2\sqrt{\mu}\alpha^3E_{0,3}(1+o(1)).\end{aligned}$$

By this, (1.5) and Theorem 1.7 (ii), we have

$$\begin{aligned}\|v_\alpha'\|_2^2 &= 2\alpha^3E_{0,3}(1+o(1))2\alpha^{-1}\left\{C_{0,3}+\frac{1}{2}H_3\alpha^2+O(\alpha^4)\right\} \\ &= 4\alpha^2E_{0,3}C_{0,3}(1+o(1)).\end{aligned}$$

By this and Theorem 1.7 (ii), we have

$$\lambda(\alpha)=4\alpha^{-2}(1+4bE_{0,3}C_{0,3}\alpha^2+o(\alpha^2))\left\{C_{0,3}+\frac{1}{2}H_3\alpha^2+O(\alpha^4)\right\}^2.$$

We next consider the case $p>3$. By (4.1), for $0<x<1/2$, we have

$$\frac{1}{2}v_\alpha'(x)^2+\mu\left\{\frac{1}{4}v_\alpha(x)^4+\frac{1}{p+1}v_\alpha(x)^{p+1}(1+o(1))\right\}=\mu\left\{\frac{1}{4}\alpha^4+\frac{1}{p+1}\alpha^{p+1}(1+o(1))\right\}.$$

By this, for $0\le x\le 1/2$, we have

$$v_\alpha'(x)=\sqrt{\frac{\mu}{2}}\sqrt{\alpha^4-v_\alpha(x)^4}\sqrt{1+\frac{4}{p+1}\frac{\alpha^{p+1}-v_\alpha(x)^{p+1}}{\alpha^4-v_\alpha(x)^4}(1+o(1))}.$$

By this, (3.5) and the same calculation as that of (4.2) and putting $v_\alpha(x)=\alpha s$, we have

$$\begin{aligned}\|v_\alpha'\|_2^2 &= \sqrt{2\mu}\alpha^3\int_0^1\sqrt{1-s^4}\sqrt{1+\frac{4}{p+1}\alpha^{p-3}\frac{1-s^{p+1}}{1-s^4}(1+o(1))}ds \\ &= \sqrt{2\mu}\alpha^3\left\{E_{0,3}+E_5\alpha^{p-3}(1+o(1))\right\} \\ &= 4\alpha^2\left\{C_{0,3}+H_2\alpha^{p-3}+H_3\alpha^2+O(\alpha^4)\right\}\left\{E_{0,p}+E_5\alpha^{p-3}(1+o(1))\right\}.\end{aligned} \tag{4.3}$$

(iii) Let $3 < p \le 5$. Then by (1.5), (3.5) and (4.3), we have

$$\lambda(\alpha) = 8\alpha^{-2}\left\{C_{0,3} + H_2\alpha^{p-3} + H_3\alpha^2 + O(\alpha^4)\right\}^2 \times \left[4b\alpha^2\left\{C_{0,3} + H_2\alpha^{p-3} + H_3\alpha^2 + O(\alpha^4)\right\}\left\{E_{0,3} + E_5\alpha^{p-3}(1+o(1))\right\} + 1\right].$$

(iv) Let $p > 5$. Then by (1.8), (3.5) and (4.3), we have

$$\lambda(\alpha) = 8\alpha^{-2}\left\{C_{0,3} + H_3\alpha^2 + o(\alpha^2)\right\}^2 \times \left[4b\alpha^2\left\{C_{0,3} + H_3\alpha^2 + o(\alpha^2)\right\}\left\{E_{0,3} + E_5\alpha^{p-3}(1+o(1))\right\} + 1\right].$$

Thus the proof of Theorem 1.8 is complete. □

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