Electronic Journal of Qualitative Theory of Differential Equations
2024, No. **30**, 1–14; https://doi.org/10.14232/ejqtde.2024.1.30 www.math.u-szeged.hu/ejqtde/

On the existence of patterns in reaction-diffusion problems with Dirichlet boundary conditions

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Received 16 October 2023, appeared 26 June 2024
Communicated by Sergei Trofimchuk

Abstract. Consider a general reaction-diffusion problem, $u_t = \Delta u + f(x, u, u_x)$, on a revolution surface or in an n-dimensional ball with Dirichlet boundary conditions. In this work, we provide conditions related to the geometry of the domain and the spatial heterogeneities of the problem that ensure the existence or not of a non-constant stationary stable solution. Several applications are presented, particularly with regard to the Allen–Cahn, Fisher–KPP and sine-Gordon equations.

Keywords: patterns, Dirichlet boundary conditions, surface of revolution.

2020 Mathematics Subject Classification: 35B35, 35B36.

1 Introduction

In this work we consider the following problem

$$\begin{cases} u_t = \Delta u + f(x, u, u_x), & (t, x) \in \mathbb{R}^+ \times \Omega, \\ u(t, x) = B, & (t, x) \in \mathbb{R}^+ \times \partial\Omega \end{cases} \tag{1.1}$$

where f is a C^1 function, $B \in \mathbb{R}$ and Ω is a surface of revolution in $\mathbb{R}^3$ or is an n-dimensional ball. We say that U is a *stationary solution* of (1.1) if U is a solution of (1.1) independent of temporal variable t, that is

$$\begin{cases} \Delta U + f(x, U, U_x) = 0, & x \in \Omega, \\ U(x) = B, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

A stationary solution U of (1.1) is called *stable* if for every $\eta > 0$ there exists $\delta > 0$ such that for every solution v to (1.1) satisfying $\|v(0, \cdot) - U(\cdot)\|_{L^\infty} < \delta$ it holds that $\|v(t, \cdot) - U(\cdot)\|_{L^\infty} < \eta$, for all $t > 0$. Finally, if U is a non-constant stable stationary solution of (1.1), then U is commonly referred to as the *spatial pattern* or simply *pattern*.

The study of reaction-diffusion equations has been a central focus in the field of mathematical modeling for several decades. These systems have wide-ranging applications in various scientific disciplines, including chemistry, biology, physics, and ecology. One of the intriguing phenomena that often arises in reaction-diffusion systems is the spontaneous formation of spatial patterns. These patterns, which can take on diverse shapes and structures, emerge as a consequence of the interplay between the underlying reaction kinetics and the diffusion of the interacting species [9,20].

In this work, we are interested in investigating the role of spatial heterogeneities as well as the domain geometry concerning the existence or not of patterns. The literature on this subject is extensive, mainly when Neumann boundary conditions are considered. The difficulty in obtaining results with Dirichlet boundary conditions leads to a reduced number of studies. Here, we cite [8,11,21], where the authors achieve results for one-dimensional problems, and [13] for problems in n-dimensional balls. Some results on surfaces of revolution can be found in [19].

Our proposal to study the problem on surfaces of revolution is motivated by the recent interest of the scientific community in these domains [3,4,14,16,18,19], and success is primarily attributed to the well-established symmetry properties of stable solutions in this domain (similar phenomena are observed in the case of balls in $\mathbb{R}^n$). Such symmetry leads us to one-dimensional problems, and this is crucial for obtaining the results.

The proposed ideas can be applied in various situations. In this study, we provide several examples involving the Allen–Cahn, Fisher–KPP, and sine-Gordon problems. These choices were made given the significant relevance of these models. However, as evident, the potential applications extend to many other cases, including reaction-convection-diffusion problems.

The work is divided as follows. In Section 2, we present preliminary results related to the existence and uniqueness of solutions for one-dimensional nonlinear second-order problems. In this section, we illustrate how to obtain results regarding the existence or not of patterns in one-dimensional problems, underscoring the significance of this section in its own right. In Section 3, we present the main results for the problem on surfaces of revolution, whereas Section 4 is dedicated to the sine-Gordon problem in an n-dimensional ball. Finally, in Section 5, we provide some concluding remarks.

2 Preliminaries and some general one-dimensional results

In this section, we will present three general results on the existence and uniqueness of solutions for certain elliptic problems in a interval. In this case, for the sake of simplicity, we will replace the notation u_x with u'. Results of this type are commonly understood when $f(x,u,u')$ satisfies a specific uniform Lipschitz condition, assuming the interval length for the variable u where the problem occurs is sufficiently small. However, in numerous scenarios, it is necessary to extend this result to include functions $f(x,u,u')$ that are Lipschitz not for all u but solely for u within a bounded interval. This is what is accomplished in the first theorem below, which considers Dirichlet conditions at the boundary.

Additionally, it is crucial to highlight that instead of the typical Lipschitz condition, we presume a set of one-sided conditions which, while not more restrictive, proves to be considerably more practical. Further elaboration on this matter can be found in [1,2,6].

The subsequent results in this section are related to a function $g \in C^1$ such that

$$g(s,0,0) = 0 \tag{2.1}$$

and

$$G_1(u-v,u'-v') \le g(s,u,u') - g(s,v,v') \le G_2(u-v,u'-v') \tag{2.2}$$

where

$$G_2(u,u') = \begin{cases} M_2u' + K_2u, & u \geq 0,\ u' \geq 0, \\ M_1u' + K_2u, & u \geq 0,\ u' \leq 0, \\ M_1u' + K_1u, & u \leq 0,\ u' \leq 0, \\ M_2u' + K_1u, & u \leq 0,\ u' \geq 0, \end{cases} \tag{2.3}$$

$$G_1(u,u') = \begin{cases} M_1u' + K_1u, & u \geq 0,\ u' \geq 0, \\ M_2u' + K_1u, & u \geq 0,\ u' \leq 0, \\ M_2u' + K_2u, & u \leq 0,\ u' \leq 0, \\ M_1u' + K_2u, & u \leq 0,\ u' \geq 0, \end{cases} \tag{2.4}$$

and $M_i, K_i \in \mathbb{R}$ $(i = 1, 2)$ are constant.

The next three theorems are crucial to all the results of this work.

Theorem 2.1 (Theorem 1 in [1]). *For $(s, u, u') \in [0, L] \times J \times \mathbb{R}$, where J is a closed interval in $\mathbb{R}$, let $g(s, u, u')$ be a continuous function satisfying (2.1) and (2.2). If the two problems $(i = 1, 2)$*

$$\begin{cases} u_i''(s) + G_i(u_i(s), u_i'(s)) = 0, & s \in (a, b), \\ u_i(a) = A', \quad u_i(b) = B', \end{cases} \tag{2.5}$$

have unique solutions on every sub-interval $[a, b]$ of $[0, L]$ for arbitrary A', B', and if for $a = 0$, $b = L$, $A' = A$, $B' = B$ the ranges of u_i $(i = 1, 2)$ are subsets of J, then the problem

$$\begin{cases} u''(s) + g(s, u(s), u'(s)) = 0, \\ u(0) = A, \quad u(L) = B, \end{cases} \tag{2.6}$$

has a unique solution $u(s)$, which remains in J and it satisfies

$$u_1(s) \leq u(s) \leq u_2(s),$$

where u_1 and u_2 are solutions of (2.5) with G_1 and G_2, respectively, and $a = 0$, $b = L$, $A' = A$, $B' = B$.

Before stating the next theorem, we define

$$\alpha(M,K) = \begin{cases} \dfrac{2}{\sqrt{4K - M^2}} \cos^{-1}\left(\dfrac{M}{2\sqrt{K}}\right), & \text{if } 4K - M^2 > 0, \\ \dfrac{2}{\sqrt{M^2 - 4K}} \cosh^{-1}\left(\dfrac{M}{2\sqrt{K}}\right), & \text{if } 4K - M^2 < 0,\ M > 0,\ K > 0, \\ \dfrac{2}{M}, & \text{if } 4K - M^2 = 0,\ M > 0, \\ +\infty, & \text{otherwise} \end{cases} \tag{2.7}$$

and

$$\beta(M,K) = \begin{cases} \dfrac{2}{\sqrt{4K - M^2}} \cos^{-1}\left(\dfrac{-M}{2\sqrt{K}}\right), & \text{if } 4K - M^2 > 0, \\ \dfrac{2}{\sqrt{M^2 - 4K}} \cosh^{-1}\left(\dfrac{-M}{2\sqrt{K}}\right), & \text{if } 4K - M^2 < 0,\ M < 0,\ K > 0, \\ \dfrac{-2}{M}, & \text{if } 4K - M^2 = 0,\ M < 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.8}$$

The next theorem is fundamental for verifying the existence and uniqueness of solution to problems in (2.5) in the theorem above.

Theorem 2.2 (Theorem 1 in [2])**.** *Let $G(y, y')$ be a continuous real valued function satisfying $G(0, 0) = 0$ and (2.2) (assuming $g(s, y, y') = G(y, y')$) with G_1 and G_2 defined in (2.4) and (2.3). If*

$$L < \alpha(M_2, K_2) + \beta(M_1, K_2),$$

then the boundary value problem

$$\begin{cases} u''(s) + G(u(s), u'(s)) = 0, & s \in (0, L), \\ u(0) = A', \quad u(L) = B' \end{cases} \tag{2.9}$$

has a unique solution for every pair of real numbers A', B'.

Below, we present a new result regarding existence and uniqueness, specifically for mixed boundary conditions. In particular, we will use this theorem to investigate the sine-Gordon problem in an n-dimensional ball.

Theorem 2.3 (Theorem 1 in [6])**.** *Let $g(s, y, y')$ be a continuous real valued function satisfying (2.2) with G_1 and G_2 defined in (2.4) and (2.3). If*

$$L < \beta(M_1, K_2),$$

then the mixed boundary value problem

$$\begin{cases} u''(s) + g(s, u(s), u'(s)) = 0, & s \in (0, L), \\ u'(0) = A, \quad u(L) = B \end{cases} \tag{2.10}$$

has a unique solution for every pair of real numbers A, B.

With the aforementioned theorems, it is not difficult to derive results regarding the non-existence of patterns in one-dimensional problems and zero Dirichlet boundary conditions. Although this is not the main objective of this work, we present a simple example below.

Example 2.4. Consider the following problem

$$\begin{cases} u_t = u_{xx} + \rho(x)u(1-u), & (t, x) \in \mathbb{R}^+ \times (0, L), \\ u(0) = 0, \quad u(L) = 0, \end{cases} \tag{2.11}$$

where ρ is a continuous function with sign-changing or not. Note that this includes the important Fisher–KPP equation which will be further elucidated in the subsequent section.

In this case we consider $J = [0, 1]$ and then, $g(x, u) = \rho(x)u(1-u)$ satisfies (2.1) and (2.2) with G_1 and G_2 given by (2.4) and (2.3) if

$$M_1 = M_2 = 0, \quad K_1 = -\sup_{x \in [0,L]} |\rho(x)| \text{ and } K_2 = \sup_{x \in [0,L]} |\rho(x)|.$$

Now, in order to use Theorem 2.1 we have to analyse

$$\begin{cases} z'' + G_i(z) = 0, & (0, L), \\ z(0) = A', \quad z(L) = B' \end{cases} \tag{2.12}$$

with $i = 1, 2$. We can use Theorem 2.2 to conclude that if $L < \alpha(M_2, K_2) + \beta(M_1, K_2)$, that is

$$\frac{2}{\sqrt{4K_2 - M_2^2}} \cos^{-1}\left(\frac{M_2}{2\sqrt{K_2}}\right) + \frac{2}{\sqrt{4K_2 - M_1^2}} \cos^{-1}\left(\frac{-M_1}{2\sqrt{K_2}}\right) = \frac{\pi}{2\sqrt{K_2}} + \frac{\pi}{2\sqrt{K_2}} = \frac{\pi}{\sqrt{K_2}} > L$$

or

$$K_2 < (\pi/L)^2,$$

then (2.12) has a unique solution (for $i = 1, 2$) for any $A', B' \in \mathbb{R}$. In particular, it is easy to see that if $A' = B' = 0$, then $z \equiv 0 \in J = [0, 1]$ is the unique solution. Finally, Theorem 2.1 yields that, in these conditions, $u \equiv 0$ is the unique stationary solution of (2.11), and thus, (2.11) does not admit patterns.

Remark 2.5. Note that the interval J ($= [0, 1]$ in the above example) is associated with the range of variation of u and with the inequalities in (2.2). Evidently, its choice affects the values of M_i and K_i, and consequently the application of the results of existence and uniqueness of solution to determine whether patterns emerge or not.

Below, we present a simple example of pattern existence for a problem with mixed boundary conditions. In this case, the chosen nonlinearity is related to the sine-Gordon equation.

Example 2.6. Consider the following problem with mixed boundary conditions

$$\begin{cases} u_t = (e^{5x} u_x)_x + (x + 6)\sin(u), & (t, x) \in \mathbb{R}^+ \times (0, 1), \\ u_x(t, 0) = 1/2, \quad u(t, 1) = 1/4. \end{cases} \tag{2.13}$$

This problem can be written as

$$\begin{cases} \dfrac{u_t}{e^{5x}} = u_{xx} + 5u_x + \dfrac{(x + 6)}{e^{5x}} \sin(u), & (t, x) \in \mathbb{R}^+ \times (0, 1), \\ u_x(t, 0) = 1/2, \quad u(t, 1) = 1/4, \end{cases} \tag{2.14}$$

and the corresponding stationary problem is

$$\begin{cases} u_{xx} + 5u_x + \dfrac{(x + 6)}{e^{5x}} \sin(u) = 0, & x \in (0, 1), \\ u_x(0) = 1/2, \quad u(1) = 1/4. \end{cases} \tag{2.15}$$

We note that $h(x, u, u_x) = 5u_x + \frac{(x+6)}{e^{5x}} \sin(u)$ satisfies (2.2) with G_1 and G_2 defined in (2.4) and (2.3) with

$$M_1 = M_2 = 5, \quad K_1 = -6, \quad K_2 = 6.$$

A simple analysis of (2.8) shows that

$$\beta(M_1, K_2) = \infty.$$

From Theorem 2.3, it follows that (2.15) has a unique solution U. Now note that $E : \{u \in H^1(0, 1); u(1) = 1/4\} \to \mathbb{R}$ defined by

$$E[u] = \int_0^1 \frac{e^{5x}}{2}(u_x)^2 - F(u, x)dx + \frac{u(0)}{2},$$

where $F(u,x) = (x+6)\int_0^u \sin(\sigma)d\sigma$ is the energy functional associated with (2.15), its critical points are solutions to (2.15). Now, we state that E serves as a strict Lyapunov functional for (2.13) i.e., except at stationary states, $E[u(t,\cdot)]$ is strictly decreasing on orbits. To verify this, we take a solution u of (2.13) and a function $v \in H^1(0,1)$ such that $v(1) = 0$. Then

$$u_t = (e^{5x}u_x)_x + (x+6)\sin(u)$$

and we can multiply this equation by v, integrate on $(0,1)$ and use integration by parts to achieve

$$\begin{aligned}
\int_0^1 vu_t dx &= \int_0^1 ve^{5x}u_{xx}dx + \int_0^1 v5e^{5x}u_x dx + \int_0^1 v(x+6)\sin(u)dx \\
&= \int_0^1 v5e^{5x}u_x dx + v(1)e^5 u_x(t,1) - v(0)u_x(t,0) - \int_0^1 (ve^{5x})_x u_x dx \\
&\quad + \int_0^1 v(x+6)\sin(u)dx \\
&= \int_0^1 v5e^{5x}u_x dx - \frac{v(0)}{2} - \int_0^1 v_x e^{5x}u_x dx - \int_0^1 v5e^{5x}u_x dx \\
&\quad + \int_0^1 v(x+6)\sin(u)dx.
\end{aligned}$$

We can cancel the first and fourth term of the last equality to obtain

$$\int_0^1 vu_t dx = -\frac{v(0)}{2} - \int_0^1 v_x e^{5x}u_x dx + \int_0^1 v(x+6)\sin(u)dx. \tag{2.16}$$

Now observe that if u is a solution of (2.13), then $u_t(t,\cdot) \in H^1(0,1)$, $u(t,1) = 1/4$ for all t, and thus $u_t(t,1) = 0$. If we differentiate $E[u(t,\cdot)]$ with respect to t, we obtain

$$\frac{d}{dt}E[u(t,\cdot)] = \int_0^1 e^{5x}u_x u_{tx}dx - \int_0^1 (x+6)\sin(u)u_t dx + \frac{u_t(t,0)}{2}. \tag{2.17}$$

We can compare (2.16) and (2.17) to get

$$\frac{d}{dt}E[u(t,\cdot)] = -\int_0^1 (u_t)^2 dx.$$

Therefore, we have a system with a gradient structure, and then the bounded trajectories of (2.13) approach the set of stationary solutions (for the reader's convenience, we cite [7] for topics related to the dynamics of (2.13) and [15, Chapter 2] for results related to the existence and boundedness of solutions of (2.13)). Since U is non-constant and the only stationary solution of the problem, we conclude that U is a pattern as defined above.

3 Surfaces of revolution

Considering a smooth curve C in $\mathbb{R}^3$ parameterized by $(\psi(s), 0, \chi(s))$, where $s \in [l_1, l_2]$ ($[0,1] \subset (l_1, l_2)$), with $\psi(l_1) = \psi(l_2) = 0$, we can generate a borderless surface of revolution $\mathcal{M}$. This surface can be parametrized by

$$x = (\psi(s)\cos(\theta), \psi(s)\sin(\theta), \chi(s)), \quad (s,\theta) \in [l_1, l_2] \times [0, 2\pi). \tag{3.1}$$

Let $\mathcal{M}$ be the surface of revolution parametrized by (3.1). We also assume that $\psi, \chi \in C^2$, $\psi > 0$ in (l_1, l_2), $(\psi_s)^2 + (\chi_s)^2 = 1$ and $\chi_s(s) \geq 0$ in $[l_1, l_2]$. Moreover, $\psi_s(l_1) = -\psi_s(l_2) = 1$, and as stated above, we assume $\psi(l_1) = \psi(l_2) = 0$.

By setting $x^1 = s$ and $x^2 = \theta$, we can conclude that the surface of revolution $\mathcal{M}$, with the above parametrization, is a 2-dimensional Riemannian manifold with the metric

$$g = ds^2 + \psi^2(s)d\theta^2. \tag{3.2}$$

$\mathcal{M}$ has no boundary, and we always assume that $\mathcal{M}$ and the Riemannian metric g on it are smooth (see [5], for instance). The area element on $\mathcal{M}$ is given by $d\sigma = \psi d\theta ds$, and the gradient of u with respect to the metric g is given by

$$\nabla_g u = \left(\partial_s u, \frac{1}{\psi^2}\partial_\theta u\right).$$

The Laplace–Beltrami operator Δ_g on $\mathcal{M}$ can be expressed as

$$\Delta_g u = u_{ss} + \frac{\psi_s}{\psi}u_s + \frac{1}{\psi^2}u_{\theta\theta}. \tag{3.3}$$

We consider $\mathcal{S} \subset \mathcal{M}$ as a surface of revolution with a boundary parameterized by

$$x = (\psi(s)\cos(\theta), \psi(s)\sin(\theta), \chi(s)), \quad (s, \theta) \in [0,1] \times [0, 2\pi). \tag{3.4}$$

Hence, $\partial\mathcal{S} = \mathcal{C}_0 \cup \mathcal{C}_1$, where $\mathcal{C}_0$ and $\mathcal{C}_1$ are two circles parameterized by $(\theta \in [0, 2\pi))$

$$(\psi(0)\cos(\theta), \psi(0)\sin(\theta), \chi(0))$$

and

$$(\psi(1)\cos(\theta), \psi(1)\sin(\theta), \chi(1)),$$

respectively.

Theorem 3.1. *Consider the following problem on a surface $\mathcal{S}$ as defined above*

$$\begin{cases} u_t = \Delta_g u + h(x,u), \quad (t,x) \in \mathbb{R}^+ \times \mathcal{S}, \\ u(t,x) = 0, \quad (t,x) \in \mathbb{R}^+ \times \partial\mathcal{S} = \mathbb{R}^+ \times (\mathcal{C}_0 \cup \mathcal{C}_1), \end{cases} \tag{3.5}$$

where h is a function of class C^1 and $h(\cdot, \eta)$ is independent of angular variation. Suppose that

(a) $\tilde{h}(s,u,u_s) := \frac{\psi_s}{\psi}u_s + h(s,u)$ satisfies (2.1) and (2.2) with G_i ($i = 1,2$) given by (2.3), (2.4) for $(s,u,u_s) \in [0,1] \times J \times \mathbb{R}$ where $J \subset \mathbb{R}$ is a closed interval containing 0;

(b) $\alpha(M_2, K_2) + \beta(M_1, K_2) > 1$ where α and β are numbers defined in (2.7) and (2.8).

Then problem (3.5) does not admit patterns.

Proof. First, we observe that stable stationary solutions of (3.5) must be independent of angular variation. This is a well-known result that can be seen in [3, 14]. Thus, due to (3.3), we can conclude that if u is a stable stationary solution of (3.5), then u satisfies:

$$\begin{cases} u_{ss} + \dfrac{\psi_s}{\psi}u_s + h(s,u) = 0, \quad s \in [0,1], \\ u(0) = u(1) = 0. \end{cases} \tag{3.6}$$

Our goal now is to prove that problem (3.6) has a unique solution $u \equiv 0$. To achieve this, we use Theorem 2.1.

According to hypothesis (a), $\tilde{h}(s,u,u_s) = \frac{\psi_s}{\psi}u_s + h(s,u)$ satisfies (2.1) and (2.2) with G_i given by (2.3) and (2.4), and it is the first part of Theorem 2.1. Note that, by hypothesis (b), we can use Theorem 2.2 twice (with $L = 1$ and $G = G_1$ and again with $G = G_2$) to conclude that the two problems $(i = 1,2)$

$$\begin{cases} u_i'' + G_i(u_i(s), u_i'(s)) = 0, \\ u_i(a) = A', \quad u_i(b) = B' \end{cases} \tag{3.7}$$

have unique solutions on every sub-interval $[a,b] \subset [0,1]$ for arbitrary A', B'.

Finally, the problems $(i = 1,2)$

$$\begin{cases} z'' + G_i(z,z') = 0, \quad s \in [0,1], \\ z(0) = z(1) = 0, \end{cases} \tag{3.8}$$

has $z = 0 \in J$ as solution, which allows us to utilize Theorem 2.1 and conclude that $u \equiv 0$ is the unique solution of (3.6). Hence, it follows that problem (3.5) does not admit the existence of patterns, and the theorem is proved. □

3.1 The Allen–Cahn problem

The aim now is to apply Theorem 3.1 to some relevant cases commonly found in the literature. In this subsection, we address the nonlinearity of Allen–Cahn. In this case, we consider problem (3.5) with $h(x,u) = u - u^3$, i.e.

$$\begin{cases} u_t = \Delta_g u + u - u^3, \quad (t,x) \in \mathbb{R}^+ \times \mathcal{S}, \\ u(t,x) = 0, \quad (t,x) \in \mathbb{R}^+ \times \partial\mathcal{S} = \mathbb{R}^+ \times (\mathcal{C}_0 \cup \mathcal{C}_1). \end{cases} \tag{3.9}$$

As we know, a stable solution of (3.9) must satisfy

$$\begin{cases} u_{ss} + \dfrac{\psi_s}{\psi}u_s + u - u^3 = 0, \quad s \in [0,1], \\ u(0) = u(1) = 0. \end{cases} \tag{3.10}$$

A simple computation shows that $\tilde{h}(s,u,u_s) = \frac{\psi_s}{\psi}u_s + u - u^3$ satisfies (2.2) if we consider, for example: $J = [0,1]$, G_1 and G_2 given by (2.4) and (2.3), respectively, with

$$M_1 = \inf_{s\in[0,1]} \left\{ \frac{\psi_s}{\psi} \right\}, \quad M_2 = \sup_{s\in[0,1]} \left\{ \frac{\psi_s}{\psi} \right\}, \quad K_1 = -2 \text{ and } K_2 = 1. \tag{3.11}$$

Hence, if we assume ψ such that

$$\alpha(M_2, K_2) + \beta(M_1, K_2) > 1,$$

we have the hypothesis (b) and we can use Lemma 2.2 to ensure that the problems $(i = 1,2)$

$$\begin{cases} z'' + G_i(z,z') = 0, \quad s \in [0,1], \\ z(0) = z(1) = 0, \end{cases} \tag{3.12}$$

have unique solutions $u_i \equiv 0 \in [0,1]$ $(i = 1,2)$.

Remark 3.2. The conditions (a) and (b) of Theorem 3.1 involve the geometry of the domain (represented by the function ψ) along with the reaction term of the problem. In particular, $\frac{\psi'}{\psi}$ (see also (3.11)) represents the geodesic curvature of the parallel circles $s =$ constant on $\mathcal{S}$.

Example 3.3. Consider the problem (3.9) where $\mathcal{S}_1$ is a finite straight cylinder, that is, $\psi(s) = 1$ ($\chi(s) = s + 1$) for all $s \in [0, 1]$. In this case, $M_1 = M_2 = 0$, $K_1 = -2$ and $K_2 = 1$. Thus, it easy to see that

$$\alpha(0, 1) + \beta(0, 1) = \frac{\pi}{2} + \frac{\pi}{2} = \pi > 1$$

and we can conclude that there are no patterns for this case.

Similarly, if $\psi(s) = s^2/4 + 1/2$ and $\chi(s) = \frac{s}{4}\sqrt{4 - s^2} + \arcsin(s/2)$ for $s \in [0, 1]$, then $\mathcal{S}_2$ resembles a frustum of a hyperboloid (see figure below) and we have $M_1 = 0$, $M_2 = 2/3$, $K_1 = -2$ and $K_2 = 1$. It follows that

$$\alpha(2/3, 1) + \beta(0, 1) > 1$$

and again there are no patterns for the problem (3.9) in this case.

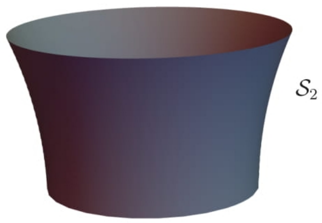

Figure 3.1: Surface of revolution $\mathcal{S}_2$

Our results can also be applied to spatially heterogeneous problems. For instance, we can consider $a \in C^1(\mathcal{S})$ as a positive diffusivity coefficient and $b \in C^1(\mathcal{S})$ as a positive reaction coefficient multiplying $u - u^3$. In this case, the problem becomes:

$$\begin{cases} u_t = \mathrm{div}_g(a(x)\nabla u) + b(x)(u - u^3), \quad (t, x) \in \mathbb{R}^+ \times \mathcal{S}, \\ u(t, x) = 0, \quad (t, x) \in \mathbb{R}^+ \times \partial\mathcal{S} = \mathbb{R}^+ \times (\mathcal{C}_0 \cup \mathcal{C}_1). \end{cases} \tag{3.13}$$

If we assume the functions a and b are independent of angular variation, then we have

$$\begin{cases} u_t = au_{ss} + \dfrac{(a\psi)_s}{\psi} u_s + \dfrac{a}{\psi^2} u_{\theta\theta} + b(u - u^3), \quad (t, s, \theta) \in \mathbb{R}^+ \times [0, 1] \times [0, 2\pi), \\ u(t, 0, \theta) = u(t, 1, \theta) = 0, \quad (t, \theta) \in \mathbb{R}^+ \times [0, 2\pi), \end{cases} \tag{3.14}$$

and the stable solutions satisfy

$$\begin{cases} u_{ss} + \tilde{h}(s, u, u_s) = 0, \quad s \in [0, 1], \\ u(0) = u(1) = 0, \end{cases} \tag{3.15}$$

where $\tilde{h}(s, u, u_s) = \frac{(a(s)\psi(s))_s}{a(s)\psi(s)} u_s(s) + \frac{b(s)}{a(s)}(u(s) - u^3(s))$.

In this case, $\tilde{h}$ satisfies (2.1) and satisfies (2.2) if we consider

$$M_1 = \inf_{s\in[0,1]} \left\{ \frac{(a\psi)_s}{a\psi} \right\}, \tag{3.16}$$

$$M_2 = \sup_{s\in[0,1]} \left\{ \frac{(a\psi)_s}{a\psi} \right\}, \tag{3.17}$$

$$K_1 = \inf_{s\in[0,1]} \left\{ \frac{-2b(s)}{a(s)} \right\} \tag{3.18}$$

and

$$K_2 = \sup_{s\in[0,1]} \left\{ \frac{b(s)}{a(s)} \right\}. \tag{3.19}$$

Now, if we proceed as before, we find that if a, b, ψ are taken such that

$$\alpha(M_2, K_2) + \beta(M_1, K_2) > 1$$

occurs, we can also conclude the non-existence of patterns for this spatially heterogeneous problem.

3.2 The Fisher–KPP problem

A similar analysis can also be conducted for the Fisher–KPP problem. In this case, considering $a, b \in C^1(\mathcal{S})$, we have

$$\begin{cases} u_t = \mathrm{div}_g(a(x)\nabla u) + b(x)(u - u^2), & (t,x) \in \mathbb{R}^+ \times \mathcal{S}, \\ u(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \partial\mathcal{S} = \mathbb{R}^+ \times (\mathcal{C}_0 \cup \mathcal{C}_1). \end{cases} \tag{3.20}$$

Similar to the previous case, several instability results can be derived from the relationship between functions a and b, and the geometry of $\mathcal{S}$ represented by the function ψ. However, now, we will demonstrate with examples how we can utilize the ideas developed here to obtain results of the existence of patterns for problems with non-zero Dirichlet boundary conditions. In this case, we once again make use of the symmetry of stable solutions, and as usual (see [3, 14, 18]), we analyze the existence of stable solution to the problem

$$\begin{cases} u_t = au_{ss} + \dfrac{(a(s)\psi)_s}{\psi} u_s + b(s)(u - u^2), & (t,s) \in \mathbb{R}^+ \times [0,1], \\ u(t,0) = A, & t \in \mathbb{R}^+, \\ u(t,1) = B, & t \in \mathbb{R}^+. \end{cases} \tag{3.21}$$

Example 3.4. Consider $a \equiv b \equiv 1$ and $\mathcal{S}$ again a finite straight cylinder ($\psi(s) = 1$ and $\chi(s) = s + 1$ for all $s \in [0,1]$). Then we consider the following Fisher–KPP problem with non-zero Dirichlet boundary conditions

$$\begin{cases} u_t = \Delta_g u + (u - u^2), & (t,x) \in \mathbb{R}^+ \times \mathcal{S}, \\ u(t,x) = 1/3, & (t,x) \in \mathbb{R}^+ \times \mathcal{C}_0, \\ u(t,x) = 1/2, & (t,x) \in \mathbb{R}^+ \times \mathcal{C}_1. \end{cases} \tag{3.22}$$

In this case, we have to analyze the following problem

$$\begin{cases} u_{ss} + (u - u^2) = 0, & s \in [0,1], \\ u(0) = 1/3, & u(1) = 1/2. \end{cases} \tag{3.23}$$

It is not difficult to see that $h(u) = u - u^2$ satisfies (2.1) and (2.2) with $J = [0,1]$ and

$$M_1 = M_2 = 0, \quad K_1 = -1, \quad K_2 = 1.$$

Moreover,

$$\begin{aligned}\alpha(M_2, K_2) + \beta(M_1, K_2) &= \frac{2}{\sqrt{4K_2 - M_2^2}} \cos^{-1}\left(\frac{M_2}{2\sqrt{K_2}}\right) + \frac{2}{\sqrt{4K_2 - M_1^2}} \cos^{-1}\left(\frac{-M_1}{2\sqrt{K_2}}\right) \\ &= \frac{\pi}{2} + \frac{\pi}{2} = \pi > 1.\end{aligned}$$

Hence, we can use Lemma 2.2 to ensure that

$$\begin{cases} z'' + z = 0, & s \in (0,1), \\ z(a) = A', & z(b) = B' \end{cases} \tag{3.24}$$

and

$$\begin{cases} z'' - z = 0, & s \in (0,1), \\ z(a) = A', & z(b) = B' \end{cases} \tag{3.25}$$

have unique solutions on every sub-interval $[a,b]$ of $[0,1]$ for arbitrary A', B'. Finally, in order to apply Theorem 2.1 we have to consider the problems (3.24) and (3.25) with $a = 0$, $b = 1$, $A' = 1/3$ e $B' = 1/2$. After a few calculations, it's not hard to see that

$$z_1(s) = \frac{2\cos(s) - 2\cot(1)\sin(s) + 3\csc(1)\sin(s)}{6}$$

and

$$z_2(s) = \frac{e^{-s}(-3e + 2e^2 - 2e^{2s} + 3e^{(1+2s)})}{6(e^2 - 1)}$$

are solutions of (3.24) and (3.25) respectively (with $a = 0$, $b = 1$, $A' = 1/3$ e $B' = 1/2$), and both solutions have range contained in $[0,1]$. According to Theorem 2.1, problem (3.23) has a unique solution U. Now we proceed as in the Example 2.6. The energy functional $E : \{u \in H^1(\mathcal{S}); u(x) = 1/3 \text{ for } x \in \mathcal{C}_0,\ u(x) = 1/2 \text{ for } x \in \mathcal{C}_1\} \to \mathbb{R}$ associated with the problem (3.23) is defined by

$$E[u] = \int_{\mathcal{S}} \frac{1}{2}|\nabla_g u|^2 + F(u)dx$$

where $F(u) = \int_0^u s - s^2 ds$. It is routine to verify that (3.22) is a gradient system (see Example 2.6), so we can conclude that U is a pattern.

Once again, depending on computational capacity, one can contemplate more general problems involving heterogeneities, different boundary values, and alternative surfaces.

4 Sine-Gordon equation in an n-dimensional ball

This section is dedicated to studying the sine-Gordon equation. In this equation, we have $f(u) = \sin(u)$ and since f is globally bounded, we can, in this case, analyze the problem in an n-dimensional ball $\mathcal{B}$ centered at the origin with a radius equal to 1. Hence, we consider the following problem

$$\begin{cases} u_t = \operatorname{div}(a(x)\nabla u) + b(x)\sin(u), & (t,x) \in \mathbb{R}^+ \times \mathcal{B}, \\ u(t,x) = B, \quad (t,x) \in \mathbb{R}^+ \times \partial\mathcal{B}, \end{cases} \tag{4.1}$$

where a and b are functions of class C^1 with radial symmetry, and $B \in \mathbb{R}$.

It is well-known that stable solutions of (4.1) are radially symmetric; thus, if u is a stable solution it satisfies (for simplicity, we consider $n = 2$)

$$\begin{cases} u_{rr} + \dfrac{(a(r)r)_r}{a(r)r} u_r + \dfrac{b(r)}{a(r)} \sin(u) = 0, & r \in (0,1), \\ u_r(0) = 0, \quad u(1) = B. \end{cases} \tag{4.2}$$

We can state the following theorem.

Theorem 4.1. *Consider (4.1) and suppose that*

$$h(r,u,u_r) := \frac{(a(r)r)_r}{a(r)r} u_r + \frac{b(r)}{a(r)} \sin(u)$$

satisfies (2.2) with G_1 and G_2 defined in (2.4) and (2.3), respectively, and $\beta(M_1, K_2) > 1$. Then, if $B = 2k\pi$ ($k \in \mathbb{Z}$) (4.1) does not admit patterns.

Proof. A direct application of Theorem 2.3 gives us that the problem (4.2) with $B = 2k\pi$ ($k \in \mathbb{Z}$) has $u \equiv 2k\pi$ as its unique solution. Therefore, (4.1) does not admit patterns. □

Example 4.2. If a and b are taken such that $a(r) = e^{5r}/r$ and $b(r) = (r+6)e^{5r}/r$ then $h(r,u,u_r) = 5u_r + (r+6)\sin(u)$. Hence h satisfies (2.2) with

$$M_1 = M_2 = 5, \quad K_1 = -6, \quad K_2 = 6.$$

It follows that $\beta(M_1, K_2) = \infty$ e therefore (4.1) does not admit patterns if $B = 2k\pi$ ($k \in \mathbb{Z}$).

5 Concluding remarks

In this paper, we present a straightforward and efficient approach to studying pattern formation in problems with Dirichlet boundary conditions. The symmetry of the domains under consideration, along with the well-known properties of stable solutions, enabled us to leverage results on the existence, uniqueness, and stability of solutions in one-dimensional problems to achieve our objectives. Below, we provide some concluding remarks that complement the ideas discussed thus far.

(i) Evidently, the problem of sine-Gordon could be considered on revolution surfaces as before, and then results of existence or non-existence of patterns would also be generated for this case. On the other hand, the absence of a result like Theorem 2.1 for problems with mixed boundary conditions prevents us from considering the nonlinearities of Allen–Cahn and Fisher–KPP in an n-dimensional ball.

(ii) The examples presented in this work serve the purpose of illustrating how one can apply the developed theory. In this regard, the parameters (surfaces and heterogeneities) were chosen in a way to simplify the computations. Particularly, in Example 4.2, the choice of the diffusion coefficient $a(r) = e^{5r}/r$ made the problem more straightforward and allowed us to use Theorem 4.1.

(iii) Similarly, other equations can be considered beyond those highlighted in this work (namely Allen–Cahn, Fisher–KPP and sine-Gordon equations). For instance, with a nonlinearity of the form $f(u,x) = u(u-\theta(x))(1-u)$, where $0 < \theta(x) < 1$, which is related to the *Fife–Greenlee equation* [10], or the *perturbed sine-Gordon equation* where $f(u) = \sin(u) - g(u)$ [17], or even in problems with advection terms, that is, in reaction–convection–diffusion problems, see [12] and references therein.

Acknowledgements

The author is partially supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Grant/Award Number: 311893/2022-8; Fundação de Amparo à Pesquisa do Estado de Minas Gerais (FAPEMIG), Grant/Award Number: RED-00133-21 and CAPES/Humboldt Research Scholarship Program, Number: 88881.876233/2023-01.

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