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Topological dimensions of random attractors for a stochastic reaction-diffusion equation with delay

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Abstract. The aim of this paper is to obtain an estimation of Hausdorff as well as fractal dimensions of random attractors for a stochastic reaction-diffusion equation with delay. The stochastic equation is firstly transformed into a delayed random partial differential equation by means of a random conjugation, which is then recast into an auxiliary Hilbert space. For the obtained equation, it is firstly proved that it generates a random dynamical system (RDS) in the auxiliary Hilbert space. Then it is shown that the equation possesses random attractors by a uniform estimate of the solution and the asymptotic compactness of the generated RDS. After establishing the variational equation in the auxiliary Hilbert space and the almost surely differentiable properties of the RDS, upper estimates of both Hausdorff and fractal dimensions of the random attractors are obtained.

Keywords: Hausdorff dimension, fractal dimension, random dynamical system, random attractors, delay, stochastic reaction-diffusion equations.

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1 Introduction

Existence and estimation of topological dimensions of attractors play important roles in the study of the long time behavior of deterministic or random dynamical systems. For many infinite dimensional systems generated by deterministic or stochastic partial differential equations and delay differential equations, the existence of attractors can reduce the essential part of the flow to a compact set. The finite dimensionality of the attractors, which represents the

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number of degrees of freedom presented in the long term dynamics of the system can further simplify global dynamics of complex nonlinear systems and hence it is of great significance.

The theory of attractors for deterministic infinite dimensional dynamical systems has been well established (see the monograph [26]). On the other hand, the study of random attractors for RDSs dates back to the pioneer works [15,16,24], where H. Crauel, F. Flandoli, B. Schmalfuß, amongst others, generalized the concept of global attractors of infinite dimensional dissipative systems and established the basic framework of random attractors for infinite dimensional RDSs. Since then, the existence, dimension estimation and qualitative properties of random attractors for various stochastic nonlinear evolution equations or stochastic functional differential equations have been investigated by many researchers. For example, for the stochastic reaction-diffusion equation without time delay, Caraballo et al. [7], Gao et al. [25] and Li and Guo [33] explored the existence of global attractors on bounded domains. In [2], [42] and [45], the authors obtained the existence of global attractors on unbounded domains. For the stochastic reaction diffusion equation with delay, the existence of random attractors and their structure have been studied in [5,8,12,32,41] and the references therein.

Criteria for the finite Hausdorff dimensionality of attractors for deterministic fluid dynamics models have been derived by Douady and Oesterle [20], which were later generalized by Constantin, Foiaș and Temam [13] (see also Temam [40]). Then, it was further extended to the stochastic case in [17] and [37], where the RDS is first linearized and the global Lyapunov exponents of the linearized mapping is later examined. The main difficulty of this method lies in controlling the difference between the original nonlinear RDS and its linearization, since in the stochastic case, the attractor is a random set which is not uniformly bounded. Debussche showed that the random attractors of many random dynamical systems generated by dissipative evolution equations have finite Hausdorff dimension by an ergodicity argument in [18] and further derived a precise bound on the dimension by combining the method of linearization and Lyapunov exponents in [19]. With respect to the fractal dimensionality of random sets, Langa proved the finite fractal dimensionality of the random attractor associated to a model from fluid dynamics in [30]. Langa and Robinson generalized the method in [19] to the fractal dimension by requiring differentiability of RDS in [31]. Recently, the above established framework was generalized and adopted to various stochastic and random evolution equations. For instance, Fan proved the existence of random attractor and obtained an upper bound of the Hausdorff and fractal dimension of the random attractor for a stochastic wave equations in [23] by using the method in [19]. In the recent work [46], Zhou and Zhao proved the finiteness of fractal dimension of random attractor for stochastic damped wave equation with linear multiplicative white noise.

Despite the fact that the finite Hausdorff and fractal dimensionality of attractors for abstract RDSs and applications to stochastic partial differential equations (SPDEs) have been extensively and intensively studied, to the best of our knowledge, the estimation of dimensions of SPDEs with delay, i.e., the stochastic partial functional differential equations (SPFDEs) have not been extensively studied. There are only some early results on the existence and local stability of solutions [6,27,39] and recent results on the existence and qualitative properties of random attractors [28,29,32,41,45]. Indeed, the dimension estimation of attractors for delayed partial differential equations is scarce even for the deterministic case. To this respect, the only works about dimensions of attractors for partial functional differential equations (PFDEs) we could find are [38] and the very recent work [36]. In this paper, we make an attempt to estimate topological dimensions of random attractors for a stochastic delayed reaction-diffusion equation. Specifically, we consider the following SPFDE with additive noise

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \Delta u(x,t) - \mu u(x,t) + f\left(u(x,t-\tau)\right) + \sum_{j=1}^{m} g_j(x)\frac{\mathrm{d}w_j(t)}{\mathrm{d}t}, t > 0, x \in \mathcal{O}, \\ u_0(x,s) = \phi(x,s), -\tau \le s \le 0, x \in \mathcal{O} \\ u_0(x,t) = 0, -\tau \le t, x \in \partial\mathcal{O}, \end{cases} \tag{1.1}$$

where $\mathcal{O} \subseteq \mathbb{R}^N$ is a bounded open domain with smooth boundary $\partial\mathcal{O}$, $\{w_j\}_{j=1}^m$ are mutually independent two-sided real-valued Wiener process on an appropriate probability space to be specified below. Equation (1.2) can model many processes from chemistry or mathematical biology. For instance, it can be used to describe the evolution of mature populations for age-structured species, where Δu and μu represent the spatial diffusion and the death rate of mature individuals, τ is a positive number, representing the maturation time. The maturation time $f\left(u(x,t-\tau)\right)$ represents birth rate, $\sum_{j=1}^{m} g_j \frac{\mathrm{d}w_j(t)}{\mathrm{d}t}$ stands for the random perturbations or environmental effects.

Let $\mathbb{X} = L^2(\mathcal{O})$ be the space of square Lesbegue integrable functions on $\mathcal{O}$ with its usual norm $\|\cdot\|_{\mathbb{X}}$ and inner product $(\cdot,\cdot)_{\mathbb{X}}$, $\mathcal{C} = C([-\tau,0],\mathbb{X})$ be the space of continuous function from $[-\tau,0]$ to $\mathbb{X}$ with the usual supremum norm $\|\cdot\|_{\mathcal{C}}$ and $A = \Delta$. Let $u \in C([-\tau,T],\mathbb{X})$ and for each $t \in [0,T]$ define the function $u_t : [-\tau,0] \to \mathbb{X}$ by $u_t(\xi) = u(t+\xi)$ for $\xi \in [0,T]$. Then, we can rewrite the term $\tilde{f}(u(t-\tau)) = f(u_t)$ for any $u \in C([-\tau,T]$, by simply defining $f(\phi) = \tilde{f}(\phi(-\tau))$, for $\phi \in \mathcal{C}$ (notice that we are identifying the function $\tilde{f}$ in problem (1.1) with its associated Nemitskii operator: $\tilde{f}(u(x,t-\tau) \equiv \tilde{f}(u(t-\tau))(x)$ for all $x \in \mathcal{O}$ and $t \in [0,T]$). However, in order to deal with weak solutions of problem (1.1), we need to have the functional $\tilde{f}$ well defined in a bigger space than $\mathcal{C}$, namely we will extend the definition of $\tilde{f}$ to the Hilbert space $\mathcal{L} \triangleq L^2([-\tau,0],\mathbb{X})$ of all square Lebesgue integral functions from $[-\tau,0]$ to $\mathbb{X}$ equipped with the inner product $(\varphi,\psi)_{\mathcal{L}} = [\int_{-\tau}^{0}(\varphi(s),\psi(s))_{\mathbb{X}}^2 ds]^{1/2}$ and norm $\|\varphi\|_{\mathcal{L}} = [\int_{-\tau}^{0}\|\varphi(s)\|_{\mathbb{X}}^2 ds]^{1/2}$ for all $\varphi \in \mathcal{L}$. This can be done by imposing appropriate assumptions on the function f (or equivalently on $\tilde{f}$). This is explained in details in the next section (see also [10,11]). From now on we will identify the notation of f and its extension to the space $\mathcal{L}$.

Then (1.1) can be written as the following abstract SPFDE in $\mathbb{X} = L^2(\mathcal{O})$

$$\frac{du(t)}{dt} = Au(t) - \mu u(t) + f\left(u_t\right) + \sum_{j=1}^{m} g_j \frac{\mathrm{d}w_j(t)}{\mathrm{d}t}. \tag{1.2}$$

The main difficulty for studying the topological dimensions of (1.2) lies in the fact that the natural phase spaces for deterministic or stochastic PFDEs are Banach spaces while all the above mentioned theories are established for dynamical systems in Hilbert spaces. Hence, in [38], the authors associated the deterministic PFDE with a nonlinear semigroup on a product space, i.e. a Hilbert space. In this paper, we extend the method established in [38] and [36] to the stochastic case. Nevertheless, the extension is not trivial since the RDSs are nonautonomous in nature and the random attractor is not uniformly bounded. In [38], the authors assumed that the deterministic PFDEs are dissipative which directly implies the existence of attractors in the auxiliary Hilbert space. In this paper, we will firstly prove the existence of a random attractor for (1.2) in the auxiliary Hilbert space and then provide explicit upper bounds of the Hausdorff and fractal dimensionality for the obtained attractor.

The rest of this paper is organized as follows. In Section 2, we introduce some notation, hypotheses and recast (1.2) into a Hilbert space. In Section 3, we prove the obtained auxiliary equation admits a global mild solution which generates a RDS and possesses a random attractor under certain conditions. In Section 4, we obtain an upper bound of the Hausdorff and fractal dimensions for the random attractor of the auxiliary equation, which directly implies the finite dimensionality of the original equation (1.2). Finally, we conclude the paper by pointing out some potential research directions.

2 Auxiliary equation

In this paper, we consider the canonical probability space $(\Omega, \mathcal{F}, P)$ with

$$\Omega = \{\omega = (\omega_1, \omega_2, \ldots, \omega_m) \in C(\mathbb{R}; \mathbb{R}^m) : \omega(0) = 0\}$$

and $\mathcal{F}$ being the Borel σ-algebra induced by the compact open topology of Ω, while P being the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Then, we identify $W(t)$ with $\omega(t)$, i.e.,

$$W(t) \equiv (\omega_1(t), \omega_2(t), \ldots, \omega_m(t)) \quad \text{for } t \in \mathbb{R},$$

and the time shift by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \qquad t \in \mathbb{R}.$$

We now follow the idea of [21] to transform (1.2) into a pathwise deterministic equation. The same idea has been adopted by many authors when dealing with random attractors or invariant manifolds for various stochastic evolution equations, such as [22,28,32,34]. Consider the stochastic stationary solution of the one dimensional Ornstein–Uhlenbeck equation

$$\mathrm{d}z_j + \mu z_j \mathrm{d}t = \mathrm{d}w_j(t), \qquad j = 1, \ldots, m, \tag{2.1}$$

which is given by

$$z_j(t) \triangleq z_j(\theta_t\omega_j) = -\mu \int_{-\infty}^{0} e^{\mu s} (\theta_t\omega_j)(s)\mathrm{d}s, \qquad t \in \mathbb{R}. \tag{2.2}$$

By Definition 3.4 (in Section 3), one can see that the random variable $|z_j(\omega_j)|$ is tempered and $z_j(\theta_t\omega_j)$ is P-a.e. ω continuous. Therefore, Proposition 4.3.3 in [1] implies that there exists a tempered function $0 < r(\omega) < \infty$ such that

$$\sum_{j=1}^{m} |z_j(\omega_j)|^2 \le r(\omega), \tag{2.3}$$

where $r(\omega)$ satisfies, for P-a.e. $\omega \in \Omega$,

$$r(\theta_t\omega) \le e^{\frac{\mu}{2}|t|} r(\omega), \qquad t \in \mathbb{R}. \tag{2.4}$$

Combining (3.11) with (2.4), we obtain that for P-a.e. $\omega \in \Omega$,

$$\sum_{j=1}^{m} |z_j(\theta_t\omega_j)|^2 \le e^{\frac{\mu}{2}|t|} r(\omega), \qquad t \in \mathbb{R}. \tag{2.5}$$

Moreover, we have

$$\sum_{j=1}^{m} |z_j(\theta_\xi\omega_j)|^2 \le e^{\frac{\mu\tau}{2}} r(\omega), \tag{2.6}$$

for any $\xi \in [-\tau, 0]$ and P-a.e. $\omega \in \Omega$. Putting $z(\theta_t\omega) = \sum_{j=1}^{m} g_j z_j(\theta_t\omega_j)$, we have

$$\mathrm{d}z + \mu z \mathrm{d}t = \sum_{j=1}^{m} g_j \mathrm{d}w_j.$$

Take the transformation $v(t) = u(t) - z(\theta_t\omega)$. Then, simple computation gives

$$\frac{dv(t)}{dt} = Av(t) - \mu v(t) + f(v_t + z(\theta_{t+\cdot}\omega)) + Az(\theta_t\omega), \tag{2.7}$$

where $\theta_{t+\cdot}\omega$ is defined as $\theta_{t+\xi}\omega$ for $\xi \in [-\tau, 0]$.

Throughout the remaining part of this paper, we always impose the following assumptions on A and the nonlinear term f:

Hypothesis A1 $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a densely defined linear operator that generates a strongly continuous compact semigroup $S(t)$ on $\mathbb{X}$. Moreover, $\varrho \triangleq s(\tilde{A}) - \mu < 0$, where $s(\tilde{A})$ is defined by $s(\tilde{A}) := \sup\{\Re\lambda : \lambda \in \sigma(\tilde{A})\}$ representing the spectral bound of the linear operator $\tilde{A}$.

Hypothesis A2 $f : \mathcal{C} \to \mathbb{X}$ is Lipschitz continuous with $\mathbf{0}$ being a fixed point, that is, $f(\mathbf{0}) = \mathbf{0}$ and there exists $L_f > 0$ such that

$$\|f(\phi) - f(\varphi)\|_{\mathbb{X}} \leq L_f \|\phi - \varphi\|_{\mathcal{C}},$$

for any $\phi, \varphi \in \mathcal{C}$. Moreover, there exists $m_0 \geqslant 0$ and $C_f > 0$ such that for all $l \in [0, m_0]$, $0 \leqslant t, u$ and $v \in C([-\tau, t]; \mathbb{X})$, the following inequality holds

$$\int_0^t e^{ls} \|f(u_s) - f(v_s)\|_{\mathbb{X}}^2 \,\mathrm{d}s \leqslant C_f^2 \int_{-\tau}^t e^{ls} \|u(s) - v(s)\|_{\mathbb{X}}^2 \,\mathrm{d}s. \tag{2.8}$$

Remark 2.1. Notice that, thanks to Hypothesis A2, given $u \in C^0([-\tau, T]; \mathbb{X})$, the function $f_u : t \in [0, T] \to \mathbb{X}$ defined by $f_u(t) = f(u_t)\ \forall\, t \in [0, T]$, is measurable (see Bensoussan et al. [4]) and, in fact, belongs to $L^\infty(0, T; \mathbb{X})$. Then, thanks to (2.8), the mapping

$$\mathcal{F} : u \in C^0([-\tau, T]; X) \to f_u \in L^2(0, T; \mathbb{X})$$

has a unique extension to a mapping $\widetilde{\mathcal{F}}$ which is uniformly continuous from $L^2(-\tau, T; \mathbb{X})$ into $L^2(0, T; \mathbb{X})$. From now on, we will denote $f(u_t) = \widetilde{\mathcal{F}}(u)(t)$ for each $u \in L^2(-h, T; \mathbb{X})$, and thus, $\forall\, t \in [0, T]$, $\forall\, u, v \in L^2(-\tau, T; \mathbb{X})$, we will have

$$\int_0^t e^{ls} \|f(u_s) - f(v_s)\|_{\mathbb{X}}^2 \,\mathrm{d}s \leqslant C_f^2 \int_{-\tau}^t e^{ls} \|u(s) - v(s)\|_{\mathbb{X}}^2 \,\mathrm{d}s.$$

Remark 2.2. Observe that considering the abstract formulation of our original problem with a functional f satisfying Assumption A2, we not only are considering the case of constant delay ($f(u_t) = f(u(t - \tau))$ but also the distributed delay one as well, that is, when $f(u_t) = \int_\tau^0 g(s, u(t+s))ds$, for an appropriate Lipschitz function g (see Caraballo and Real [9] for more information).

Since for P-a.e. $\omega \in \Omega$, (2.7) is a path-wise deterministic equation, by similar techniques as [10, Theorem 2.3] and [43, Theorem 8], we have the following results on the existence of solutions to (2.7).

Lemma 2.3. *Assume that **Hypotheses A1–A2** hold. Then, for any initial condition $(\phi, \phi(0)) \in \mathcal{L} \times \mathbb{X}$, there exists a solution $v(\cdot, \omega, \phi)$ to problem (2.7) with $v(\cdot, \omega, \phi) \in L^2(-r, T; \mathbb{X}) \cap L^\infty(0, T; \mathbb{X}) \cap C([-r, T]; \mathbb{X}), \forall T > 0$ and P-a.e. $\omega \in \Omega$.*

In order to estimate topological dimensions of the random attractors of (1.2), unlike previous works [5, 8, 12, 32, 41], where v_t is taken as the state and $\mathcal{L}$ as state space for the above obtained pathwise deterministic delayed equation (2.7), we take $V(t) = (v_t, v(t))$ as state space and recast the equation into an auxiliary product space $H = \mathcal{L} \times \mathbb{X}$ equipped with the inner product

$$\langle(\phi, l), (\psi, k)\rangle = \int_{-\tau}^{0} (\phi(s), \psi(s))_{\mathbb{X}} ds + (h, k)_{\mathbb{X}} \quad \text{for } (\phi, l), (\psi, k) \in H$$

and norm

$$\|(\phi, l)\| = \langle(\phi, l), (\phi, l)\rangle^{1/2} \quad \text{for } (\phi, l) \in H,$$

making H a Hilbert space and hence we can overcome the lack of Hilbert space geometry in applying the abstract theory established in [18, 19, 30, 31]. Furthermore, recasting (1.2) into the Hilbert space H also facilitates us to construct an appropriate variational equation. Take $V(t) = (v_t, v(t))$,

$$\hat{f}(t, \theta_t\omega, v_t) \triangleq Az(\theta_t\omega) + f\left(v_t + z(\theta_{t+\cdot}\omega)\right) \tag{2.9}$$

and

$$F(t, \theta_t\omega, V(t)) = (0, \hat{f}(t, \theta_t\omega, v_t)). \tag{2.10}$$

We consider the following auxiliary random partial differential equation on H.

$$\begin{cases} \dfrac{dV(t)}{dt} = \tilde{A}V(t) - \tilde{L}V(t) + F(t, \theta_t\omega, V(t)), \\ V(0) = (\phi, l), \quad (\phi, l) \in H, \end{cases} \tag{2.11}$$

where operator $\tilde{A}$ is defined as

$$\tilde{A} := \begin{pmatrix} \frac{d}{dt} & 0 \\ 0 & A \end{pmatrix}, \tag{2.12}$$

with domain

$$D(\tilde{A}) = \left\{(\phi, l) \in H : \phi \text{ is differentiable on } [-\tau, 0], \dot{\phi} \in \mathcal{L} \text{ and } h = \phi(0) \in D(A)\right\}.$$

The linear operator $\tilde{L}$ is defined by

$$\tilde{L} := \begin{pmatrix} 0 & 0 \\ 0 & \mu I \end{pmatrix}.$$

It follows from the definition of $\tilde{L}$ that

$$\|\tilde{L}\| \triangleq \sup_{\varphi \in H, \|\varphi\| = 1} \|\tilde{L}\varphi\| \leq \mu. \tag{2.13}$$

It follows from **Hypothesis A1**, Lemma 3.6, Theorem 3.25 in [3] that the operator $(\tilde{A}, D(\tilde{A}))$ is closed and densely defined on H, and generates a strongly continuous semigroup $\tilde{S}(t)$ given by

$$\tilde{S}(t) := \begin{pmatrix} S(t) & 0 \\ S_t & T_0(t) \end{pmatrix},$$

where $(T_0(t))_{t\geq 0}$ is the nilpotent left shift semigroup on $\mathcal{L}$, and $S_t : \mathbb{X} \to \mathcal{L}$ is defined by

$$(S_t x)(\xi) := \begin{cases} S(t+\xi)x & \text{if } -t < \xi \leq 0, \\ 0 & \text{if } -\tau \leq \xi \leq -t. \end{cases}$$

Moreover, by Theorem 4.11 in [3], we have

$$\|\tilde{S}(t)\| \leqslant e^{s(\tilde{A})t}, \qquad t \geqslant 0.$$

Furthermore, it follows from Pazy [35, Theorem 6.1.5] that (2.11) admits a global classical solution which can be represented by a integral equation based on the variation of constants formula.

Theorem 2.4. *Assume that* **Hypothesis A1** *holds and f is continuously differentiable. Then, for each $(\phi, l) \in H$, there exists a continuous function $V(\cdot, \omega, (\phi, l)) : [0, \infty) \to H$ such that*

$$V(t,\omega,(\phi,l)) = e^{-\tilde{L}t}\tilde{S}(t)(\phi,l) + \int_0^t e^{-\tilde{L}(t-s)}\tilde{S}(t-s)F(s,\theta_s\omega, V(s,\omega,(\phi,l)))ds, \qquad t \geqslant 0 \tag{2.14}$$

for P-a.e. $\omega \in \Omega$. Moreover, if $(\phi, l) \in D(\tilde{A})$, then $V(t, \omega, (\phi, l))$ is a strong solution of (2.11).

3 Random attractors

This section is devoted to showing the existence of random attractors for the auxiliary equation (2.11). In the sequel, we first introduce the concept of random attractor and random dynamical systems following [1] and [15, 16, 24]. Subsequently, we prove the existence of tempered pullback attractors for the auxiliary equation (2.11) by first establishing a uniform estimation for the solution and then proving that the RDS generated by (1.2) is pullback asymptotically compact. Unlike the previous works [5, 8, 12, 41], we prove the uniform a priori estimates of the solution by using the semigroup approach instead of taking inner product.

Definition 3.1. Let $\{\theta_t : \Omega \to \Omega, t \in \mathbb{R}\}$ be a family of measure preserving transformations such that $(t, \omega) \mapsto \theta_t\omega$ is measurable and $\theta_0 = \text{id}$, $\theta_{t+s} = \theta_t\theta_s$, for all $s, t \in \mathbb{R}$. The flow θ_t together with the probability space $\left(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}}\right)$ is called a metric dynamical system.

It follows from Definition 3.1 that $\left(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}}\right)$ is a metric dynamical system, where $(\Omega, \mathcal{F}, P)$ is defined in Section 2. Moreover, θ is ergodic. For a given separable Hilbert space $(H, \|\cdot\|_H)$, denote by $\mathcal{B}(H)$ the Borel σ-algebra generated by open subsets in H.

Definition 3.2. A mapping $\Phi : \mathbb{R}^+ \times \Omega \times H \to H$ is said to be a random dynamical system (RDS) on a complete separable metric space (H, d) with Borel σ-algebra $\mathcal{B}(H)$ over the metric dynamical system $\left(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}^+}\right)$ if

(i) $\Phi(\cdot,\cdot,\cdot) : \mathbb{R}^+ \times \Omega \times H \to H$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(H), \mathcal{B}(H))$-measurable;

(ii) $\Phi(0, \omega, \cdot)$ is the identity on H for P-a.e. $\omega \in \Omega$;

(iii) $\Phi(t+s, \omega, \cdot) = \Phi(t, \theta_s\omega, \cdot) \circ \Phi(s, \omega, \cdot)$, for all $t, s \in \mathbb{R}^+$ for P-a.e. $\omega \in \Omega$.

A RDS Φ is continuous or differentiable if $\Phi(t, \omega, \cdot) : H \to H$ is continuous or differentiable for all $t \in \mathbb{R}^+$ and P-a.e. $\omega \in \Omega$.

Definition 3.3. A set-valued map $\Omega \ni \omega \mapsto D(\omega) \in 2^H$, such that $D(\omega)$ is closed, is said to be a random set in H if the mapping $\omega \mapsto d(x, D(\omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for any $x \in H$, where $d(x, D(\omega)) \triangleq \inf_{y \in D(\omega)} \mathrm{d}(x, y)$ is the distance in H between the element x and the set $D(\omega) \subset H$.

Definition 3.4. A random set $\{D(\omega)\}_{\omega \in \Omega}$ of H is called tempered with respect to $(\theta_t)_{t \in \mathbb{R}}$ if for P-a.e. $\omega \in \Omega$,

$$\lim_{t \to \infty} e^{-\beta t} d\left(D\left(\theta_{-t}\omega\right)\right) = 0, \quad \text{for all } \beta > 0,$$

where $d(D) = \sup_{x \in D} \|x\|_H$.

Definition 3.5. Let $\mathcal{D} = \{D(\omega) \subset H, \omega \in \Omega\}$ be a family of random sets. A random set $K(\omega) \in \mathcal{D}$ is said to be a $\mathcal{D}$-pullback absorbing set for Φ if for P-a.e. $\omega \in \Omega$ and for every $B \in \mathcal{D}$, there exists $T = T(B, \omega) > 0$ such that

$$\Phi\left(t, \theta_{-t}\omega, B\left(\theta_{-t}\omega\right)\right) \subseteq K(\omega) \quad \text{for all } t \geq T.$$

If, in addition, for all $\omega \in \Omega, K(\omega)$ is measurable in Ω with respect to $\mathcal{F}$, then we say K is a closed measurable $\mathcal{D}$-pullback absorbing set for Φ.

Definition 3.6. A RDS Φ is said to be $\mathcal{D}$-pullback asymptotically compact in H if for P-a.e. $\omega \in \Omega$, $\{\Phi\left(t_n, \theta_{-t_n}\omega, x_n\right)\}_{n \geq 1}$ has a convergent subsequence in H whenever $t_n \to \infty$ and $x_n \in D\left(\theta_{-t_n}\omega\right)$ for any given $D \in \mathcal{D}$.

Definition 3.7. A compact random set $\mathcal{A}(\omega)$ is said to be a $\mathcal{D}$-pullback random attractor associated to the RDS Φ if it satisfies the invariance property

$$\Phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}\left(\theta_t\omega\right), \quad \text{for all } t \geq 0$$

and the pullback attracting property

$$\lim_{t \to \infty} \operatorname{dist}\left(\Phi\left(t, \theta_{-t}\omega, D\left(\theta_{-t}\omega\right)\right), \mathcal{A}(\omega)\right) = 0, \quad \text{for all } t \geq 0, D \in \mathcal{D}, P - a.e.\ \omega \in \Omega.$$

where $\operatorname{dist}(\cdot, \cdot)$ denotes the Hausdorff semidistance

$$\operatorname{dist}(A, B) = \sup_{x \in A} \inf_{y \in B} \mathrm{d}(x, y), \qquad A, B \subset H.$$

Lemma 3.8 ([15, Theorem 3.11])**.** *Let (θ, Φ) be a continuous random dynamical system. Suppose that Φ is $\mathcal{D}$-pullback asymptotically compact and has a closed pullback $\mathcal{D}$-absorbing random set $K = \{K(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$. Then it possesses a random attractor $\{\mathcal{A}(\omega)\}_{\omega \in \Omega}$, where*

$$\mathcal{A}(\omega) = \cap_{\tau \geq 0} \overline{\cup_{t \geq \tau} \Phi\left(t, \theta_{-t}\omega, K\left(\theta_{-t}\omega\right)\right)}.$$

For convenience, we introduce the following Gronwall inequality in [5] that will be frequently used in our subsequent proofs.

Lemma 3.9. *Let $T > 0$ and u, α, f and g be non-negative continuous functions defined on $[0, T]$ such that*

$$u(t) \leq \alpha(t) + f(t) \int_0^t g(r) u(r) dr, \quad \textit{for } t \in [0, T].$$

Then,

$$u(t) \leq \alpha(t) + f(t) \int_0^t g(r) \alpha(r) e^{\int_r^t f(\tau) g(\tau) d\tau} dr, \quad \textit{for } t \in [0, T].$$

Apparently, under the conjugation transformation induced by (2.2), no exceptional sets appear in the equation (2.11). By the uniqueness of solution to (2.11) for each $\omega \in \Omega$, we can see the mapping $\Phi(\cdot,\cdot,\cdot) : \mathbb{R}^+ \times \Omega \times H \to H$ defined by

$$\Phi(t,\omega,(\phi,\phi(0))) = V(t,\omega,(\phi,\phi(0))) \tag{3.1}$$

generates a RDS, which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(H), \mathcal{B}(H))$-measurable. Let P_1 and P_2 be the projections of H onto $\mathcal{L}$ and $\mathbb{X}$ respectively. Then, by Theorem 3.1 and Proposition 3.2 in [38], we have

$$v_t(\cdot,\omega,\phi) = P_1 V(t,\omega,(\phi,\phi(0))) \tag{3.2}$$

and

$$v(t,\omega,\phi) = P_2 V(t,\omega,(\phi,\phi(0))) \tag{3.3}$$

for $t \geqslant 0$ and P-a.e. $\omega \in \Omega$, where $v(t,\omega,\phi)$ is the solution to (2.7). Therefore, the solution of (1.2) can be represented by

$$\begin{aligned} u_t(\cdot,\omega,\phi) &= v_t(\cdot,\omega,\phi) + z(\theta_{t+\cdot}\omega) = P_1[V(t,\omega,(\phi,\phi(0))) + (z(\theta_{t+\cdot}\omega), z(\theta_t\omega))] \\ &\triangleq P_1 \Psi(t,\omega,(\psi,\psi(0))) \end{aligned} \tag{3.4}$$

where the mapping $\Psi : \mathbb{R}^+ \times \Omega \times H \to H$ is defined by

$$\begin{aligned} \Psi(t,\omega,(\psi,\psi(0))) &\triangleq \Phi(t,\omega,(\phi,\phi(0))) + (z(\theta_{t+\cdot}\omega), z(\theta_t\omega)) \\ &= V(t,\omega,(\phi,\phi(0))) + (z(\theta_{t+\cdot}\omega), z(\theta_t\omega)) \end{aligned} \tag{3.5}$$

and $(\psi,\psi(0)) = (\phi,\phi(0))) + (z(\theta_{\cdot}\omega), z(\omega))$. By the cocycle property of z and Φ, we can see that Ψ is a RDS on H. In the following, we show the existence of random attractor for Ψ.

Lemma 3.10. *Assume that **Hypotheses A1–A2** hold and $\varrho \triangleq s(\tilde{A}) - \mu < \frac{-\mu}{2}$, $\varrho + L_f < 0$, then there exists $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ satisfying that, for any $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there is $T_B(\omega) > 0$ such that*

$$\Psi(t,\theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq K(\omega) \quad \text{for all } t \geqslant T_B(\omega),$$

that is, $\{K(\omega)\}_{\omega\in\Omega}$ is a random absorbing set for Ψ in $\mathcal{D}$.

Proof. We first derive uniform estimates of V by (2.14) and then obtain the existence of an absorbing set for Ψ given by $\Psi(t,\omega,(\phi,\phi(0)) = V(t,\omega,(\phi,\phi(0))) + (z(\theta_{t+\cdot}\omega), z(\theta_t\omega))$. It follows from (2.14) that, for any $t > 0$,

$$\begin{aligned} &\|V(t,\omega,(\phi,\phi(0)))\| \\ &\quad = \left\| e^{-\tilde{L}t}\tilde{S}(t)(\phi,\phi(0)) + \int_0^t e^{-\tilde{L}(t-s)}\tilde{S}(t-s)F(s,\theta_s\omega, V(s,\omega,(\phi,\phi(0))))ds \right\| \\ &\quad \le e^{\varrho t}\|(\phi,\phi(0))\| + \int_0^t e^{\varrho(t-s)}\|\tilde{f}(s,\theta_s\omega, v_s(\cdot,\omega,\phi)))\|_{\mathbb{X}} ds \\ &\quad \le e^{\varrho t}\|(\phi,\phi(0))\| + \int_0^t e^{\varrho(t-s)}(\|Az(\theta_s\omega)\|_{\mathbb{X}} + L_f\|z(\theta_{s+\cdot}\omega)\|_{\mathcal{L}})ds \\ &\qquad + L_f \int_0^t e^{\varrho(t-s)}\|v_s(\cdot,\omega,\phi)\|_{\mathcal{L}} ds \\ &\quad \le e^{\varrho t}\|(\phi,\phi(0))\| + \int_0^t e^{\varrho(t-s)}(\|Az(\theta_s\omega)\|_{\mathbb{X}} + L_f\|z(\theta_{s+\cdot}\omega)\|_{\mathcal{L}})ds \\ &\qquad + L_f \int_0^t e^{\varrho(t-s)}\|V(s,\omega,(\phi,\phi(0)))\| ds \end{aligned} \tag{3.6}$$

for P-a.e. $\omega \in \Omega$. For the sake of simplicity, we denote $\varpi(\omega) = (\phi(\cdot,\omega), \phi(0,\omega))$. By replacing ω by $\theta_{-t}\omega$, we derive from (3.6) that, for all $t \geq 0$,

$$\begin{aligned} \|V(t,\theta_{-t}\omega,\varpi(\theta_{-t}\omega))\| \leq e^{\varrho t}\|\varpi(\theta_{-t}\omega)\| + L_f \int_0^t e^{\varrho(t-s)}\|V(s,\theta_{-t}\omega,\varpi(\theta_{-t}\omega)\|ds \\ + \int_0^t e^{\varrho(t-s)}(\|Az(\theta_{-t}\theta_s\omega)\|_{\mathbb{X}} + L_f\|z(\theta_{-t}\theta_{s+\cdot}\omega)\|_{\mathcal{L}})ds. \end{aligned} \tag{3.7}$$

Since $g_j \in \mathbb{X}$, $Ag_j \in \mathbb{X}$ and $z(\omega) = \sum_{j=1}^m g_j z_j(\omega_j)$, it follows from (2.5) and (2.6) that there exists a constant c such that $p_1(\omega) \triangleq \|Az(\omega)\|_{\mathbb{X}} + L_f\|z(\theta.\omega)\|_{\mathcal{L}} \leq c\sum_{j=1}^m |z_j(\omega_j)|^2$. Therefore, it follows from (2.4) and (2.5) that

$$\int_0^t e^{\varrho(t-s)} p_1(\theta_{s-t}\omega)\,\mathrm{ds} \leq c\int_0^t e^{(\varrho+\frac{\mu}{2})(t-s)} r(\omega)\mathrm{ds} \leq cr(\omega), \tag{3.8}$$

where the second inequality follows from the assumption that $\rho + \frac{\mu}{2} < 0$. Incorporating (3.8) into (3.7) gives rise to

$$\|V(t,\theta_{-t}\omega,\varpi(\theta_{-t}\omega)\| \leq e^{\varrho t}\|\varpi(\theta_{-t}\omega)\| + L_f\int_0^t e^{\varrho(t-s)}\|V(s,\theta_{-t}\omega,\varpi(\theta_{-t}\omega))\|ds + cr(\omega). \tag{3.9}$$

Multiplying both sides of (3.9) by $e^{-\varrho t}$,

$$\begin{aligned} &e^{-\varrho t}\|V(t,\theta_{-t}\omega,\varpi(\theta_{-t}\omega))\| \\ &\quad\leq \|\varpi(\theta_{-t}\omega)\| + L_f\int_0^t e^{-\varrho s}\|V(s,\theta_{-t}\omega,\varpi(\theta_{-t}\omega))\|ds + ce^{-\varrho t}r(\omega). \end{aligned} \tag{3.10}$$

Hence, by the Gronwall inequality (Lemma 3.9), we have

$$\begin{aligned} e^{-\varrho t}\|V(t,\theta_{-t}\omega,\varpi(\theta_{-t}\omega))\| &\leq \|\varpi(\theta_{-t}\omega)\| + ce^{-\varrho t}r(\omega) \\ &\quad + L_f\int_0^t e^{L_f(t-s)}(\|\varpi(\theta_{-s}\omega)\| + ce^{-\varrho s}r(\omega))ds \\ &\leq \|\varpi(\theta_{-t}\omega)\| + ce^{-\varrho t}r(\omega) + L_f\|\varpi(\theta_{-t}\omega)\|\int_0^t e^{L_f(t-s)}\mathrm{ds} \\ &\quad + cL_f r(\omega)\int_0^t e^{L_f(t-s)}e^{-\varrho s}\mathrm{ds}. \end{aligned} \tag{3.11}$$

Therefore, we have

$$\begin{aligned} \|V(t,\theta_{-t}\omega,\varpi(\theta_{-t}\omega))\| \leq e^{\varrho t}\|\varpi(\theta_{-t}\omega)\| + cr(\omega) + e^{\varrho t}(e^{L_f t} - 1)\|\varpi(\theta_{-t}\omega)\| \\ + \frac{cL_f}{\varrho + L_f}[e^{(L_f+\varrho)t} - 1]r(\omega). \end{aligned} \tag{3.12}$$

Note that $\Psi(t,\omega,\chi(\omega)) = V(t,\omega,\varpi(\omega)) + (z(\theta_{t+\cdot}\omega), z(\theta_t\omega))$ and $\chi(\omega) = \varpi(\omega) + (z(\theta.\omega), z(\omega))$. The above estimate (3.12) implies that, for all $t \geq 0$

$$\begin{aligned} \|\Psi(t,\theta_{-t}\omega,\chi(\theta_{-t}\omega))\| &\leq \|V(t,\theta_{-t}\omega,\varpi(\theta_{-t}\omega))\| + \|(z(\theta_{-t}\theta_{t+\cdot}\omega), z(\theta_{-t}\theta_t\omega))\| \\ &\leq e^{\varrho t}\|\varpi(\theta_{-t}\omega)\| + 2cr(\omega) + e^{\varrho t}(e^{L_f t} - 1)\|\varpi(\theta_{-t}\omega)\| \\ &\quad + \frac{cL_f}{\varrho + L_f}[e^{(L_f+\varrho)t} - 1]r(\omega). \end{aligned} \tag{3.13}$$

Therefore, if $\chi \in \mathcal{D}(\theta_{-t}\omega)$ and $L_f + \varrho < 0$, then there exists a $T_{\mathcal{D}} > 0$ such that, for all $t \geq T_D(\omega)$,

$$e^{\varrho t}\|\varpi(\theta_{-t}\omega)\| + e^{\varrho t}(e^{L_f t} - 1)\|\varpi(\theta_{-t}\omega)\| + \frac{cL_f}{\varrho + L_f} e^{(L_f+\varrho)t} r(\omega) \leq c_1(\omega), \tag{3.14}$$

which, along with (3.13) shows that, for all $t \geq T_{\mathcal{D}}(\omega)$

$$\|\Psi(t, \theta_{-t}\omega, \chi(\theta_{-t}\omega))\| \leq 2cr(\omega) + \frac{-cL_f}{\varrho + L_f} r(\omega) + c_1(\omega). \tag{3.15}$$

Given $\omega \in \Omega$, define

$$K(\omega) = \{\varphi \in H : \|\varphi\| \leq 2cr(\omega) + \frac{-cL_f}{\varrho + L_f} r(\omega) + c_1(\omega)\}. \tag{3.16}$$

Then, $K = \{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Furthermore, (3.15) implies that $K(\omega)$ is a random absorbing set for the RDS Ψ in $\mathcal{D}$. □

Lemma 3.11. *Assume that **Hypotheses A1–A2** are satisfied and $\varrho \triangleq s(\tilde{A}) - \mu < \frac{-\mu}{2}$, $\varrho + L_f < 0$. Then, the RDS Ψ is $\mathcal{D}$-pullback asymptotically compact for $t > \tau$ (the time delay), i.e., for P-a.e. $\omega \in \Omega$, the sequence $\{\Psi(t_n, \theta_{-t_n}\omega, \phi_n(\theta_{-t_n}\omega))\}$ has a convergent subsequence provided $t_n \to \infty$, $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $\phi_n(\theta_{-t_n}\omega) \in B(\theta_{-t_n}\omega)$.*

Proof. Take an arbitrary random set $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, a sequence $t_n \to +\infty$ and $\phi_n \in B(\theta_{-t_n}\omega)$. We have to prove that $\{\Psi(t_n, \theta_{-t_n}\omega, \phi_n)\}$ is precompact. Since $\{K(\omega)\}$ is a random absorbing set for Ψ, there exists $T > 0$ such that, for all $\omega \in \Omega$,

$$\Psi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega) \tag{3.17}$$

for all $t \geq T$. Because $t_n \to +\infty$, we can choose $n_1 \geq 1$ such that $t_{n_1} - 1 \geq T$. Applying (3.17) for $t = t_{n_1} - 1$ and $\omega = \theta_{-1}\omega$, we find that

$$\eta_1 \triangleq \Psi\left(t_{n_1} - 1, \theta_{-t_{n_1}}\omega, \phi_{n_1}\right) \in K(\theta_{-1}\omega) \tag{3.18}$$

Similarly, we can choose a subsequence $\{n_k\}$ of $\{n\}$ such that $n_1 < n_2 < \cdots < n_k \to +\infty$ with $t_{n_k} \geq k$ and

$$\eta_k \triangleq \Psi\left(t_{n_k} - k, \theta_{-t_{n_k}}\omega, \phi_{n_k}\right) \in K(\theta_{-k}\omega) \tag{3.19}$$

Hence, by the assumptions we conclude that the sequence

$$\{\Psi(k, \theta_{-k}\omega, \eta_k)\} \text{ is precompact.} \tag{3.20}$$

On the other hand, by (3.19) we have

$$\Psi(k, \theta_{-k}\omega, \eta_k) = \Psi(k, \theta_{-k}\omega, \Psi(t_{n_k} - k, \theta_{-t_{n_k}}\omega, \phi_{n_k})) = \Psi\left(t_{n_k}, \theta_{-t_{n_k}}\omega, \phi_{n_k}\right), \tag{3.21}$$

for all $k \geq 1$. Combining (3.20) and (3.21), we obtain that the sequence $\left\{\Psi\left(t_{n_k}, \theta_{-t_{n_k}}\omega, \phi_{n_k}\right)\right\}$ is precompact. Therefore, $\{\Psi(t_n, \theta_{t_n}\omega, \phi_n)\}$ is precompact, which completes the proof. □

Lemma 3.10 says that the continuous RDS Ψ has a random absorbing set while Lemma 3.11 tells us that (θ, Ψ) is pullback asymptotically compact in H. Thus, it follows from Lemma 3.8 that the continuous RDS (θ, Ψ) possesses a random attractor. Namely, we obtain the following result.

Theorem 3.12. *Assume that **Hypotheses A1–A2** are satisfied and $\varrho \triangleq s(\tilde{A}) - \mu < \frac{-\mu}{2}$, $\varrho + L_f < 0$, then the continuous RDS Ψ admits a $\mathcal{D}$-pullback attractor $\mathcal{A}_\Psi(\omega)$ in H belonging to the class $\mathcal{D}$.*

Moreover, by Theorem 3.12, the relationship between the RSDs Φ and Ψ defined by (3.5) as well as Proposition 3.2 in [38], we have the following result about the existence of random attractors for equation (1.2).

Corollary 3.13. *Assume that **Hypotheses A1–A2** are satisfied and $\varrho \triangleq s(\tilde{A}) - \mu < \frac{-\mu}{2}$, $\varrho + L_f < 0$. Then, the continuous RDS $P_1\Psi$ generated by (1.2) admits a pullback attractor $P_1\mathcal{A}_\Psi(\omega)$ in P_1H. Moreover, $\mathcal{A}_\Phi(\omega) \triangleq \{\zeta | \zeta = \chi - (z(\theta_{t+\cdot}\omega), z(\theta_t\omega)), \chi \in \mathcal{A}_\Psi(\omega)\}$ is a random attractor of Φ.*

4 Topological dimensions of random attractors

The aim of this section is to estimate the Hausdorff and fractal dimensions for the attractor of (1.2). Denote by $d_H(\mathcal{A}_\Psi(\omega))$ and $d_F(\mathcal{A}_\Psi(\omega))$ the Hausdorff and fractal dimensions of a random set $\mathcal{A}_\Psi(\omega)$ respectively. We only need to prove that there exist constants d_H and d_F such that $d_H(\mathcal{A}_\Psi(\omega)) \leq d_H$ and $d_F(\mathcal{A}_\Psi(\omega)) \leq d_F$, since by Theorem 3.1 and Proposition 3.2 in [38], the topological dimensions of attractor $P_1\mathcal{A}_\Psi(\omega)$ for (1.2) satisfy $d_H(P_1\mathcal{A}_\Psi(\omega)) \leq d_H$ and $d_F(P_1\mathcal{A}_\Psi(\omega)) \leq d_F$, i.e., the random attractors of (1.2) have finite Hausdorff and fractal dimensions less than those of (2.11). In the sequel, we investigate the Hausdorff and fractal dimensions for the random attractor $\mathcal{A}_\Psi(\omega)$ of (2.11).

We first recall the concepts of Hausdorff and fractal dimensions of the attractor $\mathcal{A}_\Psi(\omega) \subset H$. More details can be found in [19] and [31]. The Hausdorff dimension of the compact set $\mathcal{A}_\Psi(\omega) \subset H$ is

$$d_H(\mathcal{A}_\Psi(\omega)) = \inf\{d : \mu_H(\mathcal{A}_\Psi(\omega), d) = 0\}$$

where, for $d \geq 0$,

$$\mu_H(\mathcal{A}_\Psi(\omega), d) = \lim_{\varepsilon \to 0} \mu_H(\mathcal{A}_\Psi(\omega), d, \varepsilon)$$

denotes the d-dimensional Hausdorff measure of the set $\mathcal{A}_\Psi(\omega) \subset H$, where

$$\mu_H(\mathcal{A}_\Psi(\omega), d, \varepsilon) = \inf \sum_i r_i^d$$

and the infimum is taken over all coverings $\mathcal{K} = \{B_i\}_{i \in I}$ of $\mathcal{A}_\Psi(\omega)$ by balls B_i of radius $r_i \leq \varepsilon$ and the sum is over all balls of $\mathcal{K}$. It can be shown that there exists $d_H(\mathcal{A}_\Psi(\omega)) \in [0, +\infty]$ such that $\mu_H(\mathcal{A}_\Psi(\omega), d) = 0$ for $d > d_H(\mathcal{A}_\Psi(\omega))$ and $\mu_H(\mathcal{A}_\Psi(\omega), d) = \infty$ for $d < d_H(\mathcal{A}_\Psi(\omega))$. $d_H(\mathcal{A}_\Psi(\omega))$ is called the Hausdorff dimension of $\mathcal{A}_\Psi(\omega)$.

The fractal dimension (or capacity) of $\mathcal{A}_\Psi(\omega)$ is defined as

$$d_F(\mathcal{A}_\Psi(\omega)) = \inf\{d > 0 : \mu_F(\mathcal{A}_\Psi(\omega), d) = 0\},$$

where

$$\mu_F(\mathcal{A}_\Psi(\omega), d) = \limsup_{\varepsilon \to 0} \varepsilon^d n_F(\mathcal{A}_\Psi(\omega), \varepsilon)$$

and $n_F(\mathcal{A}_\Psi(\omega), \varepsilon)$ is the minimum number of balls of radius $\leqslant \varepsilon$ which is necessary to cover $\mathcal{A}_\Psi(\omega)$.

Take a covering of $\mathcal{A}_\Psi(\omega)$ by balls of radii less than ε:

$$\mathcal{A}_\Psi(\omega) \subset \bigcup_{i=1} B(u_i, r_i), \qquad r_i \leq \varepsilon,\ u_i \in H$$

where $B(u_i, r_i)$ denotes the ball in H of center u_i and radius r_i. Let $\theta = \theta_1$ and define

$$\Psi(\omega)\phi = \Psi(1, \omega, \phi) \tag{4.1}$$

for any $\phi \in H$ and P-a.e. $\omega \in \Omega$. Then, it follows from the invariance of $\mathcal{A}_\Psi(\omega)$ that

$$\mathcal{A}_\Psi(\theta\omega) \subset \bigcup_{i=1} \Psi(\omega) B(u_i, r_i).$$

In order to approximate $\Psi(\omega)$ by a linear map, we impose the following almost surely uniformly differentiable assumption of $\Psi(\omega)$ on the attractor $\mathcal{A}_\Psi(\omega)$.

Hypothesis A3 The mapping $\Psi(\omega)$ is $\mathbb{P}$ almost surely differentiable on $\mathcal{A}_\Psi(\omega)$, that is, $\mathbb{P}$ almost surely, for every u in $\mathcal{A}_\Psi(\omega)$, there exists a continuous linear operator $D\Psi(\omega, u) : H \to H$, such that if $u, u+h \in \mathcal{A}_\Psi(\omega)$, then

$$\|\Psi(\omega)(u+h) - \Psi(\omega)u - D\Psi(\omega, u) \cdot h\| \leq K(\omega)\|h\|^{1+\alpha},$$

where $K(\omega)$ is a random variable such that

$$K(\omega) \geq 1, \quad \text{for all } \omega \in \Omega,$$

$\mathbb{E}(\ln K) < \infty$ and $\alpha > 0$.

We follow [40, Chapter 5] to give following definitions. For the bounded linear operator $D\Psi(\omega, u)$ on H and $n \in \mathbb{N}$, we set

$$\alpha_n(D\Psi(\omega, u)) = \sup_{\substack{G \subset H \\ \dim \leqslant n}} \inf_{\substack{\phi \in G \\ \|\phi\|=1}} \|D\Psi(\omega, u)\phi\|$$

and

$$\omega_n(D\Psi(\omega, u)) = \alpha_1(D\Psi(\omega, u)) \ldots \alpha_n(D\Psi(\omega, u)),$$

where $\alpha_n(D\Psi(\omega, u))$ are the square roots of the eigenvalues of $D\Psi(\omega, u)^* D\Psi(\omega, u)$ corresponding to orthogonal eigenvectors e_n, which are in decreasing order. We set

$$\alpha_\infty(D\Psi(\omega, u)) = \inf_n \alpha_n(D\Psi(\omega, u))$$

and further make the following assumptions.

Hypothesis A4 For every $d \in \mathbb{N}$, there exists an integrable random variable $\bar{\omega}_d$, such that $\mathbb{P}$ almost surely,

$$\omega_d(D\Psi(\omega, u)) \leq \bar{\omega}_d(\omega)$$

for any $u \in \mathcal{A}_\Psi(\omega)$ and

$$\mathbb{E}(\ln(\bar{\omega}_d)) < 0.$$

Under the above assumptions, we have the following results concerning the dimension estimation of random attractors $\mathcal{A}_\Psi(\omega)$ for Ψ, of which the proof is given in [19, 31].

Lemma 4.1. *Assume that* ***Hypotheses A3–A4*** *are satisfied. Then,* $\mathbb{P}$*-a.s.*

$$d_{\mathrm{H}}(\mathcal{A}_\Psi(\omega)) \leqslant d$$

and

$$d_{\mathrm{F}}(\mathcal{A}_\Psi(\omega)) \leqslant \gamma$$

for any γ *such that*

$$\gamma > \frac{\mathbb{E}\left[\max_{1 \leqslant j \leqslant d}(dq_j - jq_d)\right]}{-\mathbb{E} q_d},$$

where $q_j = \log \bar{\omega}_j$.

In the following, we verify **Hypothesis A3–A4**. We first establish the following result, which is a key ingredient to prove the $\mathbb{P}$ almost surely uniform differentiability results of $\Psi(\omega)$.

Proposition 4.2. *If $f : \mathcal{L} \to H$ is twice continuously differentiable, then for each $\varpi \in \mathcal{A}_\Psi(\omega)_\Phi$ and $h \in H$, there exists a continuous function $U^{\varpi,h}(t,\omega) : [0,\infty) \times \Omega \to H$ such that*

$$U^{\varpi,h}(t,\omega) = e^{-\tilde{L}t}\tilde{S}(t)h + \int_0^t e^{-\tilde{L}(t-s)}\tilde{S}(t-s)(0, D\tilde{f}\,(P_1\Phi(s,\omega,\varpi))\,P_1U(s,\omega))ds, \quad t \geqslant 0. \quad (4.2)$$

Moreover, if $h \in D(\tilde{A})$, then $U(t,\omega)$ is a strong solution of the following variational equation on H.

$$\begin{cases} \dfrac{dU(t,\omega)}{dt} = \tilde{A}U(t,\omega) - \tilde{L}U(t,\omega) + (0, D\tilde{f}\,(P_1\Phi(s,\omega,\varpi))\,P_1U(t,\omega)), \\ U(0,\omega) = h \in H, \end{cases} \quad (4.3)$$

where operators $\tilde{A}$ and $\tilde{f}$ are defined by (2.12) and (2.9), $\Phi(t,\omega,\varpi)$ is the RDS defined by (3.5) with initial condition h.

Proof. Let

$$L_1(\omega) = \sup_{\varsigma \in P_1\mathcal{A}_\Phi(\omega)} |D\tilde{f}(\varsigma)|, \quad (4.4)$$

where

$$|D\tilde{f}(\varsigma)| = \sup_{\|\eta\|_{\mathcal{L}} \leqslant 1} \|D\tilde{f}(\varsigma)\eta\|_{\mathbb{X}}. \quad (4.5)$$

Since $\tilde{f}$ is C^1 and $P_1\mathcal{A}_\Psi(\omega)$ is compact, then $L_1(\omega) < \infty$. Given any $h \in D(\tilde{A})$, define $F_\chi : H \to H$ by

$$F_\varpi(h) = (0, D\tilde{f}\,(P_1\Phi(t,\omega,\varpi))\,P_1h), \qquad t \geqslant 0,\ h \in H.$$

It follows from the invariance of $\mathcal{A}_\Phi(\omega)$ under Φ and $\varpi \in \mathcal{A}_\Phi(\omega)$ that $\Phi(s,\omega,\varpi) \in \mathcal{A}_\Phi(\omega)$ and hence $P_1\Phi(t,\omega,\varpi) \in P_1\mathcal{A}_\Phi(\omega)$ and $|Df\,(P_1\Phi(t,\omega,\varpi))| \leqslant L_1(\omega) < \infty$, for all $t \geqslant 0$. This implies that $F_\varpi(\cdot)$ is Lipschitz continuous on H. Therefore the conclusion follows from Pazy [35, Theorem 6.1.5]. □

Now, we establish the following almost surely uniform differentiability results of $\Psi(\omega)$ on the random attractor $\mathcal{A}_\Psi(\omega)$.

Theorem 4.3. *The mapping $\Psi(\omega)$ is $\mathbb{P}$ almost surely differentiable on $\mathcal{A}_\Psi(\omega)$, that is, $\mathbb{P}$ almost surely, for every u in $\mathcal{A}_\Psi(\omega)$, there exist a continuous linear operator $D\Psi(\omega,u) : H \to H$, such that if $u, u+h \in \mathcal{A}_\Psi(\omega)$, then*

$$\|\Psi(\omega)(u+h) - \Psi(\omega)u - D\Psi(\omega,u)\cdot h\| \leq K(\omega)\|h\|^{1+\alpha}$$

where $K(\omega)$ is a random variable such that

$$K(\omega) \geq 1, \qquad w \in \Omega$$

and $\alpha > 0$ is a number.

Proof. We first claim that for any constant $t > 0$ and $\chi, \chi + h \in \mathcal{A}_\Psi(\omega)$, there exists a constant $L(t) > 0$ such that

$$\|\Psi(t,\omega,\chi) - \Psi(t,\omega,\chi+h)\|_X \leqslant L(t)\|h\|.$$

By Theorem 2.4 and the relationship $\Psi(t,\omega,\chi(\omega)) = \Phi(t,\omega,\varpi(\omega)) + (z(\theta_{t+\cdot}\omega), z(\theta_t\omega))$ with $\varpi(\omega) = \chi(\omega) - (z(\theta_\cdot\omega), z(\omega))$, we have

$$\Psi(t,\omega,\chi) = e^{-\tilde{L}t}\tilde{S}(t)\chi + \int_0^t e^{-\tilde{L}(t-s)}\tilde{S}(t-s)F(s,\theta_s\omega,\Phi(s,\omega,\varpi))ds + (z(\theta_{t+\cdot}\omega), z(\theta_t\omega)), \quad (4.6)$$

$$\begin{aligned}\Psi(t,\omega,\chi+h) = e^{-\tilde{L}t}\tilde{S}(t)(\chi+h) + \int_0^t e^{-\tilde{L}(t-s)}\tilde{S}(t-s)F(s,\theta_s\omega,\Phi(s,\omega,\varpi+h)))ds \\ + (z(\theta_{t+\cdot}\omega), z(\theta_t\omega)),\end{aligned} \quad (4.7)$$

from which it follows that

$$\begin{aligned}\Psi(t,\omega,\chi+h) - \Psi(t,\omega,\chi) = e^{-\tilde{L}t}\tilde{S}(t)h + \int_0^t e^{-\tilde{L}(t-s)}\tilde{S}(t-s)[F(s,\theta_s\omega,\Phi(s,\omega,\varpi+h)) \\ - F(s,\theta_s\omega,\Phi(s,\omega,\varpi))]ds.\end{aligned} \quad (4.8)$$

Since $\|\tilde{S}(t)\| \leqslant e^{s(\tilde{A})t}, t \geqslant 0$, we have

$$\begin{aligned}&\|\Psi(t,\omega,\chi+h) - \Psi(t,\omega,\chi)\| \\ &\leq e^{\varrho t}\|h\| + L_f\int_0^t e^{\varrho(t-s)}\|P_1[\Phi(s,\omega,\varpi+h) - \Phi(s,\omega,\varpi)]\|ds \\ &= e^{\varrho t}\|h\| + L_f\int_0^t e^{\varrho(t-s)}\|P_1[\Psi(s,\omega,\chi+h) - \Psi(s,\omega,\chi)]\|ds \\ &\leq e^{\varrho t}\|h\| + L_f\int_0^t e^{\varrho(t-s)}\|\Psi(s,\omega,\chi+h) - \Psi(s,\omega,\chi)\|ds.\end{aligned} \quad (4.9)$$

Multiplying both sides of (4.9) by $e^{-\varrho t}$ and taking into account the Gronwall inequality, we obtain

$$e^{-\varrho t}\|\Psi(t,\omega,\chi+h) - \Psi(t,\omega,\chi)\| \leq e^{L_f t}\|h\|, \quad (4.10)$$

and hence

$$\|\Psi(t,\omega,\chi+h) - \Psi(t,\omega,\chi)\| \leq e^{(L_f+\varrho)t}\|h\|. \quad (4.11)$$

Therefore, the claim holds by taking $L(t) = e^{(L_f+\varrho)t}$.

Next we prove that, for any $t > 0$, there exist $K(\omega) \geq 1$ and $\alpha > 0$ such that, if $\chi, \chi + h \in \mathcal{A}_\Psi(\omega)$, then

$$\|\Psi(t,\omega,\chi+h) - \Psi(t,\omega,\chi) - U^{\chi+h,\chi}(t,\omega)\| \leq K(\omega)\|h\|^{1+\alpha}. \quad (4.12)$$

Let

$$L_2(\omega) := \sup_{\xi\in\overline{co}\mathcal{A}_\Psi(\omega)(w)} \left|D^2 f(P_1\xi)\right|, \quad (4.13)$$

where $\overline{co}\mathcal{A}_\Psi(\omega)$ represents the closed convex hull of $\mathcal{A}_\Psi(\omega)$. Since f is C^2 and $\mathcal{A}_\Psi(\omega)$ is compact, $L_2 < \infty$. By Proposition 4.2, we have

$$U^{\chi,h}(t,\omega) = e^{-\tilde{L}t}\tilde{S}(t)h + \int_0^t e^{-\tilde{L}(t-s)}\tilde{S}(t-s)(0, D\tilde{f}\,(P_1\Phi(s,\omega,\varpi))\,P_1U(s))ds, \qquad t \geqslant 0.$$

For notation simplicity, we denote $y(t,\omega) \triangleq \Psi(t,\omega,\chi+h) - \Psi(t,\omega,\chi) = \Phi(s,\omega,\varpi+h) - \Phi(s,\omega,\varpi)$ and $w(t,\omega) \triangleq \Psi(t,\omega,\chi+h) - \Psi(t,\omega,\chi) - U^{\varpi,h}(t,\omega)$. Then, it follows from (4.6) and (4.7) that

$$\begin{aligned}
&\|w(t,\omega)\| \\
&\quad= \Big\| \int_0^t e^{-\tilde{L}(t-s)} \tilde{S}(t-s)\{0, \tilde{f}\left(P_1\Phi(s,\omega,\varpi+h)\right) \\
&\qquad - \tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right) - D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right) P_1 U(s,\omega)\} ds \| \\
&\quad\le \int_0^t e^{\varrho(t-s)} \|\tilde{f}\left(P_1\Phi(s,\omega,\varpi+h)\right) - \tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right) - D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right) P_1 U(s,\omega)\|_{\mathbb{X}} ds \\
&\quad\le \int_0^t e^{\varrho(t-s)} \int_0^1 |D\tilde{f}\left(P_1(\Phi(s,\omega,\varpi)+\vartheta y(s,\omega))\right) - D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right)| d\vartheta \|P_1 y(s,\omega)\|_{\mathcal{L}} ds \\
&\qquad + \int_0^t e^{\varrho(t-s)} \|D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right) P_1 w(s,\omega)\|_{\mathbb{X}} ds \\
&\quad\le \int_0^t e^{\varrho(t-s)} \int_0^1 \int_0^1 |D^2\tilde{f}\left(P_1(\Phi(s,\omega,\varpi)+\lambda\vartheta y(s,\omega))\right)| \lambda\vartheta d\lambda d\vartheta \|P_1 y(s,\omega)\|_{\mathcal{L}}^2 ds \\
&\qquad + \int_0^t e^{\varrho(t-s)} \|D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right) P_1 w(s,\omega)\|_{\mathbb{X}} ds \\
&\quad\le \int_0^t e^{\varrho(t-s)} \int_0^1 \int_0^1 |D^2\tilde{f}\left(P_1(\Phi(s,\omega,\varpi)+\lambda\vartheta y(s,\omega))\right)| d\lambda d\vartheta \|y(s,\omega)\|^2 ds \\
&\qquad + \int_0^t e^{\varrho(t-s)} |D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right)| \|w(s,\omega)\| ds.
\end{aligned} \tag{4.14}$$

Since $\chi, \chi+h \in \mathcal{A}_\Psi(\omega)$, it follows from the invariance Corollary 3.13 that $\varpi, \varpi+h \in \mathcal{A}_\Phi(\omega)$. Therefore, the invariance of $\mathcal{A}_\Phi(\omega)$ under Φ implies that $\Phi(t,\omega,\varpi), \Phi(t,\omega,\varpi+h) \in \mathcal{A}_\Psi(\omega)$ for all $t \geqslant 0$. Therefore, $P_1\Phi(t,\omega,\varpi)+\lambda\vartheta y(s,\omega)) \in \overline{co}\left(\mathcal{A}_\Phi(\omega)\right)$, for all $\vartheta,\lambda \in [0,1]$, where $\overline{co}\mathcal{A}_\Phi(\omega)$ represents the closed convex hull of $\mathcal{A}_\Phi(\omega)$. Thus, it follows from (4.13) and (4.10) and the fact f is C^2 that

$$\|w(t,\omega)\| \leqslant L_2(\omega) \int_0^t e^{[2(L_f+\varrho)+\varrho](t-s)} \|h\|^2 ds + L_1(\omega) \int_0^t e^{\varrho(t-s)} \|w(s,\omega)\| ds. \tag{4.15}$$

Multiplying both sides of (4.15) by $e^{-\varrho t}$ yields that

$$e^{-\varrho t}\|w(t,\omega)\| \leqslant \frac{-L_2(\omega)e^{-\varrho t}}{2(L_f+\varrho)+\varrho}\|h\|^2 + L_1(\omega) \int_0^t e^{-\varrho s} \|w(s,\omega)\| ds, \tag{4.16}$$

which implies, by the Gronwall inequality, that

$$e^{-\varrho t}\|w(t,\omega)\| \leqslant \frac{-L_2(\omega)e^{-\varrho t}}{2(L_f+\varrho)+\varrho}\|h\|^2 + \frac{L_1(\omega)e^{L_1(\omega)t}}{-(L_f+\varrho)}(e^{-(\varrho+L_1)t}-1)\|h\|^2. \tag{4.17}$$

Therefore, we have

$$\|w(t,\omega)\| \leqslant \frac{-L_2(\omega)}{2(L_f+\varrho)+\varrho}(1+\frac{L_1(\omega)(1-e^{(L_1(\omega)+\varrho)t}}{-(L_f+\varrho)})\|h\|^2. \tag{4.18}$$

Take $D\Psi(\omega)h \triangleq U^{\varpi,h}(1,\omega)$, then it follows from (4.2) that $D\Psi(\omega)$ is linear and continuous.

Moreover, we have

$$\begin{aligned}&\|\Psi(\omega)(\chi+h)-\Psi(\omega)\chi-D\Psi(\omega,\chi)\cdot h\|\\&\quad=\|\Psi(1,\omega,\chi+h)-\Psi(1,\omega\chi)-U^{\chi,h}(1,\omega)\|\\&\quad\le\frac{-L_2(\omega)}{2(L_f+\varrho)+\varrho}\left(1+\frac{L_1(\omega)(1-e^{(L_1(\omega)+\varrho)}}{-(L_f+\varrho)}\right)\|h\|^2,\end{aligned}\tag{4.19}$$

what implies that the statement of Theorem 4.3 holds by taking $\alpha = 1$ and $K(\omega) = \frac{-L_2(\omega)}{2(L_f+\varrho)+\varrho}(1+\frac{L_1(\omega)(1-e^{(L_1(\omega)+\varrho)}}{-(L_f+\varrho)})$. □

We can now prove the main results of this paper.

Theorem 4.4. *Assume that **Hypotheses A1–A4** as well as conditions of Theorem 3.12 are satisfied, $f:\mathcal{L}\to H$ is twice continuously differentiable and there exists $L_3(\omega)$ such that for all $\phi\in\mathcal{L}$ and a.e. $\omega\in\Omega$ such that*

$$\left(D\tilde{f}\left(P_1\Phi(t,\omega,\phi)\right)\varphi_j,\varphi_j(0)\right)<L_3(\omega),\tag{4.20}$$

where $\varphi_j\in\mathcal{C}, j=1,2,\dots,m$ is a sequence of unit orthogonal vectors. Then, the Hausdorff and fractal dimensions of the random attractor $\mathcal{A}_\Psi(\omega)$ of (2.11) are bounded by

$$d<[\frac{L_1(\omega)}{c}]^{\frac{1}{1+2/N}},\tag{4.21}$$

where c is a positive constant, N is dimension of the spatial domain, $L_1(\omega)$ is defined by (4.4).

Proof. The existence of random attractor $\mathcal{A}_\Psi(\omega)$ has been proved in Theorem 2.4 under **Hypotheses A1–A2**. We show in the sequel the finite dimensionality of $\mathcal{A}_\Psi(\omega)$. Let $\varpi\in\mathcal{A}_\Psi(\omega)$, $h_i=(\phi_i,\phi_i(0))\in D(\bar{A})$ and $U_i^{\varpi,h_i}(t,\omega)$ be defined by

$$U_i^{\varpi,h_i}(t,\omega)=e^{-\tilde{L}t}\tilde{S}(t)h_i+\int_0^t e^{-\tilde{L}(t-s)}\tilde{S}(t-s)(0,D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right)P_1U_i^{\varpi,h_i}(s,\omega))ds,\quad t\geqslant 0.\tag{4.22}$$

It follows from Proposition 4.2 that $U_i^{\varpi,h_i}(t,\omega)$ satisfies the following variational equation on H.

$$\begin{cases}\dfrac{dU_i^{\varpi,h_i}(t,\omega)}{dt}=\tilde{A}U_i^{\varpi,h_i}(t,\omega)-\tilde{L}U_i^{\varpi,h_i}(t,\omega)+(0,D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right)P_1U_i^{\varpi,h_i}(t,\omega)),\\U_i^{\varpi,h_i}(0,\omega)=h_i\in H.\end{cases}\tag{4.23}$$

Define a family of random maps $U_i(t,\omega):H\to H, i=1,\dots,m$ by $U_i(t,\omega)h_i=U^{\varpi,h_i}(t,\omega)$. By a similar argument to that for (2.40) in [40] Chapter V, we obtain

$$\frac{1}{2}\frac{d}{dt}\left|U_1(t,\omega)\wedge\cdots\wedge U_m(t,\omega)\right|^2_{\wedge^m H}=\left|U_1(t,\omega)\wedge\cdots\wedge U_m(t,\omega)\right|^2_{\wedge^m H}\operatorname{Tr}\left(G(t)\circ Q_m(t)\right),\tag{4.24}$$

where $|\cdot|_{\wedge^m H}$ represents the exterior product and

$$Q_m(t)=Q_m\left(t,\varpi;h_1,\dots,h_m\right)\tag{4.25}$$

is the orthogonal projection of H onto the space spanned by $\{U_j(t,\omega)\}_{j=1,2,\dots,m}$ and $G(t)=G(t,\omega):H\to H$ is defined by

$$G(t,\omega)h_i=\tilde{A}h_i-\tilde{L}h_i+(0,D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right)P_1h_i).\tag{4.26}$$

Therefore

$$\begin{aligned}&|U_1(t,\omega)\wedge\cdots\wedge U_m(t,\omega)|_{\wedge^m H}\\&\quad=|U_1(0,\omega)\wedge\cdots\wedge U_m(0,\omega)|_{\wedge^m H}\exp\left(\int_0^t \operatorname{Tr}\left(G(s,\omega)\circ Q_m(s)\right)ds\right)\\&\quad=|h_1\wedge\cdots\wedge h_m|_{\wedge^m H}\exp\left(\int_0^t \operatorname{Tr}\left(G(s,\omega)\circ Q_m(s)\right)ds\right).\end{aligned}\tag{4.27}$$

Let us fix an orthonormal basis $\{e_j=(\varphi_j,\varphi_j(0))\}_{j=1,2,\dots,m}$ of the span $\{U_j(t,\omega)\}_{j=1,2,\dots,m}$. Then, we have

$$\begin{aligned}\operatorname{Tr}\left(G(s,\omega)\circ Q_m(s)\right)&=\sum_{j=1}^m\langle G(s,\omega)e_j,e_j\rangle\\&=\sum_{j=1}^m\langle \tilde{A}e_j-\tilde{L}e_j+(0,D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right)P_1e_j),e_j\rangle\\&=\sum_{j=1}^m\langle(\dot{\varphi}_j,A\varphi_j(0)-\mu\varphi_j(0)+D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right)\varphi_j),(\varphi_j,\varphi_j(0))\rangle.\end{aligned}\tag{4.28}$$

By the definition of the inner product,

$$\begin{aligned}&\left\langle(\dot{\varphi}_j-L\varphi_j,A\varphi_j(0)-\mu\varphi_j(0)+D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right)\varphi_j),(\varphi_j,\varphi_j(0))\right\rangle\\&\quad=\int_{-\tau}^0\left(\frac{d}{dr}\varphi_j(s+r),\varphi_j(s+r)\right)dr+\left(A\varphi_j(0),\varphi_j(0)\right)-\mu\|\varphi_j(0)\|_{\mathbb{X}}\\&\quad\quad+\left(D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right)\varphi_j,\varphi_j(0)\right)\\&\quad\leqslant\|\varphi_j(s)\|_{\mathbb{X}}^2-\|\varphi_j(s-\tau)\|_{\mathbb{X}}^2-\left(-A\varphi_j(0),\varphi_j(0)\right)+L_3(\omega).\end{aligned}\tag{4.29}$$

Incorporating (4.29) into (4.28) yields

$$\operatorname{Tr}\left(G(s,\omega)\circ Q_m(s)\right)\leqslant\sum_{j=1}^m(\|\varphi_j(s)\|_{\mathbb{X}}^2-\|\varphi_j(s-\tau)\|_{\mathbb{X}}^2)-\sum_{j=1}^m\left(-A\varphi_j(0),\varphi_j(0)\right)+L_3(\omega).\tag{4.30}$$

In order to estimate the evolution of the volume under the random maps $U_i(t,\omega):H\to H, i=1,\dots,m$, i.e., the norm of $|U_1(t,\omega)\wedge\cdots\wedge U_m(t,\omega)|_{\wedge^m H}$ defined by (4.27), we introduce two quantities $q_m(t,\omega)$ and $q_m(\omega)$, which are defined by

$$q_m(t,\omega)\triangleq\sup_{\chi\in\mathcal{A}_\Psi(\omega),h_i\in D(\tilde{A}),\|h_i\|_H\leq 1}\frac{1}{t}\int_0^t\operatorname{Tr}\left(G(s,\omega)\circ Q_m(s)\right)ds$$

and

$$q_m(\omega)\triangleq\lim_{t\to\infty}q_m(t,\omega)$$

respectively. Now we keep in mind that

$$q_m(t,\omega)\leqslant\frac{1}{t}\int_0^t\left[\sum_{j=1}^m\left(\|\varphi_j(s)\|_{\mathbb{X}}^2-\|\varphi_j(s-\tau)\|_{\mathbb{X}}^2\right)-\sum_{j=1}^m\left(-A\varphi_j(0),\varphi_j(0)\right)+L_3(\omega)\right]ds,\tag{4.31}$$

and

$$\frac{1}{t}\int_0^t\sum_{j=1}^m\left(\|\varphi_j(s)\|_{\mathbb{X}}^2-\|\varphi_j(s-\tau)\|_{\mathbb{X}}^2\right)ds=0.\tag{4.32}$$

Moreover, by [40] Chapter VI Section 2.1, there exists a $c > 0$ such that

$$\sum_{i=1}^{m} \left(-A\varphi_j(0), \varphi_j(0)\right) \geq cm^{1+\frac{2}{N}}.$$

Then, we have $q_m \leq -cm^{1+\frac{2}{N}} + L_3(\omega)$. Therefore, the Hausdorff and fractal dimensions of the random attractor $\mathcal{A}_\Psi(\omega)$ of (2.11) obtained in Theorem 3.12 are bounded by the first integer such m that $q_m \leq 0$, that is,

$$d < \left[\frac{L_3(\omega)}{c}\right]^{\frac{1}{1+2/N}}, \tag{4.33}$$

completing the proof. □

5 Conclusions

In this paper, we have estimated the topological dimensions of random attractor for the stochastic delayed semilinear partial differential equation (1.2). In order to overcome the difficulty caused by the lack of Hilbert geometry, we recast the equation into a Hilbert space. One naturally wonders, whether we can estimate the dimension of attractors for SPFDEs in their natural phase space, i.e. Banach spaces. This requires to establish the general framework to estimate the dimension of attractors of RDS in Banach spaces, which will be studied in the near future. Moreover, there are also SPFDEs on infinite domains which can model the spatial-temporal patterns for the mature population of age-structured species under random perturbations. The existence of random attractors for a stochastic nonlocal delayed reaction-diffusion equation on a semi-infinite interval have been studied in [29]. However, little attention has been paid to the estimation of topological dimensions of random attractor for the equation therein, which also deserves much effort in the future.

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