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Multiple positive radial solutions for Dirichlet problem of the prescribed mean curvature spacelike equation in a Friedmann–Lemaître–Robertson–Walker spacetime

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Abstract. In this paper, we consider the radially symmetric spacelike solutions of a nonlinear Dirichlet problem for the prescribed mean curvature spacelike equation in a Friedmann–Lemaître–Robertson–Walker spacetime. By using a conformal change of variable, this problem can be translated an equivalent problem in the Minkowski spacetime. By using the lower and upper solution method, fixed point, a priori bounds and topological degree method, we obtain the existence, nonexistence and multiplicity of radially symmetric spacelike solutions.

Keywords: topological degree, radially symmetric spacelike solutions, Dirichlet problem, prescribed mean curvature spacelike equation, Friedmann–Lemaître–Robertson–Walker spacetime.

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1 Introduction

Let $I \subseteq \mathbb{R}$ be an open interval in $\mathbb{R}$ with the metric $-dt^2$. Denote by $\mathcal{M}$ the $(N+1)$-dimensional product manifold $I \times \mathbb{R}^N$ with $N \geq 1$ endowed with the Lorentzian metric

$$g = -dt^2 + f^2(t)dx^2,$$

where $f \in C^\infty(I)$, $f > 0$, is called the *scale factor* or *warping function* in the related literature. Clearly, $\mathcal{M}$ is a Lorentzian warped product with base $(I, -dt^2)$, fiber $(\mathbb{R}^N, dx^2)$ and warping function f, we refer it as a Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime. In the fiber space $(\mathbb{R}^N, dx^2)$, the metric dx^2 is an arbitrary Riemannian metric in a Generalized FLRW spacetime. In cosmology, the FLRW spacetime is the accepted model for a spatially homogeneous and isotropic Universe. In this context, the warping function $f(t)$ is interpreted as the radius of the Universe at time t, and the sign of its derivative indicates if the Universe

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is expanding or contracting at given time, for more details of FLRW spacetime, we refer the reader to [11,21,22,27,34–37] and the references therein. Observe that for the particular case $f(t) \equiv 1$ we recover the Minkowski spacetime.

Given $f \in C^\infty(I)$, $f > 0$, for each $u \in C^\infty(\Omega)$, where Ω is a domain of $\mathbb{R}^N$, such that $u(\Omega) \subseteq I$, we can consider its graph $M = \{(x, u(x)) : x \in \Omega\}$ in the FLRW spacetime $\mathcal{M}$. The graph is spacelike whenever

$$|\operatorname{grad} u| < f(u) \quad \text{in } \Omega, \tag{1.1}$$

where $\operatorname{grad} u$ is the gradient of u in $\mathbb{R}^N$ and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$, in this case, the unit timelike normal vector field in the same time orientation of ∂_t is given by

$$A = \frac{f(u)}{\sqrt{f(u) - |\operatorname{grad} u|^2}} \left(\frac{1}{f^2(u)} \operatorname{grad} u + \partial_t \right),$$

and the corresponding mean curvature associated to A, is defined by

$$\frac{1}{N} \left\{ \operatorname{div} \left(\frac{\operatorname{grad} u}{f(u)\sqrt{f^2(u) - |\operatorname{grad} u|^2}} \right) + \frac{f'(u)}{\sqrt{f^2(u) - |\operatorname{grad} u|^2}} \left(N + \frac{|\operatorname{grad} u|^2}{f^2(u)} \right) \right\},$$

where div denotes the divergence operator of $\mathbb{R}^N$, $f'(u) := f' \circ u$, it can be seen as a quasilinear elliptic operator Q, because of (1.1). We are interested in the existence of spacelike graphs with a prescribed mean curvature function in the FLRW spacetime $\mathcal{M}$. The general problem of the curvature prescription is, given a function $H : I \times \mathbb{R}^N \to \mathbb{R}$, to obtain solutions of the quasilinear elliptic equation

$$Q(u) = H(u, x), \quad |\operatorname{grad} u| < f(u) \quad \text{in } \Omega, \tag{1.2}$$

and (1.2) is called the prescribed mean curvature spacelike equation in FLRW spacetime. Specially relevant is the case when H is constant, then it is called the prescribed constant mean curvature spacelike equation (if $H = 0$ it is also called the maximal spacelike graph equation).

In the recent years, most of the efforts have been directed to the prescribed mean curvature spacelike equation in Minkowski spacetime ($f(t) \equiv 1$), in this context, we mention the seminal work of R. Bartnik and L. Simon [1], E. Calabi [8], S.-Y. Cheng and S.-T. Yau [10] and A. E. Treibergs [39], in these papers, the spacelike graphs having the property that their mean curvature is zero or constant are considered. More recently, Dirichlet problems for prescribed mean curvature spacelike equation in Minkowski spacetime have been widely concerned by many scholars, and their attention is mainly focused on their positive solutions, we refer the reader to [3–6,12–16,23,24,28–32,41,42] and the references therein. In particular, based on the detailed analysis of time map, some exact multiplicity of positive solutions have been obtained in [24,42], for the radially symmetric solutions on a ball, some existence, nonexistence and multiplicity results have been established in [4,5], and some bifurcation results have been obtained in [14,28] via bifurcation technique, and when Ω is a general domain in $\mathbb{R}^N$, some existence and bifurcation results have been obtained in the papers [13,15,16,31]. In addition to, these concern discrete problems associated with the prescribed mean curvature spacelike equation in Minkowski spacetime, we refer the reader to [7,9,25,26] and the references therein.

In comparison with the study in Minkowski spacetime, the number of references devoted to the prescribed mean curvature spacelike equation in FLRW spacetime is appreciably lower. Only in the recent years, C. Bereanu, D. de la Fuente, A. Romero and P. J. Torres [2,20] have considered the existence and multiplicity of radially symmetric spacelike solutions of the Dirichlet problem by using the Schauder fixed point Theorem with approximation process, J. Mawhin and P. J. Torres [33, 38] have provided some sufficient conditions for the existence of radially symmetric spacelike solutions of the Neumann problem by the Leray–Schauder degree theory, G. Dai, A. Romero and P. J. Torres [17–19] have obtained the existence and multiplicity of radially symmetric spacelike positive solutions of the equation with 0-Dirichlet boundary condition on a ball and studied the global structure of the solution set via the Rabinowitz's global bifurcation method. Xu and Ma [40] have considered the differential and difference problems associated with the discrete approximation of radially symmetric spacelike solutions of the Dirichlet problem, by using lower and upper solutions, they proved the existence of solutions of the corresponding differential and difference problems, and based on the ideas of a prior bound showed the solutions of the discrete problem converge to the solutions of the continuous problem.

In this paper we are concerned with the mixed boundary value problem

$$\begin{cases} -(r^{N-1}\phi(v'))' = \lambda N r^{N-1}\left[\dfrac{f'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} - f(\varphi^{-1}(v))H(\varphi^{-1}(v),r)\right], \quad r\in(0,R),\\ |v'|<1, \quad r\in(0,R),\\ v'(0)=v(R)=0, \end{cases} \tag{1.3}$$

where $\phi(s)=\frac{s}{\sqrt{1-s^2}}$, and $\phi:(-1,1)\to\mathbb{R}$ is an increasing homeomorphism with $\phi(0)=0$, such an ϕ is called *singular*, λ is a positive parameter, R is a positive constant, $f\in C^\infty(I)$ and $f>0$, I is an open interval in $\mathbb{R}$, $\varphi(s)=\int_0^s\frac{dt}{f(t)}$, φ^{-1} is the inverse function of φ, $H: I\times[0,R]\to\mathbb{R}$ is a continuous function. The aim of this paper is to investigate the intervals of the λ in which the (1.3) has zero, one or two positive radial solutions.

This study mainly motivated by the numerical approximation of radially symmetric spacelike solutions of the nonlinear Dirichlet problem for the prescribed mean curvature spacelike equation in FLRW spacetime:

$$\begin{cases} \operatorname{div}\left(\frac{\operatorname{grad} u}{f(u)\sqrt{f^2(u)-|\operatorname{grad} u|^2}}\right)+\frac{f'(u)}{\sqrt{f^2(u)-|\operatorname{grad} u|^2}}\left(N+\frac{|\operatorname{grad} u|^2}{f^2(u)}\right)=NH(u,|x|) \quad \text{in } \mathcal{B},\\ |\operatorname{grad} u|<f(u) \quad \text{in } \mathcal{B},\\ u=0 \quad \text{on } \partial\mathcal{B}, \end{cases} \tag{1.4}$$

where $\mathcal{B}=\{x\in\mathbb{R}^N:|x|<R\}$, $f\in C^\infty(I)$, $f>0$ and $H: I\times[0,+\infty)\to\mathbb{R}$ is the prescribed mean curvature function. We follow the method developed in [20], let us define the function $\varphi: I\to\mathbb{R}$ by $\varphi(s)=\int_0^s\frac{dt}{f(t)}$, and φ is an increasing diffeomorphism from I onto $J:=\varphi(I)$ such that $\varphi(0)=0$. Doing the change $v=\varphi(u)$ and taking radial coordinates, we can reduce the Dirichlet problem (1.4) to the mixed boundary value problem (1.3) with $\lambda=1$, and the solutions of (1.3) with $\lambda=1$ are just the radially symmetric spacelike solutions of (1.4).

We say that a function $v\in C^1[0,R]$ is a solution of (1.3) if $\|v'\|_\infty<1$, $r^{N-1}\phi(v')\in C^1[0,R]$, and (1.3) is satisfied. For (1.3), since the graph associate to v is spacelike, i.e. $\|v'\|_\infty<1$, we deduce that $\|v\|_\infty<R$, this implies the image of nonnegative v is in $[0,R]$, therefore, when discussing the nonnegative solutions of (1.3), we always assume $\varphi^{-1}([0,R])\subset I$, which is equivalent to

$$I_f R := \left[0, \int_0^R f(\varphi^{-1}(s))ds\right]\subset I.$$

In Section 2, we present a lower and upper solution result for continuous problem (1.3) with $\lambda = 1$. In Section 3, we give some notations and fixed point reformulation of (1.3) with $\lambda = 1$ and prove all possible solutions and their first differences have a prior bounds, based on this, we calculate some topological degrees. Using the results of these two parts and the estimate of the first derivative of a concave function, in Section 4, we show that there is a $\Lambda > 0$ such that problem (1.3) has zero, at least one or at least two positive solutions when $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$, $\lambda > \Lambda$. Finally in Section 5, for the convenience of readers and integrity of the paper, we give the detailed derivation process of problem (1.3) with $\lambda = 1$.

The main result is as follows.

Theorem 1.1. *Assume that $I_f R \subset I$ and $f'(t) \geq 0$, $H(t,r) < \frac{f'}{f}(t)$ for all $r \in [0,R]$, $t \in I_f R$ and assume also that*

$$\begin{cases} \lim\limits_{t \to 0^+} \frac{Nf'(t)}{\varphi(t)} = f_0, \\ \lim\limits_{t \to 0^+} \frac{Nf(t)H(t,r)}{\varphi(t)} = H_0, \\ f_0 - H_0 = 0. \end{cases} \tag{A_{fH}}$$

Then there is a $\Lambda > \frac{2NM_0}{R^3}$ such that problem (1.3) has zero, at least one or at least two positive solutions when $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$, $\lambda > \Lambda$.

Notations: The space $C := C[0,R]$ will be endowed with the usual sup-norm $\|\cdot\|_\infty$ and $C^1 := C^1[0,R]$ will considered with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$. $C^1_M := \{u \in C^1 : u'(0) = u(R) = 0\}$ is the closed subspace of C^1. For $u_0 \in C^1_M$, we set $B(u_0, \rho) := \{u \in C^1_M : \|u\| < \rho\}$ $(\rho > 0)$ and B_ρ is used to represent $B(0,\rho)$.

2 Lower and upper solutions

In this section, we develop the lower and upper solution method for the mixed boundary value problem

$$\begin{cases} -(r^{N-1}\phi(v'))' = Nr^{N-1}\left[\dfrac{f'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} - f(\varphi^{-1}(v))H(\varphi^{-1}(v),r)\right], \quad r \in (0,R), \\ |v'| < 1, \quad r \in (0,R), \\ v'(0) = v(R) = 0. \end{cases} \tag{2.1}$$

Definition 2.1. A lower solution α of (2.1) is a function $\alpha \in C^1$ such that $\|\alpha'\|_\infty < 1$, $r^{N-1}\phi(\alpha') \in C^1$, $I_f R \subset I$ and

$$-(r^{N-1}\phi(\alpha'))' \leq Nr^{N-1}\left[\frac{f'(\varphi^{-1}(\alpha))}{\sqrt{1-\alpha'^2}} - f(\varphi^{-1}(\alpha))H(\varphi^{-1}(\alpha),r)\right], \quad r \in (0,R), \quad \alpha(R) \leq 0.$$

An upper solution β of (2.1) is a function $\beta \in C^1$ such that $\|\beta'\|_\infty < 1$, $r^{N-1}\phi(\beta') \in C^1$, $I_f R \subset I$ and

$$-(r^{N-1}\phi(\beta'))' \geq Nr^{N-1}\left[\frac{f'(\varphi^{-1}(\beta))}{\sqrt{1-\beta'^2}} - f(\varphi^{-1}(\beta))H(\varphi^{-1}(\beta),r)\right], \quad r \in (0,R), \quad \beta(R) \geq 0.$$

Such a lower or an upper solution is called strict if the above inequalities are strict.

Theorem 2.2. *Assume that $I_f R \subset I$ and $f'(t) \geq 0$, $H(t,r) < \frac{f'}{f}(t)$ for all $r \in [0,R]$, $t \in I_f R$. If (2.1) has a lower solution α and an upper solution β such that $\alpha(r) \leq \beta(r)$ for all $r \in [0,R]$, then (2.1) has at least one solution v such that $\alpha(r) \leq v(r) \leq \beta(r)$ for all $r \in [0,R]$.*

Proof. Let $\gamma : [0,R] \times \mathbb{R} \to \mathbb{R}$ be the continuous function defined by

$$\gamma(r,v) = \begin{cases} \alpha(r), & \text{if } v < \alpha(r), \\ v, & \text{if } \alpha(r) \leq v \leq \beta(r), \\ \beta(r), & \text{if } v > \beta(r). \end{cases}$$

We consider the modified problem

$$\begin{cases} (r^{N-1}\phi(v'))' + Nr^{N-1}\Bigg[\dfrac{f'(\varphi^{-1}(\gamma(r,v)))}{\sqrt{1-v'^2}} \\ \qquad -H(\varphi^{-1}(\gamma(r,v)),r)f(\varphi^{-1}(\gamma(r,v))) - v + \gamma(r,v)\Bigg] = 0, \quad r \in (0,R), \\ |v'| < 1, \quad r \in (0,R), \\ v'(0) = 0 = v(R). \end{cases} \tag{2.2}$$

It follows from [2] that the problem (2.2) has at least one solution.

We show that if v is a solution (2.2), then $\alpha(r) \leq v(r) \leq \beta(r)$ for all $r \in [0,R]$. This will conclude the proof.

Suppose by contradiction that there is some $r_0 \in [0,R]$ such that

$$\max_{[0,R]}[\alpha - v] = \alpha(r_0) - v(r_0) > 0.$$

If $r_0 \in (0,R)$, then $\alpha'(r_0) = v'(r_0)$ and there are sequences $\{r_k\}$ in $(0,r_0)$ converging to r_0 such that $\alpha'(r_k) - v'(r_k) \geq 0$. Since ϕ is an increasing homeomorphism then we can have

$$r_k^{N-1}\phi(v'(r_k)) - r_0^{N-1}\phi(v'(r_0)) \leq r_k^{N-1}\phi(\alpha'(r_k)) - r_0^{N-1}\phi(\alpha'(r_0)),$$

which means

$$(r_0^{N-1}\phi(\alpha'(r_0)))' \leq (r_0^{N-1}\phi(v'(r_0)))'.$$

Therefore, since α is a lower solution of (2.1) we have

$$\begin{aligned} &(r_0^{N-1}\phi(\alpha'(r_0)))' \\ &\quad \leq (r_0^{N-1}\phi(v'(r_0)))' \\ &\quad = Nr_0^{N-1}\left[-\frac{f'(\varphi^{-1}(\alpha(r_0)))}{\sqrt{1-(\alpha'(r_0))^2}} + H(\varphi^{-1}(\alpha(r_0)),r_0)f(\varphi^{-1}(\alpha(r_0))) + v(r_0) - \alpha(r_0)\right] \\ &\quad < Nr_0^{N-1}\left[-\frac{f'(\varphi^{-1}(\alpha(r_0)))}{\sqrt{1-(\alpha'(r_0))^2}} + H(\varphi^{-1}(\alpha(r_0)),r_0)f(\varphi^{-1}(\alpha(r_0)))\right] \\ &\quad \leq (r_0^{N-1}\phi(\alpha'(r_0)))', \end{aligned}$$

but this a contradiction.

If $\max_{[0,R]}[\alpha - v] = \alpha(R) - v(R) > 0$, then by definition of lower solutions, we obtain a contradiction again. If $\max_{[0,R]}[\alpha - v] = \alpha(0) - v(0) > 0$, then there exists $r_1 \in (0,R]$ such that $\alpha(r) - v(r) > 0$ for all $r \in [0,r_1]$ and $\alpha'(r_1) - v'(r_1) \leq 0$. It follows that

$$(r_1^{N-1}\phi(\alpha'(r_1)))' \leq (r_1^{N-1}\phi(v'(r_1)))'.$$

Note that $I_f R \subset I$ and $f'(t) \geq 0$ for all $t \in I_f R$. By using the fact and integrating (2.2) from 0 to r_1, we have that

$$\begin{aligned} r_1^{N-1}\phi(\alpha'(r_1)) &\leq r_1^{N-1}\phi(v'(r_1)) \\ &< N\int_0^{r_1} r^{N-1}\left[-\frac{f'(\varphi^{-1}(\alpha(r)))}{\sqrt{1-(v'(r))^2}} + H(\varphi^{-1}(\alpha(r)), r) f(\varphi^{-1}(\alpha(r)))\right] dr \\ &\leq N\int_0^{r_1} r^{N-1}\left[-\frac{f'(\varphi^{-1}(\alpha(r)))}{\sqrt{1-(\alpha'(r))^2}} + H(\varphi^{-1}(\alpha(r)), r) f(\varphi^{-1}(\alpha(r)))\right] dr \\ &\leq r_1^{N-1}\phi(\alpha'(r_1)). \end{aligned}$$

But this is a contradiction. Hence, $\alpha(r) \leq v(r)$ for all $r \in [0, R]$. Analogously, we can show that $v(r) \leq \beta(r)$ for all $r \in [0, R]$. □

Remark 2.3. The Theorem 2.2 still holds for $f(t) \equiv 1$.

3 Fixed point, a priori bounds and degree

In this section, we consider problems of type

$$\begin{cases} (r^{N-1}\phi(v'))' + r^{N-1}g(r, v, v') = 0, & r \in (0, R), \\ |v'| < 1, & r \in (0, R), \\ v'(0) = v(R) = 0, \end{cases} \tag{3.1}$$

where $N \geq 1$ is an integer, $R > 0$ is a constant, and we also assume that

(A_ϕ) $\phi : (-1, 1) \to \mathbb{R}$ is an odd, increasing homeomorphism;

(A_g) $g : [0, R] \times [0, \alpha) \times (-1, 1) \to [0, +\infty)$ is a continuous function with $0 < \alpha \leq +\infty$.

Recall, by a solution of (3.1) we mean a function $v \in C^1$ with $\|v'\|_\infty < 1$, such that $r^{N-1}\phi(v') \in C^1$ and (3.1) is satisfied.

Setting

$$\sigma(r) := 1/r^{N-1},$$

we introduce the linear operators

$$S : C \to C, \quad Sv(r) = \sigma(r)\int_0^r t^{N-1}v(t)\, dt \quad (r \in [0, R]), \quad Sv(0) = 0;$$

$$K : C \to C^1, \quad Kv(r) = \int_r^R v(t)\, dt \quad (r \in [0, R]).$$

It is easy to see the standard argument that K is bounded and that S is compact by the Arzelà–Ascoli theorem. This means that the nonlinear operator $K \circ \phi^{-1} \circ S : C \to C^1$ is compact. Moreover, for a given function $h \in C$, the problem

$$(r^{N-1}\phi(v'))' + r^{N-1}h(r) = 0, \quad r \in (0, R), \quad |v'| < 1, \quad v'(0) = v(R) = 0$$

has a unique solution

$$v = K \circ \phi^{-1} \circ S \circ h.$$

Next, let N_g be the Nemytskii operator associated with g, i.e.,

$$N_g : C \to C, \quad N_g = g(\cdot, v(\cdot), v'(\cdot)).$$

Noticing that N_g is continuous and maps a bounded set to a bounded set. So problem (3.1) has the following reformulation about fixed points.

Lemma 3.1. *A function $v \in C^1_M$ is a solution of problem (3.1) if and only if the compact nonlinear operator*

$$\mathcal{N}_g : C^1_M \to C^1_M, \quad \mathcal{N}_g = K \circ \phi^{-1} \circ S \circ N_g$$

has a fixed point, and furthermore the fixed point of $\mathcal{N}_g$ satisfies

$$\|v'\|_\infty < 1, \quad \|v\|_\infty < R \tag{3.2}$$

and

$$d_{LS}[I - \mathcal{N}_g, B_\rho, 0] = 1 \quad \textit{for all } \rho \geq (R+1).$$

Proof. Since the range of ϕ^{-1} is $(-1, 1)$, the inequality (3.2) holds. Next, consider the compact homotopy

$$\mathcal{H} : [0,1] \times C^1_M, \quad \mathcal{H}(\tau, \cdot) = \tau \mathcal{N}_g$$

and

$$\mathcal{H}([0,1] \times C^1_M) \subset B_{(R+1)}.$$

Then, from the invariance under homotopy of the Leray–Schauder degree it follows that

$$d_{LS}[I - \mathcal{H}(0,\cdot), B_\rho, 0] = d_{LS}[I - \mathcal{H}(1,\cdot), B_\rho, 0] = d_{LS}[I - \mathcal{N}_g, B_\rho, 0] = 1,$$

for all $\rho \geq (R+1)$. □

In view of Theorem 2.2 and Remark 2.3, we have the following result.

Lemma 3.2. *Assume that (3.1) has a lower solution α and an upper solution β such that $\alpha(r) \leq \beta(r)$ for all $r \in [0, R]$, and let $\Omega_{\alpha,\beta} := \{v \in C^1_M : \alpha \leq v \leq \beta\}$. Assume also that (3.1) has an unique solution v_0 in $\Omega_{\alpha,\beta}$ and there exists $\rho_0 > 0$ such that $\overline{B}(v_0, \rho_0) \subset \Omega_{\alpha,\beta}$. Then*

$$d_{LS} = [I - \mathcal{N}_g, B(v_0, \rho), 0] = 1 \quad \textit{for all } 0 < \rho \leq \rho_0,$$

where $\mathcal{N}_g$ is the fixed point operator associated to (3.1).

Proof. Let $\mathcal{N}_g$ be the fixed point operator associated with (3.1). The proof of Theorem 2.2 shows that any fixed point v of $\mathcal{N}_g$ is contained in $\Omega_{\alpha,\beta}$, and this means that v_0 is the unique fixed of $\mathcal{N}_g$ and there exists $\rho_0 > 0$ such that $\overline{B}(v_0, \rho_0) \subset \Omega_{\alpha,\beta}$. From Lemma 3.1 and the excision property of the Leray–Schauder degree there is

$$d_{LS}[I - \mathcal{N}_g, B(v_0, \rho_0), 0] = 1,$$

which is

$$d_{LS}[I - \mathcal{N}_g, B(v_0, \rho), 0] = 1 \quad \text{for all } 0 < \rho \leq \rho_0.$$ □

Lemma 3.3. *Assume that (A_ϕ), (A_g) and*

(A'_g) *$g(r, v, v') > 0$ for all $(r, v, v') \in (0, R] \times (0, \alpha) \times (-1, 1)$.*

Let v be a nontrivial solution of (3.1). Then $v > 0$ on $[0,R)$ and v is strictly decreasing.

Proof. Let's first integrate both sides of (3.1) from 0 to r, which is

$$v'(r) = -\phi^{-1}\left(\frac{1}{r^{N-1}}\int_0^r s^{N-1}g(s,v,v')ds\right). \tag{3.3}$$

Then integrate both sides of (3.3) from r to R to get

$$v(r) = \int_r^R \phi^{-1}\left(\frac{1}{t^{N-1}}\int_0^t s^{N-1}g(s,v,v')ds\right)dt. \tag{3.4}$$

So if $g(r,v,v') > 0$, we have $v > 0$ on $[0,R)$ and v is strictly decreasing. □

In the next lemma we assume that g is sublinear with respect to ϕ at zero.

Lemma 3.4. *Assume that conditions (A_ϕ), (A_g) and (A'_g) hold. Assume also that*

$$\lim_{s\to 0^+}\frac{g(r,s,s')}{\phi(s)} = 0 \quad \textit{uniformly for } r\times s' \in [0,R]\times(-1,1) \tag{3.5}$$

and

$$\liminf_{s\to 0^+}\frac{\phi(\sigma s)}{\phi(s)} > 0 \quad \textit{for all } \sigma > 0. \tag{3.6}$$

Then there exists $\rho_0 > 0$ such that

$$d_{LS}[I-\mathcal{N}_g, B_\rho, 0] = 1 \quad \textit{for all } 0 < \rho \le \rho_0.$$

Proof. Using (3.6) we can find $\varepsilon > 0$ such that

$$R\varepsilon/N < \liminf_{s\to 0}\frac{\phi(s/R)}{\phi(s)}. \tag{3.7}$$

Using (3.5) we can find $s_\varepsilon > 0$ such that

$$g(r,s,s') \le \varepsilon\phi(s) \quad \text{for all } (r,s,s') \in [0,R]\times[0,s_\varepsilon]\times(-1,1). \tag{3.8}$$

Next, we consider the compact homotopy

$$\mathcal{H}: [0,1]\times C^1_M \to C^1_M, \quad \mathcal{H}(\tau,v) = \tau\mathcal{N}_g(v).$$

Let's we say have $\rho_0 > 0$ such that

$$v \ne \mathcal{H}(\tau,v) \quad \text{for all } (\tau,v) \in [0,1]\times(\overline{B}_{\rho_0}\setminus\{0\}). \tag{3.9}$$

In fact, suppose there exists

$$v_k = \tau_k\mathcal{N}_g(v_k), \quad \tau_k \in [0,1],$$

where $v_k \in C^1_M\setminus\{0\}$, $k\in\mathbb{N}$, $\|v_k\| \to 0$. From the previous lemma, v is strictly monotonically decreasing and strictly positive on $[0,R)$.

Asuming $\|v_k\| \le s_\varepsilon$, $k\in\mathbb{N}$, we can see from (3.8)

$$g(r,v_k(r),v'_k(r)) \le \varepsilon\phi(\|v_k\|_\infty) \quad \text{for all } r\in[0,R],\ k\in\mathbb{N}.$$

Then for any $k \in \mathbb{N}$, there is

$$\begin{aligned}\|v_k\|_\infty &\le \int_0^R \phi^{-1}\left(\sigma(t)\int_0^t r^{N-1}g(r,v_k,v_k')dr\right)dt\\ &\le R\phi^{-1}\left(\frac{\varepsilon R}{N}\phi(\|v_k\|_\infty)\right).\end{aligned}$$

That is, there is

$$\frac{\phi\left(\frac{\|v_k\|_\infty}{R}\right)}{\phi(\|v_k\|_\infty)} \le \frac{\varepsilon R}{N}.$$

This contradicts (3.7) and so (3.9) is true. That is, for any $\rho \in (0,\rho_0]$, there is

$$d_{LS}[I-\mathcal{H}(1,\cdot),B_\rho,0] = d_{LS}[I-\mathcal{N}_g,B_\rho,0] = d_{LS}[I-\mathcal{H}(0,\cdot),B_\rho,0] = d_{LS}[I,B_\rho,0] = 1. \qquad \square$$

4 Proof of main result

First of all there is an important lemma before the main result of this paper.

Lemma 4.1. *Let $k \in (0,1), \beta_0 \in (0, \frac{1-k}{8}R)$ be given. Let $I_{k,\beta_0} := [\frac{4\beta_0}{1-k}, R-\frac{4\beta_0}{1-k}]$. Then*

$$\frac{R}{2} \in I_{k,\beta_0}$$

and

$$|v'(s)| \le 1-k, \quad \forall v \in \mathcal{A}, \quad \forall s \in I_{k,\beta_0},$$

where $\mathcal{A} := \{\, v \mid v \text{ is concave in } [0,R],\ v'(0) < 1,\ v'(R) > -1,\ \|v\|_\infty \le 4\beta_0\}$.

Proof. Let $a = 1-k,\ b = \frac{4\beta_0}{1-k}$, then

$$0 < a < 1, \quad b \in \left(0, \frac{R}{2}\right), \quad I := I_{k,\beta_0} = [b, R-b].$$

Since $v \in C^1[0,R]$, v is concave in $[0,R]$ and v' is decreasing. If there exists $s \in I$ such that $|v'(s)| > 1-k = a$, then $v'(s) > a$ or $v'(s) < -a$. If $v'(s) < -a$, then $\frac{v(s)-v(R)}{s-R} = v'(t)$, for some $t \in (s,R)$. So we have $\frac{v(s)}{s-R} \le v'(s) < -a$. Therefore $v(s) > a(R-s) \ge ab = 4\beta_0 \ge \|v\|_\infty$. This is a contradiction. Analogously, we can get a contradiction for other case. $\square$

Proof of Theorem 1.1. Let us say

$$S_j := \{\lambda > 0 : (1.3) \text{ at least } j \text{ positive solutions}\}, \quad (j = 1,2).$$

1. *The existence of* Λ.

Let $\lambda > 0$ and v be a positive solution of (1.3). Firstly, using hypothesis (A_{fH}), we have: $\forall \varepsilon_0 > 0$, $\exists\, \delta_1$, for $|\varphi^{-1}(v) - 0| < \delta_1$, there can be $\left|\frac{Nf'(\varphi^{-1}(v))}{v} - f_0\right| < \varepsilon_0$. For the above ε_0, $\exists\, \delta_2$, when $|\varphi^{-1}(v) - 0| < \delta_2$, there is $\left|\frac{Nf(\varphi^{-1}(v))H(\varphi^{-1}(v),r)}{v} - H_0\right| < \varepsilon_0$.

Secondly, integrating (1.3) from 0 to $r \in (0, R]$ and using that v is a positive solution of (1.3) such that we obtain

$$
\begin{aligned}
-r^{N-1}\phi(v') &= \int_0^r \lambda t^{N-1}\left(\frac{Nf'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} - Nf(\varphi^{-1}(v))H(\varphi^{-1}(v),t)\right)dt \\
&< \lambda \int_0^r t^{N-1}\left(\frac{f_0 v}{\sqrt{1-v'^2}} - H_0 v\right)dt \\
&= \lambda \int_0^r t^{N-1}\left(\frac{f_0 v}{\sqrt{1-v'^2}} - f_0 v\right)dt.
\end{aligned}
$$

Using Lemma 4.1, let $k = a_0$, $\beta_0 = \frac{(1-a_0)\eta}{8}R \in (0, \frac{1-a_0}{8}R)$, a_0 is the constant that satisfies the definition and $\eta \in (0,1)$ is the given constant, then there is $I = [\frac{\eta}{2}R, R - \frac{\eta}{2}R]$. Hence, $\|v\|_\infty \le \frac{(1-a_0)\eta}{2}R$, $|v'(s)| \le 1 - a_0$, for all $s \in I$.

Therefore,

$$
\begin{aligned}
-r^{N-1}\phi(v') &< \lambda \int_0^r t^{N-1}\left(\frac{f_0 v}{\sqrt{1-v'^2}} - f_0 v\right)dt \\
&\le \lambda \int_0^r t^{N-1} f_0 \frac{(1-a_0)\eta}{2} R\left(\frac{1}{\sqrt{1-(1-a_0)^2}} - 1\right)dt \\
&\le \lambda MR \int_0^r t^{N-1}dt \\
&= \frac{\lambda MR r^N}{N},
\end{aligned}
$$

where $M = f_0 \frac{(1-a_0)\eta}{2}\left(\frac{1}{\sqrt{1-(1-a_0)^2}} - 1\right)$.

Therefore, there is

$$
-v'(r) \le -\frac{v'(r)}{\sqrt{1-v'^2}} < \frac{\lambda MRr}{N}. \tag{4.1}
$$

Integrating (4.1) from 0 to R we obtain

$$
v(0) < \frac{\lambda MR^3}{2N}. \tag{4.2}
$$

Next, using $v(0) > 0$, we obtain

$$
\lambda > \frac{2NM_0}{R^3},
$$

where $M_0 := v(0)/M$.

We know from [18] that the problem (1.3) has at least one positive solution for $\lambda > 0$. Specially, $S_1 \neq \emptyset$ and we can define

$$
\Lambda = \Lambda(R) := \inf S_1.
$$

Clearly, we have $\Lambda \ge \frac{2NM_0}{R^3}$. We claim that $\Lambda \in S_1$. Indeed, let $\lambda_k \subset S_1, \lambda_k \to \Lambda$ $(k \to \infty)$. Since $v_k \in C^1_M$, v_k is positive on $[0, R)$, then

$$
v_k = K \circ \phi^{-1} \circ S \circ \left(\lambda_k \left(\frac{Nf'(\varphi^{-1}(v_k))}{\sqrt{1-v_k'^2}} - Nf(\varphi^{-1}(v_k))H(\varphi^{-1}(v_k), r)\right)\right).
$$

Using (3.2) and the Arzelà–Ascoli theorem can have $v \in C$ and has a subsequence such that $\{v_k\} \to v$. So, it follows that $v \geq 0$ and

$$v = K \circ \phi^{-1} \circ S \circ \left(\Lambda \left(\frac{N f'(\varphi^{-1}(v))}{\sqrt{1 - v'^2}} - N f(\varphi^{-1}(v)) H(\varphi^{-1}(v), r) \right) \right).$$

With (4.2), we can see that there is a constant $c_1 > 0$ such that $v_k(0) > c_1$, $\forall k \in \mathbb{N}$. This ensures that $v(0) \geq c_1$, according to Lemma 3.3, has $v > 0$ on $[0, R)$. Hence, $\Lambda \in S_1$. Obviously, $\Lambda > \frac{2NM_0}{R^3}$.

Next, let $\lambda_0 > \Lambda$, where λ_0 is arbitrary. Here $\lambda_0 \in S_1$ is proved by Theorem 2.2. Let v_1 be a positive solution for (1.3) corresponding to $\lambda = \Lambda$. It is now easy to know that v_1 is a lower solution to problem (1.3) when $\lambda = \lambda_0$. Construct the upper solution, let $H > 0$, $\widetilde{R} > R$, while considering the problem

$$\left(r^{N-1} \frac{v'}{\sqrt{1 - v'^2}} \right)' + r^{N-1} H = 0, \quad v'(0) = v(\widetilde{R}) = 0. \tag{4.3}$$

By integrating the above formula, we get

$$v(r) = \frac{N}{H} \left[\sqrt{1 + \frac{H^2}{N^2} \widetilde{R}^2} - \sqrt{1 + \frac{H^2}{N^2} r^2} \right].$$

For fixed $\lambda_2 > \lambda_0$, let v_2 is the solution of problem (4.3) corresponding to $H = \lambda_2 M \widetilde{R}$. By $v_2(R) > 0$ and

$$\lambda_0 \left(\frac{N f'(\varphi^{-1}(v_2))}{\sqrt{1 - v_2'^2}} - N f(\varphi^{-1}(v_2)) H(\varphi^{-1}(v_2), r) \right) \leq \lambda_2 M \widetilde{R}, \quad r \in [0, R].$$

Then we can see that v_2 is an upper solution of problem (1.3) when $\lambda = \lambda_0$, then

$$v_2(R) = N \left[\sqrt{\frac{1}{(\lambda_2 M \widetilde{R})^2} + \frac{\widetilde{R}^2}{N^2}} - \sqrt{\frac{1}{(\lambda_2 M \widetilde{R})^2} + \frac{R^2}{N^2}} \right].$$

Then there is $v_1(0) < v_2(R)$ when $\widetilde{R}$ is sufficiently large. Consider that v_1, v_2 is strictly decreasing, then there is $v_1 < v_2$ on $[0, R]$. Thus, from Theorem 2.2 we know that $\lambda_0 \in S_1$, therefore $S_1 \in [\Lambda, \infty]$.

2. *Multiplicity.*

Let $\lambda_0 > \Lambda$. Let us prove $\lambda_0 \in S_2$ by Lemma 3.1, 3.2, 3.4. Let v_1, v_2 be constructed as above. When $\lambda = \lambda_0$, let v_0 be a solution to problem (1.3) such that $v_1 \leq v_0 \leq v_2$, i.e., $v_0 \in \Omega_{v_1, v_2} := \{v_0 \in C^1_M : v_1 \leq v_0 \leq v_2\}$.

First, we claim that exists $\varepsilon > 0$ with $\overline{B}(v_0, \varepsilon) \subset \Omega_{v_1, v_2}$. For all $r \in [0, R]$, there is

$$\begin{aligned}
v_2(r) &= \int_r^{\widetilde{R}} \phi^{-1} \left(\sigma(t) \int_0^t s^{N-1} \lambda_2 M \widetilde{R} ds \right) dt \\
&> \int_r^R \phi^{-1} \left(\sigma(t) \int_0^t s^{N-1} \lambda_2 \left(\frac{N f'(\varphi^{-1}(v_2))}{\sqrt{1 - {v_2}'^2}} - N f(\varphi^{-1}(v_2)) H(\varphi^{-1}(v_2), s) \right) ds \right) dt \\
&\geq \int_r^R \phi^{-1} \left(\sigma(t) \int_0^t s^{N-1} \lambda_0 \left(\frac{N f'(\varphi^{-1}(v_0))}{\sqrt{1 - {v_0}'^2}} - N f(\varphi^{-1}(v_0)) H(\varphi^{-1}(v_0), s) \right) ds \right) dt \\
&= v_0(r).
\end{aligned}$$

Therefore, there exists $\varepsilon_2 > 0$ such that $v \le v_2$ for all $v \in \overline{B}(v_0, \varepsilon_2)$. Similarly on $[0, R/2]$ there is $v_1 < v_0$. Therefore $\varepsilon_1' > 0$ can be found such that

$$v \in C^1_M \text{ and } \|v - v_0\|_\infty \le \varepsilon_1' \Rightarrow v \ge v_1 \text{ on } [0, R/2]. \tag{4.4}$$

On the other hand, we have

$$-v_0' = \phi^{-1} \circ S \circ \lambda_0 \left(\frac{Nf'(\varphi^{-1}(v_0))}{\sqrt{1 - v_0'^2}} - Nf(\varphi^{-1}(v_0)) H(\varphi^{-1}(v_0), r) \right)$$

and

$$-v_1' = \phi^{-1} \circ S \circ \Lambda \left(\frac{Nf'(\varphi^{-1}(v_1))}{\sqrt{1 - v_1'^2}} - Nf(\varphi^{-1}(v_1)) H(\varphi^{-1}(v_1), r) \right),$$

yielding $v_0' < v_1'$ on $[R/2, R]$. So we can find a sufficiently small $\varepsilon_1 \in (0, \varepsilon_1')$ such that $v' < v_1'$ on $[R/2, R]$, where $v \in \overline{B}(v_0, \varepsilon_1)$. It follows from $v_0(R) = 0 = v(R)$ that for all $v \in \overline{B}(v_0, \varepsilon_1)$ has $v > v_1$ on [0,R]. Considering (4.4), we claim $\varepsilon \in (0, \min\{\varepsilon_1, \varepsilon_2\})$. Next, if the problem (1.3) has a second solution in Ω_{v_1, v_2}, then the proof of the multiplicity is completed.

If not, using Lemma 3.2 we get

$$d_{LS}[I - \mathcal{N}_{\lambda_0}, B(v_0, \rho), 0] = 1 \quad \text{for all } 0 < \rho \le \varepsilon,$$

where $\mathcal{N}_{\lambda_0}$ is the fixed point operator associated to (1.3) with $\lambda = \lambda_0$.

In addition, using Lemma 3.1 we have

$$d_{LS}[I - \mathcal{N}_{\lambda_0}, B_\rho, 0] = 1 \quad \text{for all } \rho \ge (R + 1).$$

From Lemma 3.4 one has

$$d_{LS}[I - \mathcal{N}_{\lambda_0}, B_\rho, 0] = 1 \quad \text{for all sufficiently small } \rho.$$

When ρ_1, ρ_2 is sufficiently small and $\rho_3 \ge R + 1$ such that $\overline{B}(v_0, \rho_1) \cap \overline{B}_{\rho_2} = \emptyset$ and $\overline{B}(v_0, \rho_1) \cup \overline{B}_{\rho_2} \subset B_{\rho_3}$. Then, from the additivity-excision property of the Leray–Schauder degree it follows that

$$d_{LS}[I - \mathcal{N}_{\lambda_0}, B_{\rho_3} \setminus [\overline{B}(v_0, \rho_1) \cup \overline{B}_{\rho_2}], 0] = -1,$$

which, together with the existence property of the Leray–Schauder degree, imply that $\mathcal{N}_{\lambda_0}$ has a fixed point $\widetilde{v}_0 \in B_{\rho_3} \setminus [\overline{B}(v_0, \rho_1) \bigcup \overline{B}_{\rho_2}]$. We infer that (1.3) has a second positive solution, and the proof is complete. □

Appendix: derivation process of problem (1.3)

To the best of our knowledge, problem (1.3) was first given in [20], but they did not given derivation process. For the convenience of readers and integrity of the paper, here we give the detailed derivation.

Without loss of generality, let us consider the radially symmetric spacelike solutions of the Dirichlet problem with the mean curvature operator in FLRW spacetime

$$\begin{cases} \operatorname{div}\left(\frac{\operatorname{grad} u}{f(u)\sqrt{f^2(u) - |\operatorname{grad} u|^2}} \right) + \frac{f'(u)}{\sqrt{f^2(u) - |\operatorname{grad} u|^2}} \left(N + \frac{|\operatorname{grad} u|^2}{f^2(u)} \right) = NH(u, |x|) \quad \text{in } B(R), \\ |\operatorname{grad} u| < f(u) \quad \text{in } B(R), \\ u = 0 \quad \text{on } \partial B(R), \end{cases} \tag{A.1}$$

where $B(R) = \{x \in \mathbb{R}^N : |x| < R\}$ and $N \geq 1$.

Step 1. If $N = 1$.

Then (A.1) reduces to

$$\begin{cases} \left(\dfrac{u'}{f(u)\sqrt{f^2(u) - u'^2}} \right)' + \dfrac{f'(u)}{\sqrt{f^2(u) - u'^2}} \left(1 + \dfrac{u'^2}{f^2(u)} \right) = H(u, |x|), \quad x \in (0, R), \\ |u'| < f(u), \quad x \in (0, R), \\ u'(0) = u(R) = 0. \end{cases} \tag{A.2}$$

In fact (A.2) can be converted to the following

$$\begin{cases} \left(\dfrac{1}{f(u)} \cdot \dfrac{u'}{f(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}} \right)' + \dfrac{f'(u)(f^2(u) + u'^2)}{f^3(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}} = H(u, |x|), \quad x \in (0, R), \\ |u'| < f(u), \quad x \in (0, R), \\ u'(0) = u(R) = 0. \end{cases} \tag{A.3}$$

Let $v(r) = \varphi(u(x))$ and $r = |x|$. Then

$$v'(r) = \varphi'(u)u'(x) = \frac{u'(x)}{f(u(x))}, \quad \left(\varphi(s) = \int_0^s \frac{dt}{f(t)} \right),$$

and accordingly,

$$u(x) = \varphi^{-1}(v(r)), \quad u'(x) = f(u(x))v'(r). \tag{A.4}$$

Since

$$\begin{aligned} &\left(\frac{1}{f(u)} \cdot \frac{u'}{f(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}} \right)' + \frac{f'(u)(f^2(u) + u'^2)}{f^3(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}} \\ &= \frac{-f'(u)u'}{f^2(u)} \cdot \frac{u'}{f(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}} + \frac{1}{f(u)} \cdot \left(\frac{u'}{f(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}} \right)' \\ &\quad + \frac{f'(u)}{f(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}} + \frac{f'(u)u'^2}{f^3(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}} \\ &= \frac{1}{f(u)} \cdot \left(\frac{u'}{f(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}} \right)' + \frac{f'(u)}{f(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}}. \end{aligned} \tag{A.5}$$

Then, this fact together with (A.4), problem (A.3) can be converted to the following

$$\begin{cases} -\left(\dfrac{v'}{\sqrt{1 - v'^2}} \right)' = \dfrac{f'(\varphi^{-1}(v))}{\sqrt{1 - v'^2}} - f(\varphi^{-1}(v))H(\varphi^{-1}(v), r), \quad r \in (0, R), \\ |v'| < 1, \quad r \in (0, R), \\ v'(0) = v(R) = 0. \end{cases} \tag{A.6}$$

Step 2. If $N \geq 2$.

Given $u(x),\ x = (x_1, \ldots, x_N)$.

Let $v(r) = \varphi(u(x))$ and $r = |x| = \left(\sum\limits_{i=1}^{N} x_i^2\right)^{\frac{1}{2}}$. Then

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}\left(\sum_{i=1}^{N} x_i^2\right)^{-\frac{1}{2}} 2x_i = \frac{x_i}{r}. \tag{A.7}$$

$$\frac{\partial v}{\partial x_i} = v'(r)\frac{\partial r}{\partial x_i} = v'(r)\cdot\frac{x_i}{r} = \varphi'(u)\cdot\frac{\partial u}{\partial x_i} = \frac{1}{f(u)}\cdot\frac{\partial u}{\partial x_i}.$$

Hence

$$\frac{\partial u}{\partial x_i} = f(u)\cdot v'(r)\cdot\frac{x_i}{r}. \tag{A.8}$$

Since

$$\operatorname{grad} u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right),$$

then

$$|\operatorname{grad} u|^2 = \sum_{i=1}^{N}\left(\frac{\partial u}{\partial x_i}\right)^2 = \sum_{i=1}^{N}\left(f(u)\cdot v'(r)\cdot\frac{x_i}{r}\right)^2 = (f(u)v'(r))^2\sum_{i=1}^{N}\left(\frac{x_i}{r}\right)^2 = (f(u)v'(r))^2, \tag{A.9}$$

that is

$$\left(\frac{|\operatorname{grad} u|}{f(u)}\right)^2 = (v'(r))^2,$$

and accordingly, from this and (A.8), we have that

$$\begin{aligned}
&\operatorname{div}\left(\frac{\operatorname{grad} u}{f(u)\sqrt{f^2(u) - |\operatorname{grad} u|^2}}\right) + \frac{f'(u)}{\sqrt{f^2(u) - |\operatorname{grad} u|^2}}\left(N + \frac{|\operatorname{grad} u|^2}{f^2(u)}\right)\\
&= \operatorname{div}\left(\frac{1}{f(u)}\cdot\frac{\operatorname{grad} u}{f(u)\sqrt{1 - \left(\frac{|\operatorname{grad} u|}{f(u)}\right)^2}}\right) + \frac{f'(u)\left(Nf^2(u) + |\operatorname{grad} u|^2\right)}{f^3(u)\sqrt{1 - \left(\frac{|\operatorname{grad} u|}{f(u)}\right)^2}}\\
&= \operatorname{div}\left(\frac{1}{f(u)}\cdot\frac{\operatorname{grad} u}{f(u)\sqrt{1 - (v'(r))^2}}\right) + \frac{f'(u)\left(Nf^2(u) + (f(u)v'(r))^2\right)}{f^3(u)\sqrt{1 - (v'(r))^2}}\\
&= \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{1}{f(u)}\cdot\frac{1}{f(u)\sqrt{1 - (v'(r))^2}}\cdot f(u)\cdot v'(r)\cdot\frac{x_i}{r}\right)\\
&\quad + \frac{f'(u)\left(Nf^2(u) + (f(u)v'(r))^2\right)}{f^3(u)\sqrt{1 - (v'(r))^2}}\\
&= \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{1}{f(u)}\cdot\frac{v'(r)}{\sqrt{1 - (v'(r))^2}}\cdot\frac{x_i}{r}\right) + \frac{f'(u)(Nf^2(u) + (f(u)v'(r))^2)}{f^3(u)\sqrt{1 - (v'(r))^2}}.
\end{aligned} \tag{A.10}$$

From now on, let us fixed the notation $\phi(s) = \frac{s}{\sqrt{1-s^2}}$.

From (A.7), (A.8), it follows that

$$\begin{aligned}
&\frac{\partial}{\partial x_i}\left(\frac{1}{f(u)}\cdot\frac{v'(r)}{\sqrt{1-(v'(r))^2}}\cdot\frac{x_i}{r}\right)\\
&\quad=\frac{-f'(u)\cdot f(u)\cdot v'(r)\cdot\frac{x_i}{r}}{f^2(u)}\cdot\phi(v'(r))\cdot\frac{x_i}{r}\\
&\qquad+\frac{1}{f(u)}\left[\phi'(v'(r))\cdot\frac{x_i}{r}\cdot\frac{x_i}{r}+\phi(v'(r))\cdot\frac{r-x_i\cdot\frac{x_i}{r}}{r^2}\right]\\
&\quad=\frac{-f'(u)\cdot v'(r)\cdot\left(\frac{x_i}{r}\right)^2}{f(u)}\cdot\phi(v'(r))\\
&\qquad+\frac{1}{f(u)}\phi'(v'(r))\cdot\left(\frac{x_i}{r}\right)^2+\frac{1}{f(u)}\cdot\phi(v'(r))\cdot\frac{r^2-x_i^2}{r^3}.
\end{aligned}\tag{A.11}$$

Hence

$$\begin{aligned}
&\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{1}{f(u)}\cdot\frac{v'(r)}{\sqrt{1-(v'(r))^2}}\cdot\frac{x_i}{r}\right)\\
&\quad=\frac{-f'(u)\cdot v'(r)}{f(u)}\cdot\phi(v'(r))+\frac{1}{f(u)}\phi'(v'(r))+\frac{1}{f(u)}\cdot\phi(v'(r))\cdot\frac{N-1}{r}.
\end{aligned}\tag{A.12}$$

From this and (A.10), we have that

$$\begin{aligned}
&\operatorname{div}\left(\frac{\operatorname{grad}u}{f(u)\sqrt{f^2(u)-|\operatorname{grad}u|^2}}\right)+\frac{f'(u)}{\sqrt{f^2(u)-|\operatorname{grad}u|^2}}\left(N+\frac{|\operatorname{grad}u|^2}{f^2(u)}\right)\\
&=\frac{-f'(u)\cdot v'(r)}{f(u)}\cdot\phi(v'(r))+\frac{1}{f(u)}\phi'(v'(r))+\frac{1}{f(u)}\cdot\phi(v'(r))\cdot\frac{N-1}{r}\\
&\quad+\frac{Nf'(u)}{f(u)\sqrt{1-(v'(r))^2}}+\frac{f'(u)v'(r)}{f(u)}\cdot\phi(v'(r))\\
&=\frac{1}{f(u)}\phi'(v'(r))+\frac{1}{f(u)}\cdot\phi(v'(r))\cdot\frac{N-1}{r}+\frac{Nf'(u)}{f(u)\sqrt{1-(v'(r))^2}}\\
&=NH(u,r).
\end{aligned}\tag{A.13}$$

Hence, we have

$$\phi'(v'(r))+\frac{N-1}{r}\phi(v'(r))=-\frac{Nf'(u)}{\sqrt{1-(v'(r))^2}}+Nf(u)H(u,r),$$

multiplying both sides of the equation by r^{N-1}, we get that

$$r^{N-1}\phi'(v'(r))+(N-1)r^{N-2}\phi(v'(r))=Nr^{N-1}\left[-\frac{f'(u)}{\sqrt{1-(v'(r))^2}}+f(u)H(u,r)\right],$$

that is

$$-(r^{N-1}\phi(v'(r)))'=Nr^{N-1}\left[\frac{f'(u)}{\sqrt{1-(v'(r))^2}}-f(u)H(u,r)\right].\tag{A.14}$$

From this and the fact

$$u(x)=\varphi^{-1}(v(r)),$$

problem (A.1) can be converted to

$$\begin{cases} -(r^{N-1}\phi(v'))' = Nr^{N-1}\left[\dfrac{f'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} - f(\varphi^{-1}(v))H(\varphi^{-1}(v),r)\right], \quad r \in (0,R), \\ |v'| < 1, \quad r \in (0,R), \\ v'(0) = v(R) = 0. \end{cases} \tag{A.15}$$

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References

[1] R. Bartnik, L. Simon, Spacelike hypersurfaces with prescribed boundary values and mean curvature, *Comm. Math. Phys.* **87**(1982), 131–152. https://doi.org/10.1007/BF01211061; MR0680653; Zbl 0512.53055

[2] C. Bereanu, D. de la Fuente, A. Romero, P. J. Torres, Existence and multiplicity of entire radial spacelike graphs with prescribed mean curvature function in certain Friedmann–Lemaître–Robertson–Walker spacetimes, *Commun. Contemp. Math.* **19**(2017), No. 2, 18. https://doi.org/10.1142/S0219199716500061; MR3611658; Zbl 1368.35154

[3] C. Bereanu, P. Jebelean, J. Mawhin, Radial solutions for some nonlinear problems involving mean curvature operators in Euclidean and Minkowski spaces, *Proc. Amer. Math. Soc.* **137**(2009), No. 1, 161–169. https://doi.org/10.1090/S0002-9939-08-09612-3; MR2439437; Zbl 1161.35024

[4] C. Bereanu, P. Jebelean, P. J. Torres, Positive radial solutions for Dirichlet problems with mean curvature operators in Minkowski space, *J. Funct. Anal.* **264**(2013), No. 1, 270–287. https://doi.org/10.1016/j.jfa.2012.10.010; MR2995707; Zbl 1336.35174

[5] C. Bereanu, P. Jebelean, P. J. Torres, Multiple positive radial solutions for a Dirichlet problem involving the mean curvature operator in Minkowski space, *J. Funct. Anal.* **265**(2013), No. 4, 644–659. https://doi.org/10.1016/j.jfa.2013.04.006; MR3062540; Zbl 1285.35051

[6] C. Bereanu, J. Mawhin, Existence and multiplicity results for some nonlinear problems with singular ϕ-Laplacian, *J. Differential Equations* **243**(2007), No. 2, 536–557. https://doi.org/10.1016/j.jde.2007.05.014; MR2371799; Zbl 1148.34013

[7] C. Bereanu, J. Mawhin, Boundary value problems for second-order nonlinear difference equations with discrete ϕ-Laplacian and singular ϕ, *J. Difference Equ. Appl.* **14**(2008), No. 10–11, 1099–1118. https://doi.org/10.1080/10236190802332290; MR2447187; Zbl 1161.39003

[8] E. Calabi, *Examples of Bernstein problems for some nonlinear equations*, Proc. Sympos. Pure Math., Vol. XIV-XVI, Amer. Math. Soc., Providence, RI, 1970. MR0264210; Zbl 0211.12801

[9] T. Chen, R. Ma, Y. Liang, Multiple positive solutions of second-order nonlinear difference equations with discrete singular ϕ-Laplacian, *J. Difference Equ. Appl.* **25**(2019), No. 1, 38–55. https://doi.org/10.1080/10236198.2018.1554064; MR3911294; Zbl 1407.39002

[10] S.-Y. Cheng, S.-T. Yau, Maximal spacelike hypersurfaces in the Lorentz–Minkowski spaces, *Ann. of Math. (2)* **104**(1976), No. 3, 407–419. https://doi.org/10.2307/1970963; MR0431061; Zbl 0352.53021

[11] Y. Choquet-Bruhat, *General relativity and the Einstein equations*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2009. https://doi.org/10.1093/acprof:oso/9780199230723.001.0001; MR2473363; Zbl 1157.83002

[12] I. Coelho, C. Corsato, F. Obersnel, P. Omari, Positive solutions of the Dirichlet problem for the one-dimensional Minkowski-curvature equation, *Adv. Nonlinear Stud.* **12**(2012), No. 3, 621–638. https://doi.org/10.1515/ans-2012-0310; MR2976056; Zbl 1263.34028

[13] C. Corsato, F. Obersnel, P. Omari, S. Rivetti, Positive solutions of the Dirichlet problem for the prescribed mean curvature equation in Minkowski space, *J. Math. Anal. Appl.* **405**(2013), No. 1, 227–239. https://doi.org/10.1016/j.jmaa.2013.04.003; MR3053503; Zbl 1310.35140

[14] G. Dai, Bifurcation and positive solutions for problem with mean curvature operator in Minkowski space, *Calc. Var. Partial Differential Equations* **55**(2016), No. 4, 17. https://doi.org/10.1007/s00526-016-1012-9; MR3513210; Zbl 1356.35117

[15] G. Dai, Bifurcation and nonnegative solutions for problems with mean curvature operator on general domain, *Indiana Univ. Math. J.* **67**(2018), No. 6, 2103–2121. https://doi.org/10.1512/iumj.2018.67.7546; MR3900363; Zbl 1420.35101

[16] G. Dai, Global structure of one-sign solutions for problem with mean curvature operator, *Nonlinearity* **31**(2018), No. 11, 5309–5328. https://doi.org/10.1088/1361-6544/aadf43; MR3867236

[17] G. Dai, Some results on surfaces with different mean curvatures in $\mathbb{R}^{N+1}$ and L^{N+1}, *Ann. Mat. Pura Appl. (4)* **201**(2022), No. 1, 335–357. https://doi.org/10.1007/s10231-021-01118-1; MR4375012; Zbl 1486.35035

[18] G. Dai, A. Romero, P. J. Torres, Global bifurcation of solutions of the mean curvature spacelike equation in certain Friedmann–Lemaître–Robertson–Walker spacetimes, *J. Differential Equations* **264**(2018), No. 12, 7242–7269. https://doi.org/10.1016/j.jde.2018.02.014; MR3779636; Zbl 1391.35043

[19] G. Dai, A. Romero, P. J. Torres, Global bifurcation of solutions of the mean curvature spacelike equation in certain standard static spacetimes, *Discrete Contin. Dyn. Syst. Ser. S* **13**(2020), No. 11, 3047–3071. https://doi.org/10.3934/dcdss.2020118; MR4147382; Zbl 1473.35031

[20] D. de la Fuente, A. Romero, P. J. Torres, Radial solutions of the Dirichlet problem for the prescribed mean curvature equation in a Robertson–Walker spacetime, *Adv. Nonlinear Stud.* **15**(2015), No. 1, 171–181. https://doi.org/10.1515/ans-2015-0109; MR3299388; Zbl 1312.35122

[21] A. Friedmann, On the curvature of space, *Gen. Relativity Gravitation* **31**(1999), No. 12, 1991–2000. https://doi.org/10.1023/A:1026751225741; MR1728515; Zbl 1051.83500

[22] A. Friedmann, On the possibility of a world with constant negative curvature of space, *Gen. Relativity Gravitation* **31**(1999), No. 12, 2001–2008. https://doi.org/10.1023/A:1026755309811; MR1728514; Zbl 1051.83501

[23] D. Gurban, P. Jebelean, C. Serban, Non-potential and non-radial Dirichlet systems with mean curvature operator in Minkowski space, *Discrete Contin. Dyn. Syst.* **40**(2020), No. 1, 133–151. https://doi.org/10.3934/dcds.2020006; MR4026955; Zbl 1439.35165

[24] S. Y. Huang, Classification and evolution of bifurcation curves for the one-dimensional Minkowski curvature problem and its applications, *J. Differential Equations* **264**(2018), 5977–6011. https://doi.org/10.1016/j.jde.2018.01.021; MR3765772; Zbl 1390.34051

[25] P. Jebelean, C. Popa, Numerical solutions to singular ϕ-Laplacian with Dirichlet boundary conditions, *Numer. Algorithms* **67**(2014), No. 2, 305–318. https://doi.org/10.1007/s11075-013-9792-x; MR3264579; Zbl 1323.65084

[26] P. Jebelean, C. Popa, C. Serban, Numerical extremal solutions for a mixed problem with singular ϕ-Laplacian, *NoDEA Nonlinear Differential Equations Appl.* **21**(2014), No. 2, 289–304. https://doi.org/10.1007/s00030-013-0247-9; MR3180885; Zbl 1305.34037

[27] G. Lemaître, Republication of: A homogeneous universe of constant mass and increasing radius accounting for the radial velocity of extra-galactic nebulae, *Gen. Relativity Gravitation* **45**(2013), No. 8, 1635–1646. https://doi.org/10.1007/s10714-013-1548-3; MR3084107; Zbl 1273.83189

[28] R. Ma, H. Gao, Y. Lu, Global structure of radial positive solutions for a prescribed mean curvature problem in a ball, *J. Funct. Anal.* **270**(2016), No. 7, 2430–2455. https://doi.org/10.1016/j.jfa.2016.01.020; MR3464046; Zbl 1342.34044

[29] R. Ma, L. Wei, Z. Chen, Evolution of bifurcation curves for one-dimensional Minkowski-curvature problem, *Appl. Math. Lett.* **103**(2020), 8. https://doi.org/10.1016/j.aml.2019.106176; MR4046672; Zbl 1441.53010

[30] R. Ma, M. Xu, Positive rotationally symmetric solutions for a Dirichlet problem involving the higher mean curvature operator in Minkowski space, *J. Math. Anal. Appl.* **460**(2018), No. 1, 33–46. https://doi.org/10.1016/j.jmaa.2017.11.049; MR3739891; Zbl 1384.53018

[31] R. Ma, M. Xu, Connected components of positive solutions for a Dirichlet problem involving the mean curvature operator in Minkowski space, *Discrete Contin. Dyn. Syst. Ser. B* **24**(2019), No. 6, 2701–2718. https://doi.org/10.3934/dcdsb.2018271; MR3960600; Zbl 1422.35053

[32] R. Ma, M. Xu, Z. He, Nonconstant positive radial solutions for Neumann problem involving the mean extrinsic curvature operator, *J. Math. Anal. Appl.* **484**(2020), No. 2, 13. https://doi.org/10.1016/j.jmaa.2019.123728; MR4041218; Zbl 1433.35122

[33] J. Mawhin, P. J. Torres, Prescribed mean curvature graphs with Neumann boundary conditions in some FLRW spacetimes, *J. Differential Equations* **261**(2016), No. 12, 7145–7156. https://doi.org/10.1016/j.jde.2016.09.013; MR3562322; Zbl 1351.35219

[34] B. O'Neill, *Semi-Riemannian geometry with application to relativity*, Pure and Applied Mathematics, Vol. 103, Academic Press, New York, 1983. MR0719023; Zbl 0531.53051

[35] H. P. Robertson, Kinematics and world structure, *Astrophys. J.* **82**(1935), 284–301. https://doi.org/10.1086/143681; Zbl 0013.03905

[36] H. P. Robertson, Kinematics and world structure II, *Astrophys. J.* **83**(1936), 187–201. https://doi.org/10.1086/143716; Zbl 0014.08701

[37] H. P. Robertson, Kinematics and world structure III, *Astrophys. J.* **83**(1936), 257–271. https://doi.org/10.1086/143726; Zbl 0014.08702

[38] P. J. Torres, The prescribed mean curvature problem with Neumann boundary conditions in FLRW spacetimes, *Rend. Istit. Mat. Univ. Trieste* **49**(2017), 19–25. https://doi.org/10.13137/2464-8728/16202; MR3748500; Zbl 1445.35188

[39] A. E. Treibergs, Entire spacelike hypersurfaces of constant mean curvature in Minkowski space, *Invent. Math.* **66**(1982), No. 1, 39–56. https://doi.org/10.1007/BF01404755; MR0652645; Zbl 0483.53055

[40] M. Xu, R. Ma, Nonspurious solutions of the Dirichlet problem for the prescribed mean curvature spacelike equation in a Friedmann–Lemaître–Robertson–Walker spacetime, *Rocky Mountain J. Math.* **53**(2023), No. 4, 1291–1311. https://doi.org/10.1216/rmj.2023.53.1291; MR4635003

[41] R. Yang, Y.-H. Lee, I. Sim, Bifurcation of nodal radial solutions for a prescribed mean curvature problem on an exterior domain, *J. Differential Equations* **268**(2020), No. 8, 4464–4490. https://doi.org/10.1016/j.jde.2019.10.035; MR4066026; Zbl 1436.34015

[42] X. Zhang, M. Feng, Bifurcation diagrams and exact multiplicity of positive solutions of one-dimensional prescribed mean curvature equation in Minkowski space, *Commun. Contemp. Math.* **21**(2019), No. 3, 17. https://doi.org/10.1142/S0219199718500037; MR3947060; Zbl 1422.34101