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A modified zero energy critical point theory with applications to several nonlocal problems

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Abstract. In this paper, we devote ourselves to considering a modified zero energy critical point theory for a specific set of functionals denoted as Φ_μ, defined within the confines of a uniformly convex Banach space. Integrating the nonlinear generalized Rayleigh quotient approach with Ljusternik–Schnirelman category, we establish the nonexistence and multiplicity of zero energy critical points of the involved functionals. In particular, the modified zero energy critical point theory can be applied to more nonlocal problems. Our main results improve and complement the existing results in the related literature.

Keywords: Ljusternik–Schnirelman category, nonlinear generalized Rayleigh quotient, zero energy critical points.

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1 Introduction

In the past decades, researchers have used classical variational methods to deal with various nonlocal problems and obtained various properties of their solutions, such as existence, multiplicity, asymptotic behavior and so on. However, although the classical variational methods have been properly modified, it seems still difficult to be directly effective for some complicated or special nonlocal problems. Based on this situation, the research on new variational methods has aroused increasing interest. It is worth mentioning that in this process, the existence of the number and index theory makes the Ljusternik–Schnirelman category theory more widely used. For more detailed applications of this theory, we refer to [18,33] and references therein. Along this direction, in this paper we employ the Ljusternik–Schnirelman category theory and the nonlinear generalized Rayleigh (NG-Rayleigh) quotient method to forge a critical point theory at zero energy levels for the energy functional (1.1). By means of this theory, we deal with several kinds of nonlocal problems, and present the nonexistence and multiplicity of their solutions at zero energy levels. More precisely, we consider $\Phi_\mu : X \to \mathbb{R}$

$$\Phi_\mu(u) := \frac{1}{\eta}N(u) - \frac{\mu}{\eta}A(u) - \frac{1}{\beta}B(u) + \frac{1}{\gamma}R(u), \tag{1.1}$$

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the setting is within X, a uniformly convex Banach space endowed with the norm $|\cdot|_X$, where $1 < \eta < \gamma < \beta$. Moreover, the functionals N, A, B, and R are considered to be homogeneous, nonnegative, and even, belonging to the class $C^1(X)$.

Throughout this paper, the following assumptions are imposed on the above-mentioned nonnegative even functionals:

$(\mathcal{M}_1)$ For any $u \in X \backslash \{0\}$, there exists $C > 0$ such that the following inequalities hold:

$$C\|u\|_X^\eta \geq A(u) > 0, \qquad C\|u\|_X^\beta \geq B(u) > 0, \qquad R(u) > 0, \qquad N(u) \geq C^{-1}\|u\|_X^\eta;$$

$(\mathcal{M}_2)$ $N(tu) = t^\eta N(u), A(tu) = t^\eta A(u), B(tu) = t^\beta B(u), R(tu) = t^\gamma R(u)$, for any $t > 0$;

$(\mathcal{M}_3)$ If $u_n \rightharpoonup u$ in X, then $A'(u_n) \to A'(u)$ and $B'(u_n) \to B'(u)$ in X^*. Moreover, for any $u_n, u \in X$, it holds that $R'(u_n)(u_n - u) \geq 0$.

$(\mathcal{M}_4)$ Let N be weakly lower semicontinuous, and there exists $C > 0$ such that for every $u_n, u \in X$, the inequality

$$(N'(u_n) - N'(u))(u_n - u) \geq C(\|u_n\|^{\eta-1} - \|u\|^{\eta-1})(\|u_n\| - \|u\|)$$

holds true.

According to assumption $(\mathcal{M}_1)$, we know that $A(u) > 0$ for all $u \in X \setminus \{0\}$. Therefore, $\Phi_\mu(u) = 0$ is equivalent to

$$\mu = \mu_0(u) := \frac{N(u) - \frac{\eta}{\beta}B(u) + \frac{\eta}{\gamma}R(u)}{A(u)}, \quad \text{for any } u \in X \setminus \{0\},$$

where $\mu_0(u)$ is called the Rayleigh quotient, the functional is derived using the NG-Rayleigh quotient approach. For any $u \in X \setminus \{0\}$,

$$\mu_0'(u) = \frac{\Phi'_{\mu_0(u)}(u)}{A(u)} = 0,$$

if and only if $\Phi'_\mu(u) = 0$.

We will search for the critical points of μ_0 by considering the fibering map $t \mapsto \mu_0(tu)$. Obviously, $\mu_0(tu) \in C^2(0,\infty)$ for every $u \in X \setminus \{0\}, u \mapsto \mu_0'(tu) \in C^1(X \setminus \{0\})$ for every $t > 0$. In order to get the critical point of $\mu_0(tu)$, from $(\mathcal{M}_2)$ it follows that

$$\mu_0(tu) = \frac{N(u)}{A(u)} - \frac{\eta}{\beta}\frac{B(u)}{A(u)}t^{\beta-\eta} + \frac{\eta}{\gamma}\frac{R(u)}{A(u)}t^{\gamma-\eta}, \quad \text{for any } u \in X \setminus \{0\},\ t > 0.$$

Let

$$\mu_0'(tu) = -(\beta-\eta)\frac{\eta}{\beta}\frac{B(u)}{A(u)}t^{\beta-\eta-1} + (\gamma-\eta)\frac{\eta}{\gamma}\frac{R(u)}{A(u)}t^{\gamma-\eta-1} = 0.$$

Then

$$t_0(u) = \left(\frac{\gamma-\eta}{\beta-\eta}\frac{\beta}{\gamma}\frac{R(u)}{B(u)}\right)^{\frac{1}{\beta-\gamma}},$$

that is $\mu_0'(t_0(u)u) = 0$. Since $\mu_0''(t_0(u)u) < 0$, we obtain that $t_0(u)$ is a non-degenerate global maximum point of $\mu_0(tu)$.

Define

$$\Lambda(u) := \mu_0(t_0(u)u) = \left(\frac{N(u)}{A(u)} - \frac{\eta}{\beta}\frac{B(u)}{A(u)}t_0(u)^{\beta-\eta} + \frac{\eta}{\gamma}\frac{R(u)}{A(u)}t_0(u)^{\gamma-\eta}\right)$$
$$= \frac{N(u)}{A(u)} + C_0\frac{R(u)^{\frac{\beta-\eta}{\beta-\gamma}}}{A(u)B(u)^{\frac{\gamma-\eta}{\beta-\gamma}}},$$

as a NG-Rayleigh quotient. It is obvious that $t_0(u)u$ can be considered as the zero energy critical point of Φ_μ, where $\mu = \Lambda(u)$. One may easily check that $\Lambda \in C^1(X\backslash\{0\})$,

$$\Lambda'(u)v = \frac{A(u)N'(u)v - N(u)A'(u)v}{A(u)^2}$$
$$+ C_0 Q(u)\left(\frac{\beta-\eta}{\beta-\gamma}B(u)A(u)R'(u)v - \frac{\gamma-\eta}{\beta-\gamma}R(u)A(u)B'(u)v - B(u)R(u)A'(u)v\right),$$

for every $u \in X\backslash\{0\}, v \in X$, where

$$C_0 = \frac{\eta}{\gamma}\frac{\beta-\gamma}{\beta-\eta}\left(\frac{\gamma-\eta}{\beta-\eta}\frac{\beta}{\gamma}\right)^{\frac{\gamma-\eta}{\beta-\gamma}} > 0, \qquad Q(u) = \frac{R(u)^{\frac{\gamma-\eta}{\beta-\gamma}}}{A(u)^2 B(u)^{\frac{\beta-\eta}{\beta-\gamma}}}.$$

To better utilize the Ljusternik–Schnirelman category theory, we first denote $\widetilde{\Lambda}$ as Λ on $S = \{u \in X\backslash\{0\} : \|u\| = 1\}$, where S is considered as a unit sphere in X and is a symmetric C^1 manifold. According to [28, Proposition 2.3], the critical point of $\widetilde{\Lambda}$ is also the critical point of Λ. Since $N(u), A(u), B(u), R(u)$ are even functionals, $\widetilde{\Lambda}$ is also an even functional. Now, let us recall the concept of Krasnoselskii genus. Given a set $F \subset S$, it is closed, nonempty and symmetric. We define

$$\gamma(F) := \inf\{n \in \mathbb{N} : \exists h : F \to \mathbb{R}^n \setminus \{0\} \text{ odd and continuous}\}.$$

to represent the Krasnoselskii genus of F. Setting

$$\mathcal{F}_n = \{F \subset S : F \text{ is compact, symmetric, and } \gamma(F) \geq n\},$$

for every $n \in \mathbb{N}$. Define the critical value of $\widetilde{\Lambda}$:

$$\mu_n := \inf_{F\in\mathcal{F}_n}\sup_{u\in F}\widetilde{\Lambda}, \text{ if } \widetilde{\Lambda} \text{ is bounded from below on } S.$$

It is well-known that the Krasnoselskii genus of the unit sphere in an infinite dimensional Banach space is infinite (cf. [13, Corollary 2.3]), namely, $\gamma(S) = \infty$.

Next, let us sketch some recent advances concerning the zero energy critical point theory. Recently, Quoirin et al. studied qualitative properties of zero energy critical points in [28], which means that at this point, the energy function and its derivatives are both zero. Furthermore, the authors in [28] established a new zero energy critical point theory using the NG-Rayleigh quotient method and Ljusternik–Schnirelman critical theory [2], and effectively applied it to several types of elliptic partial differential equations, resulting in the existence, nonexistence, and multiplicity of zero energy critical points. For more details on the NG-Rayleigh method, we refer to [19, 20] and references therein. Undoubtedly, the zero energy critical point theory established in paper [28] provides us with a new idea and perspective for solving nonlinear partial differential equations. As one of the advantages of this theory, it has a wide range of theoretical applications, that is, it can directly handle many types of nonlinear non-local problems, such as concave-convex problem, Schrödinger–Poisson problem, Kirchhoff-type problem, (p,q)-Laplace problem, and other elliptic problems. As a pioneer paper on zero energy critical point theory, the authors in [28] applied this theory to solve several local and non-local problems. For example, the following p-Laplacian problem with concave and convex nonlinearity was investigated in a bounded domain $\Omega \subset \mathbb{R}^N$:

$$\begin{cases} -\Delta_p u = \mu |u|^{q-2}u + f|u|^{r-2}u, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $1 < q < p < r < p^*$, $f, g \in L^\infty(\Omega)$ with $g > 0$ in Ω, Δ_p is p-Laplaian operator, and $f > 0$ in some subdomain $\Omega' \subset \Omega$. With the help of the NG-Rayleigh quotient method and Ljusternik–Schnirelman theory, the authors obtained the existence, non-existence and multiplicity of zero energy solutions for concave and convex problems in [28].

On the other hand, the authors in [28] also considered the properties of the zero energy solution for the Schrödinger-Poisson system, which is physically meaningful. To elaborate, the authors conducted a comprehensive investigation into the intricacies of the following system:

$$\begin{cases} -\Delta u + \omega u + \mu\phi u = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta u + a^2\Delta^2 u = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $p \in (2,3), \omega > 0$, and $a \geq 0$. In particular, the authors established the existence, non-existence, multiplicity and sign-changing properties of the zero energy radial solution of system (1.3). Subsequently, Quoirin et al. in [29] established the existence, multiplicity and bifurcation results of the critical points for a class of functionals with prescribed energy along the same technical route as in [28]. The authors first applied the corresponding critical point theory of prescribed energy to eigenvalue problems involving nonhomogeneous perturbations in [29], and its energy functional can be given by:

$$\Phi_\mu(u) = \frac{1}{p}(|\nabla u|_p^p - \mu|u|_p^p) - \frac{1}{r}|u|_r^r,\ u \in W_0^{1,p}(\Omega), \quad \text{where } 1 < r < p^*.$$

The authors made a noteworthy discovery regarding the Schrödinger–Poisson system. Specifically, for $c > 0$ (respectively, $c < 0$) and by choosing $p < r$ (respectively, $p > r$), the study revealed the existence of an infinite number of pairs $(\mu_{n,c}, u_{n,c}) \in \mathbb{R} \times W_0^{1,p}(\Omega) \setminus \{0\}$ satisfying $\Phi_{\mu_{n,c}}(\pm u_{n,c}) = c$ and $\Phi'_{\mu_{n,c}}(\pm u_{n,c}) = 0$. In other words, $\pm u_n$ represent prescribed energy critical points.

The authors next investigated the prescribed energy critical point of the Schrödinger–Bopp–Podolsky problem in [29]. The following represents the energy functional for this particular problem:

$$\Phi_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\omega}{2}\int_{\mathbb{R}^3} |u|^2 dx + \frac{\mu}{4}\int_{\mathbb{R}^3} \phi u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx,\ \text{ where } p \in (2,3), \omega > 0.$$

The authors obtained the existence and multiplicity of the critical point with prescribed energy by the critical point theory. Besides, the authors also conducted relevant research on the concave-convex problem in [29], and we will not elaborate on it here.

Motivated by [28, 29], we are interested in making appropriate modifications and extensions according to the existing zero energy critical point theory so that it can be applied to some specific situations. Along this direction, we propose a suitable class of energy functional (1.1). We point out that the purpose of this paper is to address the problem of energy functional (1.1) in the following situation:

$$\Phi'_\mu(u) = 0, \qquad \Phi_\mu(u) = 0.$$

Let $\delta_0 = \inf_{X\setminus\{0\}} \Lambda(u)$. Then the primary result of our article can be stated as:

Theorem 1.1. *Suppose that* $(\mathcal{M}_1)$*–*$(\mathcal{M}_4)$ *hold.*

(i) If $\mu < \delta_0$*, then there is no critical point having zero energy for the energy functional* Φ_μ*.*

(ii) If $\mu > \delta_0$*, then the energy functional* Φ_μ *has infinitely many zero energy critical points which change sign.*

Remark 1.2. At the beginning, we attempt to investigate the qualitative properties for some nonlocal problems in $\mathbb{R}^N$ by employing Theorem 1.1. But the embedding $H^1_r(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is compact only for $2 < p < 2^*$, we cannot verify assumption $(\mathcal{M}_3)$ if problems under consideration are involved the critical exponents. Inspired by [28], we give a modified version of Theorem 3.1 in [28], so that we are able to deal with Schrödinger–Poisson systems with critical nonlinearity.

Remark 1.3. In order to deal with Kirchhoff-type problems with critical growth in bounded domains, we follow the idea of proof in Lemma 3.3 of [21] to detour the compact embedding theorem, hence assumption $(\mathcal{M}_3)$ can be verified, which leads to the nonexistence and multiplicity of zero energy critical points for the following Kirchhoff problem with critical nonlinearity.

Notations. Throughout this paper, the following notions are employed:

- Denote $|\cdot|_q$ for the $L^q(\mathbb{R}^N)$-norm for $q \in [1, \infty]$;
- Denote various positive constants by $C, C_0, C_1, C_2, C_3, \ldots$;
- Let $D^{1,2}(\mathbb{R}^3)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with the norm $\|u\|_{D^{1,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^{\frac{1}{2}}$.

2 Proof of Theorem 1.1

In this proof, we refer to the technical approach demonstrated by [28] to prove Theorem 1.1. In order to supplement and enrich the zero energy critical point theory, we give a modified result, which can be applied to a wider range of nonlocal Lapalcian equations. Based on the Ljusternik–Schnirelman category (see [32, Theorem 5.7]), we only need to prove that $\widetilde{\Lambda}$ satisfies the Palais–Smale condition and is bounded from below on the unit sphere S.

Lemma 2.1. *Assume* $(\mathcal{M}_1)$ *holds, then* $\widetilde{\Lambda}$ *is bounded from below on* S*.*

Proof. From $(\mathcal{M}_1)$, for all $u \in S$,

$$\frac{N(u)}{A(u)} \geq \frac{C^{-1}\|u\|^\eta}{C\|u\|^\eta} = \frac{1}{C^2}, \qquad \frac{R(u)^{\frac{\beta-\eta}{\beta-\gamma}}}{B(u)^{\frac{\gamma-\eta}{\beta-\gamma}} A(u)} > 0.$$

Then

$$\widetilde{\Lambda} = \frac{N(u)}{A(u)} + C_0 \frac{R(u)^{\frac{\beta-\eta}{\beta-\gamma}}}{A(u)B(u)^{\frac{\gamma-\eta}{\beta-\gamma}}} > \frac{1}{C^2}.$$

Therefore, $\widetilde{\Lambda}$ is bounded from below on S. □

A crucial proposition required to validate the Palais–Smale condition for $\widetilde{\Lambda}(u)$ is as follows:

Lemma 2.2. *Assume that $(u_n) \subset S$, $\widetilde{\Lambda}'(u_n) \to 0$, then $\Lambda'(u_n)(u_n - u) \to 0$ as $n \to \infty$.*

Proof. Since $(u_n) \subset S = \{u \in X \backslash \{0\} : \|u\| = 1\}$, we see that $\|u_n\| = 1$. According to S is weakly closed and (u_n) is bounded in S, we can attain $u_n \rightharpoonup u$ in S. Let $\mathcal{T}_S(u) = \{v \in X : i'(u)v = 0\}$, at the point u, $\mathcal{T}_S(u)$ represents the tangent space to the set S, where $i(u) = \frac{1}{2}\|u\|^2$. Note that, for any $w \in X$ and any $n \in \mathbb{N}$, the pair $(t_n, v_n) \in \mathbb{R} \times \mathcal{T}_S(u_n)$ is uniquely identified, ensuring $w = v_n + t_n u_n$ and subsequently, $i'(u_n)w = i'(u_n)v_n + i'(u_n)t_n u_n$. According to the definition of $\mathcal{T}_S(u)$, we can obtain that $i'(u_n)v_n = 0, i'(u_n)u_n = \|u_n\|^2 = 1$. Therefore, $i'(u_n)w = t_n i'(u_n)u_n = t_n$. Then, (t_n) is bounded, consequently, (v_n) is also bounded. Since $\widetilde{\Lambda}'(u_n) \to 0$, namely, $|\Lambda'(u_n)v_n| \leq \varepsilon_n \|v_n\|$ for any $v_n \in \mathcal{T}_S(u_n)$ with $\varepsilon_n \to 0$, we have $\Lambda'(u_n)v_n \to 0$. According to the Lemma 2.1 of [28] we obtain that $\Lambda'(u_n)u_n = 0$. We conclude that $\Lambda'(u_n)w \to 0$ for any $w \in X$. Taking $w = u_n - u$, we get that $\Lambda'(u_n)(u_n - u) \to 0$, as $n \to \infty$. □

Lemma 2.3. *Assume that $(\mathcal{M}_1), (\mathcal{M}_3), (\mathcal{M}_4)$ hold, then $\widetilde{\Lambda}$ satisfies the Palais–Smale condition.*

Proof. Choose $(u_n) \subset S$ such that $(\widetilde{\Lambda}(u_n))$ is bounded and $\widetilde{\Lambda}'(u_n) \to 0$, i.e. $|\Lambda'(u_n)v| \leq \varepsilon_n \|v\|$ for any $v \in \mathcal{T}_S(u_n)$, with $\varepsilon_n \to 0$. By Lemma 2.2, together with the fact that $(u_n) \subset S$, $\widetilde{\Lambda}'(u_n) \to 0$, we know that

$$u_n \rightharpoonup u \quad \text{in } S, \qquad \Lambda'(u_n)(u_n - u) \to 0.$$

Then, for any $u_n, u \in S$,

$$\begin{aligned} &\Lambda'(u_n)(u_n - u) \\ &\quad = \frac{A(u_n)N'(u_n)(u_n - u) - N(u_n)A'(u_n)(u_n - u)}{A(u_n)^2} \\ &\qquad + C_0 Q(u_n) \left(\frac{\beta - \eta}{\beta - \gamma} B(u_n)A(u_n)R'(u_n)(u_n - u) \right. \\ &\qquad\qquad - \frac{\gamma - \eta}{\beta - \gamma} R(u_n)A(u_n)B'(u_n)(u_n - u) \\ &\qquad\qquad \left. - B(u_n)R(u_n)A'(u_n)(u_n - u) \right) \to 0. \end{aligned} \tag{2.1}$$

According to $(\mathcal{M}_1)$, we can infer that $A(u_n), B(u_n)$ is bounded, $N(u_n), R(u_n)$ is bounded away from zero. Since $(\widetilde{\Lambda}(u_n))$ is bounded, we know that $Q(u_n)$ is bounded. In the light of (M_3), we can obtain

$$A'(u_n)(u_n - u) \to 0, \qquad B'(u_n)(u_n - u) \to 0.$$

The above analysis leads to the following conclusion:

$$\left(N'(u_n) + R'(u_n)\right)(u_n - u) \to 0.$$

According to $(\mathcal{M}_3)$, we obtain $R'(u_n)(u_n - u) \geq 0$. From $(\mathcal{M}_4)$, we have

$$(N'(u_n) - N'(u))(u_n - u) \geq C(\|u_n\|^{\eta-1} - \|u\|^{\eta-1})(\|u_n\| - \|u\|) \geq 0$$

for every $u_n, u \in S$. Moreover, $u_n \rightharpoonup u$ in S, we have $N'(u)(u_n - u) \to 0$, then $N'(u_n)(u_n - u) \geq 0$. Therefore, we can conclude that $N'(u_n)(u_n - u) \to 0$. Since $(N'(u_n) - N'(u))(u_n - u) \to 0$, we obtain $\|u_n\| \to \|u\|$. Note that X is a reflexive Banach space and $u_n \rightharpoonup u$ in S, which imply that $u_n \to u$ in S. This completes the proof. □

Proof of Theorem 1.1.

(1) We prove that there is no critical point having zero energy when $\mu < \delta_0$. Note that u is a critical point of Λ, if and only if, $t_0(u)u$ is a zero energy critical point of Φ_μ with $\mu = \Lambda(u)$. In other words, this means that

$$\delta_0 = \inf_{X\setminus\{0\}} \Lambda(u) \leq \mu \leq \sup_{X\setminus\{0\}} \Lambda(u),$$

which yields the desired conclusion.

(2) According to Lemma 2.1, we know that $\widetilde{\Lambda}$ is bounded from below on S. Moreover, Lemma 2.3 implied that $\widetilde{\Lambda}$ satisfies the Palais–Smale condition. Note that $\hat{\gamma}(S) = \infty$, we get from Ljusternik–Schnirelman category (see [32, Theorem 5.7]) that there exists a sequence $(u_n) \subset S$ such that $\widetilde{\Lambda}'(u_n) = 0, \widetilde{\Lambda}(u_n) = \mu_n$. Therefore, the energy functional Φ has infinitely many zero energy sign changing critical points $(u_n) \subset S$.

3 Applications of Theorem 1.1

In this section, we shall prove the nonexistence of solutions and the existence of infinitely many solutions for three non-local problems, we confirm that these are just a small part of applications of Theorem 1.1.

3.1 Critical Schrödinger–Poisson system in the whole space

In this subsection, let us consider a Schrödinger–Poisson system with p-Laplacian:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \lambda\phi u = |u|^{p^*-2}u + \mu|u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{3.1}$$

where $\lambda > 0$ is a constant, $12/7 < p < 3$, $p^* := 3p/(3-p)$, and $\Delta_p = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian. The p-Laplacian operator appears in nonlinear fluid dynamics, and the range of p is related to the velocity of the fluid and material. For more information on the physical origin of p-Laplacian, we refer to [9]. For any given $u \in W^{1,p}(\mathbb{R}^3)$, there exists a unique

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|}\mathrm{d}y, \qquad \phi_u \in D^{1,2}\left(\mathbb{R}^3\right),$$

satisfying $-\Delta\phi_u = |u|^2$ (see [17]).

The system (3.1) can be viewed as a perturbation of the system

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \lambda\phi u = |u|^{q-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{3.2}$$

Du, Su and Wang first considered this system in [16], they established the existence of nontrivial solutions through the mountain pass theorem. Systems like (3.2) originate from quantum mechanics models [10, 12, 23], semiconductor theory [24, 25]. They described the interaction between quantum particles and electromagnetic fields. After the seminal work of Benci and Fortunato in [7,8], many researches have been conducted on systems such as (3.2) in the past few decades, as shown in [3,4,6,14,15,22,26,27,31] and their references.

Inspired by the above work, we are committed to studying the existence of solutions for system (3.1) in $\mathbb{R}^3$. Specifically, by using the nonlinear generalized Rayleigh quotient method and Ljusternik–Schnirelman theory, we obtain that there exist infinitely many zero energy sign changing weak solutions.

First of all, we give the variational framework for system (3.1). Let $W^{1,p}(\mathbb{R}^3)$ denote the usual Sobolev space equipped with the norm

$$\|u\| := \left(\int_{\mathbb{R}^3} |\nabla u|^p + |u|^p dx\right)^{\frac{1}{p}}.$$

One may easily get that the corresponding functional of (3.1) is $\Phi_\mu(u) : W^{1,p}(\mathbb{R}^3) \to \mathbb{R}$

$$\Phi_\mu(u) = \frac{1}{p}\int_{\mathbb{R}^3}(|\nabla u|^p + |u|^p)dx - \frac{\mu}{p}\int_{\mathbb{R}^3}|u|^p dx - \frac{1}{p^*}\int_{\mathbb{R}^3}|u|^{p^*}dx + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi u^2 dx.$$

It is standard to verify that Φ_μ is C^1. Then, for every $u \in W^{1,p}(\mathbb{R}^3)\backslash\{0\}$, we have

$$\mu_0(u) = \frac{\|u\|^p}{|u|_p^p} - \frac{p}{p^*}\frac{|u|_{p^*}^{p^*}}{|u|_p^p} + \frac{\lambda p}{4}\frac{\int_{\mathbb{R}^3}\phi u^2 dx}{|u|_p^p},$$

then

$$\mu_0(tu) = \frac{\|u\|^p}{|u|_p^p} - t^{p^*-p}\frac{p}{p^*}\frac{|u|_{p^*}^{p^*}}{|u|_p^p} + t^{4-p}\frac{\lambda p}{4}\frac{\int_{\mathbb{R}^3}\phi u^2 dx}{|u|_p^p}.$$

Let

$$\mu_0'(tu) = -t^{p^*-p-1}(p^*-p)\frac{p^*}{q}\frac{|u|_{p^*}^{p^*}}{|u|_p^p} + t^{3-p}(4-p)\frac{\lambda p}{4}\frac{\int_{\mathbb{R}^3}\phi u^2 dx}{|u|_p^p} = 0.$$

It is easy to see that

$$t_0(u) = \left(\frac{4-p}{p^*-p}\frac{p^*}{4}\frac{\lambda\int_{\mathbb{R}^3}\phi u^2 dx}{|u|_{p^*}^{p^*}}\right)^{\frac{1}{p^*-4}} > 0,$$

that is $\mu_0'(t_0(u)u) = 0$. Since $\mu_0''(t_0(u)u) < 0$, $t_0(u)$ is a nondegenerate global maximum point of $\mu_0(tu)$. Therefore, we have

$$\Lambda_1(u) = \mu_0(t_0(u)u) = \frac{\|u\|^p}{|u|_p^p} + \frac{p}{4}\frac{p^*-4}{p^*-p}\left(\frac{4-p}{p^*-p}\frac{p^*}{4}\right)^{\frac{4-p}{p^*-4}}\frac{\left(\lambda\int_{\mathbb{R}^3}\phi u^2 dx\right)^{\frac{p^*-p}{p^*-4}}}{|u|_p^p|u|_{p^*}^{p^*\frac{4-p}{p^*-4}}}.$$

For simplicity's sake, we call $\widetilde{\Lambda_1}$ the restriction of Λ_1 to S_1, where

$$S_1 = \{u \in W^{1,p}(\mathbb{R}^3)\backslash\{0\} : \|u\| = 1\}.$$

Now it is in a position to state our main result in this section as follows.

Theorem 3.1. *Let $\delta_1 = \inf_{W^{1,p}(\mathbb{R}^3)\setminus\{0\}} \Lambda_1(u)$.*

(i) If $\mu < \delta_1$, then there is no nontrivial weak solution having zero energy for system (3.1);

(ii) If $\mu > \delta_1$, then system (3.1) has infinitely many zero energy sign changing weak solutions.

To prove Theorem 3.1, according to Ljusternik–Schnirelman category, we only need to prove that $\widetilde{\Lambda_1}$ is bounded from below and $\widetilde{\Lambda_1}$ satisfies the Palais-Smale condition.

Lemma 3.2. *$\widetilde{\Lambda_1}$ is bounded from below on S_1.*

Proof. Since the embedding $W^{1,p}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ is continuous, there exists $C > 0$ such that $C\|u\|^p \geqslant |u|_p^p$. Because $\frac{12}{7} < p < 3, p^* > 4$, one may check that

$$\widetilde{\Lambda_1}(u) = \frac{\|u\|^p}{|u|_p^p} + \frac{p}{4}\frac{p^*-4}{p^*-p}\left(\frac{4-p}{p^*-p}\frac{p^*}{4}\right)^{\frac{4-p}{p^*-4}} \frac{\left(\lambda\int_{\mathbb{R}^3}\phi u^2 dx\right)^{\frac{p^*-p}{p^*-4}}}{|u|_p^p|u|_{p^*}^{p^*\frac{4-p}{p^*-4}}} > \frac{1}{C}.$$

Therefore, $\widetilde{\Lambda_1}$ is bounded from below on S_1. □

In order to prove that $\widetilde{\Lambda_1}$ satisfies the Palais–Smale condition, we require the following proposition.

Lemma 3.3. *If $u_n \rightharpoonup u$ in S_1, then for any $v \in W^{1,p}(\mathbb{R}^3)$ there holds*

$$\int_{\mathbb{R}^3} |u_n|^{p-2}u_n v dx \to \int_{\mathbb{R}^3} |u|^{p-2}uv dx, \qquad \int_{\mathbb{R}^3} |u_n|^{p^*-2}u_n v dx \to \int_{\mathbb{R}^3} |u|^{p^*-2}uv dx, \qquad n\to\infty.$$

Proof. Since $u_n \rightharpoonup u$ in S_1, we derive that $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^3$. Note that the embedding $W^{1,p}(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ is continuous for $s \in [p, p^*]$, there exists a positive constant C such that

$$\int_{\mathbb{R}^3} \left||u_n|^{p-2}u_n\right|^{\frac{p}{p-1}} dx \le \int_{\mathbb{R}^3} \left||u_n|^{p-1}\right|^{\frac{p}{p-1}} dx = \int_{\mathbb{R}^3} |u_n|^p dx \le C\|u_n\|^p,$$

$$\int_{\mathbb{R}^3} \left||u_n|^{p^*-2}u_n\right|^{\frac{p^*}{p^*-1}} dx \le \int_{\mathbb{R}^3} \left||u_n|^{p^*-1}\right|^{\frac{p^*}{p^*-1}} dx = \int_{\mathbb{R}^3} |u_n|^{p^*} dx \le C\|u_n\|^{p^*}.$$

Therefore, $\{|u_n|^{p-2}u_n\}$ is bounded in $L^{\frac{p}{p-1}}(\mathbb{R}^3)$ and $\{|u_n|^{p^*-2}u_n\}$ is bounded in $L^{\frac{p^*}{p^*-1}}(\mathbb{R}^3)$. It follows from [34, Proposition 5.4.7] that $|u_n|^{p-2}u_n \rightharpoonup |u|^{p-2}u$ in $L^{\frac{p}{p-1}}(\mathbb{R}^3)$ and $|u_n|^{p^*-2}u_n \rightharpoonup |u|^{p^*-2}u$ in $L^{\frac{p^*}{p^*-1}}(\mathbb{R}^3)$. Thus for any $v \in W^{1,p}(\mathbb{R}^3)$ we have

$$\int_{\mathbb{R}^3} |u_n|^{p-2}u_n v dx \to \int_{\mathbb{R}^3} |u|^{p-2}uv dx, \qquad \int_{\mathbb{R}^3} |u_n|^{p^*-2}u_n v dx \to \int_{\mathbb{R}^3} |u|^{p^*-2}uv dx, \qquad n\to\infty,$$

as required. □

Lemma 3.4 (See [17, Proposition 2.1])**.** *For any $u \in W^{1,p}(\mathbb{R}^3)$, the following properties are applicable:*

(1) *$\phi_u \geq 0$ and for any $t \in \mathbb{R}^+, \phi_{tu} = t^p\phi_u$.*

(2) *There exists a positive constant C such that*

$$|\nabla\phi_u|_2^2 = \int_{\mathbb{R}^3} \phi_u|u|^p dx \le C\|u\|^{2p}.$$

(3) *In the case of* $u_n \rightharpoonup u$ *in* $W^{1,p}(\mathbb{R}^3)$, *it follows that* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D^{1,2}(\mathbb{R}^3)$ *and*

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^{p-2} u_n \varphi dx \to \int_{\mathbb{R}^3} \phi_u u^{p-2} u \varphi dx, \quad \text{for any } \varphi \in W^{1,p}(\mathbb{R}^3).$$

Lemma 3.5. $\widetilde{\Lambda_1}$ *satisfies the Palais–Smale condition.*

Proof. Choose a sequence $(u_n) \subset S_1$ such that $(\widetilde{\Lambda_1}(u_n))$ is bounded and $\widetilde{\Lambda_1}'(u_n) \to 0$, that is $|\Lambda_1'(u_n)v| \le \varepsilon_n \|v\|$ for any $v \in \mathcal{T}_{S_1}(u_n)$, with $\varepsilon_n \to 0$. With the help of Lemma 2.2, if there is a sequence $(u_n) \subset S_1$ and $\widetilde{\Lambda_1}'(u_n) \to 0$, we can obtain that $u_n \rightharpoonup u$ in S_1 and $\Lambda_1'(u_n)(u_n - u) \to 0$ as $n \to \infty$. Then, for any $u_n, u \in S_1$,

$$\begin{aligned}
&\Lambda_1'(u_n)(u_n-u) \qquad (3.3)\\
&= p\frac{|u_n|_p^p \int_{\mathbb{R}^3} \left(|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n-u) + |u_n|^{p-2}u_n(u_n-u)\right)dx - \|u_n\|^p \int_{\mathbb{R}^3} |u_n|^{p-2}u_n(u_n-u)dx}{|u_n|_p^{2p}}\\
&\quad + \Bigg(4\frac{p^*-p}{q-4}|u_n|_{p^*}^{p^*}|u_n|_p^p \int_{\mathbb{R}^3} \phi u_n(u_n-u)dx - p^*\frac{4-p}{p^*-4}|u_n|_p^p \int_{\mathbb{R}^3} \phi u_n^2 dx \int_{\mathbb{R}^3} |u_n|^{p^*-2}u_n(u_n-u)dx\\
&\quad - p|u_n|_{p^*}^{p^*} \int_{\mathbb{R}^3} \phi u_n^2 dx \int_{\mathbb{R}^3} |u_n|^{p-2}u_n(u_n-u)dx\Bigg) C_1 Q(u_n) \to 0, \quad \text{as } n \to \infty,
\end{aligned}$$

where

$$C_1 = \lambda \frac{p}{4}\frac{p^*-4}{p^*-p}\left(\frac{4-p}{p^*-p}\frac{p^*}{4}\right)^{\frac{4-p}{p^*-4}}, \qquad Q(u_n) = \frac{\left(\int_{\mathbb{R}^3} \phi u_n^2 dx\right)^{\frac{4-p}{p^*-4}}}{|u_n|_p^{2p}|u_n|_{p^*}^{p^*\frac{p^*-p}{p^*-4}}}.$$

Next, we claim that

$$\int_{\mathbb{R}^3} \left(|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n-u) + |u_n|^{p-2}u_n(u_n-u)\right)dx \to 0, \quad \text{as } n \to \infty.$$

Therefore, we need to show that $|u_n|_{p^*}^{p^*}$, $|u_n|_p^p$, $\int_{\mathbb{R}^3} \phi u_n^2 dx$ and $Q(u_n)$ are bounded and

$$\int_{\mathbb{R}^3} |u_n|^{p-2}u_n(u_n-u)dx \to 0, \quad \int_{\mathbb{R}^3} |u_n|^{p^*-2}u_n(u_n-u)dx \to 0, \quad \int_{\mathbb{R}^3} \phi_{u_n} u_n(u_n-u)dx \to 0$$

as $n \to \infty$. Indeed, since $(u_n) \subset S_1$ and the embedding $W^{1,p}(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ is continuous for $s \in [p, p^*]$, we can obtain

$$0 < |u_n|_{p^*}^{p^*} \le C\|u_n\|^{p^*} = C, \qquad 0 < |u_n|_p^p \le C_1\|u_n\|^p = C_1.$$

This implied that $|u_n|_{p^*}^{p^*}$ and $|u_n|_p^p$ are bounded. By means of Lemma 3.4, we have

$$0 < \int_{\mathbb{R}^3} \phi u_n^2 dx \le C_2\|u_n\|^4 = C_2.$$

Then $\int_{\mathbb{R}^3} \phi u_n^2 dx$ is bounded. We can deduce the boundedness of $Q(u_n)$ from the fact that $(\widetilde{\Lambda_1}(u_n))$ is bounded. According to Lemma 3.3, given $v = (u_n - u) \in S_1$, we can attain that

$$\int_{\mathbb{R}^3} \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n-u)dx \to 0, \qquad \int_{\mathbb{R}^3} \left(|u_n|^{p^*-2}u_n - |u|^{p^*-2}u\right)(u_n-u)dx \to 0,$$

as $n \to \infty$. Through Lemma 3.4, given $\varphi = u_n - u$, we obtain

$$\int_{\mathbb{R}^3} \left(\phi_{u_n}u_n - \phi_u u\right)(u_n-u)dx \to 0, \qquad n \to \infty.$$

Inasmuch as $u_n \rightharpoonup u$ in S_1, one can conclude that

$$\int_{\mathbb{R}^3} |u_n|^{p-2}u_n(u_n-u)dx \to 0, \int_{\mathbb{R}^3} |u_n|^{p^*-2}u_n(u_n-u)dx \to 0, \int_{\mathbb{R}^3} \phi_{u_n}u_n(u_n-u)dx \to 0,$$

as $n \to \infty$. Therefore, we obtain the following conclusion:

$$\int_{\mathbb{R}^3} \left(|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n-u) + |u_n|^{p-2}u_n(u_n-u)\right) dx \to 0, \quad \text{as } n \to \infty.$$

Notice that $p(u) = \int_{\mathbb{R}^3} |\nabla u|^p + |u|^p dx$, according to the Hölder inequality, one has

$$\begin{aligned}(p'(u_n) - p'(u), u_n - u) &\geq \|u_n\|^p + \|u\|^p - \|u_n\|^{p-1}\|u\| - \|u\|^{p-1}\|u_n\| \\ &= (\|u_n\|^{p-1} - \|u\|^{p-1})(\|u_n\| - \|u\|) \geq 0.\end{aligned}$$

Owing to $u_n \rightharpoonup u$ in S_1, it follows that $(p'(u_n) - p'(u), u_n - u) \to 0$, as $n \to \infty$. Therefore, $\|u_n\| \to \|u\|$ in S_1. According to the uniform convexity of $W^{1,p}(\mathbb{R}^3)$, we can obtain that $u_n \to u$ in S_1. Consequently, there exits a sequence (u_n) such that $u_n \to u$ in S_1 up to a subsequence. Therefore, $\widetilde{\Lambda_1}$ satisfied the Palais–Smale condition. □

Proof of Theorem 3.1.

(i) We prove that there is no critical point having zero energy when $\mu < \delta_1$. A crucial observation is that u being a critical point of Λ_1 is equivalent to $t_0(u)u$ being a zero energy critical point of Φ_μ, where $\mu = \Lambda_1(u)$. In other words, this means that

$$\delta_1 = \inf_{W^{1,p}(\mathbb{R}^3)\backslash\{0\}} \Lambda_1(u) \leq \mu \leq \sup_{W^{1,p}(\mathbb{R}^3)\backslash\{0\}} \Lambda_1(u),$$

which yields the desired conclusion.

(ii) According to Lemma 3.2, we know that $\widetilde{\Lambda_1}$ is bounded from below on S_1. In the meantime, we obtain that $\widetilde{\Lambda_1}$ satisfies the Palais–Smale condition from Lemma 3.5. Note that $\hat{\gamma}(S_1) = \infty$, Ljusternik–Schnirelman category (see[32, Theorem 5.7]) yields that there exists a sequence $(u_n) \subset S_1$ such that $\widetilde{\Lambda_1}'(u_n) = 0, \widetilde{\Lambda_1}(u_n) = \mu_n$. Therefore, the energy functional Φ has infinitely many zero energy critical points $(u_n) \subset S_1$. Since $\widetilde{\Lambda_1}(u)$ is an even functional, $\widetilde{\Lambda_1}'(\pm u_n) = 0, \widetilde{\Lambda_1}(\pm u_n) = \mu_n$. Hence, system (3.1) possesses infinitely many zero energy sign changing weak solutions. □

3.2 Critical Schrödinger–Poisson system in bounded domains

The purpose of this subsection is to study the existence and nonexistence of solutions for a Schrödinger–Poisson system with critical nonlinearity in bounded domains. Here is the system under consideration:

$$\begin{cases} -\Delta u + \lambda\phi|u|^{q-2}u = \mu u - |u|^{2^*-2}u & \text{in } \Omega, \\ -\Delta\phi = |u|^q & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where $\lambda = -1$, $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, μ is a real parameter, $1 < N/(N-2) < q < 2N/(N-2) = 2^*$. It is well known that problem (3.4) is equivalent to a nonlocal nonlinear problem related with famous Choquard equations in bounded domains. For more related results, for instance we refer to [1,5,11,30].

Now we start the analysis of problem (3.4). One may easily get that the corresponding functional of (3.4) is as follows:

$$\Phi_\mu(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\mu}{2}\int_\Omega |u|^2 dx - \frac{1}{2q}\int_\Omega \phi|u|^q dx + \frac{1}{2^*}\int_\Omega |u|^{2^*} dx, \qquad u \in H_0^1(\Omega).$$

This Hilbert space $H_0^1(\Omega)$ provides a suitable framework for our analysis, capturing the essential properties of functions under consideration. Given the norm

$$\|u\| := \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

It is apparent that the functional Φ_μ is C^1. Then, for every $u \in H_0^1(\Omega)\backslash\{0\}$,

$$\mu_0(u) = \frac{\|u\|^2}{|u|_2^2} - \frac{1}{q}\frac{\int_\Omega \phi|u|^q dx}{|u|_2^2} + \frac{2}{2^*}\frac{|u|_{2^*}^{2^*}}{|u|_2^2},$$

$$\mu_0(tu) = \frac{\|u\|^2}{|u|_2^2} - t^{2q-2}\frac{1}{q}\frac{\int_\Omega \phi|u|^q dx}{|u|_2^2} + t^{2^*-2}\frac{2}{2^*}\frac{|u|_{2^*}^{2^*}}{|u|_2^2}.$$

Let $\mu_0'(tu) = 0$, we obtain

$$t_0(u) = \left(\frac{2^*-2}{2q-2}\frac{2q}{2^*}\frac{|u|_{2^*}^{2^*}}{\int_\Omega \phi|u|^q dx}\right)^{\frac{1}{2q-2^*}} > 0,$$

that is $\mu_0'(t_0(u)u) = 0$. In the meantime, on account of $\mu_0''(t_0(u)u) < 0$, we can deduce that $t_0(u)$ is a nondegenerate global maximum point of $\mu_0(tu)$. As a result, we can obtain that

$$\Lambda_2(u) = \mu_0(t_0(u)u) = \frac{\|u\|^2}{|u|_2^2} + \frac{2}{2^*}\frac{2q-2^*}{2q-2}\left(\frac{2^*-2}{2q-2}\frac{2q}{2^*}\right)^{\frac{2^*-2}{2q-2^*}}\frac{(|u|_{2^*}^{2^*})^{\frac{2q-2}{2q-2^*}}}{|u|_2^2(\int_\Omega \phi|u|^q dx)^{\frac{2^*-2}{2q-2^*}}}.$$

For clarity, we call $\widetilde{\Lambda_2}$ the restriction of Λ_2 to S_2, where

$$S_2 = \{u \in H_0^1(\Omega)\backslash\{0\} : \|u\| = 1\}.$$

As for our central discovery in this subsection, it can be phrased as:

Theorem 3.6. *Let* $\delta_2 = \inf_{H_0^1(\Omega)\backslash\{0\}} \Lambda_2(u)$.

(i) If $\mu < \delta_2$, *then there is no nontrivial weak solution having zero energy in system* (3.4);

(ii) If $\mu > \delta_2$, *then system* (3.4) *has infinitely many zero energy sign changing weak solutions.*

As mentioned earlier, to achieve our goal, we only need to prove the boundedness from below of $\widetilde{\Lambda_2}$ on S_2 and verify that $\widetilde{\Lambda_2}$ meets the Palais–Smale condition.

Lemma 3.7. $\widetilde{\Lambda_2}$ *is bounded from below on* S_2.

Proof. According to the embedding theorem, $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, there is a constant $C > 0$ such that $|u|_2^2 \leq C\|u\|^2$. Note that $1 < N/(N-2) < q < 2^*$, one may verify that

$$\widetilde{\Lambda_2} = \frac{\|u\|^2}{|u|_2^2} + \frac{2}{2^*}\frac{2q-2^*}{2q-2}\left(\frac{2^*-2}{2q-2}\frac{2q}{2^*}\right)^{\frac{2^*-2}{2q-2^*}} \frac{(|u|_{2^*}^{2^*})^{\frac{2q-2}{2q-2^*}}}{|u|_2^2(\int_\Omega \phi|u|^q dx)^{\frac{2^*-2}{2q-2^*}}} > \frac{1}{C}.$$

That is to say, $\widetilde{\Lambda_2}$ is bounded from below on S_2. □

Lemma 3.8. *$\widetilde{\Lambda_2}$ satisfies the Palais–Smale condition.*

Proof. Choose a sequence $(u_n) \subset S_2$ such that $\widetilde{\Lambda_2}(u_n)$ is bounded and $\widetilde{\Lambda_2}'(u_n) \to 0$. In other words, for any $v \in \mathcal{T}_{S_2}(u_n)$, we have $|\Lambda_2'(u_n)v| \leq \varepsilon_n\|v\|$, where $\varepsilon_n \to 0$.

With the assistance of Lemma 2.2, assuming a sequence $(u_n) \subset S_2$ satisfies $\widetilde{\Lambda_2}'(u_n) \to 0$, we can conclude that $u_n \rightharpoonup u$ in S_2 and $\Lambda_2'(u_n)(u_n - u) \to 0$ as $n \to \infty$. Then, for any $u_n, u \in S_2$,

$$\begin{aligned} \Lambda_2'(u_n)(u_n-u) = {} & 2\frac{|u_n|_2^2\int_\Omega \nabla u_n \nabla(u_n-u)dx - \|u_n\|^2\int_\Omega u_n(u_n-u)dx}{|u_n|_2^4} \\ & + C_2Q(u_n)\Bigg(2^*\frac{2q-2}{2q-2^*}|u_n|_2^2\int_\Omega \phi|u_n|^q dx\int_\Omega |u_n|^{2^*-2}u_n(u_n-u)dx \\ & - 2q\frac{2^*-2}{2q-2^*}|u_n|_{2^*}^{2^*}|u_n|_2^2\int_\Omega \phi|u_n|^{q-2}u_n(u_n-u)dx \\ & - 2|u_n|_{2^*}^{2^*}\int_\Omega \phi|u_n|^q dx\int_\Omega u_n(u_n-u)dx\Bigg) \to 0, \end{aligned} \tag{3.5}$$

where

$$C_2 = \frac{2}{2^*}\frac{2q-2^*}{2q-2}\left(\frac{2^*-2}{2q-2}\frac{2q}{2^*}\right)^{\frac{2^*-2}{2q-2^*}}, \qquad Q(u_n) = \frac{(|u_n|_{2^*}^{2^*})^{\frac{2^*-2}{2q-2^*}}}{|u_n|_2^4(\int_\Omega \phi|u_n|^q dx)^{\frac{2q-2}{2q-2^*}}}.$$

In order to prove Lemma 3.7, we first verify that $\int_\Omega \nabla u_n \nabla(u_n-u)dx \to 0$, as $n \to \infty$. Therefore, on the one hand, we need to show that $|u_n|_2^2, |u_n|_{2^*}^{2^*}, Q(u_n)$ is bounded. On the other hand, we need to prove

$$\int_\Omega u_n(u_n-u)dx \to 0, \int_\Omega \phi_{u_n}|u_n|^{q-2}(u_n-u)dx \to 0, \quad \text{as } n \to \infty.$$

In fact, we know that $u_n \not\equiv 0$ from $(u_n) \subset S_2$. According to Lemma 3.4, we can obtain

$$0 < \int_\Omega \phi_{u_n}|u_n|^q dx \leq C_2\|u_n\|^{2q} = C_2.$$

Therefore, $\int_\Omega \phi_{u_n}|u_n|^q dx$ is bounded. On the basis of the embedding theorem, $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ is continuous for $2 \leq s \leq 2^*$, we have

$$0 < |u_n|_2^2 \leq C\|u_n\|^2 = C, \qquad 0 < |u_n|_{2^*}^{2^*} \leq C\|u_n\|^{2^*} = C,$$

which implied that $|u_n|_2^2$ and $|u_n|_{2^*}^{2^*}$ are bounded. Note that $\widetilde{\Lambda_2}(u_n)$ is bounded, one may check that $Q(u_n)$ is bounded. Thanks to the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, one has

$$\int_\Omega u_n(u_n-u)dx \to 0, \quad \text{as } n \to \infty.$$

With the aid of Lemma 3.4, it holds

$$\int_\Omega \phi_{u_n}|u_n|^{q-2}u_n\varphi dx \to \int_\Omega \phi_u|u|^{q-2}u\varphi dx, \quad \text{for any } \varphi \in H_0^1(\Omega).$$

Let $\varphi = u_n - u$. Then we obtain

$$\int_\Omega \phi_{u_n}|u_n|^{q-2}(u_n-u)dx - \int_\Omega \phi_u|u|^{q-2}(u_n-u)dx \to 0, \quad \text{as } n \to \infty.$$

Since $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, we have $\int_\Omega \phi_u|u|^{q-2}(u_n-u)dx \to 0$. Consequently,

$$\int_\Omega \phi_{u_n}|u_n|^{q-2}(u_n-u)dx \to 0, \quad \text{as } n \to \infty.$$

The analysis leads to the following conclusion:

$$\int_\Omega \nabla u_n \nabla(u_n-u)dx + \int_\Omega |u_n|^{2^*-2}u_n(u_n-u)dx \to 0.$$

In order to obtain $\int_\Omega \nabla u_n \nabla(u_n-u)dx \to 0$, we will use the Hölder inequality to derive

$$\int_\Omega \nabla u_n \nabla(u_n-u)dx \geq 0, \qquad \int_\Omega |u_n|^{2^*-2}u_n(u_n-u)dx \geq 0.$$

For convenience, put $p_1(u) = \int_\Omega |\nabla u|^2 dx$, $p_2(u) = \int_\Omega |u|^{2^*}dx$. One may check that

$$(p_1'(u_n) - p_1'(u), u_n - u) = \|u_n\|^2 + \|u\|^2 - \int_\Omega \nabla u_n \nabla u dx - \int_\Omega \nabla u \nabla u_n dx,$$

$$(p_2'(u_n) - p_2'(u), u_n - u) = \|u_n\|_{2^*}^{2^*} + \|u\|_{2^*}^{2^*} - \int_\Omega |u_n|^{2^*-2}u_n u dx - \int_\Omega |u|^{2^*-2}u u_n dx.$$

By virtue of the Hölder inequality, we have

$$\int_\Omega \nabla u_n \nabla u dx \leq \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u_n|^2 dx\right)^{\frac{1}{2}} = \|u_n\|\|u\|,$$

$$\int_\Omega |u_n|^{2^*-2}u_n u dx \leq \left(\int_\Omega |u_n|^{2^*}dx\right)^{\frac{2^*-1}{2^*}} \left(\int_\Omega |u|^{2^*}dx\right)^{\frac{1}{2^*}} = |u_n|_{2^*}^{2^*-1}|u|_{2^*},$$

$$\int_\Omega |u|^{2^*-2}u u_n dx \leq \left(\int_\Omega |u|^{2^*}dx\right)^{\frac{2^*-1}{2^*}} \left(\int_\Omega |u_n|^{2^*}dx\right)^{\frac{1}{2^*}} = |u|_{2^*}^{2^*-1}|u_n|_{2^*}.$$

Therefore,

$$(p_1'(u_n) - p_1'(u), u_n - u) \geq (\|u_n\| - \|u\|)(\|u_n\| - \|u\|) \geq 0,$$

$$(p_2'(u_n) - p_2'(u), u_n - u) \geq (\|u_n\|^{2^*-1} - \|u\|^{2^*-1})(\|u_n\| - \|u\|) \geq 0.$$

Since $u_n \rightharpoonup u$ in S_2, we have $(p_1'(u), u_n - u) \to 0, (p_1'(u), u_n - u) \to 0$, as $n \to \infty$. From which it follows that

$$\int_\Omega \nabla u_n \nabla(u_n-u)dx \geq 0, \qquad \int_\Omega |u_n|^{2^*-2}u_n(u_n-u)dx \geq 0.$$

As a result, we can draw the following conclusion:

$$\int_\Omega \nabla u_n \nabla(u_n-u)dx \to 0, \quad \text{as } n \to \infty.$$

Moreover, since $(p_1'(u_n) - p_1'(u), u_n - u) \to 0$, we have $\|u_n\| \to \|u\|$ in S_2. By the uniform convexity of $H_0^1(\Omega)$, we deduce that $u_n \to u$ in S_2. Therefore, $\widetilde{\Lambda_2}$ satisfied the Palais–Smale condition. □

Proof of Theorem 3.6.

(i) We show that there is no critical point having zero energy when $\mu < \delta_2$. Note that u is a critical point of Λ, if and only if, $t_0(u)u$ is a zero energy critical point of Φ_μ with $\mu = \Lambda(u)$, this means the desired conclusion.

(ii) According to Lemma 3.5, we know that $\widetilde{\Lambda_2}$ is bounded from below on S_2. Moreover, $\widetilde{\Lambda_2}$ satisfies the Palais–Smale condition due to Lemma 3.7. Note that $\hat{\gamma}(S_2) = \infty$, it follows from Ljusternik–Schnirelman category (see [32, Theorem 5.7]) that there exists a sequence $(u_n) \subset S_2$ such that $\widetilde{\Lambda_2}'(u_n) = 0, \widetilde{\Lambda_2}(u_n) = \mu_n$. The remainder is the same as the proof of Theorem 3.1, here we omit it. □

3.3 Kirchhoff-type problems with critical growth

Now, let us consider the following Kirchhoff-type problem:

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2)\Delta u = \mu u + |u|^4 u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $a, b > 0$, μ is a real parameter.

Inspired by the works described above, in this paper we study the existence of zero energy solutions for a class of Kirchhoff problem with critical growth in bounded domains. In recent years, Kirchhoff-type equation is an extension of the classical D'Alembert's wave equation. It was firstly proposed by Kirchhoff in 1883. Various problems of Kirchhoff-type are usually named nonlocal problems in virtue of the appearance of the nonlocal term $a + b\int_\Omega |\nabla u|^2$ and have been extensively investigated up to now. In [28], Quoirin et al. investigated qualitative properties of solutions for a Kirchhoff-type problem with subcritical growth as an application of their zero energy critical point theory. However, their theory seems difficulty to deal with the problem like (3.6) involving the critical exponent. For this purpose, we explore a new strategy (Theorem 1.1) to solve this problem.

As usual, one can get that the corresponding functional of (3.6) is $\Phi_\mu : H_0^1(\Omega) \to \mathbb{R}$:

$$\Phi_\mu(u) = \frac{a}{2}\int_\Omega |\nabla u|^2 dx - \frac{\mu}{2}\int_\Omega |u|^2 dx - \frac{1}{6}\int_\Omega |u|^6 dx + \frac{b}{4}\left(\int_\Omega |\nabla u|^2 dx\right)^2.$$

It is evident that Φ_μ is C^1. Then, according to the previous preliminaries, for every $u \in H_0^1(\Omega)\backslash\{0\}$:

$$\mu_0(u) = \frac{a\|u\|^2}{|u|_2^2} - \frac{1}{3}\frac{|u|_6^6}{|u|_2^2} + \frac{b}{2}\frac{\|u\|^4}{|u|_2^2},$$

$$\mu_0(tu) = \frac{a\|u\|^2}{|u|_2^2} - \frac{t^4}{3}\frac{|u|_6^6}{|u|_2^2} + t^2\frac{b}{2}\frac{\|u\|^4}{|u|_2^2}.$$

Let $\mu_0'(tu) = 0$, we obtain

$$t_0(u) = \left(\frac{3}{4}\frac{b\|u\|^4}{|u|_6^6}\right)^{\frac{1}{2}} > 0,$$

and $t_0(u)$ is a nondegenerate global maximum point of $\mu_0(tu)$ via $\mu_0''(t_0(u)u) < 0$. Therefore, we have

$$\Lambda_3(u) = \frac{a\|u\|^2}{|u|_2^2} + \frac{3}{16}\frac{(b\|u\|^4)^2}{|u|_2^2|u|_6^6}.$$

For simplicity, we call $\widetilde{\Lambda_3}$ the restriction of Λ_3 to S_3, where

$$S_3 = \{u \in H_0^1(\Omega)\backslash\{0\} : \|u\| = 1\}.$$

The main result we have derived in this section is expressed as:

Theorem 3.9. *Let $\delta_3 = \inf_{H_0^1(\Omega)\backslash\{0\}} \Lambda_3(u)$.*

(i) If $\mu < \delta_3$, then there is no nontrivial weak solution having zero energy in problem (3.6);

(ii) If $\mu > \delta_3$, then problem (3.6) has infinitely many zero energy sign changing weak solutions.

To verify this result, according to Ljusternik–Schnirelman category (see [32, Theorem 5.7]), it is necessary to prove that $\widetilde{\Lambda_3}$ is bounded below and satisfies the Palais–Smale condition.

Lemma 3.10. *$\widetilde{\Lambda_3}$ is bounded from below on S_3.*

Proof. Since the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, there exists $C > 0$ such that $C\|u\|^2 \geqslant |u|_2^2$. Hence, one can obtain that

$$\widetilde{\Lambda_3}(u) = \frac{a\|u\|^2}{|u|_2^2} + \frac{3}{16}\frac{b\|u\|^4}{|u|_2^2|u|_6^3} > \frac{a}{C},$$

which yields that $\widetilde{\Lambda_3}$ is bounded from below on S_3. □

Lemma 3.11. *If $(u_n) \subset S_3$, then $u_n \rightharpoonup u$ in $H_0^1(\Omega)\backslash\{0\}$ up to a subsequence. Therefore,*

$$\int_{\mathbb{R}^3} |u_n|^4 u_n \varphi dx \to \int_{\mathbb{R}^3} |u|^4 u \varphi dx, \quad \text{as } n \to \infty,$$

for any $\varphi \in C_0^\infty(\mathbb{R}^3)$.

Proof. We employ the strategy outlined in Lemma 3.3 from [21]. In fact, it is related to a result from the Lebesgue Dominated Convergence Theorem. First we notice that

$$|u_n|^4 u_n \varphi \to |u|^4 u \varphi \quad \text{as } n \to \infty,$$

almost everywhere in the compact support Ω of φ, and

$$||u_n|^4 u_n \varphi \chi_\Omega| \leq |u_n|^5 |\varphi| \chi_\Omega,$$

where χ_Ω represents the characteristic function of Ω. Given that $u_n \to u$ in $L^s_{loc}(\mathbb{R}^3)$ for all $5 < s < 6$, utilizing the Hölder inequality yields

$$\int_\Omega |u_n|^5 |\varphi| dx \leq \left(\int_\Omega |u_n|^s dx\right)^{\frac{5}{s}} \left(\int_\Omega |\varphi|^{\frac{s}{s-5}} dx\right)^{\frac{s-5}{s}},$$

which means that $|u_n|^5 |\varphi| \chi_\Omega \in L^1(\mathbb{R}^3)$. According to Lebesgue dominated convergence theorem, it follows that

$$\int_{\mathbb{R}^3} |u_n|^4 u_n \varphi dx \to \int_{\mathbb{R}^3} |u|^4 u \varphi dx, \quad \text{as } n \to \infty.$$

The proof is now complete. □

Lemma 3.12. *$\widetilde{\Lambda_3}$ satisfied the Palais–Smale condition.*

Proof. Choosing a sequence $(u_n) \subset S_3$ such that $(\widetilde{\Lambda_3}(u_n))$ is bounded and $\widetilde{\Lambda_3}'(u_n) \to 0$, that is $|\Lambda_3'(u_n)v| \leq \varepsilon_n \|v\|$ for any $v \in \mathcal{T}_{S_3}(u_n)$, with $\varepsilon_n \to 0$. In view of Lemma 2.2, if there is a sequence $(u_n) \subset S_3$ and $\widetilde{\Lambda_3}'(u_n) \to 0$, we can acquire that $u_n \rightharpoonup u$ in S_3 and $\Lambda'(u_n)(u_n - u) \to 0$ as $n \to \infty$. Then, for any $u_n, u \in S_3$,

$$\begin{aligned}\Lambda'(u_n)(u_n - u) = 2a&\frac{|u_n|_2^2 \int_\Omega \nabla u_n \nabla(u_n - u)dx - \|u_n\|^2 \int_\Omega u_n(u_n - u)dx}{|u_n|_2^4} \\ &+ \frac{3b}{16}Q(u_n)\Bigg(4|u_n|_2^2|u_n|_6^6 \int_\Omega |\nabla u_n|^2 dx \int_\Omega \nabla u_n \nabla(u_n - u)dx \\ &- 6\|u_n\|^4|u|_2^2 \int_\Omega |u_n|^4 u_n(u_n - u)dx - 2\|u_n\|^4|u_n|_6^6 \int_\Omega u_n(u_n - u)dx\Bigg) \to 0,\end{aligned}$$

as $n \to \infty$, where

$$Q(u_n) = \frac{b\|u_n\|^4}{|u_n|_2^4(|u_n|_6^6)^{\frac{1}{2}}}.$$

Since $(u_n) \subset S_3$ and the embedding $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ is continuous for $1 \leq s \leq 6$, there exists a constant $C > 0$ such that

$$0 < |u_n|_2^2 \leq C\|u_n\|^2 = C, \qquad 0 < |u_n|_6^6 \leq C\|u_n\|^6 = C.$$

Therefore, $|u_n|_2^2, |u_n|_6^6$ is bounded. Note that $(\widetilde{\Lambda_3}(u_n))$ is bounded, then $Q(u_n)$ is bounded. Since $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, one can easily deduce from the Sobolev embedding theorem that

$$\int_\Omega u_n(u_n - u)dx \to 0, \quad \text{as } n \to \infty.$$

By means of Lemma 3.11, given $\varphi = u_n - u$, then as $n \to \infty$,

$$\int_\Omega |u_n|^4 u_n(u_n - u)dx - \int_\Omega |u|^4 u(u_n - u)dx \to 0.$$

This yields that

$$\int_\Omega |u_n|^4 u_n(u_n - u)dx \to 0, \quad \text{as } n \to \infty,$$

which leads to the following conclusion:

$$\int_\Omega \nabla u_n \nabla(u_n - u)dx \to 0, \quad \text{as } n \to \infty,$$

which means that $\|u_n\| \to \|u\|$ in S_3. By the uniform convexity of $H_0^1(\Omega)$, it follows that $u_n \to u$ in S_3. In conclusion, $\widetilde{\Lambda_3}$ satisfied the Palais–Smale condition. □

Proof of Theorem 3.9.

(i) We prove that there is no critical point having zero energy when $\mu < \delta_3$. Note that u is a critical point of Λ_3, if and only if, $t_0(u)u$ is a zero energy critical point of Φ_μ with $\mu = \Lambda_3(u)$, this yields the desired conclusion.

(ii) According to Lemma 3.10, it follows that $\widetilde{\Lambda_3}$ is bounded from below on S_3. In the meantime, $\widetilde{\Lambda_3}$ satisfies the Palais–Smale condition as per the Lemma 3.12. Note that $\hat{\gamma}(S_3) = \infty$, Ljusternik–Schnirelman category (see [32, Theorem 5.7]) implied that there exists a sequence $(u_n) \subset S_3$ such that $\widetilde{\Lambda_3}'(u_n) = 0, \widetilde{\Lambda_3}(u_n) = \mu_n$. Analogous to the proof of Theorem 3.1, problem (3.6) possesses infinitely many zero-energy, sign-changing weak solutions. □

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