Electronic Journal of Qualitative Theory of Differential Equations
Proc. 9th Coll. QTDE, 2012, No. **14** 1-12;
http://www.math.u-szeged.hu/ejqtde/

Nonlinear second order evolution equations with state-dependent delays*

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Abstract

We consider second order quasilinear parabolic equations where also the main part contains functional dependence and state-dependent delay on the unknown function. Existence and some qualitative properties of the solutions are shown.

1 Introduction

It is well known that the theory of monotone type operators can be applied to first order evolution equations and as particular cases to nonlinear functional parabolic equations of the form

$$D_t u - \sum_{i=1}^{n} D_i[a_i(t, x, u, Du; u)] + a_0(t, x, u, Du; u) = f$$

where the last terms in the brackets mean "functional" (non-local) dependence on u, e.g. some integral operators applied to u or some state-dependent delays (see, e.g., [7] -[10]). It is less known that monotone type operators can be applied also to certain second order nonlinear evolution equations, including "functional" equations.

The aim is to consider some second order evolution equations with functional dependence and state dependent delays. Differential equations and systems with state-dependent delay in one variable were considered thoroughly e.g. by I. Györi, F. Hartung, T. Krisztin, J. Turi, H.-O. Walther, J. Wu in [3] - [5].

2 Existence of solutions

Denote by $\Omega \subset \mathbb{R}^n$ a bounded domain having the uniform C^1 regularity property (see [1]), $Q_T = (0, T) \times \Omega$ and $p \geq 2$ be a real number. Let $V \subset W^{1,p}(\Omega)$ be

*This paper is in final form and no version of it will be submitted for publication elsewhere.

†This work was supported by the Hungarian National Foundation for Scientific Research under grant OTKA K 81403.

a closed linear subspace of the usual Sobolev space $W^{1,p}(\Omega)$ (of real valued functions) containing $W_0^{1,p}(\Omega)$ (the closure of $C_0^\infty(\Omega)$). Denote by $L^p(0,T;V)$ the Banach space of the set of measurable functions $u:(0,T)\to V$ with the norm

$$\| u \|^p_{L^p(0,T;V)} = \int_0^T \| u(t) \|_V^p \, dt.$$

The dual space of $L^p(0,T;V)$ is $L^q(0,T;V^\star)$ where $1/p+1/q=1$ and $V^\star$ is the dual space of V (see, e.g., [11]).

By using the notations $u' = D_t u$, $u" = D_t^2 u$, we shall consider the equation

$$u" + N(u'(t), u'([\gamma_0(u)](t))) + Qu + \tag{2.1}$$

$$M(u(t), u([\gamma_1(u)](t)), Du(t), Du([\gamma_2(u)](t))) = f$$

with the initial condition

$$u(0) = u_0, \quad u'(0) = u_1 \tag{2.2}$$

where $N: L^p(0,T;V)\times L^2(Q_T) \to L^q(0,T;V^\star)$ is a nonlinear operator, $(Qu)(t) = \tilde{Q}(u(t))$ and $\tilde{Q}: V \to V^\star$ is a linear and continuous operator,

$$M: L^p(0,T;V) \times L^2(Q_T) \times L^p_n(Q_T) \times L^2_n(Q_T) \to \times L^q(Q_T)$$

is a nonlinear operator

Further, for $j=0,1,2$

(G) $\gamma_j : L^2(Q_T) \to C_a[0,T]$ are continuous (nonlinear) operators such that

$$0 \le [\gamma_j(u)](t) \le t, \quad [\gamma_j(u)]'(t) \ge c_0$$

with some constant $c_0 > 0$. ($C_a[0,T]$ denotes the set of absolutely continuous functions in $[0,T]$.)

Condition (G) is fulfilled e.g. by the operators of the form

$$[\gamma_j(u)(t) = t\beta \left(\int_{Q_t} \Gamma(t,\tau,\xi) u^2(\tau,\xi) d\tau d\xi \right)$$

where $\Gamma, \frac{\partial \Gamma}{\partial t}$ are continuous and nonnegative, $\beta \in C^1(\mathbb{R})$ satisfies $\delta_1 \le \beta \le 1$ with some constant $\delta_1 > 0$ and $\beta' \ge 0$.

(i) Assumptions on N:

$$N: L^p(0,T;V) \times L^2(Q_T) \to L^q(0,T;V^\star)$$

is bounded, demicontinuous and belongs to $(S)_+$ with respect to $D(L) = \{u \in L^p(0,T;V) : u' \in L^q(0,T;V^\star), u(0) = 0\}$, i.e.

$$(v_j) \to v \text{ weakly in } L^p(0,T;V), \quad v_j \in D(L),$$

$$(v_j') \to v' \text{ weakly in } L^q(0,T;V^\star), \quad (w_j) \to w \text{ (strongly) in } L^2(Q_T),$$

$$\limsup[N(v_j, w_j), v_j - v] \leq 0$$

imply

$$(v_j) \to v \text{ (strongly) in } L^p(0,T;V).$$

Further, there are constants $c_2 > 0, c_3$ such that

$$[N(v,w), v] \geq c_2 \parallel v \parallel^p_{L^p(0,T;V)} - c_3.$$

(ii) Assumptions on Q: $(Qu)(t) = \tilde{Q}(u(t))$ and $\tilde{Q} : V \to V^\star$ is a linear and continuous operator,

$$\langle \tilde{Q}\tilde{u}, \tilde{v}\rangle = \langle \tilde{Q}\tilde{v}, \tilde{u}\rangle, \quad \langle \tilde{Q}\tilde{u}, \tilde{u}\rangle \geq 0, \quad \tilde{u}, \tilde{v} \in V.$$

(iii) Assumptions on M:

$$M : L^p(0,T;V) \times L^2(Q_T) \times L^p_n(Q_T) \times L^2_n(Q_T) \to \times L^q(Q_T)$$

is (nonlinear) bounded, demicontinuous and

$$\lim_{\|(u,\tilde{u},w,\tilde{w}\|)\to\infty} \frac{\parallel M(u,\tilde{u},w,\tilde{w}) \parallel^q_{L^q(0,T;V^\star)}}{\parallel (u,\tilde{u},w,\tilde{w}) \parallel^p} = 0.$$

Theorem 2.1 *Assume (i) - (iii) and (G). Then for any $f \in L^q(0,T;V^\star)$, $u_0 \in V$ and $u_1 \in L^2(\Omega)$ there exists $u \in L^p(0,T;V)$ such that $u' \in L^p(0,T;V)$, $u" \in L^q(0,T;V^\star)$ and u satisfies (2.1), (2.2).*

For the definition of the generalized derivatives $u', u"$ see, e.g., [11], page 417. In the proof of the theorem we shall use

Lemma 2.2 *Assume that $\gamma : L^2(Q_T) \to C_a[0,T]$ satisfies (G). If $(u_k) \to u$ in $L^2(Q_T)$ and $(w_k) \to w$ in $L^2(Q_T)$ then*

$$w_k([\gamma(u_k)](t), x) \to w([\gamma(u)](t), x) \text{ in } L^2(Q_T).$$

Further, $w([\gamma(u)](t)$ is bounded in $L^2(Q_T)$ if u, w are bounded in $L^2(Q_T)$.

Proof of the lemma Clearly,

$$w_k([\gamma(u_k)](t), x) - w([\gamma(u)](t), x) = \tag{2.3}$$

$$\{w_k([\gamma(u_k)](t), x) - w([\gamma(u_k)](t), x)\} +$$

$$\{w([\gamma(u_k)](t), x) - w([\gamma(u)](t), x)\}.$$

For the first term in the right hand side of (2.3) we have (by using the notation $\psi^k(t) = [\gamma(u_k)](t)$, (G))

$$\int_\Omega \left\{ \int_0^T w_k([\gamma(u_k)](t), x) - w([\gamma(u_k)](t), x)|^2 dt \right\} dx \leq \tag{2.4}$$

$$\frac{1}{c_0}\int_\Omega \left\{\int_0^T |w_k(\psi_k(t),x)-w(\psi_k(t),x)|^2 \frac{\partial \psi_k}{\partial t}dt\right\}dx \leq$$

$$\frac{1}{c_0}\int_{Q_T} |w_k(\tau,x)-w(\tau,x)|^2 d\tau dx \to 0.$$

Further, approximating the function $w \in L^2(Q_T)$ by a function $\tilde{w} \in C(\overline{Q_T})$, we have for the second term on the right hand side of (2.3)

$$w([\gamma(u_k)](t),x) - w([\gamma(u)](t),x) = \tag{2.5}$$

$$\{w([\gamma(u_k)](t),x) - \tilde{w}([\gamma(u_k)](t),x)\}+$$

$$\{\tilde{w}([\gamma(u_k)](t),x) - \tilde{w}([\gamma(u)](t),x)\}+$$

$$\{\tilde{w}([\gamma(u)](t),x) - w([\gamma(u)](t),x)\}.$$

The first and third terms on the right hand side of (2.5) can be estimated similarly to (2.4). The $L^2(Q_T)$ norm of the second term on the right hand side of (2.5) is small for sufficiently large k because $\tilde{w}$ is uniformly continuous on $\overline{Q_T}$ and $(\gamma(u_k)) \to \gamma(u)$ in $C[0,T]$. By using the substitution as in (2.4), we obtain the second part of the lemma. So we have proved the lemma.

The proof of Theorem 2.1 For simplicity, consider the case $u_0 = 0, u_1 = 0$. Define operator $S : L^p(0,T;V) \to L^p(0,T;V)$ by

$$(Sv)(t) = \int_0^t v(s)ds.$$

Then S is a linear and continuous operator and u is a solution of (2.1), (2.2) with $u_0 = 0, u_1 = 0$ iff $v = u' \in L^p(0,T;V)$ satisfies

$$v' + N(v, v([\gamma_0(Sv)](t))) + QSv+$$

$$M(Sv, (Sv)([\gamma_1(Sv)]), DSv, (DSv)([\gamma_2(Sv)])) = f, \quad v(0) = 0.$$

Consider the operator $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ defined by

$$A(v) = N(v, v([\gamma_0(Sv)](t))) + QSv+$$

$$M(Sv, (Sv)([\gamma_1(Sv)](t)), DSv, (DSv)([\gamma_2(Sv)](t))).$$

By using the lemma and (i) - (iii), it is not difficult to show that A is bounded and demicontinuous. Now we show that A belongs to $(S)_+$ with respect to

$$D(L) = \{v \in L^p(0,T;V) : v' \in L^q(0,T;V^\star), v(0) = 0\}$$

The last property means:

$$v_j \in D(L), \quad (v_j) \to v \text{ weakly in } L^p(0,T;V), \tag{2.6}$$

$$(v_j') \to v' \text{ weakly in } L^q(0,T;V^\star), \tag{2.7}$$

$$\limsup [A(v_j), v_j - v] \leq 0 \tag{2.8}$$

imply

$$(v_j) \to v \text{ strongly in } L^p(0,T;V), \tag{2.9}$$

To prove that (2.6) - (2.8) imply (2.9), observe

$$[QS(v_j), v_j - v] = [QS(v_j - v), v_j - v] + [QS(v), v_j - v],$$

the first term on the right is nonnegative (see, e.g. [11]) and the second term tends to 0, thus

$$\liminf [QS(v_j), v_j - v] \geq 0. \tag{2.10}$$

Further, by compact imbedding theorem, (2.6), (2.7) imply that $(v_j) \to v$ in $L^p(Q_T)$, for a subsequence, hence

$$[M(Sv_j, (Sv_j)([\gamma_1(Sv_j)]), DSv_j, D(Sv_j)([\gamma_2(Sv_j)])), v_j - v] \to 0 \tag{2.11}$$

because the first term in $[\cdot, \cdot]$ is bounded in $L^q(Q_T)$ since M is bounded.

(2.8), (2.10), (2.11) imply that

$$\limsup [N(v_j), v_j([\gamma_0(Sv_j)](t)), v_j - v] \leq 0$$

By the lemma

$$v_j([\gamma_0(Sv_j)](t)) \to v([\gamma_0(Sv)](t)) \text{ in } L^2(Q_T).$$

Thus (i) implies

$$(v_j) \to v \text{ in } L^p(0,T;V).$$

So $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ is bounded, demicontinuous, belongs to $(S)_+$.

Finally, assumptions (i), (ii), (iii) imply that A is coercive. Because by (iii)

$$\left| \frac{[M(Sv, (Sv)([\gamma_1(Sv)](t)), DSv, (DSv)([\gamma_2(Sv)](t))), v]}{\| v \|^p_{L^p(0,T;V)}} \right| \leq$$

$$\left\{ \frac{\| M(Sv, (Sv)([\gamma_1(Sv)](t)), DSv, (DSv)([\gamma_2(Sv)](t))) \|^q_{L^q(0,T;V^\star)}}{\| v \|^p} \right\}^{1/q}$$

and the term on the right hand side in brackets can be written in the form

$$\frac{\| M(Sv, (Sv)([\gamma_1(Sv)](t)), DSv, (DSv)([\gamma_2(Sv)](t))) \|^q_{L^q(0,T;V^\star)}}{\| v \|^p + \| Sv \|^p + \| (Sv)([\gamma_1(Sv)](t)) \|^p + \| DSv \|^p + \| (DSv)([\gamma_2(Sv)](t)) \|^p} \times$$

$$\frac{\| v \|^p + \| Sv \|^p + \| (Sv)([\gamma_1(Sv)](t)) \|^p + \| DSv \|^p + \| (DSv)([\gamma_2(Sv)](t)) \|^p}{\| v \|^p}$$

where the second fraction is bounded by the lemma and for any $\varepsilon > 0$ there exists $a > 0$ such that the first fraction is less than ε if its denominator is grater than a. Thus, choosing sufficiently small $\varepsilon > 0$, by (i), (ii) we obtain

$$\frac{[A(v), v]}{\| v \|^p} \geq \frac{c_2}{2} - \frac{c_4}{\| v \|^p}$$

with some constant c_4 which implies

$$\lim_{\|v\| \to \infty} \frac{[A(v), v]}{\| v \|} = +\infty,$$

i.e. A is coercive. Consequently, there is a solution of (2.1), (2.2).

3 Examples

The following examples satisfy the assumptions of Theorem 2.1.

$$[N(v, w), z] = \sum_{i=1}^{n} \int_{Q_T} b(t, x, [H(w)](t, x))(D_i v)|Dv|^{p-2} D_i z dt dx +$$

$$\int_{Q_T} b_0(t, x, [H_0(w)](t, x)) v|v|^{p-2} z dt dx$$

where b, b_0 are Carathéodory functions, $0 < c_2 \leq b, b_0 \leq c_3$;

$$H, H_0 : L^2(Q_T) \to C(\overline{Q_T}) \text{ are continuous linear operators}$$

$$\langle \tilde{Q}\tilde{u}, \tilde{v} \rangle = \int_{\Omega} \left[\sum_{k,l=1}^{n} a_{kl} D_k \tilde{u} D_l \tilde{v} + d_0 \tilde{u} \tilde{v} \right] dx$$

where $a_{kl}, d_0 \in L^\infty(\Omega)$, $a_{kl} = a_{lk}$, $\sum_{k,l=1}^{n} a_{kl}(x) \xi_k \xi_l \geq 0$, $d_0 \geq 0$.

$$M(u, \tilde{u}, w, \tilde{w}) =$$

$$\hat{b}(t, x, [F_1(\tilde{u})](t, x), [F_2(\tilde{w})](t, x)) \cdot \alpha(t, x, u, w)$$

where $\alpha, \hat{b}$ are Carathéodory functions,

$$|\alpha(t, x, u, w)| \leq \text{const}[1 + |u|^\rho + |w|^\rho],$$

$$|\hat{b}(t, x, \theta_1, \theta_2)|^{q_1} \leq \text{const}[1 + \theta_1^2 + \theta_2^2]$$

where $0 \leq \rho < p - 1$, $q_1 = p/(p - 1 - \rho)$ and $F_j : L^2(Q_T) \to L^2(Q_T)$ are continuous operators satisfying with some $\sigma < p$

$$\int_{Q_T} |F_1(\tilde{u})|^2 \leq \text{const} \left[\int_{Q_T} |\tilde{u}|^2 \right]^{\sigma/2}, \int_{Q_T} |F_2(\tilde{w})|^2 \leq \text{const} \left[\int_{Q_T} |\tilde{w}|^2 \right]^{\sigma/2}.$$

(For $p > 2$, σ may be 2, F_j linear continuous operator.)

4 Boundedness and stabilization

Now we formulate an existence theorem in $(0,\infty)$ which can be obtained from Theorem 2.1, by using a diagonal process and the Volterra property (see, e.g. [6], [9]). Denote by $L^p_{loc}(0,\infty;V)$ the set of functions $u:(0,\infty)\to V$ such that for each fixed finite $T>0$, $u|_{(0,T)}\in L^p(0,T;V)$ and let $Q_\infty=(0,\infty)\times\Omega$, $L^\alpha_{loc}(Q_\infty)$ be the set of functions $u:Q_\infty\to\mathbb{R}$ such that $u|_{Q_T}\in L^\alpha(Q_T)$ for any finite T. On operators γ_j assume

(G_∞) Operators $\gamma_j: L^2_{loc}(Q_\infty)\to C_a[0,\infty)$ are of Volterra type, i.e. $[\gamma_j(u)](T)$ depends only on $u|_{Q_T}$, for any finite T and $\gamma_j: L^2(Q_T))\to C_a[0,T]$ is continuous for every T. Further,

$$\frac{\partial}{\partial t}[\gamma_j(u)](t,x)\geq c_0, \quad 0\leq[\gamma_j(u)](t,x)\leq t$$

with some constant $c_0>0$.

Theorem 4.1 *Assume that $\tilde{Q}:V\to V^\star$ satisfies (ii). Let*

$$N: L^p_{loc}(0,\infty;V)\times L^2_{loc}(Q_\infty)\to L^q_{loc}(0,\infty;V^\star),$$

$$M: L^p_{loc}(0,\infty;V)\times L^2_{loc}(Q_\infty)\times L^p_{n,loc}(Q_\infty)\times L^2_{n,loc}(Q_\infty)\to L^q_{loc}(0,\infty;V^\star)$$

be operators of Volterra type and assume that for each finite $T>0$ their restrictions to $(0,T)$ satisfy (i) and (iii).

Then for arbitrary $f\in L^q_{loc}(0,\infty;V^\star)$, $u_0\in V$, $u_1\in H$ there exists u such that $u\in C([0,\infty);V)$, $u'\in L^p_{loc}(0,\infty;V)$, $u"\in L^q_{loc}(0,\infty;V^\star)$ and

$$u"(t)+N(u'(t),u'([\gamma_0(u)](t)))+Qu+ \tag{4.12}$$

$$M(u(t),u([\gamma_1(u)](t)),Du(t),Du([\gamma_2(u)](t)))=f \text{ for a.a. } t\in(0,\infty),$$

$$u(0)=u_0,\quad u'(0)=u_1 \tag{4.13}$$

Now we formulate a theorem on the boundedness of the solutions of (4.12), (4.13).

Theorem 4.2 *Let the assumptions of Theorem 4.1 be satisfied such that for all $v\in L^p_{loc}(0,\infty;V)$, $w\in L^2_{loc}(Q_\infty)$*

$$\langle N(v,w),v\rangle\geq c_2\parallel v(t)\parallel^p_V,\quad t\in(0,\infty) \tag{4.14}$$

with some constant $c_2>0$ and for all $u\in L^p_{loc}(0,\infty;V)$, $\tilde{u}\in L^2_{loc}(Q_\infty)$, $w\in L^p_{n,loc}(Q_\infty)$, $\tilde{w}\in L^2_{n,loc}(Q_\infty)$

$$\parallel M(u,\tilde{u},w,\tilde{w})\parallel^q_{V^\star}\leq\Phi_1(t),\quad t\in(0,\infty) \tag{4.15}$$

with some $\Phi_1\in L^1(0,\infty)$. Finally, let $f\in L^q(0,\infty;V^\star)$.

Then for a solution u of (4.12), (4.13), $y(t) = \| u'(t) \|_H^2$ is bounded for $t \in (0,\infty)$, $u' \in L^p(0,\infty;V)$ and

$$\langle \tilde{Q}[u(t)], u(t)\rangle \text{ is bounded for } t \in (0,\infty).$$

If

$$\langle \tilde{Q}\tilde{u}, \tilde{u}\rangle \geq c_3 \| \tilde{u} \|_{W^{1,2}(\Omega)}^2 \quad \text{for } \tilde{u} \in V$$

with some constant $c_3 > 0$ then

$$\| u(t) \|_{W^{1,2}(\Omega)} \quad \text{is bounded for } t \in (0,\infty).$$

Proof Applying both sides of (4.12) to u' and integrating over $[0,T]$, we obtain

$$[u", u'] + [N(u', u'([\gamma_0(u)](t))), u'] + [Qu, u'] + \tag{4.16}$$

$$[M(u(t), u([\gamma_1(u)](t)), Du(t), Du([\gamma_2(u)](t))), u'] = [f, u'].$$

According to [11], [9] we have

$$[u", u'] = \frac{1}{2} \| u'(t) \|_H^2 - \frac{1}{2} \| u'(0) \|_H^2 = \frac{1}{2}y(t) - \frac{1}{2}y(0), \tag{4.17}$$

$$[Qu, u'] = \frac{1}{2}\langle \tilde{Q}u(T), u(T)\rangle - \frac{1}{2}\langle \tilde{Q}u(0), u(0)\rangle. \tag{4.18}$$

Further, by Young's inequality and (4.15)

$$|[M(u(t), u([\gamma_1(u)](t)), Du(t), Du([\gamma_2(u)](t))), u']| \leq \tag{4.19}$$

$$\frac{\varepsilon^p}{p} \int_0^T \| u'(t) \|_V^p \, dt + \frac{1}{\varepsilon^q q} \int_0^T \Phi_1(t) dt,$$

$$|[f, u']| \leq \frac{\varepsilon^p}{p} \int_0^T \| u'(t) \|_V^p \, dt + \frac{1}{\varepsilon^q q} \int_0^T \| f(t) \|_{V^\star}^q \, dt. \tag{4.20}$$

Choosing sufficiently small $\varepsilon > 0$, we obtain from (4.14), (4.16) - (4.20) the inequality

$$\frac{1}{2}y(T) + \frac{c_2}{2} \int_0^T \| u'(t) \|_V^p \, dt + \frac{1}{2}\langle \tilde{Q}u(T), u(T)\rangle \leq$$

$$\text{const} \left[1 + \int_0^T \Phi_1(t) dt + \int_0^T \| f(t) \|_{V^\star}^q \, dt \right]$$

which implies the statements of Theorem 4.2.

Now we prove a theorem on the stabilization of the solution as $t \to \infty$.

Theorem 4.3 *Assume that the assumptions of Theorem 4.2 are satisfied such that for all* $v \in L^p_{loc}(0,\infty;V)$, $w \in L^2_{loc}(Q_\infty)$

$$\langle [N(v,w)](t), v(t)\rangle \geq c_2(1+t)^\mu \parallel v(t) \parallel^p_V, \quad t \in (0,\infty) \tag{4.21}$$

with some constants $\mu > p-1$ $(p \geq 2)$, $c_2 > 0$. *Further, there exists* $f_\infty \in V^\star$, *a continuous function* $\Phi \in L^1(0,\infty)$ *with* $\lim_\infty \Phi = 0$ *such that*

$$\parallel f(t) - f_\infty \parallel^q_{V^\star} \leq \Phi(t), \quad t \in (0,\infty) \tag{4.22}$$

and there exists a solution $u_\infty \in V$ *of*

$$\tilde{Q}u_\infty = f_\infty. \tag{4.23}$$

Then for a solution u *of (4.12), (4.13) we have*

$$\lim_{t\to\infty} \parallel u'(t) \parallel_H = 0, \tag{4.24}$$

$$\int_0^\infty (1+t)^\beta \parallel u'(t) \parallel^2_H dt < \infty, \quad \int_0^\infty (1+t)^\mu \parallel u'(t) \parallel^p_V dt < \infty \tag{4.25}$$

where $0 \leq \beta < [2\mu - (p-2)]/p$ *and there exists* $w \in V$ *such that*

$$\parallel u(t) - w \parallel^q_V \leq \frac{const}{\lambda - 1}\frac{1}{(1+t)^{\lambda-1}} \text{ where } \lambda = \mu/(p-1) > 1. \tag{4.26}$$

Proof Applying (4.12) to $u' = (u-u_\infty)'$, we obtain by (4.23)

$$\int_0^T \langle u"(t), u'(t)\rangle dt + \int_0^T \langle N(u', u'([\gamma_0(u)](t))), u'(t)\rangle dt + \tag{4.27}$$

$$\int_0^T \langle \tilde{Q}[u(t)-u_\infty], [u(t)-u_\infty]'\rangle dt +$$

$$\int_0^T \langle M(u(t), u([\gamma_1(u)](t)), Du(t), Du([\gamma_2(u)](t))), u'(t)\rangle dt =$$

$$\int_0^T \langle f(t) - f_\infty, u'(t)\rangle dt.$$

Similarly to the proof of Theorem 4.2, equality (4.27) implies, by using Young's inequality with sufficiently small $\varepsilon > 0$, that for $y(t) = \parallel u'(t) \parallel^2_H$ the following inequality holds:

$$\frac{1}{2}y(T) + \frac{c_2}{2}\int_0^T (1+t)^\mu \parallel u'(t) \parallel^p_V dt + \frac{1}{2}\langle \tilde{Q}[u(T)-u_\infty], u(T)-u_\infty\rangle \leq \tag{4.28}$$

$$\text{const}\left[1 + \int_0^T \Phi_1(t)dt + \int_0^T \Phi(t)dt\right] + \frac{1}{2}\langle \tilde{Q}[u(0)-u_\infty], u(0)-u_\infty\rangle.$$

Since the right hand side of (4.28) is bounded for all $T > 0$, we obtain the second part of (4.25). Consequently, for any $T_1 < T_2$ we have

$$\| u(T_2) - u(T_1) \|_V = \| (Su')(T_2) - (Su')(T_1) \|_V = \| \int_{T_1}^{T_2} u'(t)dt \|_V \leq \tag{4.29}$$

$$\int_{T_1}^{T_2} \| u'(t) \|_V \, dt = \int_{T_1}^{T_2} \frac{1}{(1+t)^{\lambda/q}} (1+t)^{\lambda/q} \| u'(t) \|_V \, dt \leq$$

$$\left\{ \int_{T_1}^{T_2} \frac{1}{(1+t)^{\lambda}} dt \right\}^{1/q} \left\{ \int_{T_1}^{T_2} (1+t)^{\mu} \| u'(t) \|_V^p \, dt \right\}^{1/p}$$

where $\lambda > \mu/(p-1) > 1$ and thus $p\lambda/q = \lambda(p-1) = \mu$.

Thus for any $\varepsilon > 0$ there exists $T_0 > 0$ such that for $T_0 < T_1 < T_2$

$$\| u(T_2) - u(T_1) \|_V < \varepsilon.$$

Hence, there exists $w \in V$ such that

$$\lim_{T \to \infty} \| u(T) - w \|_V = 0. \tag{4.30}$$

In order to prove (4.26), letting $T_2 \to \infty$ in (4.29), we find

$$\| w - u(T_1) \|_V \leq \int_{T_1}^{\infty} \| u'(t) \|_V \, dt \leq$$

$$\left\{ \int_{T_1}^{\infty} \frac{1}{(1+t)^{\lambda}} dt \right\}^{1/q} \left\{ \int_{T_1}^{\infty} (1+t)^{\mu} \| u'(t) \|_V^p \, dt \right\}^{1/p} \leq$$

$$\left\{ \frac{1}{\lambda - 1} \frac{1}{(1+T_1)^{\lambda-1}} dt \right\}^{1/q} \left\{ \int_{T_1}^{\infty} (1+t)^{\mu} \| u'(t) \|_V^p \, dt \right\}^{1/p},$$

i.e. we have (4.26).

The first estimation in (4.25) can be proved as follows.

If $0 \leq \beta < [2\mu - (p-2)]/p$ then by Hölder's inequality

$$\int_0^{\infty} (1+\beta)^{\beta} \| u'(t) \|_H^2 \, dt \leq \text{const} \int_0^{\infty} (1+\beta)^{\beta} \| u'(t) \|_V^2 \, dt =$$

$$\text{const} \int_0^{\infty} (1+\beta)^{\beta - 2\mu/p} [(1+t)^{2\mu/p} \| u'(t) \|_V^2] dt \leq$$

$$\text{const} \left\{ \int_0^{\infty} (1+\beta)^{\frac{\beta p - 2\mu}{p-2}} dt \right\}^{(p-2)/p} \left\{ \int_0^{\infty} (1+t)^{\mu} \| u'(t) \|_V^p \, dt \right\}^{2/p} < \infty$$

because of the second part of (4.25) and $\frac{\beta p - 2\mu}{p-2} < -1$. In the case $p = 2$ the first multiplier in the last term is the $L^{\infty}(0,\infty)$ norm of the function $t \mapsto (1+t)^{\beta - 2\mu/p}$.

Now we apply again (4.12) to $u' = (u - u_\infty)'$ and integrate over $[T_1, T_2]$ then we obtain by (4.23) the inequality (similarly to (4.27))

$$\frac{1}{2}[y(T_2) - y(T_1)] + \frac{c_2}{2} \int_{T_1}^{T_2} (1+t)^\mu \parallel u'(t) \parallel_V^p dt +$$

$$\frac{1}{2}\langle \tilde{Q}[u(T_2) - u_\infty], u(T_2) - u_\infty \rangle - \frac{1}{2}\langle \tilde{Q}[u(T_1) - u_\infty], u(T_1) - u_\infty \rangle \leq$$

$$\text{const} \left[\int_{T_1}^{T_2} \Phi_1(t) dt + \int_{T_1}^{T_2} \Phi(t) dt \right].$$

Since $\tilde{Q} : V \to V^\star$is a continuous and linear operator, by (4.30)

$$\lim_{T_1, T_2 \to \infty} \{ \langle \tilde{Q}[u(T_2) - u_\infty], u(T_2) - u_\infty \rangle - \langle \tilde{Q}[u(T_1) - u_\infty], u(T_1) - u_\infty \rangle \} = 0,$$

thus (4.25) and $\Phi_1, \Phi \in L^1(0, \infty)$ imply

$$\lim_{T_1, T_2 \to \infty} [y(T_2) - y(T_1)] = 0.$$

Consequently, $\lim_{T \to \infty} y(T)$ exists and is finite, further, by the first estimation in (4.25) it must be 0, i.e. we have (4.24), which completes the proof of Theorem 4.3.

Remark The example in Section 2 satisfies the assumptions of Theorem 4.2 if

$$0 < c_2 \leq b(t, x, \theta) \leq B(T) < \infty, \quad 0 < c_2 \leq b_0(t, x, \theta) \leq B(T) < \infty, \quad t \in [0, T]$$

for all $T > 0$ and

$$|\hat{b}(t, x, \theta_1, \theta_2)| \leq \text{const}, \quad |\alpha(t, x, u, w)| \leq \Phi_1(t), \quad t \in (0, \infty).$$

Further, N satisfies the assumptions of Theorem 4.3 if

$$\text{const}(1+t)^\mu \leq b(t, x, \theta), \quad \text{const}(1+t)^\mu \leq b_0(t, x, \theta), \quad t \in (0, \infty)$$

is satisfied, too (with some positive constant).

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(Received July 31, 2011)