Multiple solutions for a fractional p-Kirchhoff system with singular nonlinearity

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Abstract. This paper examines a class of fractional p-Kirchhoff systems driven by a nonlocal integro-differential operator with singular nonlinearity. By making use of Nehari manifold techniques, the existence of two nontrivial solutions is established. Our results extend those in Xiang et al. [*Nonlinearity* **29**(2016), 3186–3205] for the corresponding subcritical case.

Keywords: p-Kirchhoff system, singular nonlinearity, Nehari manifold.

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1 Introduction

We look for nontrivial solutions of the following fractional p-Kirchhoff system

$$\begin{cases} \left(\sum\limits_{i=1}^{k}[u_i]_{s,p}^p\right)^{\theta-1}(-\Delta)_p^s u_j(x) = \lambda_j|u_j|^{q-2}u_j + \sum\limits_{i\neq j}\beta_{ij}|u_i|^{1-m}|u_j|^{-m} & \text{in } \Omega, \\ u_j = 0 & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases} \tag{1.1}$$

where

$$[u_j]_{s,p} = \left(\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u_j(x)-u_j(y)|^p}{|x-y|^{N+ps}}dxdy\right)^{\frac{1}{p}}, \qquad j=1,2,\ldots,k,\ k\geq 2,$$

$\theta\geq 1$, $N>ps$ with $s\in(0,1)$, $0<m<1$, $0<2-2m<\theta p<q<p_s^*=\frac{Np}{N-sp}$, $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $\lambda_j>0$ is a parameter, $\beta_{ij}>0$ for all $1\leq i<j\leq k$, $\beta_{ij}=\beta_{ji}$ for $i\neq j$, $j=1,2,\ldots,k$, and $(-\Delta)_p^s$ is the fractional p-Laplace operator which may be defined along any $v\in C_0^\infty(\mathbb{R}^N)$ as

$$(-\Delta)_p^s v = 2\lim_{\delta\to 0^+}\int_{\mathbb{R}^N\setminus B_\delta(x)}\frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{N+ps}}dy \quad \text{for } x\in\mathbb{R}^N,$$

where $B_\delta(x)$ denotes the ball in $\mathbb{R}^N$ of radius δ centered at x. For more details on the fractional p-Laplacian, we can see [8] and the references therein.

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In [9], a steady-state Kirchhoff variational model in bounded regular domains of $\mathbb{R}^N$ was proposed by Fiscella and Valdinoci. In fact, problem (1.1) is a fractional version of Kirchhoff model. Specifically, Kirchhoff proposed the following model

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L} \|\nabla u\|_{L^2([0,L])}^2 \right) \frac{\partial^2 u}{\partial x^2} = f(x, u), \tag{1.2}$$

where ρ, p_0, h, E, L are constants. As we all know, this model extends the classical D'Alambert wave equation. Set $M(y) = p_0/h + (E(2L))y$ with $y \geq 0$. If $M(0) = 0$, we call problem (1.2) degenerate, otherwise, it is called non-degenerate if $M(0) > 0$. For $M(0) = 0$, it has a very important physical significance, that is, the base tension of the string is equal to zero. Clearly, in this paper, we are concerned about the situation of degradation in the fractional p-Laplacian setting. We refer the interested reader to [2,6,12,13] for some related results.

In recent years, with the application of nonlocal operators in real life or engineering fields becoming more and more obvious, such as bridge survey, population model, image processing, etc., the fractional Laplacian operator has received extensive attention. Most recently, Sousa in [14] studied a class of fractional p-Laplacian differential operators with variable exponents. The author obtained the existence of a positive solution for the investigated fractional system of the Kirchhoff type by using the method of sub- and super- solutions, via technical assumptions on the nonlinearity. In [19] Zuo et al. considered a variational approach based on the scaling function method to solve optimization problems. Precisely, in [18] Zhao et al. studied a p-fractional Schrödinger–Kirchhoff equation with electromagnetic fields and the Hardy–Littlewood–Sobolev nonlinearity. They used the concentration-compactness principles and improved techniques to obtain Palais–Smile condition at level c. By variational methods, they obtained the existence and multiplicity of solutions. For more literature about the results for nonlocal fractional Laplacian operators and related nonlocal integro-differential equations, we can also refer to [1,7,17] and the references therein.

On the other hand, there are a lot of literature on the equation or system with singular nonlinearity. Consider the following semilinear problem

$$\begin{cases} (-\Delta)^s u = \lambda k(x) u^{-\gamma} + M u^q & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, u > 0 & \text{in } \Omega, \end{cases}$$

where $n > 2s$, $M \geq 0$, $0 < s < 1$, $\gamma > 0$, $\lambda > 0$, $1 < q < 2_s^* - 1$. The weights $k : \Omega \to \mathbb{R}$ are assumed to be nonnegative and (essentially) bounded. In [3], the authors studied the existence of distributional solutions for small λ using the uniform estimates of $\{u_n\}$ which are solutions of the regularized problems with singular term $u^{-\gamma}$ replaced by $(u + \frac{1}{m})^{-\gamma}$. This was extended for the p-fractional Laplace operator by Canino et al. in [5]. Assuming $0 < \gamma < 1$, Ghanmi and Saoudi [10] studied the existence of at least two solutions for singular equations with a positively homogeneous function by making use of variational methods. For fractional Laplacian system involving singular nonlinearity, the work [11] dealt with

$$\begin{cases} (-\Delta)^s u = \lambda a(x)|u|^{q-2}u + \frac{1-\alpha}{2-\alpha-\beta}c(x)|u|^{-\alpha}|v|^{1-\beta} & x \in \Omega, \\ (-\Delta)^s v = \mu b(x)|v|^{q-2}u + \frac{1-\alpha}{2-\alpha-\beta}c(x)|u|^{1-\alpha}|v|^{-\beta} & x \in \Omega, \\ u = v = 0 & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\lambda, \mu \in (0, \infty)$, $0 < \alpha, \beta < 1$, $N > 2s$, $1 < q < 2 < 2_s^* = \frac{2N}{N-2s}$, $s \in (0,1)$, and $a, b, c \in C(\overline{\Omega})$ are nonnegative functions. With the help of Nehari manifold, the authors obtained two nontrivial solutions to this system.

Inspired by above papers, the main purpose of this paper is to extend the following work [16]

$$
\begin{cases}
\left(\sum_{i=1}^{k} [u_i]_{s,p}^{p} \right)^{\theta-1} (-\Delta)_p^s u_j(x) = \lambda_j |u_j|^{q-2} u_j + \sum_{i \neq j} \beta_{ij} |u_i|^m |u_j|^{m-2} u_j & \text{in } \Omega, \\
u_j = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\tag{1.3}
$$

In [16], when $1 < q < \theta p < 2m < p_s^*$, the authors obtained two distinct solutions to system (1.3). We try to study whether it is possible to get similar result when replacing $\sum_{i \neq j} \beta_{ij} |u_i|^{1-m} |u_j|^{-m}$ in the place of $\sum_{i \neq j} \beta_{ij} |u_i|^m |u_j|^{m-2} u_j$. The main difficulties in dealing with this problem come from the singular nonlinearity, i.e. $0 < m < 1$. To our best knowledge, our result for the fractional p-Kirchhoff system with singular nonlinearity is new.

Before describing main result, we recall some necessary definitions. For convenience, we denote by $|u|_r := \|u\|_{L^r(\mathbb{R}^N)}$ the norm of Lebesgue space $L^r(\Omega)$ with $r \geq 1$. Define $W^{s,p}(\Omega)$ as a linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function u in $W^{s,p}(\Omega)$ belongs to $L^p(\Omega)$ and

$$
\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy < \infty.
$$

Equip $W^{s,p}(\Omega)$ with the norm

$$
\|u\|_{W^{s,p}(\Omega)} = |u|_p + \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy \right)^{\frac{1}{p}}.
$$

Obviously, $W^{s,p}(\Omega)$ is a Banach space. We shall consider the following closed linear subspace

$$
W_0^{s,p}(\Omega) = \left\{ u \in W^{s,p}(\Omega) : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.
$$

Moreover, we have that

$$
\|u_j\|_{W_j} = \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_j(x) - u_j(y)|^p}{|x - y|^{N+ps}} dx dy \right)^{\frac{1}{p}}
$$

is an equivalent norm of $W_j = W_0^{s,p}(\Omega)$. It follows from the fractional Sobolev inequality that

$$
S = \inf_{u_j \in W_j} \left(\frac{\|u_j\|_{W_j}}{|u_j|_{p_s^*}} \right)^p.
\tag{1.4}
$$

In this paper we will work in the reflexive Banach space $W = W_1 \times \cdots \times W_k$ endowed with the norm

$$
\|\mathbf{u}\|_W = \left(\|u_1\|_{W_1}^p + \cdots + \|u_k\|_{W_k}^p \right)^{\frac{1}{p}}, \qquad \forall \mathbf{u} = (u_1, \ldots, u_k) \in W.
$$

The variational functional of system (1.1) is

$$
J(\mathbf{u}) = \frac{1}{\theta p} \|\mathbf{u}\|_W^{\theta p} - \frac{1}{q} \sum_{j=1}^{k} \lambda_j |u_j|_q^q - \frac{1}{1-m} \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} |u_i u_j|_{1-m}^{1-m},
\tag{1.5}
$$

for $\mathbf{u} = (u_1, \ldots, u_k) \in W$. Note that $J \notin C^1(W, \mathbb{R})$, and classical variational methods are not applicable. Moreover, we say that function $\mathbf{u} = (u_1, \ldots, u_k) \in W$ is a weak solution of system (1.1), if

$$\|\mathbf{u}\|_W^{(\theta-1)p} \sum_{j=1}^{k} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_j(x) - u_j(y)|^{p-2}(u_j(x) - u_j(y))(w_j(x) - w_j(y))}{|x-y|^{n+ps}} dxdy$$

$$= \sum_{j=1}^{k} \lambda_j \int_{\Omega} |u_j|^{q-2} u_j w_j dx + \sum_{j=1}^{k} \sum_{i \neq j} \beta_{ij} \int_{\Omega} |u_i|^{1-m} |u_j|^{-m} w_j dx,$$

for any $\mathbf{w} = (w_1, \ldots, w_k) \in W$. It is easy to see that solutions of system (1.1) correspond to the critical points of J.

Set

$$\Lambda = \frac{\theta p - 2 + 2m}{q - \theta p} \left(\frac{q - 2 + 2m}{q - \theta p} \right)^{\frac{2-2m-q}{\theta p - 2 + 2m}} \left(\sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \right)^{\frac{\theta p - q}{\theta p - 2 + 2m}} |\Omega|^{\frac{(2m-2+q)(\theta p - p_s^*)}{p_s^*(\theta p - 2 + 2m)}} S^{\frac{\theta(2m-2+q)}{\theta p - 2 + 2m}},$$

$$\Lambda_0 = \left(\frac{\theta p}{2 - 2m} \right)^{\frac{\theta p - q}{\theta p - 2 + 2m}} \Lambda,$$

$$\Theta_\Lambda = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_k) \in (\mathbb{R}^+)^k : 0 < \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}} \right)^{\frac{\theta p - q}{\theta p}} < \Lambda \right\}$$

and

$$\Theta_{\Lambda_0} = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_k) \in (\mathbb{R}^+)^k : 0 < \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}} \right)^{\frac{\theta p - q}{\theta p}} < \Lambda_0 \right\}.$$

Obviously, $\Theta_{\Lambda_0} \subset \Theta_\Lambda$. Our main result is the following.

Theorem 1.1. *Suppose that $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Theta_{\Lambda_0}$. Then system (1.1) has two distinct solutions.*

The remainder of this paper is organized as follows. In Section 2, we state some preliminary results. Section 3 is devoted to the proof of Theorem 1.1.

2 Preliminaries

In this section, we state some basic results. Define the constraint set (Nehari maniflod)

$$\mathcal{N} = \{\mathbf{u} \in W \backslash \{0\} : \langle J'(\mathbf{u}), \mathbf{u} \rangle = 0\}.$$

Thus, $\mathbf{u} \in \mathcal{N}$ if and only if

$$\|\mathbf{u}\|_W^{\theta p} = \sum_{j=1}^{k} \lambda_j \int_{\Omega} |u_j|^q dx + 2 \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_{\Omega} |u_i u_j|^{1-m} dx. \tag{2.1}$$

Fix $\mathbf{u} \in W$ and define the function of the form $K_\mathbf{u} : t \to J(t\mathbf{u})$ for $t > 0$. Such maps are famous fibering maps, which were discussed by Brown and Wu in [4]. Precisely,

$$K_\mathbf{u}(t) = J(t\mathbf{u}) = \frac{t^{\theta p}}{\theta p} \|\mathbf{u}\|_W^{\theta p} - \frac{t^q}{q} \sum_{j=1}^{k} \lambda_j \int_{\Omega} |u_j|^q dx - \frac{t^{2-2m}}{1-m} \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_{\Omega} |u_i u_j|^{1-m} dx.$$

Therefore,

$$K'_{\mathbf{u}}(t) = t^{\theta p-1}\|\mathbf{u}\|_W^{\theta p} - t^{q-1}\sum_{j=1}^k \lambda_j \int_\Omega |u_j|^q dx - 2t^{1-2m}\sum_{j=1}^k\sum_{i<j}\beta_{ij}\int_\Omega |u_i u_j|^{1-m}dx$$

and

$$K''_{\mathbf{u}}(t) = (\theta p - 1)t^{\theta p-2}\|\mathbf{u}\|_W^{\theta p} - (q-1)t^{q-2}\sum_{j=1}^k \lambda_j \int_\Omega |u_j|^q dx$$

$$- 2(1-2m)t^{-2m}\sum_{j=1}^k\sum_{i<j}\beta_{ij}\int_\Omega |u_i u_j|^{1-m}dx.$$

Lemma 2.1. *Let $\mathbf{u} \in W\backslash\{0\}$ and $t > 0$. Then $t\mathbf{u} \in \mathcal{N}$ if and only if $K'_{\mathbf{u}}(t) = 0$.*

Proof. Note that

$$tK'_{\mathbf{u}}(t) = \|t\mathbf{u}\|_W^{\theta p} - \sum_{j=1}^k \lambda_j \int_\Omega |tu_j|^q dx - 2\sum_{j=1}^k\sum_{i<j}\beta_{ij}\int_\Omega |tu_i tu_j|^{1-m}dx.$$

By (2.1), we can easily draw the conclusion of the lemma. □

Using methods similar to those used in [15], we split $\mathcal{N}$ into three sets. Accordingly, we define

$$\mathcal{N}^+ = \{t\mathbf{u} \in W : K'_{\mathbf{u}}(t) = 0, K''_{\mathbf{u}}(t) > 0\} = \{\mathbf{u} \in \mathcal{N} : K''_{\mathbf{u}}(1) > 0\};$$
$$\mathcal{N}^- = \{t\mathbf{u} \in W : K'_{\mathbf{u}}(t) = 0, K''_{\mathbf{u}}(t) < 0\} = \{\mathbf{u} \in \mathcal{N} : K''_{\mathbf{u}}(1) < 0\};$$
$$\mathcal{N}^0 = \{t\mathbf{u} \in W : K'_{\mathbf{u}}(t) = 0, K''_{\mathbf{u}}(t) = 0\} = \{\mathbf{u} \in \mathcal{N} : K''_{\mathbf{u}}(1) = 0\}.$$

In the next, we state some basic properties of submanifold.

Lemma 2.2. *Let $\mathbf{u}_0$ be a local minimizer for J such that $\mathbf{u}_0 \notin \mathcal{N}^0$. Then $\mathbf{u}_0$ is a critical point for J.*

Proof. Since $\mathbf{u}_0$ is a local minimizer of J on $\mathcal{N}$, it is a solution of the optimization problem

$$\text{minimize } J \text{ subject to } F(\mathbf{u}) = 0,$$

where

$$F(\mathbf{u}) = \|\mathbf{u}\|_W^{\theta p} - \sum_{j=1}^k \lambda_j \int_\Omega |u_j|^q dx - 2\sum_{j=1}^k\sum_{i<j}\beta_{ij}\int_\Omega |u_i u_j|^{1-m}dx.$$

Then, applying the theory of Lagrange multipliers, we can find a $\mu \in \mathbb{R}$ such that $J'(\mathbf{u}_0) = \mu F'(\mathbf{u}_0)$ which implies

$$0 = \langle J'(\mathbf{u}_0), \mathbf{u}_0\rangle = \mu\langle F'(\mathbf{u}_0), \mathbf{u}_0\rangle.$$

Further, from $\mathbf{u}_0 \in \mathcal{N}$ and $\mathbf{u}_0 \notin \mathcal{N}^0$ it is easy to know $\langle F'(\mathbf{u}_0), \mathbf{u}_0\rangle \neq 0$. So we obtain $\mu = 0$ and the proof is complete. □

Lemma 2.3. *The functional J is coercive and bounded below on $\mathcal{N}$.*

Proof. For any $\mathbf{u} \in \mathcal{N}$, by (1.4), (2.1), the Young and Hölder inequalities, we obtain

$$J(\mathbf{u}) = \left(\frac{1}{\theta p} - \frac{1}{q}\right) \|\mathbf{u}\|_W^{\theta p} - \left(\frac{1}{1-m} - \frac{2}{q}\right) \sum_{j=1}^k \sum_{i<j} \beta_{ij} |u_i u_j|_{1-m}^{1-m}$$

$$\geq \left(\frac{1}{\theta p} - \frac{1}{q}\right) \|\mathbf{u}\|_W^{\theta p} - \frac{1}{2}\left(\frac{1}{1-m} - \frac{2}{q}\right) \sum_{j=1}^k \sum_{i<j} \beta_{ij} \left(|u_i|_{2-2m}^{2-2m} + |u_j|_{2-2m}^{2-2m}\right)$$

$$\geq \left(\frac{1}{\theta p} - \frac{1}{q}\right) \|\mathbf{u}\|_W^{\theta p} - \frac{1}{2}\left(\frac{1}{1-m} - \frac{2}{q}\right) \sum_{j=1}^k \sum_{i<j} \beta_{ij} |\Omega|^{\frac{p_s^*+2m-2}{p_s^*}} S^{\frac{2m-2}{p}} \|\mathbf{u}\|_W^{2-2m},$$

which together with $2 - 2m < \theta p$ yields that J is coercive and bounded below on $\mathcal{N}$. $\square$

Set

$$I_{\mathbf{u}}(t) = t^{\theta p - q} \|\mathbf{u}\|_W^{\theta p} - 2t^{2-2m-q} \sum_{j=1}^k \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^{1-m} dx.$$

Clearly, $t\mathbf{u} \in \mathcal{N}$ if and only if

$$I_{\mathbf{u}}(t) = \sum_{j=1}^k \lambda_j \int_\Omega |u_j|^q dx. \tag{2.2}$$

Moreover, $I_{\mathbf{u}}$ satisfies the following properties.

Lemma 2.4. *Suppose that $\mathbf{u} \in W\backslash\{0\}$. One has*

(i) the function $I_{\mathbf{u}}$ possesses a unique maximum at

$$t = t_{max} = \left(\frac{2(q-2+2m)\sum_{j=1}^k \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^{1-m} dx}{(q-\theta p)\|\mathbf{u}\|_W^{\theta p}}\right)^{\frac{1}{\theta p-2+2m}};$$

(ii) $I_{\mathbf{u}}'(t) > 0$ for $t \in (0, t_{max})$ and $I_{\mathbf{u}}'(t) < 0$ for $t \in (t_{max}, +\infty)$;

(iii) $\lim_{t\to 0^+} I_{\mathbf{u}}(t) = -\infty$, $\lim_{t\to+\infty} I_{\mathbf{u}}(t) = 0$.

Proof. Note that

$$I_{\mathbf{u}}'(t) = (\theta p - q)t^{\theta p - q - 1} \|\mathbf{u}\|_W^{\theta p} - 2(2-2m-q)t^{1-2m-q} \sum_{j=1}^k \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^{1-m} dx.$$

Set $I_{\mathbf{u}}'(t) = 0$. Obviously, $I_{\mathbf{u}}'(t_{max}) = 0$ and $I_{\mathbf{u}}''(t_{max}) < 0$, with unique

$$t_{max} = \left(\frac{2(q-2+2m)\sum_{j=1}^k \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^{1-m} dx}{(q-\theta p)\|\mathbf{u}\|_W^{\theta p}}\right)^{\frac{1}{\theta p-2+2m}}.$$

Moreover, it is easy to see that (ii) and (iii) follow from the structure of $I_{\mathbf{u}}$. $\square$

Lemma 2.5. *Suppose that $t\mathbf{u} \in \mathcal{N}$. Then $t\mathbf{u} \in \mathcal{N}^+$ or $(\mathcal{N}^-)$ if and only if $I_u'(t) > 0$ or (< 0).*

Proof. If $t\mathbf{u} \in \mathcal{N}$, by (2.1), we get

$$K''_{\mathbf{u}}(t) = (\theta p - q)t^{\theta p - 2}\|\mathbf{u}\|_W^{\theta p} - 2(2 - 2m - q)t^{-2m}\sum_{j=1}^k\sum_{i<j}\beta_{ij}\int_\Omega|u_iu_j|^{1-m}dx.$$

Note that

$$t^{1-q}I'_{\mathbf{u}}(t) = K''_{\mathbf{u}}(t),$$

which yields that $t\mathbf{u} \in \mathcal{N}^+$ or $(\mathcal{N}^-)$ if and only if $I'_u(t) > 0$ or (< 0). $\square$

Lemma 2.6. *Suppose that* $\mathbf{u} \in W\backslash\{0\}$*. Then for* $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Theta_\Lambda$*, there exist* $t^+, t^- > 0$ *such that* $t^+ < t_{max} < t^-$*,* $t^+\mathbf{u} \in \mathcal{N}^+$*,* $t^-\mathbf{u} \in \mathcal{N}^-$ *and*

$$J(t^+\mathbf{u}) = \inf_{0\leq t\leq t_{max}}J(t\mathbf{u}), \qquad J(t^-\mathbf{u}) = \sup_{t\geq 0}J(t\mathbf{u}).$$

Proof. By (1.4), the Young and Hölder inequalities, we have

$$I_{\mathbf{u}}(t_{max}) = \frac{\theta p - 2 + 2m}{q - \theta p}\left(\frac{q - 2 + 2m}{q - \theta p}\right)^{\frac{2-2m-q}{\theta p - 2 + 2m}}\frac{\left(2\sum_{j=1}^k\sum_{i<j}\beta_{ij}\int_\Omega|u_iu_j|^{1-m}dx\right)^{\frac{\theta p - q}{\theta p - 2 + 2m}}}{\left(\|\mathbf{u}\|_W^{\theta p}\right)^{\frac{2-2m-q}{\theta p - 2 + 2m}}}$$

$$\geq \frac{\theta p - 2 + 2m}{q - \theta p}\left(\frac{q - 2 + 2m}{q - \theta p}\right)^{\frac{2-2m-q}{\theta p - 2 + 2m}}\frac{\left(\sum_{j=1}^k\sum_{i<j}\beta_{ij}|\Omega|^{\frac{p_s^*+2m-2}{p_s^*}}S^{\frac{2m-2}{p}}\|\mathbf{u}\|_W^{2-2m}\right)^{\frac{\theta p - q}{\theta p - 2 + 2m}}}{\left(\|\mathbf{u}\|_W^{\theta p}\right)^{\frac{2-2m-q}{\theta p - 2 + 2m}}}$$

$$= \frac{\theta p - 2 + 2m}{q - \theta p}\left(\frac{q - 2 + 2m}{q - \theta p}\right)^{\frac{2-2m-q}{\theta p - 2 + 2m}}\left(\sum_{j=1}^k\sum_{i<j}\beta_{ij}|\Omega|^{\frac{p_s^*+2m-2}{p_s^*}}S^{\frac{2m-2}{p}}\right)^{\frac{\theta p - q}{\theta p - 2 + 2m}}\|\mathbf{u}\|_W^q.$$

It follows from $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Theta_\Lambda$ that

$$0 < \sum_{j=1}^k\lambda_j\int_\Omega|u_j|^qdx \leq |\Omega|^{\frac{p_s^*-q}{p_s^*}}\sum_{j=1}^k\lambda_j|u_j|_{p_s^*}^q$$

$$\leq |\Omega|^{\frac{p_s^*-q}{p_s^*}}S^{-\frac{q}{p}}\sum_{j=1}^k\lambda_j\|u_j\|_{W_j}^q \leq |\Omega|^{\frac{p_s^*-q}{p_s^*}}S^{-\frac{q}{p}}\left(\sum_{j=1}^k\lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}}\left(\sum_{j=1}^k\|u_j\|_{W_j}^{\theta p}\right)^{\frac{q}{\theta p}}$$

$$\leq |\Omega|^{\frac{p_s^*-q}{p_s^*}}S^{-\frac{q}{p}}\left(\sum_{j=1}^k\lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}}\|\mathbf{u}\|_W^q < I_{\mathbf{u}}(t_{max}),$$

which implies that there exist $t^+, t^- > 0$ such that $t^+ < t_{max} < t^-$,

$$I_{\mathbf{u}}(t^+) = \sum_{j=1}^k\lambda_j\int_\Omega|u_j|^qdx = I_{\mathbf{u}}(t^-),$$

$I'_{\mathbf{u}}(t^+) > 0$ and $I'_{\mathbf{u}}(t^-) < 0$. Then, by (2.2) and Lemma 2.5, we obtain $t^+\mathbf{u} \in \mathcal{N}^+$ and $t^-\mathbf{u} \in \mathcal{N}^-$. Combining Lemma 2.4 and

$$K'_{\mathbf{u}}(t) = t^{q-1}\left(I_{\mathbf{u}}(t) - \sum_{j=1}^k\lambda_j\int_\Omega|u_j|^qdx\right),$$

we get that $J(t\mathbf{u})$ is strictly decreasing on $(0, t^+)$, strictly increasing on (t^+, t^-) and strictly decreasing on $(t^-, +\infty)$. Hence

$$J(t^+\mathbf{u}) = \inf_{0 \le t \le t_{max}} J(t\mathbf{u}), \qquad J(t^-\mathbf{u}) = \sup_{t \ge 0} J(t\mathbf{u}).$$

The proof is completed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 2.7. *Suppose that* $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Theta_\Lambda$. *Then* $\mathcal{N}^0 = \emptyset$.

Proof. Arguing by contradiction, we assume that $\mathcal{N}^0 \neq \emptyset$. Then for $\mathbf{u} \in \mathcal{N}^0$, we have by (2.1) that

$$0 = K''_{\mathbf{u}}(1) = (\theta p - q)\|\mathbf{u}\|_W^{\theta p} - 2(2 - 2m - q) \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^{1-m} dx$$

$$= (\theta p + 2m - 2)\|\mathbf{u}\|_W^{\theta p} - (q + 2m - 2) \sum_{j=1}^{k} \lambda_j \int_\Omega |u_j|^q dx.$$

Hence, by (1.4), the Young and Hölder inequalities, we get

$$\|\mathbf{u}\|_W^{\theta p} = \frac{2(2 - 2m - q)}{\theta p - q} \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^{1-m} dx$$

$$\le \frac{2 - 2m - q}{\theta p - q} \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} |\Omega|^{\frac{p_s^* + 2m - 2}{p_s^*}} S^{\frac{2m-2}{p}} \|\mathbf{u}\|_W^{2-2m},$$

which implies that

$$\|\mathbf{u}\|_W \le \left(\frac{2 - 2m - q}{\theta p - q} \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} |\Omega|^{\frac{p_s^* + 2m - 2}{p_s^*}} S^{\frac{2m-2}{p}} \right)^{\frac{1}{\theta p - 2 + 2m}}. \tag{2.3}$$

Moreover, by (1.4) and the Hölder inequality, we get

$$\|\mathbf{u}\|_W^{\theta p} = \frac{q + 2m - 2}{\theta p + 2m - 2} \sum_{j=1}^{k} \lambda_j \int_\Omega |u_j|^q dx$$

$$\le \frac{q + 2m - 2}{\theta p + 2m - 2} |\Omega|^{\frac{p_s^* - q}{p_s^*}} S^{-\frac{q}{p}} \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}} \right)^{\frac{\theta p - q}{\theta p}} \|\mathbf{u}\|_W^q,$$

which implies that

$$\|\mathbf{u}\|_W \ge \left(\frac{q + 2m - 2}{\theta p + 2m - 2} |\Omega|^{\frac{p_s^* - q}{p_s^*}} S^{-\frac{q}{p}} \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}} \right)^{\frac{\theta p - q}{\theta p}} \right)^{\frac{1}{\theta p - q}}. \tag{2.4}$$

Combining (2.3) and (2.4), we obtain

$$\left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}} \right)^{\frac{\theta p - q}{\theta p}}$$

$$\ge \frac{\theta p - 2 + 2m}{q - \theta p} \left(\frac{q - 2 + 2m}{q - \theta p} \right)^{\frac{2 - 2m - q}{\theta p - 2 + 2m}} \left(\sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \right)^{\frac{\theta p - q}{\theta p - 2 + 2m}} |\Omega|^{\frac{(2m - 2 + q)(\theta p - p_s^*)}{p_s^*(\theta p - 2 + 2m)}} S^{\frac{\theta(2m - 2 + q)}{\theta p - 2 + 2m}},$$

which contradicts

$$0 < \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}} \right)^{\frac{\theta p - q}{\theta p}} < \Lambda.$$

This ends the proof. □

3 Proof of Theorem 1.1

By Lemmas 2.3 and 2.7, for $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Theta_\Lambda$, we obtain $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$ and J is bounded from below on $\mathcal{N}^+$ and $\mathcal{N}^-$. Set

$$\alpha^+ = \inf_{\mathbf{u} \in \mathcal{N}^+} J(\mathbf{u}) \quad \text{and} \quad \alpha^- = \inf_{\mathbf{u} \in \mathcal{N}^-} J(\mathbf{u}).$$

Lemma 3.1. $\alpha^+ < 0$.

Proof. For $\mathbf{u} \in \mathcal{N}^+$, we have $K'_{\mathbf{u}}(1) = 0$ and $K''_{\mathbf{u}}(1) > 0$. Then

$$(\theta p - q)\|\mathbf{u}\|_W^{\theta p} > 2(2 - 2m - q) \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^{1-m} dx.$$

This yields that

$$
\begin{aligned}
J(\mathbf{u}) &= \left(\frac{1}{\theta p} - \frac{1}{q} \right) \|\mathbf{u}\|_W^{\theta p} - \left(\frac{1}{1-m} - \frac{2}{q} \right) \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} |u_i u_j|_{1-m}^{1-m} \\
&\leq \left[\left(\frac{1}{\theta p} - \frac{1}{q} \right) - \left(\frac{1}{1-m} - \frac{2}{q} \right) \frac{q - \theta p}{2(q - 2 + 2m)} \right] \|\mathbf{u}\|_W^{\theta p} \\
&= \frac{(\theta p - q)(\theta p - 2 + 2m)}{(2 - 2m)q\theta p} \|\mathbf{u}\|_W^{\theta p} < 0,
\end{aligned}
$$

due to $0 < 2 - 2m < \theta p < q$. Therefore $\alpha^+ < 0$ follows from the definition α^+. □

Lemma 3.2. *The minimization problem*

$$\alpha^+ = \inf_{\mathbf{u} \in \mathcal{N}^+} J(\mathbf{u})$$

is achieved at a point $\mathbf{u}^+ \in \mathcal{N}^+$.

Proof. Let $\{\mathbf{u}_n\}$ be a minimizing sequence of the minimization problem, i.e. $\{\mathbf{u}_n\} \subset \mathcal{N}^+$ and $\lim_{n\to\infty} J(\mathbf{u}_n) = \alpha^+$. By Lemma 2.3, it is easy to see that $\{\mathbf{u}_n\}$ is bounded, we can find a $\mathbf{u}^+$ such that $\mathbf{u}_n \rightharpoonup \mathbf{u}^+$ weakly in W, $\mathbf{u}_n \to \mathbf{u}^+$ strongly in $L^r(\Omega)$, $1 \leq r < p_s^*$. Now, we prove

$$\lim_{n\to\infty} \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |(\mathbf{u}_n)_i (\mathbf{u}_n)_j|^{1-m} dx = \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |(\mathbf{u}^+)_i (\mathbf{u}^+)_j|^{1-m} dx \qquad (3.1)$$

and

$$\lim_{n\to\infty} \sum_{j=1}^{k} \lambda_j \int_\Omega |(\mathbf{u}_n)_j|^q dx = \sum_{j=1}^{k} \lambda_j \int_\Omega |(\mathbf{u}^+)_j|^q dx. \qquad (3.2)$$

By the Vitali theorem, we claim that

$$\lim_{n\to\infty} \int_\Omega |(\mathbf{u}_n)_i (\mathbf{u}_n)_j|^{1-m} dx = \int_\Omega |(\mathbf{u}^+)_i (\mathbf{u}^+)_j|^{1-m} dx.$$

In fact, by the Young inequality, we have

$$\int_\Omega |(\mathbf{u}_n)_i(\mathbf{u}_n)_j|^{1-m}dx \leq \frac{1}{2}\int_\Omega |(\mathbf{u}_n)_i|^{2-2m}dx + \frac{1}{2}\int_\Omega |(\mathbf{u}_n)_j|^{2-2m}dx.$$

By the Sobolev embedding theorem and boundedness of $\{(\mathbf{u}_n)_i\}$, we can find a constant $C > 0$ such that $|(\mathbf{u}_n)_i|_{p_s^*} \leq C$. Moreover, it follows from the Hölder inequality that

$$\int_\Omega |(\mathbf{u}_n)_i|^{2-2m}dx \leq |\Omega|^{\frac{p_s^*+2m-2}{p_s^*}}|(\mathbf{u}_n)_i|_{p_s^*}^{2-2m}. \tag{3.3}$$

From (3.3), for every $\epsilon > 0$, setting

$$\delta = \left(\frac{\epsilon}{C^{2-2m}}\right)^{\frac{p_s^*}{p_s^*+2m-2}},$$

when $A \subset \Omega$ with meas $A < \delta$, we obtain

$$\int_A |(\mathbf{u}_n)_i|^{2-2m}dx \leq |A|^{\frac{p_s^*+2m-2}{p_s^*}}C^{2-2m} < \epsilon.$$

Similarly, $\int_A |(\mathbf{u}_n)_j|^{2-2m}dx < \epsilon$. This yields that

$$\left\{\int_\Omega |(\mathbf{u}_n)_i(\mathbf{u}_n)_j|^{1-m}dx, n \in \mathbb{N}\right\}$$

is equi-absolutely-continuous. Thus, our claim is true. This implies that (3.1) holds. On the other hand, for $1 \leq j \leq k$, it follows from the Hölder inequality and $\mathbf{u}_n \to \mathbf{u}^+$ strongly in $L^q(\Omega)$ that

$$\int_\Omega \left||(\mathbf{u}_n)_j|^q - |(\mathbf{u}^+)_j|^q\right|dx = q\int_\Omega (|(\mathbf{u}^+)_j| + \tau(|(\mathbf{u}_n)_j - (\mathbf{u}^+)_j|))^{q-1}|(\mathbf{u}_n)_j - (\mathbf{u}^+)_j|dx$$

$$\leq q|(\mathbf{u}_n)_j + (\mathbf{u}^+)_j|_q^{q-1}|(\mathbf{u}_n)_j - (\mathbf{u}^+)_j|_q$$

$$\leq C|(\mathbf{u}_n)_j - (\mathbf{u}^+)_j|_q \to 0,$$

as $n \to \infty$, where $\tau \in (0,1)$ and $C > 0$ denotes various constants. Therefore,

$$\lim_{n\to\infty}\int_\Omega \left||(\mathbf{u}_n)_j|^q - |(\mathbf{u}^+)_j|^q\right|dx = 0, \qquad \forall j \in \{1,2,,\ldots k\},$$

which implies that (3.2) holds. Furthermore, we can prove that $\mathbf{u}_n \to \mathbf{u}^+$ strongly in W. Arguing by contradiction, we assume $\mathbf{u}_n \nrightarrow \mathbf{u}^+$ strongly in W. Then,

$$\|\mathbf{u}^+\|_W^{\theta p} < \lim_{n\to\infty}\inf \|\mathbf{u}_n\|_W^{\theta p}.$$

By Lemma 2.6, there exists $t^+ > 0$ such that $t^+\mathbf{u}^+ \in \mathcal{N}^+$. Then, for $\mathbf{u}_n \in \mathcal{N}^+$, one has

$$\lim_{n\to\infty} K'_{\mathbf{u}_n}(t^+)$$

$$= \lim_{n\to\infty}\left((t^+)^{\theta p-1}\|\mathbf{u}_n\|_W^{\theta p} - (t^+)^{q-1}\sum_{j=1}^k \lambda_j\int_\Omega |(\mathbf{u}_n)_j|^q dx - 2(t^+)^{1-2m}\sum_{j=1}^k\sum_{i<j}\beta_{ij}\int_\Omega |(\mathbf{u}_n)_i(\mathbf{u}_n)_j|^{1-m}dx\right)$$

$$> (t^+)^{\theta p-1}\|\mathbf{u}^+\|_W^{\theta p} - (t^+)^{q-1}\sum_{j=1}^k \lambda_j\int_\Omega |(\mathbf{u}^+)_j|^q dx - 2(t^+)^{1-2m}\sum_{j=1}^k\sum_{i<j}\beta_{ij}\int_\Omega |(\mathbf{u}^+)_i(\mathbf{u}^+)_j|^{1-m}dx$$

$$= K'_{\mathbf{u}^+}(t^+) = 0.$$

This yields $K'_{\mathbf{u}_n}(t^+) > 0$ for large enough n. Note that $K'_{\mathbf{u}_n}(1) = 0$ for each n and $K'_{\mathbf{u}_n}(t) < 0$ for $t \in (0,1)$. It follows that $t^+ > 1$. Moreover, from that fact that $K_{\mathbf{u}^+}(t)$ is decreasing on $(0, t^+)$, we have

$$J(t^+\mathbf{u}^+) \leq J(\mathbf{u}^+) < \lim_{n\to\infty} J(\mathbf{u}_n) = \alpha^+,$$

which contradicts the fact that $\alpha^+ = \inf_{\mathbf{u}\in\mathcal{N}^+} J(\mathbf{u})$. Thus, we conclude that $\mathbf{u}_n \to \mathbf{u}^+$ strongly in W. By $K'_{\mathbf{u}_n}(1) = 0$ and $K''_{\mathbf{u}_n}(1) > 0$, we get that $K'_{\mathbf{u}^+}(1) = 0$ and $K''_{\mathbf{u}^+}(1) \geq 0$. Note that $\mathcal{N}^0 = \varnothing$. Then $K''_{\mathbf{u}^+}(1) > 0$, which implies $\mathbf{u}^+ \in \mathcal{N}^+$. Above all, by $J(\mathbf{u}^+) = \inf_{\mathbf{u}\in\mathcal{N}^+} J(\mathbf{u}) < 0$, $\mathbf{u}^+$ is a minimizer of J on $\mathcal{N}^+$. The proof is completed. $\qquad\square$

Lemma 3.3. *Suppose that* $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Theta_{\Lambda_0}$. *Then* $\alpha^- > \alpha_0$ *for some* $\alpha_0 > 0$.

Proof. For $\mathbf{u} \in \mathcal{N}^-$, we have $K'_{\mathbf{u}}(1) = 0$ and $K''_{\mathbf{u}}(1) < 0$. Then

$$(\theta p - 2 + 2m)\|\mathbf{u}\|_W^{\theta p} < (q - 2 + 2m) \sum_{j=1}^{k} \lambda_j \int_{\Omega} |u_j|^q dx.$$

By (1.4) and the Hölder inequality, we have

$$\sum_{j=1}^{k} \lambda_j \int_{\Omega} |u_j|^q dx \leq |\Omega|^{\frac{p_s^* - q}{p_s^*}} S^{-\frac{q}{p}} \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}} \right)^{\frac{\theta p - q}{\theta p}} \|\mathbf{u}\|_W^q.$$

Hence,

$$\|\mathbf{u}\|_W > \left[\frac{q - 2 + 2m}{\theta p - 2 + 2m} |\Omega|^{\frac{p_s^* - q}{p_s^*}} S^{-\frac{q}{p}} \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}} \right)^{\frac{\theta p - q}{\theta p}} \right]^{\frac{1}{\theta p - q}}. \tag{3.4}$$

Then we obtain

$$J(\mathbf{u}) = \left(\frac{1}{\theta p} - \frac{1}{q} \right) \|\mathbf{u}\|_W^{\theta p} - \left(\frac{1}{1-m} - \frac{2}{q} \right) \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} |u_i u_j|_{1-m}^{1-m}$$

$$\geq \left(\frac{1}{\theta p} - \frac{1}{q} \right) \|\mathbf{u}\|_W^{\theta p} - \frac{1}{2} \left(\frac{1}{1-m} - \frac{2}{q} \right) \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} |\Omega|^{\frac{p_s^* + 2m - 2}{p_s^*}} S^{\frac{2m-2}{p}} \|\mathbf{u}\|_W^{2-2m}$$

$$= \|\mathbf{u}\|_W^{2-2m} \left[\left(\frac{1}{\theta p} - \frac{1}{q} \right) \|\mathbf{u}\|_W^{\theta p - 2 + 2m} - \frac{1}{2} \left(\frac{1}{1-m} - \frac{2}{q} \right) \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} |\Omega|^{\frac{p_s^* + 2m - 2}{p_s^*}} S^{\frac{2m-2}{p}} \right]$$

$$> \|\mathbf{u}\|_W^{2-2m} \left\{ \left(\frac{1}{\theta p} - \frac{1}{q} \right) \left[\frac{q - 2 + 2m}{\theta p - 2 + 2m} |\Omega|^{\frac{p_s^* - q}{p_s^*}} S^{-\frac{q}{p}} \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}} \right)^{\frac{\theta p - q}{\theta p}} \right]^{\frac{\theta p - 2 + 2m}{\theta p - q}} \right.$$

$$\left. - \frac{1}{2} \left(\frac{1}{1-m} - \frac{2}{q} \right) \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} |\Omega|^{\frac{p_s^* + 2m - 2}{p_s^*}} S^{\frac{2m-2}{p}} \right\} \geq \alpha_0 > 0,$$

thanks to $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Theta_{\Lambda_0}$ and (3.4). $\qquad\square$

Lemma 3.4. *The minimization problem*

$$\alpha^- = \inf_{\mathbf{u}\in\mathcal{N}^-} J(\mathbf{u})$$

is achieved at a point $\mathbf{u}^- \in \mathcal{N}^-$.

Proof. Let $\{\mathbf{u}_n\}$ be a minimizing sequence of the minimization problem, i.e. $\{\mathbf{u}_n\} \subset \mathcal{N}^-$ and $\lim_{n\to\infty} J(\mathbf{u}_n) = \alpha^-$. By Lemma 2.3, it is easy to see that $\{\mathbf{u}_n\}$ is bounded, we can find a $\mathbf{u}^-$ such that $\mathbf{u}_n \rightharpoonup \mathbf{u}^-$ weakly in W, $\mathbf{u}_n \to \mathbf{u}^-$ strongly in $L^r(\Omega)$, $1 \leq r < p_s^*$. Similar to Lemma 3.2, we have

$$\lim_{n\to\infty} \sum_{j=1}^k \sum_{i<j} \beta_{ij} \int_\Omega |(\mathbf{u}_n)_i (\mathbf{u}_n)_j|^{1-m} dx = \sum_{j=1}^k \sum_{i<j} \beta_{ij} \int_\Omega |(\mathbf{u}^-)_i (\mathbf{u}^-)_j|^{1-m} dx$$

and

$$\lim_{n\to\infty} \sum_{j=1}^k \lambda_j \int_\Omega |(\mathbf{u}_n)_j|^q dx = \sum_{j=1}^k \lambda_j \int_\Omega |(\mathbf{u}^-)_j|^q dx.$$

Furthermore, we can prove that $\mathbf{u}_n \to \mathbf{u}^-$ strongly in W. Arguing by contradiction, we assume $\mathbf{u}_n \not\to \mathbf{u}^-$ strongly in W. Then,

$$\|\mathbf{u}^-\|_W^{\theta p} < \liminf_{n\to\infty} \|\mathbf{u}_n\|_W^{\theta p}.$$

By Lemma 2.6, there exists $t^- > 0$ such that $t^- \mathbf{u}^- \in \mathcal{N}^-$. Thus, since $\{\mathbf{u}_n\} \subset \mathcal{N}^-$ and $J(t\mathbf{u}_n) \leq J(\mathbf{u}_n)$, for all $t > 0$ we have

$$J(t^- \mathbf{u}^-) < \lim_{n\to\infty} J(t^- \mathbf{u}_n) \leq \lim_{n\to\infty} J(\mathbf{u}_n) = \alpha^-,$$

which contradicts the fact that $\alpha^- = \inf_{\mathbf{u}\in\mathcal{N}^-} J(\mathbf{u})$. Thus, we conclude that $\mathbf{u}_n \to \mathbf{u}^-$ strongly in W. By $K'_{\mathbf{u}_n}(1) = 0$ and $K''_{\mathbf{u}_n}(1) < 0$, we get that $K'_{\mathbf{u}^-}(1) = 0$ and $K''_{\mathbf{u}^-}(1) \leq 0$. Note that $\mathcal{N}^0 = \emptyset$. Then $K''_{\mathbf{u}^+}(1) < 0$, which implies $\mathbf{u}^- \in \mathcal{N}^+$. Above all, by $J(\mathbf{u}^-) = \inf_{\mathbf{u}\in\mathcal{N}^-} J(\mathbf{u})$, $\mathbf{u}^-$ is a minimizer of J on $\mathcal{N}^-$. The proof is completed. $\qquad\square$

Proof of Theorem 1.1. For all $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Theta_{\Lambda_0}$, by Lemmas 3.2 and 3.4, we conclude that there exist $\mathbf{u}^+ \in \mathcal{N}^+$ and $\mathbf{u}^- \in \mathcal{N}^-$ satisfying $J(\mathbf{u}^+) = \inf_{\mathbf{u}\in\mathcal{N}^+} J(\mathbf{u})$ and $J(\mathbf{u}^-) = \inf_{\mathbf{u}\in\mathcal{N}^-} J(\mathbf{u})$. In view of Lemma 2.2, $\mathbf{u}^+$ and $\mathbf{u}^-$ are two solutions of system (1.1). Moreover, since $J(\mathbf{u}^+) = J(|\mathbf{u}^+|)$ and $|\mathbf{u}^+| \in \mathcal{N}^+$ and similarly $J(\mathbf{u}^-) = J(|\mathbf{u}^-|)$ and $|\mathbf{u}^-| \in \mathcal{N}^+$, so we may assume $\mathbf{u}^\pm \geq 0$. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$, two solutions of system (1.1) are distinct. And by Lemmas 3.1 and 3.3, we have $J(\mathbf{u}^+) < 0$ and $J(\mathbf{u}^-) > 0$. Hence we provided the existence of two nontrivial nonnegative solutions to our system (1.1). $\qquad\square$

Acknowledgements

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