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On the logarithmic fractional Schrödinger–Poisson system with saddle-like potential

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Abstract. In this paper, we use variational methods to prove the existence of a positive solution for the following class of logarithmic fractional Schrödinger–Poisson system:

$$\begin{cases} \epsilon^{2s}\left(-\Delta\right)^s u + V(x)u - \phi(x)u = u\log u^2 & \text{in } \mathbb{R}^3, \\ \epsilon^{2t}\left(-\Delta\right)^t \phi = |u|^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\epsilon > 0$, $s,t \in (0,1)$, $(-\Delta)^\alpha$ is the fractional Laplacian and V is a saddle-like potential.

Keywords: fractional Schrödinger–Poisson system, logarithmic nonlinearity, variational methods.

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1 Introduction and main result

In this article, we consider the following fractional Schrödinger–Poisson system:

$$\begin{cases} \epsilon^{2s}\left(-\Delta\right)^s u + V(x)u - \phi(x)u = u\log u^2 & \text{in } \mathbb{R}^3, \\ \epsilon^{2t}\left(-\Delta\right)^t \phi = |u|^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\epsilon > 0$ is a small parameter, $s,t \in (0,1)$ and $(-\Delta)^\alpha$, with $\alpha \in \{s,t\}$, is the fractional Laplacian operator which may be defined for any $u : \mathbb{R}^3 \to \mathbb{R}$ belonging to the Schwartz class by

$$(-\Delta)^\alpha u(x) = C(3,\alpha) P.V. \int_{\mathbb{R}^3} \frac{u(x)-u(y)}{|x-y|^{3+2\alpha}} \mathrm{d}y \quad (x \in \mathbb{R}^3),$$

where P.V. stands for the Cauchy principal value and $C(3,\alpha)$ is a normalizing constant; see Di Nezza–Palatucci–Valdinoci [13]. In recent years, there has been a surge of interest in studying partial differential equations involving nonlocal fractional Laplace operators. This type of nonlocal operator comes up naturally in the real world in many different applications, such as

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phase transitions, game theory, finance, image processing, Lévy processes, and optimization. For more details and applications, we refer the interested reader to the works of Applebaum [6], Bahrouni–Rădulescu–Winkert [7], Caffarelli–Silvestre [9], Di Nezza–Palatucci–Valdinoci [13], Molica Bisci–Rădulescu–Servadei [21], Pucci–Xiang–Zhang [23,24] and their references.

In the fractional scenario, there are many results for the fractional Schrödinger–Poisson system. Teng [29] studied the existence of ground state solutions for the fractional Schrödinger–Poisson system with the critical Sobolev exponent. Yang–Yu–Zhao [31] were concerned with the existence and concentration behavior of ground state solutions for the fractional Schrödinger–Poisson system with critical nonlinearity. Ambrosio [5] used penalization techniques and Ljusternik–Schnirelmann theory to deal with the multiplicity and concentration of positive solutions for a fractional Schrödinger–Poisson type system with critical growth. Meng–Zhang–He [20] dealt with the existence of a positive and a sign-changing least energy solution for a class of fractional Schrödinger–Poisson system with critical growth and vanishing potentials. Finally, other interesting results in this direction can be found in the papers of Chen–Li–Peng [10], Ji [15], Murcia–Siciliano [22], Qu–He [25] and the references therein.

The case where potential V has a saddle-like geometry was considered in del Pino–Felmer–Miyagaki [12], essentially they assumed the potential V is bounded and $V \in C^2(\mathbb{R}^3)$, which verifies the following conditions:

Fix two subspaces $X, Y \subset \mathbb{R}^3$ such that $\mathbb{R}^3 = X \oplus Y$, then fix $c_0, c_1 > 0$ such that

$$c_0 = \inf_{z\in\mathbb{R}^3} V(z) > 0 \quad \text{and} \quad c_1 = \sup_{x\in X} V(x),$$

satisfying the following geometric condition

(V_1) There exists a number $\lambda \in (0,1)$, such that

$$c_0 = \inf_{R>0} \sup_{x\in\partial B_R(0)\cap X} V(x) < \inf_{y\in Y_\lambda} V(y).$$

where Y_λ is the cone about Y given by

$$Y_\lambda = \left\{ z \in \mathbb{R}^3 : |z\cdot y| > \lambda |z||y|, \text{ for some } y \in Y \right\}.$$

In addition to the above hypotheses, they imposed the conditions below:

(V_2) The functions $V, \frac{\partial V}{\partial x_i}, \frac{\partial^2 V}{\partial x_i \partial x_j}$ are bounded in $\mathbb{R}^3$, for all $i, j \in \{1,2,3\}$;

(V_3) V satisfies the Palais–Smale condition, that is, if $(x_n) \subset \mathbb{R}^3$, such that $(V(x_n))$ is limited and $\nabla V(x_n) \to 0$, then (x_n) possesses a convergent subsequence in $\mathbb{R}^3$.

Using the above conditions on V, and supposing that

$$c_1 < 2^{\frac{2(p-1)}{N+2-p(N-2)}} c_0,$$

the authors studied the existence of positive solutions for the following Schrödinger equation:

$$-\epsilon^2 \Delta u + V(z)u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

where $p \in (2, 2^*)$ if $N \geq 3$ and $p \in (2, +\infty)$ if $N = 1, 2$, for $\epsilon > 0$ small enough. After, Alves [1] showed the existence of a positive solution for the following elliptic equation with exponential critical growth in $\mathbb{R}^2$:

$$-\epsilon^2 \Delta u + V(z)u = f(u) \quad \text{in } \mathbb{R}^2,$$

Alves and Miyagaki [4] considered the following nonlinear fractional elliptic equation with critical growth in $\mathbb{R}^N$:

$$\epsilon^{2s} (-\Delta)^s u + V(z)u = \lambda |u|^{q-2} u + |u|^{2_s^*-2} u \quad \text{in } \mathbb{R}^N,$$

where $\lambda > 0$ is a positive parameter, $q \in (2, 2_s^*)$. Recently, under the same assumptions on the potential V, Alves and Ji [3] used the variational method to prove the existence of a positive solution for the following logarithmic Schrödinger equation:

$$-\epsilon^2 \Delta u + V(z)u = u \log u^2 \quad \text{in } \mathbb{R}^N.$$

Motivated by the above papers, in this work we consider the correlation result of the fractional Schrödinger–Poisson system. Now, we state the main result.

Theorem 1.1. *Suppose that V satisfies (V_1)–(V_3). If $V(0) > c_0$ and $c_1 < c_0 + 1$, then there exists $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0)$, the system (1.1) has a positive solution.*

Remark 1.2. Noting that in this paper we consider the nonlocal term ϕ with negative coefficient. We want to point out that if we deal with the positive nonlocal term, i.e.,

$$\begin{cases} \epsilon^{2s} (-\Delta)^s u + V(x)u + \phi(x)u = u \log u^2 & \text{in } \mathbb{R}^3, \\ \epsilon^{2t} (-\Delta)^t \phi = |u|^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

it is not easy to obtain the boundedness of Palais–Smale sequence (u_n). In fact, by the logarithmic Sobolev inequality, the key point is to prove the boundedness of (u_n), where the negative coefficient plays an important role, see Lemma 3.7. In contrast, if we study (1.2), the inequality may not necessarily hold true, then the boundedness of (u_n) fails to obtain. However, we believe system (1.2) is an interesting problem, we shall consider it further in our future work.

The paper is organized as follows. In Section 2, we recall some lemmas which we will use in the paper. In Section 3, we show some estimates and prove a technical result. In Section 4, we apply the deformation lemma to provide the proof of Theorem 1.1.

2 Preliminaries

If $A \subset \mathbb{R}^3$, we denote by $|u|_{L^q(A)}$ the $L^q(A)$-norm of a function $u : \mathbb{R}^3 \to \mathbb{R}$, and by $|u|_q$ its $L^q(\mathbb{R}^3)$-norm. Let us define $D^{s,2}(\mathbb{R}^3)$ as the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to

$$[u]^2 = \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \mathrm{d}x\mathrm{d}y.$$

Then, we consider the fractional Sobolev space

$$H^s(\mathbb{R}^3) = \left\{ u \in L^2\left(\mathbb{R}^3\right) : [u] < \infty \right\},$$

endowed with the norm

$$\|u\|^2 = [u]^2 + |u|_2^2.$$

Now, we recall the following main embeddings for the fractional Sobolev spaces, see Di Nezza–Palatucci–Valdinoci [13].

Lemma 2.1. *Let $s \in (0,1)$. Then $H^s(\mathbb{R}^3)$ is continuously embedded in $L^p(\mathbb{R}^3)$ for any $p \in [2, 2_s^*]$ and compactly in $L^p_{loc}(\mathbb{R}^3)$ for any $p \in [1, 2_s^*)$ with $2_s^* = \frac{6}{3-2s}$.*

We also recall a version of the well-known concentration-compactness principle, see Felmer–Quaas–Tan [14].

Lemma 2.2. *If (u_n) is a bounded sequence in $H^s(\mathbb{R}^3)$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^2 \, \mathrm{d}x = 0,$$

where $R > 0$, then $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for all $r \in (2, 2_s^*)$.

By Lemma 2.1, we have

$$H^s(\mathbb{R}^3) \subset L^{\frac{12}{3+2t}}(\mathbb{R}^3). \tag{2.1}$$

For any fixed $u \in H^s(\mathbb{R}^3)$, $L_u : D^{t,2}(\mathbb{R}^3) \to \mathbb{R}$ be the functional given by

$$L_u(v) = \int_{\mathbb{R}^3} u^2 v \mathrm{d}x,$$

which is continuous in view of the Hölder inequality and (2.1). Indeed

$$|L_u(v)| \le \left(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2t}} \mathrm{d}x \right)^{\frac{3+2t}{6}} \left(\int_{\mathbb{R}^3} |v|^{2_t^*} \mathrm{d}x \right)^{\frac{1}{2_t^*}} \le C\|u\|^2 \|v\|_{D^{t,2}},$$

where

$$\|v\|^2_{D^{t,2}} = \iint_{\mathbb{R}^6} \frac{|v(x) - v(y)|^2}{|x-y|^{3+2t}} \mathrm{d}x\mathrm{d}y.$$

Then, by the Lax–Milgram Theorem there is a unique $\phi_u^t \in D^{t,2}(\mathbb{R}^3)$, such that $\langle \phi_u^t, v \rangle$ for each $v \in D^{t,2}(\mathbb{R}^3)$, where $\langle \cdot, \cdot \rangle$ is the inner product on $D^{t,2}(\mathbb{R}^3)$. Thus, we obtain the t-Riesz formula

$$\phi_u^t(x) = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} \mathrm{d}y, \quad \text{where } c_t = \pi^{-\frac{3}{2}} 2^{-2t} \frac{\Gamma(3-2t)}{\Gamma(t)},$$

is the only weak solution of the problem

$$(-\Delta)^t \phi_u^t = u^2 \quad \text{in } \mathbb{R}^3.$$

Then, we state the following useful properties whose proofs can be found in Liu–Zhang [19] and Teng [29]:

Lemma 2.3. *For all $u \in H^s(\mathbb{R}^3)$, then the following properties hold:*

(1) $\|\phi_u^t\|_{D^{t,2}} \le C|u|^2_{\frac{12}{3+2t}} \le C\|u\|^2$ *and* $\int_{\mathbb{R}^3} \phi_u^t u^2 \mathrm{d}x \le C_t |u|^4_{\frac{12}{3+2t}}$. *Moreover* $\phi_u^t : H^s(\mathbb{R}^3) \to D^{t,2}(\mathbb{R}^3)$ *is continuous and maps bounded sets into bounded sets;*

(2) $\phi_u^t \ge 0$ *in* $\mathbb{R}^3$*;*

(3) *if* $y \in \mathbb{R}^3$ *and* $\bar{u}(x) = u(x+y)$*, then* $\phi^t_{\bar{u}}(x) = \phi^t_u(x+y)$ *and* $\int_{\mathbb{R}^3} \phi^t_{\bar{u}} \bar{u}^2 \mathrm{d}x = \int_{\mathbb{R}^3} \phi^t_u u^2 \mathrm{d}x$;

(4) $\phi^t_{ru} = r^2 \phi^t_u$ *for all* $r \in \mathbb{R}$;

(5) *if* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$*, then* $\phi^t_{u_n} \rightharpoonup \phi^t_u$ *in* $D^{t,2}(\mathbb{R}^3)$;

(6) *if* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$*, then* $\int_{\mathbb{R}^3} \phi^t_{u_n} u^2 \mathrm{d}x = \int_{\mathbb{R}^3} \phi^t_{(u_n-u)} (u_n-u)^2 \mathrm{d}x + \int_{\mathbb{R}^3} \phi^t_u u^2 \mathrm{d}x + o_n(1)$;

(7) *if* $u_n \to u$ *in* $H^s(\mathbb{R}^3)$*, then* $\phi^t_{u_n} \to \phi^t_u$ *in* $D^{t,2}(\mathbb{R}^3)$ *and* $\int_{\mathbb{R}^3} \phi^t_{u_n} u^2 \mathrm{d}x \to \int_{\mathbb{R}^3} \phi^t_u u^2 \mathrm{d}x$.

In order to study system (1.1), we use the change of variable $x \to \epsilon x$, and the system (1.1) is equivalent to the easier handle system

$$\begin{cases} (-\Delta)^s u + V(\epsilon x) u - \phi(\epsilon x) u = u \log u^2 & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = |u|^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.2}$$

Substituting $\phi^t = \phi^t_u$ into system (2.2), we can rewrite (2.2) as a single equation

$$(-\Delta)^s u + V(\epsilon x) u - \phi^t_u u = u \log u^2 \quad \text{in } \mathbb{R}^3. \tag{2.3}$$

We shall use the variational method to study the problem (2.3). Note that, a weak solution of (2.3) in $H^s(\mathbb{R}^3)$ is a critical point of the associated energy functional

$$\mathcal{I}_\epsilon(u) := \frac{1}{2}\|u\|^2_\epsilon - \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_u |u|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} u^2 \log u^2 \mathrm{d}x,$$

defined for all $u \in \mathcal{H}_\epsilon$ where

$$\mathcal{H}_\epsilon := \left\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\epsilon x) u^2 \mathrm{d}x < \infty \right\}$$

is endowed with the norm

$$\|u\|^2_\epsilon := [u]^2 + \int_{\mathbb{R}^3} (V(\epsilon x) + 1) u^2 \mathrm{d}x.$$

Obviously, $\mathcal{H}_\epsilon$ is a Hilbert space with inner product

$$(u,v)_\epsilon = \iint_{\mathbb{R}^6} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} (V(\epsilon x)+1) uv \mathrm{d}x.$$

Definition 2.4. A solution of the problem (2.3) is a function $u \in H^s(\mathbb{R}^3)$ such that $u^2 \log u^2 \in L^1(\mathbb{R}^3)$ and

$$\iint_{\mathbb{R}^6} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} V(\epsilon x) uv \mathrm{d}x - \int_{\mathbb{R}^3} \phi^t_u uv \mathrm{d}x = \int_{\mathbb{R}^3} uv \log u^2 \mathrm{d}x, \quad \forall u, v \in C^\infty_0(\mathbb{R}^3).$$

Due to the lack of smoothness of $\mathcal{I}_\epsilon$, we shall use the approach explored in Ji–Szulkin [16] and Squassina–Szulkin [26]. Let us decompose it into a sum of a C^1 functional plus a convex lower semicontinuous functional, respectively. For $\delta > 0$, let us define the following functions:

$$F_1(\xi) = \begin{cases} 0, & \text{if } \xi = 0, \\ -\frac{1}{2}\xi^2 \log \xi^2 & \text{if } 0 < |\xi| < \delta, \\ -\frac{1}{2}\xi^2 (\log \delta^2 + 3) + 2\delta|\xi| - \frac{1}{2}\delta^2, & \text{if } |\xi| \geq \delta \end{cases}$$

and

$$F_2(\xi) = \begin{cases} 0, & \text{if } |\xi| < \delta, \\ \frac{1}{2}\xi^2 \log\left(\xi^2/\delta^2\right) + 2\delta|\xi| - \frac{3}{2}\xi^2 - \frac{1}{2}\delta^2, & \text{if } |\xi| \geq \delta. \end{cases}$$

Then,

$$F_2(\xi) - F_1(\xi) = \frac{1}{2}\xi^2 \log \xi^2, \quad \forall \xi \in \mathbb{R},$$

and the functional $\mathcal{I}_\epsilon : \mathcal{H}_\epsilon \to (-\infty, +\infty]$ may be rewritten as

$$\mathcal{I}_\epsilon(u) = \Phi_\epsilon(u) + \Psi(u), \quad u \in \mathcal{H}_\epsilon, \tag{2.4}$$

where

$$\Phi_\epsilon(u) = \frac{1}{2}\|u\|_\epsilon^2 - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t |u|^2 \mathrm{d}x - \int_{\mathbb{R}^3} F_2(u)\mathrm{d}x,$$

and

$$\Psi(u) = \int_{\mathbb{R}^3} F_1(u)\mathrm{d}x,$$

As proven in Ji–Szulkin [16] and Squassina–Szulkin [26], $F_1, F_2 \in C^1(\mathbb{R}, \mathbb{R})$. If $\delta > 0$ is small enough, F_1 is convex, even,

$$F_1(\xi) \geq 0 \quad \text{and} \quad F_1'(\xi)\xi \geq 0, \quad \forall \xi \in \mathbb{R}.$$

For each fixed $p \in (2, 2_s^*)$, there exists $C > 0$ such that

$$|F_2'(\xi)| \leq C|\xi|^{p-1}, \quad \forall \xi \in \mathbb{R}.$$

If potential V in (2.3) is replaced by a constant $A > -1$, we have the following problem

$$(-\Delta)^s u + Au - \phi_u^t u = u \log u^2 \quad \text{in } \mathbb{R}^3. \tag{2.5}$$

And the corresponding energy functional associated to (2.5) will be denoted by $\mathcal{I}_A : \mathcal{H}_\epsilon \to (-\infty, +\infty]$ and defined as

$$\mathcal{I}_A(u) = \frac{1}{2}[u]^2 + \frac{1}{2}\int_{\mathbb{R}^3}(A+1)u^2\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3}\phi_u^t|u|^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} u^2 \log u^2 \mathrm{d}x.$$

Moreover, let us denote by $m(A)$ the mountain pas level associated with $\mathcal{I}_A$, which possesses the following characterizations

$$m(A) = \inf_{u \in \mathcal{H}_\epsilon \setminus \{0\}} \left\{ \max_{t \geq 0} \mathcal{I}_A(tu) \right\} = \inf_{u \in \mathcal{M}_A} \mathcal{I}_A(u),$$

where $\mathcal{M}_A$ is the Nehari Manifold associated with $\mathcal{I}_A$, given by

$$\mathcal{M}_A = \left\{ u \in \mathcal{H}_\epsilon \setminus \{0\} : \mathcal{I}_A'(u)u = 0 \right\}.$$

3 Technical results

In the section, we recall some definitions that can be found in Szulkin [28].

Definition 3.1. Let E be a Banach space, E' be the dual space of E and $\langle \cdot, \cdot \rangle$ be the duality paring between E' and E. Let $J : E \to \mathbb{R}$ be a functional of the form $J(u) = \Phi(u) + \Psi(u)$, where $\Phi \in C^1(E, \mathbb{R})$ and Ψ is convex and lower semicontinuous. Let us list some definitions:

(i) The sub-differential $\partial J(u)$ of the functional J at a point $u \in E$ is the following set

$$\left\{w \in E' : \left\langle \Phi'(u), v-u \right\rangle + \Psi(v) - \Psi(u) \geq \langle w, v-u \rangle, \forall v \in E\right\}; \tag{3.1}$$

(ii) A critical point of J is a point $u \in E$ such that $J(u) < +\infty$ and $0 \in \partial J(u)$, i.e.

$$\left\langle \Phi'(u), v-u \right\rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in E; \tag{3.2}$$

(iii) A Palais–Smale sequence at level d for J is a sequence $(u_n) \subset E$ such that $J(u_n) \to d$ and there exists a numerical sequence $\tau_n \to 0^+$ with

$$\left\langle \Phi'(u_n), v-u_n \right\rangle + \Psi(v) - \Psi(u_n) \geq -\tau_n \|v-u_n\|, \quad \forall v \in E;$$

(iv) The functional J satisfies the Palais–Smale condition at level d ($(\mathrm{PS})_d$ condition, for short) if all Palais–Smale sequences at level d have a convergent subsequence;

(v) The effective domain of J is the set $D(J) = \{u \in E : J(u) < +\infty\}$.

In what follows, for each $u \in D(\mathcal{I}_\epsilon)$, we set the functional $\mathcal{I}'_\epsilon(u) : \mathcal{H}_{\epsilon,c} \to \mathbb{R}$ given by

$$\left\langle \mathcal{I}'_\epsilon(u), z \right\rangle = \left\langle \Phi'_\epsilon(u), z \right\rangle - \int F'_1(u) z \mathrm{d}x, \quad \forall z \in \mathcal{H}_{\epsilon,c},$$

where

$$\mathcal{H}_{\epsilon,c} = \{u \in \mathcal{H}_\epsilon : u \text{ has compact support }\},$$

and define

$$\|\mathcal{I}'_\epsilon(u)\| = \sup\left\{\left\langle \mathcal{I}'_\epsilon(u), z \right\rangle : z \in \mathcal{H}_{\epsilon,c} \text{ and } \|z\|_\epsilon \leq 1\right\}.$$

If $\|\mathcal{I}'_\epsilon(u)\|$ is finite, then $\mathcal{I}'_\epsilon(u)$ may be extended to a bounded operator in $\mathcal{H}_\epsilon$, and so, it can be seen as an element of $\mathcal{H}'_\epsilon$.

Lemma 3.2. *Let $\mathcal{I}_\epsilon$ satisfy (2.4), then:*

(i) *If $u \in D(\mathcal{I}_\epsilon)$ is a critical point of $\mathcal{I}_\epsilon$. Then, the following hold:*

$$\left\langle \Phi'_\epsilon(u), v-u \right\rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in \mathcal{H}_\epsilon;$$

(ii) *For each $u \in D(\mathcal{I}_\epsilon)$ such that $\|\mathcal{I}'_\epsilon(u)\| < +\infty$, we have $\partial \mathcal{I}_\epsilon(u) \neq \emptyset$, that is, there exists $w \in \mathcal{H}'_\epsilon$, which is denoted by $w = \mathcal{I}'_\epsilon(u)$, such that*

$$\left\langle \Phi'_\epsilon(u), v-u \right\rangle + \int_{\mathbb{R}^3} F_1(v) \mathrm{d}x - \int_{\mathbb{R}^3} F_1(u) \mathrm{d}x \geq \langle w, v-u \rangle, \quad \forall v \in \mathcal{H}_\epsilon;$$

(iii) *If a function $u \in D(\mathcal{I}_\epsilon)$ is a critical point of $\mathcal{I}_\epsilon$, then u is a solution of (2.3);*

(iv) *If $(u_n) \subset \mathcal{H}_\epsilon$ is a Palais–Smale sequence, then*

$$\left\langle \mathcal{I}'_\epsilon(u_n), z \right\rangle = o_n(1) \|z\|_\epsilon, \quad \forall z \in \mathcal{H}_{\epsilon,c};$$

(v) *If Ω is a bounded domain with regular boundary, then Ψ (and hence $\mathcal{I}_\epsilon$) is of class C^1 in $H^s(\Omega)$. More precisely, the functional*

$$\Psi(u) = \int_\Omega F_1(u) \mathrm{d}x, \quad \forall u \in H^s(\Omega)$$

belongs to $C^1(H^s(\Omega), \mathbb{R})$.

Proof. (i) follows from (3.2). (ii) can be obtained arguing as in the proof of Squassina–Szulkin [27] and recalling that $C_c^\infty(\mathbb{R}^3)$ is dense in $\mathcal{H}_\epsilon$. (iii) and (iv) follow the same lines of the proofs of Ji–Szulkin [16]. To verify (v), since $|F_1'(\tau)| \le C(1+|\tau|^{q-1})$ with $q \in (2, 2_s^*)$, it is enough to proceed as in the proof of Willem [30]. □

As a consequence of the above proprieties, we have the following result.

Lemma 3.3. *If $u \in D(\mathcal{I}_\epsilon)$ and $\|\mathcal{I}_\epsilon'(u)\| < +\infty$, then $F_1'(u)u \in L^1(\mathbb{R}^3)$.*

Proof. Let $\varpi \in C_c^\infty(\mathbb{R}^3)$ be such that $0 \le \varpi \le 1$ in $\mathbb{R}^3$, $\varpi(x) = 1$ for $|x| \le 1$ and $\varpi(x) = 0$ for $|x| \ge 2$. For $R > 0$ and $u \in D(\mathcal{I}_\epsilon)$, let $\varpi_R(x) = \varpi(\frac{x}{R})$ and $u_R(x) = \varpi_R(x)u(x)$. Let us prove that

$$\lim_{R\to\infty} \|u_R - u\|_\epsilon = 0. \tag{3.3}$$

Clearly, $u_R \to u$ in $L^2(\mathbb{R}^3)$. On the other hand,

$$\begin{aligned}[u_R - u]^2 &\le 2\left[\iint_{\mathbb{R}^6} \frac{|\varpi_R(x) - \varpi_R(y)|^2}{|x-y|^{3+2s}}|u(x)|^2 \mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2s}}|\varpi_R(x) - 1|^2 \mathrm{d}x\mathrm{d}y\right] \\ &= 2\left[\mathcal{A}_R + \mathcal{B}_R\right].\end{aligned}$$

Since

$$\begin{aligned}\mathcal{A}_R &= \iint_{\mathbb{R}^6} \frac{|\varpi_R(x) - \varpi_R(y)|^2}{|x-y|^{3+2s}}|u(x)|^2 \mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^3} |u(x)|^2 \left(\int_{|x-y|>R} \frac{|\varpi_R(x) - \varpi_R(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x + \int_{|x-y|\le R} \frac{|\varpi_R(x) - \varpi_R(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\right)\mathrm{d}y \\ &\le \int_{\mathbb{R}^3} |u(x)|^2 \left(\int_{|x-y|>R} \frac{4\|\varpi\|^2_{L^\infty(\mathbb{R}^3)}}{|x-y|^{3+2s}}\mathrm{d}x + R^{-2}\int_{|x-y|\le R} \frac{\|\nabla\varpi\|^2_{L^\infty(\mathbb{R}^3)}}{|x-y|^{3+2s}}\mathrm{d}x\right)\mathrm{d}y \\ &\le C\int_{\mathbb{R}^3} |u(x)|^2 \mathrm{d}y \left(\int_R^\infty \frac{1}{r^{2s+1}}\mathrm{d}r + R^{-2}\int_0^R \frac{1}{r^{2s-1}}\mathrm{d}r\right) \\ &\le \frac{C}{R^{2s}},\end{aligned}$$

it follows that $0 \le \mathcal{A}_R \to 0$. Moreover, $\mathcal{B}_R \to 0$ by the dominated convergence theorem. Then, (3.3) holds.

From Lemma 3.2-(ii),

$$\langle \Phi_\epsilon'(u), u_R\rangle + \int_{\mathbb{R}^3} F_1'(u_R)u_R \mathrm{d}x = \langle w, u_R\rangle, \quad \forall w \in \mathcal{H}_\epsilon'. \tag{3.4}$$

Then, combining (3.3), (3.4) with Lemma 3.2-(v), we can see that $\int_{\mathbb{R}^3} F_1'(u)u_R \mathrm{d}x \le C$ for large $R > 0$. From $u_R \to u$ a.e. in $\mathbb{R}^3$ as $R \to \infty$ and Fatou's lemma, we derive that

$$\int_{\mathbb{R}^3} F_1'(u)u\mathrm{d}x \le \liminf_{R\to\infty} \int_{\mathbb{R}^3} F_1'(u)u_R \mathrm{d}x \le \liminf_{R\to\infty} \int_{\mathbb{R}^3} F_1'(u)\varpi_R \mathrm{d}x \le C$$

The proof has been completed. □

An immediate consequence of the last lemma is the following.

Corollary 3.4. *For each $u \in D(\mathcal{I}_\epsilon)\backslash\{0\}$ with $\|\mathcal{I}_\epsilon'(u)\| < +\infty$, we have that*

$$\mathcal{I}_\epsilon'(u)u = [u]^2 + \int_{\mathbb{R}^3} V(\epsilon x)u^2 \mathrm{d}x - \int_{\mathbb{R}^3} \phi_u^t u^2 \mathrm{d}x - \int_{\mathbb{R}^3} u^2 \log u^2 \mathrm{d}x,$$

and

$$\mathcal{I}_\epsilon(u) - \frac{1}{2}\mathcal{I}_\epsilon'(u)u = \frac{1}{2}\int_{\mathbb{R}^3} u^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 \mathrm{d}x.$$

Corollary 3.5. *If $(u_n) \subset \mathcal{H}_\epsilon$ is a* (PS) *sequence for $\mathcal{I}_\epsilon$, then $\mathcal{I}_\epsilon'(u_n)u_n = o_n(1)\|u_n\|_\epsilon$. If (u_n) is bounded, we have*

$$\begin{aligned}\mathcal{I}_\epsilon(u_n) &= \mathcal{I}_\epsilon(u_n) - \frac{1}{2}\mathcal{I}_\epsilon'(u_n)u_n + o_n(1)\|u_n\|_\epsilon \\ &= \frac{1}{2}\int_{\mathbb{R}^3} u_n^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 \mathrm{d}x + o_n(1)\|u_n\|_\epsilon, \forall n \in \mathbb{N}.\end{aligned}$$

Corollary 3.6. *If $u \in \mathcal{H}_\epsilon$ is a critical point of $\mathcal{I}_\epsilon$ and $v \in \mathcal{H}_\epsilon$ verifies $F_1'(u)v \in L^1(\mathbb{R}^3)$, then $\mathcal{I}_\epsilon'(u)v = 0$.*

Now, we will prove some results that will be useful in the proof of Theorem 1.1.

Lemma 3.7. *For any $\epsilon > 0$, all* (PS) *sequences of $\mathcal{I}_\epsilon$ are bounded in $\mathcal{H}_\epsilon$.*

Proof. Let (u_n) be a $(\mathrm{PS})_d$ sequence. By Corollary 3.5, one concludes

$$\begin{aligned}\int_{\mathbb{R}^3} u_n^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 \mathrm{d}x &= 2\mathcal{I}_\epsilon(u_n) - \mathcal{I}_\epsilon'(u_n)u_n \\ &= 2d + o_n(1) + o_n(1)\|u_n\|_\epsilon \\ &\le C + o_n(1)\|u_n\|_\epsilon,\end{aligned}$$

for some $C > 0$. Consequently,

$$\|u_n\|_\epsilon^2 \le C + o_n(1)\|u_n\|_\epsilon. \tag{3.5}$$

Let us employ the following logarithmic Sobolev inequality found in Lieb–Loss [17],

$$\int_{\mathbb{R}^3} u^2 \log u^2 \mathrm{d}x \le \frac{a^2}{\pi}|\nabla u|_2^2 + \left(\log |u|_2^2 - 3(1+\log a)\right)|u|_2^2, \tag{3.6}$$

for all $a > 0$. Fixing $\frac{a^2}{\pi} = \frac{1}{4}$ and $\xi \in (0,1)$, the inequalities (3.5) and (3.6) yield that

$$\begin{aligned}\int_{\mathbb{R}^3} u_n^2 \log u_n^2 \mathrm{d}x &\le \frac{1}{4}|\nabla u_n|_2^2 + C\left(\log |u_n|_2^2 + 1\right)|u_n|_2^2 \\ &\le \frac{1}{4}|\nabla u_n|_2^2 + C_1\left(1+\|u_n\|_\epsilon\right)^{1+\xi}.\end{aligned} \tag{3.7}$$

Then by (3.7), we have that

$$\begin{aligned}d + o_n(1) &= \mathcal{I}_\epsilon(u_n) - \frac{1}{4}\mathcal{I}_\epsilon'(u_n)u_n \\ &\ge \frac{1}{4}\|u_n\|_\epsilon^2 - \frac{1}{4}\int_{\mathbb{R}^3} u_n^2 \log u_n^2 \mathrm{d}x \\ &\ge C\left(\|u_n\|_\epsilon^2 - (1+\|u_n\|_\epsilon)^{1+\xi}\right),\end{aligned}$$

which shows that the sequence (u_n) is bounded. □

Lemma 3.8. *Suppose that V satisfies (V_1)–(V_3). For each $\sigma > 0$, there exists $\epsilon_0 = \epsilon_0(\sigma) > 0$ such that, if (u_n) is a $(\mathrm{PS})_c$ sequence for $\mathcal{I}_\epsilon$ with $c \in (m(c_0) + \sigma/2, 2m(c_0) - \sigma)$ and $\epsilon \in (0, \epsilon_0)$, then (u_n) has a weak limit $u_0 \neq 0$.*

Proof. We shall prove the lemma arguing by contradiction, by supposing that there exists $\sigma > 0$, a sequence $\epsilon_n \to 0$ and $(u_m^n) \subset \mathcal{H}_\epsilon$ such that

$$\lim_{m\to+\infty} \mathcal{I}_{\epsilon_n}(u_m^n) = c_n \quad \text{and} \quad \lim_{m\to+\infty} \|\mathcal{I}'_{\epsilon_n}(u_m^n)\| = 0,$$

with $u_m^n \rightharpoonup 0$, as $m \to +\infty$.

Claim I: There exists $\delta > 0$, such that

$$\liminf_{m\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_m^n|^2 \,\mathrm{d}x \geq \delta, \quad \forall n \in \mathbb{N}.$$

Indeed, if the Claim does not hold, there is $(n_j) \subset \mathbb{N}$ satisfying

$$\liminf_{m\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} \left|u_m^{n_j}\right|^2 \mathrm{d}x \leq \frac{1}{j}, \quad \forall j \in \mathbb{N}.$$

Then, for each $j \in \mathbb{N}$, there is m_j large enough such that

$$\sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_{m_j}^{n_j}|^2 \mathrm{d}x \leq \frac{2}{j}, \quad |\mathcal{I}_{\epsilon_n}(u_{n_m}^{n_j}) - c_{n_j}| \leq \frac{1}{j}, \quad \text{and} \quad \left\|\mathcal{I}'_{\epsilon_n}(u_{m_j}^{n_j})\right\| \leq \frac{1}{j}, \quad \forall j \in \mathbb{N}. \tag{3.8}$$

Setting $w_j = u_{m_j}^{n_j}$, it shows that (w_j) is a bounded sequence, and by Lions [18],

$$\limsup_{j\to+\infty} |w_j|_p = 0, \quad \forall p \in (2, 2_s^*).$$

Then, we can see

$$\limsup_{j\to+\infty} \int_{\mathbb{R}^3} F_2'(w_j)\, w_j \mathrm{d}x = 0 \quad \text{and} \quad \limsup_{j\to+\infty} \int_{\mathbb{R}^3} \phi_{w_j}^t w_j^2 \mathrm{d}x = 0.$$

On the other hand, it follows from (3.8) that

$$\begin{aligned} \|w_j\|_{\epsilon_{n_j}}^2 + \int_{\mathbb{R}^3} F_1'(w_j) w_j \mathrm{d}x &= \mathcal{I}'_{\epsilon_{n_j}}(w_j) w_j + \int_{\mathbb{R}^3} F_2'(w_j) w_j \mathrm{d}x + \int_{\mathbb{R}^3} \phi_{w_j}^t w_j^2 \mathrm{d}x \\ &\leq o_j(1) \|w_j\|_{\epsilon_{n_j}}, \end{aligned}$$

where it follows that

$$\limsup_{j\to+\infty} \|w_j\|_{\epsilon_{n_j}}^2 = 0 \quad \text{and} \quad \limsup_{j\to+\infty} \int_{\mathbb{R}^3} F_1'(w_j) w_j \mathrm{d}x = 0.$$

Combining this fact with convexity of F_1, we can see that

$$\limsup_{j\to+\infty} \int_{\mathbb{R}^3} F_1(w_j) \mathrm{d}x = 0.$$

The above analysis imply that $\mathcal{I}_{\epsilon_{n_j}}(w_j) \to 0$ as $j \to +\infty$, and so, $c_{n_j} \to 0$ as $j \to +\infty$, which is contradictory because $c_{n_j} > m(c_0) + \sigma/2$ for all $j \in \mathbb{N}$. This proves the Claim I.

For each $n \in \mathbb{N}$, there exists $(z_m^n) \subset \mathbb{R}^3$ such that

$$\int_{B_R(z_m^n)} |u_m^n|^2 \, \mathrm{d}x \geq \frac{\delta}{2}, \quad \forall n \in \mathbb{N}.$$

Since $u_m^n \rightharpoonup 0$ as $m \to +\infty$, we have that $|z_m^n| \to +\infty$ as $m \to +\infty$. From the above study, for each $n \in \mathbb{N}$, we fix $m_n \in \mathbb{N}$ large enough satisfying

$$\int_{B_R(z_m^n)} |u_m^n|^2 \mathrm{d}x \geq \frac{\delta}{2}, \quad |\epsilon_n z_{m_n}^n| \geq n, \quad \|\mathcal{I}'_{\epsilon_n}(u_{m_n}^n)\|_\epsilon \leq \frac{1}{n} \quad \text{and} \quad |\mathcal{I}_{\epsilon_n}(u_{m_n}^n) - c_n| \leq \frac{1}{n}.$$

In what follows, we denote by (z_n) and (u_n) the sequences $(z_{m_n}^n)$ and $(u_{m_n}^n)$ respectively. Then,

$$\int_{B_R(z_n)} |u_n|^2 \mathrm{d}x \geq \frac{\delta}{2}, \quad |\epsilon_n z_n| \geq n, \quad \|\mathcal{I}'_{\epsilon_n}(u_n)\|_\epsilon \leq \frac{1}{n} \quad \text{and} \quad |\mathcal{I}_{\epsilon_n}(u_n) - c_n| \leq \frac{1}{n}.$$

The boundedness of (u_n) follows by standard arguments. Then, for some subsequence, there exists $u \in \mathcal{H}_\epsilon$ such that

$$u_n \rightharpoonup u \quad \text{in } \mathcal{H}_\epsilon.$$

Considering $\omega_n = u_n(\cdot + z_n)$, we have that (ω_n) is bounded in $\mathcal{H}_\epsilon$. Thus, there exists $\omega \in \mathcal{H}_\epsilon$ such that

$$\omega_n \rightharpoonup \omega \quad \text{in } \mathcal{H}_\epsilon,$$

and

$$\int_{B_R(0)} |\omega|^2 \mathrm{d}x = \liminf_{n \to +\infty} \int_{B_R(0)} |\omega_n|^2 \, \mathrm{d}x = \liminf_{n \to +\infty} \int_{B_R(z_n)} |u_n|^2 \mathrm{d}x \geq \frac{\delta}{2},$$

which implies that $\omega \neq 0$.

Now, for each $\varphi \in C_0^\infty(\mathbb{R}^3)$, we have the equality below

$$\begin{aligned} &\iint_{\mathbb{R}^6} \frac{(\omega_n(x) - \omega_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} V\left(\epsilon_n z_n + \epsilon_n x\right) \omega_n \varphi \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} \phi_{\omega_n}^t \omega_n \varphi \mathrm{d}x - \int_{\mathbb{R}^3} \omega_n \varphi \log \omega_n^2 \mathrm{d}x = o_n(1) \|\varphi\|_\epsilon, \end{aligned} \tag{3.9}$$

showing that ω is a nontrivial solution of the problem

$$(-\Delta)^s u + \alpha_1 u - \phi_u^t u = u \log u^2 \quad \text{in } \mathbb{R}^3, \tag{3.10}$$

where

$$\alpha_1 = \lim_{n \to +\infty} V(\epsilon_n z_n).$$

From Cabré–Sire [8], Caffarelli–Silvestre [9] and d'Avenia–Montefusco–Squassina [11], we can see that $\omega \in C^2(\mathbb{R}^3) \cap \mathcal{H}_\epsilon$.

For each $k \in \mathbb{N}$, there is $\varphi_k \in C_0^\infty(\mathbb{R}^3)$ such that

$$\|\varphi_k - \omega\|_\epsilon \to 0 \quad \text{as } k \to +\infty,$$

that is,

$$\|\varphi_k - \omega\|_\epsilon = o_k(1).$$

Using $\frac{\partial \varphi_k}{\partial x_i}$ as a test function of (3.9), we have

$$\iint_{\mathbb{R}^6} \frac{(\omega_n(x)-\omega_n(y))(\frac{\partial \varphi_k}{\partial x_i}(x)-\frac{\partial \varphi_k}{\partial x_i}(y))}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} V(\epsilon_n z_n+\epsilon_n x)\,\omega_n \frac{\partial \varphi_k}{\partial x_i}\mathrm{d}x$$
$$-\int_{\mathbb{R}^3} \phi^t_{\omega_n}\omega_n \frac{\partial \varphi_k}{\partial x_i}\mathrm{d}x - \int_{\mathbb{R}^3} \omega_n \frac{\partial \varphi_k}{\partial x_i}\log \omega_n^2 \mathrm{d}x = o_n(1).$$

Observing that

$$\iint_{\mathbb{R}^6} \frac{(\omega_n(x)-\omega_n(y))(\frac{\partial \varphi_k}{\partial x_i}(x)-\frac{\partial \varphi_k}{\partial x_i}(y))}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y$$
$$= \iint_{\mathbb{R}^6} \frac{(\omega(x)-\omega(y))(\frac{\partial \varphi_k}{\partial x_i}(x)-\frac{\partial \varphi_k}{\partial x_i}(y))}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y + o_n(1),$$

$$\int_{\mathbb{R}^3} \phi^t_{\omega_n}\omega_n \frac{\partial \varphi_k}{\partial x_i}\mathrm{d}x = \int_{\mathbb{R}^3} \phi^t_{\omega_n}\omega \frac{\partial \varphi_k}{\partial x_i}\mathrm{d}x + o_n(1),$$

and

$$\int_{\mathbb{R}^3} \omega_n \frac{\partial \varphi_k}{\partial x_i}\log \omega_n^2 \mathrm{d}x = \int_{\mathbb{R}^3} \omega \frac{\partial \varphi_k}{\partial x_i}\log \omega^2 \mathrm{d}x + o_n(1).$$

Gathering the above limit with (3.10), we derive that

$$\limsup_{n\to+\infty}\left|\int_{\mathbb{R}^3} (V(\epsilon_n z_n+\epsilon_n x) - V(\epsilon_n z_n))\,\omega_n \frac{\partial \varphi_k}{\partial x_i}\mathrm{d}x\right| = 0.$$

As φ_k has compact support, the above limit gives

$$\limsup_{n\to+\infty}\left|\int_{\mathbb{R}^3} (V(\epsilon_n z_n+\epsilon_n x) - V(\epsilon_n z_n))\,\omega \frac{\partial \varphi_k}{\partial x_i}\mathrm{d}x\right| = 0.$$

Recalling that $\frac{\partial \omega}{\partial x_i} \in L^2(\mathbb{R}^3)$, we have that $(\frac{\partial \varphi_k}{\partial x_i})$ is bounded in $L^2(\mathbb{R}^3)$. Then,

$$\limsup_{n\to+\infty}\left|\int_{\mathbb{R}^3} (V(\epsilon_n z_n+\epsilon_n x) - V(\epsilon_n z_n))\,\varphi_k \frac{\partial \varphi_k}{\partial x_i}\mathrm{d}x\right| = o_k(1),$$

and so,

$$\limsup_{n\to+\infty}\left|\frac{1}{2}\int_{\mathbb{R}^3} (V(\epsilon_n z_n+\epsilon_n x) - V(\epsilon_n z_n)) \frac{\partial (\varphi_k^2)}{\partial x_i}\mathrm{d}x\right| = o_k(1).$$

Using Green's Theorem together with the fact that φ_k has compact support, we get the limit below

$$\limsup_{n\to+\infty}\left|\int_{\mathbb{R}^3} \frac{\partial V}{\partial x_i}(\epsilon_n z_n+\epsilon_n x)\,\varphi_k^2 \mathrm{d}x\right| = o_k(1),$$

which combined with (V_2) loads to

$$\limsup_{n\to+\infty}\left|\frac{\partial V}{\partial x_i}(\epsilon_n z_n)\int_{\mathbb{R}^3} |\varphi_k|^2 \mathrm{d}x\right| = o_k(1).$$

As

$$\int_{\mathbb{R}^3} |\varphi_k|^2 \mathrm{d}x \to \int_{\mathbb{R}^3} |\omega|^2 \mathrm{d}x > 0 \quad \text{as } k \to +\infty,$$

it shows that

$$\limsup_{n\to+\infty}\left|\frac{\partial V}{\partial x_i}(\epsilon_n z_n)\right| = o_k(1), \quad \forall i \in \{1,2,3\}.$$

Since k is arbitrary, we obtain

$$\nabla V(\epsilon_n z_n) \to 0 \quad \text{and} \quad V(\epsilon_n z_n) \to \alpha_1,$$

as $n \to \infty$. Therefore, $(\epsilon_n z_n)$ is a $(\mathrm{PS})_{\alpha_1}$ sequence for V, which is a contradiction, because by hypotheses V satisfies the (PS) condition and $(\epsilon_n z_n)$ does not have any convergent subsequence in $\mathbb{R}^3$. Thus, the proof is completed. □

Hereafter, we denote by $\mathcal{N}_\epsilon$ the Nehari manifold associated with $\mathcal{I}_\epsilon$, that is,

$$\mathcal{N}_\epsilon = \{u \in \mathcal{H}_\epsilon \backslash \{0\} : \mathcal{I}'_\epsilon(u)u = 0\}.$$

The next lemma will be crucial in our study to show an important estimate.

Lemma 3.9. *Let $\epsilon_n \to 0$ and $(u_n) \subset \mathcal{N}_{\epsilon_n}$ such that $\mathcal{I}_{\epsilon_n}(u_n) \to m(c_0)$. Then, there are $(z_n) \subset \mathbb{R}^3$ with $|z_n| \to +\infty$ and $u_1 \in \mathcal{H}_\epsilon \backslash \{0\}$ such that*

$$u_n(\cdot + z_n) \to u_1 \quad \text{in } \mathcal{H}_\epsilon.$$

Moreover,

$$\liminf_{n\to+\infty} |\epsilon_n z_n| > 0.$$

Proof. Since $u_n \in \mathcal{N}_{\epsilon_n}$, we can see that $\mathcal{I}'_{c_0}(u_n)\, u_n \le 0$ and $\mathcal{I}_{c_0}(u) \le \mathcal{I}_{\epsilon_n}(u)$ for all $u \in \mathcal{H}_\epsilon$ and $n \in \mathbb{N}$. Then, there exists $\tau_n \in (0,1]$ such that

$$(\tau_n u_n) \subset \mathcal{N}_{c_0} \quad \text{and} \quad \mathcal{I}_{c_0}(\tau_n u_n) \to m(c_0).$$

Since (τ_n) is bounded, by Alves–de Morais Filho [2], there exist $(z_n) \subset \mathbb{R}^3$, $u_1 \in \mathcal{H}_\epsilon \backslash \{0\}$, and a subsequence of (u_n), still denote by (u_n), verifying

$$u_n(\cdot + z_n) \to u_1 \quad \text{in } \mathcal{H}_\epsilon.$$

Claim II:

$$\liminf_{n\to+\infty} |\epsilon_n z_n| > 0.$$

Indeed, since $u_n \in \mathcal{N}_{\epsilon_n}$ for all $n \in N$, the function $u_n^1 = u_n(\cdot + z_n)$ must verify

$$\begin{aligned}
&[u_n^1]^2 + \int_{\mathbb{R}^3} V(\epsilon_n z_n + \epsilon_n x) \left|u_n^1\right|^2 \mathrm{d}x - \int_{\mathbb{R}^3} \phi^t_{u_n^1} |u_n^1|^2 \mathrm{d}x + \int_{\mathbb{R}^3} F_1'\left(u_n^1\right) u_n^1 \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^3} F_2'\left(u_n^1\right) u_n^1 \mathrm{d}x.
\end{aligned} \tag{3.11}$$

Since F_1 is convex, even and $F_1(\tau) \ge F_1(0) = 0$ for all $\tau \in \mathbb{R}$, we can derive that $0 \le F_1(\tau) \le F_1'(\tau)\tau$ for all $\tau \in \mathbb{R}$. Supposing by contradiction that for some subsequence

$$\lim_{n\to+\infty} \epsilon_n z_n = 0.$$

Taking the limit of $n \to +\infty$ in (3.11), we have

$$[u_1]^2 + \int_{\mathbb{R}^3} V(0) |u_1|^2 \mathrm{d}x - \int_{\mathbb{R}^3} \phi^t_{u_1} |u_1|^2 \mathrm{d}x + \int_{\mathbb{R}^3} F_1'(u_1) u_1 \mathrm{d}x \le \int_{\mathbb{R}^3} F_2'(u_1) u_1 \mathrm{d}x.$$

Then, there is $\tau_1 \in (0,1]$ such that $\tau_1 u_1 \in \mathcal{M}_{V(0)}$. Thus, since $V(0) > c_0$, we derive that

$$\mathcal{I}_{V(0)}(\tau_1 u_1) \geq m(V(0)) > m(c_0) > 0. \tag{3.12}$$

On the other hand,

$$\mathcal{I}_{\epsilon_n}(u_n) \to \mathcal{I}_{V(0)}(\tau_1 u_1),$$

which leads to

$$m(c_0) \geq \mathcal{I}_{V(0)}(\tau_1 u_1). \tag{3.13}$$

From (3.12) and (3.13), we can find a contradiction, which finishes the proof. □

4 A special minimax level

To prove Theorem 1.1, we shall consider a special minimax level. To do that, we begin fixing the barycenter function by

$$\beta(u) = \frac{\int_{\mathbb{R}^3} \frac{x}{|x|}|u|^2 \mathrm{d}x}{\int_{\mathbb{R}^3} |u|^2 \mathrm{d}x}, \quad \forall u \in \mathcal{H}_\epsilon \backslash \{0\}.$$

For each $z \in \mathbb{R}^N$ and $\epsilon > 0$, let us define the function

$$\varphi_{\epsilon,z}(x) = \tau_{\epsilon,z} u_0 \left(x - \frac{z}{\epsilon}\right),$$

where $\tau_{\epsilon,z} > 0$ is such that $\varphi_{\epsilon,z} \in \mathcal{N}_\epsilon$ and u_0 is a radial positive ground state solution for $\mathcal{I}_{c_0}$, that is,

$$\mathcal{I}_{c_0}(u_0) = m(c_0) \quad \text{and} \quad \mathcal{I}'_{c_0}(u_0) = 0.$$

In what follows, we set $\Upsilon_\epsilon(z) = \varphi_{\epsilon,z}$ for all $z \in \mathbb{R}^3$.

Lemma 4.1. *The function $\Upsilon_\epsilon : \mathbb{R}^3 \to \mathcal{N}_\epsilon$ is a continuous function.*

Proof. Let $(z_n) \subset \mathbb{R}^3$ and $z \in \mathbb{R}^3$ with $z_n \to z$ in $\mathbb{R}^3$. We must prove that

$$\Upsilon_\epsilon(z_n) \to \Upsilon_\epsilon(z) \quad \text{in } \mathcal{H}_\epsilon.$$

Here, the main point is to prove that

$$\tau_{\epsilon,z_n} \to \tau_{\epsilon,z} \quad \text{in } \mathbb{R}.$$

By definition of τ_{ϵ,z_n} and $\tau_{\epsilon,z}$, they are the unique numbers that satisfy

$$\mathcal{I}_\epsilon\left(\tau_{\epsilon,z_n} u_0\left(\cdot - \frac{z_n}{\epsilon}\right)\right) = \frac{1}{2}\int_{\mathbb{R}^3}\left|\tau_{\epsilon,z_n} u_0\left(x - \frac{z_n}{\epsilon}\right)\right|^2 \mathrm{d}x,$$

and

$$\mathcal{I}_\epsilon\left(\tau_{\epsilon,z} u_0\left(\cdot - \frac{z}{\epsilon}\right)\right) = \frac{1}{2}\int_{\mathbb{R}^3}\left|\tau_{\epsilon,z} u_0\left(x - \frac{z}{\epsilon}\right)\right|^2 \mathrm{d}x,$$

that is,

$$\begin{aligned} &\frac{1}{2}[\tau_{\epsilon,z_n} u_0]^2 + \frac{1}{2}\int_{\mathbb{R}^3}(V(\epsilon x + z_n) + 1)|\tau_{\epsilon,z_n} u_0|^2 \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3}\phi^t_{\tau_{\epsilon,z_n} u_0}|\tau_{\epsilon,z_n} u_0|^2 \mathrm{d}x \\ &+ \int_{\mathbb{R}^3} F_1(\tau_{\epsilon,z_n} u_0)\mathrm{d}x - \int_{\mathbb{R}^3} F_2(\tau_{\epsilon,z_n} u_0)\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^3}|\tau_{\epsilon,z_n} u_0|^2 \mathrm{d}x, \end{aligned} \tag{4.1}$$

and

$$\frac{1}{2}[\tau_{\epsilon,z}u_0]^2 + \frac{1}{2}\int_{\mathbb{R}^3} (V(\epsilon x + z) + 1)|\tau_{\epsilon,z}u_0|^2 \, dx - \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{\tau_{\epsilon,z}u_0}|\tau_{\epsilon,z}u_0|^2 dx$$
$$+ \int_{\mathbb{R}^3} F_1(\tau_{\epsilon,z}u_0) \, dx - \int_{\mathbb{R}^3} F_2(\tau_{\epsilon,z}u_0) \, dx = \frac{1}{2}\int_{\mathbb{R}^3} |\tau_{\epsilon,z}u_0|^2 \, dx.$$

A simple calculation gives that (τ_{ϵ,z_n}) is bounded, thus for some subsequence, we can assume that $\tau_{\epsilon,z_n} \to \tau_*$. Since F_1 is increasing in $[0,+\infty)$ and $F_1(\zeta u_0) \in L^1(\mathbb{R}^3)$ for all $\zeta > 0$, taking the limit of $n \to +\infty$ in (4.1) and using the Lebesgue Theorem, we have

$$\frac{1}{2}[\tau_* u_0]^2 + \frac{1}{2}\int_{\mathbb{R}^3} (V(\epsilon x + z) + 1)|\tau_* u_0|^2 \, dx - \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{\tau_* u_0}|\tau_* u_0|^2 dx$$
$$+ \int_{\mathbb{R}^3} F_1(\tau_* u_0) \, dx - \int_{\mathbb{R}^3} F_2(\tau_* u_0) \, dx = \frac{1}{2}\int_{\mathbb{R}^3} |\tau_* u_0|^2 \, dx,$$

By uniqueness of $\tau_{\epsilon,z}$, it shows that $\tau_{\epsilon,z} = \tau_*$, and so, $\tau_{\epsilon,z_n} \to \tau_{\epsilon,z}$. Then, since

$$u_0\left(\cdot - \frac{z_n}{\epsilon}\right) \to u_0\left(\cdot - \frac{z}{\epsilon}\right) \quad \text{in } \mathcal{H}_\epsilon,$$

the proof is completed. □

We establish several properties involving β and Υ_ϵ.

Lemma 4.2. *For each $r > 0$, we have*

$$\lim_{\epsilon \to 0}\left(\sup\left\{\left|\beta(\Upsilon_\epsilon(z)) - \frac{z}{|z|}\right| : |z| \geq r\right\}\right) = 0.$$

Proof. It is enough to show that for any $(z_n) \subset \mathbb{R}^3$ with $|z_n| \geq r$ and $\epsilon_n \to 0$, we have that

$$\left|\beta(\Upsilon_{\epsilon_n}(z_n)) - \frac{z_n}{|z_n|}\right| \to 0 \quad \text{as } n \to +\infty.$$

By change of variables,

$$\left|\beta(\Upsilon_{\epsilon_n}(z_n)) - \frac{z_n}{|z_n|}\right| = \frac{\int_{\mathbb{R}^3}\left|\frac{\epsilon_n x + z_n}{|\epsilon_n x + z_n|} - \frac{z_n}{|z_n|}\right| |u_0(x)|^2 \, dx}{\int_{\mathbb{R}^3} |u_0(x)|^2 \, dx}.$$

Since for each $x \in \mathbb{R}^3$, we have

$$\left|\frac{\epsilon_n x + z_n}{|\epsilon_n x + z_n|} - \frac{z_n}{|z_n|}\right| \to 0 \quad \text{as } n \to +\infty,$$

by the Lebesgue Dominated Convergence Theorem, we get that

$$\int_{\mathbb{R}^3}\left|\frac{\epsilon_n x + z_n}{|\epsilon_n x + z_n|} - \frac{z_n}{|z_n|}\right| |u_0(x)|^2 \, dx \to 0 \quad \text{as } n \to +\infty,$$

which completes the proof. □

As a by-product of the arguments explored in the proof of the last lemma, we have

Corollary 4.3. *Fixed $r > 0$, there is $\epsilon_0 > 0$ such that*

$$(\beta(\Upsilon_\epsilon(z)), z) > 0, \quad \forall |z| \geq r \text{ and } \epsilon \in (0, \epsilon_0).$$

Proof. By Lemma 4.2, for fixed $r > 0$, there is $\epsilon_0 > 0$ such that

$$\left|\beta\left(\Upsilon_\epsilon(z)\right) - \frac{z}{|z|}\right| < \frac{1}{2}, \quad \forall |z| \geq r \text{ and } \epsilon \in (0, \epsilon_0).$$

On the other hand,

$$\begin{aligned}(\beta\left(\Upsilon_\epsilon(z)\right), z) &= \left(\beta\left(\Upsilon_\epsilon(z)\right) - \frac{z}{|z|}, z\right) + \left(\frac{z}{|z|}, z\right) \\ &= \left(\beta\left(\Upsilon_\epsilon(z)\right) - \frac{z}{|z|}, z\right) + |z|, \quad \forall z \in \mathbb{R}^3\backslash\{0\}.\end{aligned}$$

Therefore, for $|z| \geq r$, we have

$$(\beta\left(\Upsilon_\epsilon(z)\right), z) \geq |z|\left(1 - \left|\beta\left(\Upsilon_\epsilon(z)\right) - \frac{z}{|z|}\right|\right) > \frac{|z|}{2} \geq \frac{r}{2} > 0,$$

showing the corollary. □

In the sequel, we define the set

$$\mathcal{B}_\epsilon = \{u \in \mathcal{N}_\epsilon : \beta(u) \in Y\}.$$

Note that $\mathcal{B}_\epsilon \neq \emptyset$, since $\beta\left(\varphi_{\epsilon,0}\right) = 0 \in Y$, for all $\epsilon > 0$. Associated with the above set, let us consider the real number D_ϵ given by

$$D_\epsilon = \inf_{u \in \mathcal{B}_\epsilon} \mathcal{I}_\epsilon(u).$$

The next lemma establishes an important relation between the levels D_ϵ and $m(c_0)$.

Lemma 4.4. *The following conclusions are valid:*

(a) *There are ϵ_0 and $\sigma > 0$ such that*

$$D_\epsilon \geq m\left(c_0\right) + \sigma, \quad \forall \epsilon \in (0, \epsilon_0).$$

(b)

$$\limsup_{\epsilon \to 0}\left\{\sup_{x \in X} \mathcal{I}_\epsilon(\Upsilon_\epsilon(x))\right\} < 2m\left(c_0\right) - \sigma.$$

Proof. (a) By the definition of D_ϵ, we can see

$$D_\epsilon \geq m(c_0), \quad \forall \epsilon > 0.$$

Supposing by contradiction that the lemma does not hold, there is $\epsilon_n \to 0$ satisfying

$$D_{\epsilon_n} \to m(c_0).$$

Hence, there exists $u_n \in \mathcal{N}_{\epsilon_n}$ with $\beta(u_n) \in Y$ such that

$$\mathcal{I}_{\epsilon_n}(u_n) \to m(c_0).$$

Thereby, by Lemma 3.9, there are $u_1 \in \mathcal{H}_\epsilon\backslash\{0\}$ and a sequence $(z_n) \subset \mathbb{R}^3$ with

$$\liminf_{n \to +\infty} |\epsilon_n z_n| > 0$$

verifying

$$u_n(\cdot + z_n) \to u_1 \quad \text{in } \mathcal{H}_\epsilon,$$

that is

$$u_n = u_1\left(\cdot - z_n\right) + \omega_n \quad \text{with } \omega_n \to 0 \text{ in } \mathcal{H}_\epsilon.$$

From the definition of β,

$$\beta\left(u_1(\cdot - z_n)\right) = \frac{\int_{\mathbb{R}^3} \frac{\epsilon_n x + \epsilon_n z_n}{|\epsilon_n x + \epsilon_n z_n|} \left|u_1\right|^2 \mathrm{d}x}{\int_{\mathbb{R}^3} \left|u_1\right|^2 \mathrm{d}x}.$$

Repeating the same arguments explored in the proof of Lemma 4.2, we know that

$$\beta\left(u_1(\cdot - z_n)\right) = \frac{z_n}{|z_n|} + o_n(1),$$

and so,

$$\beta\left(u_n\right) = \beta\left(u_1(\cdot - z_n)\right) + o_n(1) = \frac{z_n}{|z_n|} + o_n(1).$$

Since $\beta\left(u_n\right) \in Y$, we conclude that $\frac{z_n}{|z_n|} \in Y_\lambda$ for n large enough. Consequently, $z_n \in Y_\lambda$ for n large enough, implying that

$$\liminf_{n\to\infty} V(\epsilon_n z_n) > c_0.$$

If $A = \liminf_{n\to\infty} V\left(\epsilon_n z_n\right)$, the last inequality and the Fatou's lemma show that

$$m(c_0) = \liminf_{n\to\infty} \mathcal{I}_{\epsilon_n}(u_n) \geq \liminf_{n\to\infty} \mathcal{I}_{\epsilon_n}(\tau u_n) \geq \mathcal{I}_A(\tau u_1) \geq m(A) > m(c_0),$$

which is a contradiction, recalling that there exists $\tau \in (0,1]$ such that $\mathcal{I}'_A(\tau u_1)\tau u_1 = 0$ and $u_1 \neq 0$.

(b) By $V(0) > c_0$, $c_1 < c_0 + 1$ and the fact that u_0 is a ground state solution associated with $\mathcal{I}_{c_0}$, we infer that

$$\begin{aligned}
\limsup_{\epsilon\to 0} \left\{ \sup_{x\in X} \mathcal{I}_\epsilon\left(\Upsilon_\epsilon(x)\right) \right\} &\leq \frac{1}{2}[u_o]^2 + \frac{1}{2}\int_{\mathbb{R}^3} (c_1+1)\left|u_0\right|^2 \mathrm{d}x \\
&\quad - \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_0}^t |u_0|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} u_0^2 \log u_0^2 \mathrm{d}x \\
&= \mathcal{I}_{c_0}\left(u_0\right) + \frac{(c_1 - c_0)}{2}\int_{\mathbb{R}^3} \left|u_0\right|^2 \mathrm{d}x \\
&= \mathcal{I}_{c_0}\left(u_0\right) + \left(c_1 - c_0\right)\mathcal{I}_{c_0}\left(u_0\right) \\
&= \left(1 + c_1 - c_0\right) m(c_0) \\
&< 2m(c_0),
\end{aligned}$$

which completes the proof. □

Now, we are ready to show the minimax level. We fix $\epsilon \in (0, \epsilon_0)$ and the following sets

$$\mathcal{I}_\epsilon^d = \left\{u \in \mathcal{H}_\epsilon : \mathcal{I}_\epsilon(u) \leq d\right\}, \quad Q = \bar{B}_R(0) \cap X \quad \text{and} \quad \partial Q = \partial \bar{B}_R(0) \cap X.$$

By the above notations, we define the class of the functions

$$\Gamma = \left\{h \in C\left(Q, K_r\right) : h(x) = \Upsilon_\epsilon(x), \quad \forall x \in \partial Q\right\},$$

where $r > 0, K = \Upsilon_\epsilon(Q)$ and $K_r = \{u \in \mathcal{H}_\epsilon : \text{dist}(u, K) < r\}$. Note that $\Gamma \neq \emptyset$, since Lemma 4.1 ensures that $\Upsilon_\epsilon \in \Gamma$. Then, we set

$$\Omega_r = \{u \in K_r : \beta(u) \in Y\},$$

which is not empty because $\Upsilon_\epsilon(0) = \varphi_{\epsilon,0} \in K_r$ for all $r > 0$. Here we have used the fact that $\Upsilon_\epsilon(0) \in \Upsilon_\epsilon(Q)$ and $\beta(\Upsilon_\epsilon(0)) = 0 \in Y$.

Lemma 4.5. *There exists $r_0 > 0$ such that*

$$\Theta_r = \inf_{u \in \Omega_r} \mathcal{I}_\epsilon(u) > m(c_0) + \sigma/2, \quad \forall r \in (0, r_0).$$

Moreover, there is $R > 0$ such that

$$\mathcal{I}_\epsilon(\Upsilon_\epsilon(x)) \leq \frac{1}{2}\left(m(c_0) + \Theta_r\right), \quad \forall x \in \partial B_R(0) \cap X.$$

Proof. Assume by contradiction that the lemma does not hold. Then, there exist $r_n \to 0$ and $u_n \in \Omega_{r_n}$ such that $\mathcal{I}_\epsilon(u_n) \leq m(c_0) + \sigma/2$. By definition of Ω_{r_n}, there exists $v_n \in K$ such that $\|u_n - v_n\| \leq r_n$. Since K is compact, there are a subsequence of (v_n), still denoted by itself, and $v \in K$ such that $v_n \to v$ in $\mathcal{H}_\epsilon$, then $u_n \to v$ in $\mathcal{H}_\epsilon$ and $\beta(v) \in Y$, from where it follows that $v \in \mathcal{B}_\epsilon$, then by Lemma 4.4-$(a)$, $\mathcal{I}_\epsilon(v) \geq m(c_0) + \sigma$. On the other hand, since $\mathcal{I}_\epsilon$ is lower semicontinuous, we have

$$\liminf_{n \to +\infty} \mathcal{I}_\epsilon(u_n) \geq \mathcal{I}_\epsilon(v),$$

which is a contradiction.

By (V_1), given $\delta > 0$, there exist $\epsilon_0 > 0$ and $R > 0$ such that

$$\sup\{\mathcal{I}_\epsilon(\Upsilon_\epsilon(x)) : x \in \partial B_R(0) \cap X\} \leq m(c_0) + \delta, \quad \forall \epsilon \in (0, \epsilon_0).$$

Fixing $\delta = \frac{\sigma}{4}$, where σ was given in Lemma 4.4-(a), we derive

$$\sup\{\mathcal{I}_\epsilon(\Upsilon_\epsilon(x)) : x \in \partial B_R(0) \cap X\} \leq \frac{1}{2}\left(2m(c_0) + \frac{\sigma}{2}\right) < \frac{1}{2}\left(m(c_0) + \Theta_r\right), \quad \forall \epsilon \in (0, \epsilon_0),$$

which completes the proof. □

Lemma 4.6. *If $h \in \Gamma$, then $h(Q) \cap \Omega_r \neq \emptyset$ for all $r \in (0, r_0)$.*

Proof. It is enough to show that for all $h \in \Gamma$, there exists $x_* \in Q$ such that

$$\beta(h(x_*)) \in Y.$$

For each $h \in \Gamma$, we set the function $g : Q \to \mathbb{R}^3$ given by

$$g(x) = \beta(h(x)) \quad \forall x \in Q,$$

and the homotopy $\mathcal{F} : [0,1] \times Q \to X$ as

$$\mathcal{F}(\tau, x) = \tau P_X(g(x)) + (1 - \tau)x,$$

where P_X is the projection onto $X = \{(x, 0) : x \in \mathbb{R}^3\}$. By Corollary 4.3, fixed $R > 0$ and $\epsilon > 0$ small enough, one has

$$(\mathcal{F}(\tau, x), x) > 0, \quad \forall (\tau, x) \in [0,1] \times \partial Q.$$

Applying the homotopy invariance property of the topological degree, we have

$$d(g, Q, 0) = 1,$$

which implies that there is $x_* \in Q$ such that $\beta(h(x_*)) = 0$. □

Now, we define the minimax value

$$C_\epsilon = \inf_{h\in\Gamma}\sup_{x\in Q}\mathcal{I}_\epsilon(h(x)).$$

By Lemmas 4.5 and 4.6,

$$C_\epsilon \geq \Theta_r = \inf_{u\in\Omega_r}\mathcal{I}_\epsilon(u) \geq m(c_0) + \sigma/2, \tag{4.2}$$

for ϵ is small enough. On the other hand,

$$C_\epsilon \leq \sup_{x\in Q}\mathcal{I}_\epsilon(\Upsilon_\epsilon(x)).$$

Then, by Lemma 4.4-(b), if ϵ is small enough,

$$C_\epsilon \leq \sup_{x\in Q}\mathcal{I}_\epsilon(\Upsilon_\epsilon(x)) < 2m\left(c_0\right) - \sigma. \tag{4.3}$$

From (4.2) and (4.3), there is ϵ_0 such that

$$C_\epsilon \in \left(m(c_0) + \sigma/2, 2m(c_0) - \sigma\right), \quad \forall \epsilon \in (0, \epsilon_0).$$

Proof of Theorem 1.1. Before proving Theorem 1.1, we first propose the following claim.

Claim III: For a given $\tau > 0$ small enough, there exists $u_\tau \in E$ such that

$$\Phi'_\epsilon\left(u_\tau\right)\cdot\left(v - u_\tau\right) + \Psi(v) - \Psi\left(u_\tau\right) \geq -3\tau\left\|v - u_\tau\right\|_\epsilon, \quad \forall v \in E,$$

and

$$\mathcal{I}_\epsilon(u_\tau) \in \left[C_\epsilon - \tau, C_\epsilon + \tau\right].$$

In fact, to prove the claim, we follow the ideas explored in Alves–de Morais Filho [2] and Szulkin [28]. Have this in mind, by Lemma 4.5, we can fix $\tau > 0$ small enough such that

$$C_\epsilon - \tau/2 > \frac{1}{2}\left(m(c_0) + \Theta_r\right),$$

and we set

$$\Gamma_1 = \left\{h \in C(Q, K_r) : h|_{\partial Q} \approx \Upsilon_\epsilon|_{\partial Q} \text{ in } \mathcal{I}_\epsilon^{C_\epsilon - \tau/4}, \sup_{x\in\partial Q}\mathcal{I}_\epsilon(h(x)) \leq C_\epsilon - \tau/2\right\},$$

where $\approx$ denotes the homotopy relation and the number

$$C^* = \inf_{h\in\Gamma_1}\sup_{x\in Q}\mathcal{I}_\epsilon(h(x)).$$

Arguing as in Szulkin [28], we have that $C^* = C_\epsilon$, and so, it is enough to prove that Claim III holds for C^* instead C_ϵ. In order to show this, firstly let us fix $\tau > 0$ small enough and $h \in \Gamma_1$ such that

$$\Pi(h) \leq C^* + \tau \quad \text{and} \quad \Pi(g) - \Pi(h) \geq -\tau d(g, h), \quad \forall g \in \Gamma_1, \tag{4.4}$$

where

$$\Pi(g) = \sup_{x\in Q}\mathcal{I}_\epsilon(g(x)), \quad \forall g \in \Gamma_1,$$

and

$$d(g,h) = \sup_{x\in Q} \|g(x) - h(x)\|.$$

Supposing by contradiction that Claim III does not hold and arguing as in Alves–de Morais Filho [2], we can apply Proposition 2.3 of Szulkin [28] with $A = h(Q)$ to find a closed subset W containing A in it interior and a deformation $\alpha_s : W \to \mathcal{H}_\epsilon$ having the following properties:

$$\begin{cases} \|u - \alpha_s(u)\|_\epsilon \le s, & \forall u \in W \text{ and } s \approx 0^+, \\ \mathcal{I}_\epsilon(\alpha_s(u)) - \mathcal{I}_\epsilon(u) \le 2s, & \forall u \in W, \\ \mathcal{I}_\epsilon(\alpha_s(u)) - \mathcal{I}_\epsilon(u) \le -2\tau s, & \forall u \in W \text{ with } \mathcal{I}_\epsilon(u) \ge C^* - \tau, \end{cases} \tag{4.5}$$

and

$$\sup_{u\in A} \mathcal{I}_\epsilon(\alpha_s(u)) - \sup_{u\in A} \mathcal{I}_\epsilon(u) \le -2\tau s. \tag{4.6}$$

It is easy to see that $g = \alpha_s \circ h \in \Gamma_1$, for s small enough. However, by(4.4), (4.5) and (4.6), we have

$$-\tau s \le -\tau d(g,h) \le \Pi(g) - \Pi(h) \le -2\tau s,$$

which is a contradiction. This contradiction shows that Claim III is true.

From Claim III, there exists a $(\mathrm{PS})_{C_\epsilon}$ sequence for $\mathcal{I}_\epsilon$, which will denoted by (u_n). By Lemma 3.8, we can assume that $u_n \rightharpoonup u_\epsilon$ for some $u_\epsilon \in \mathcal{H}_\epsilon \backslash \{0\}$. On the other hand, it follows from Lemma 3.2 that for each $v \in C_0^\infty(\mathbb{R}^3)$, there holds the limit $\langle \mathcal{I}'_\epsilon(u_n), v\rangle = o_n(1)\|v\|_\epsilon$, from where it shows that $\langle \mathcal{I}'_\epsilon(u_\epsilon), v\rangle = 0$, or equivalently,

$$\begin{aligned} &\iint_{\mathbb{R}^6} \frac{(u_\epsilon(x) - u_\epsilon(y))(v(x) - v(y))}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} V(\epsilon x) u_\epsilon \cdot v \mathrm{d}x - \int_{\mathbb{R}^3} \phi^t_{u_\epsilon} u_\epsilon v \mathrm{d}x \\ &\quad = \int u_\epsilon v \log u_\epsilon^2 \mathrm{d}x, \quad \forall v \in C_0^\infty(\mathbb{R}^3). \end{aligned}$$

Moreover, a similar computation also gives that

$$[u_\epsilon]^2 + \int_{\mathbb{R}^3} V(\epsilon x)|u_\epsilon|^2 \mathrm{d}x - \int_{\mathbb{R}^3} \phi^t_{u_\epsilon} |u_\epsilon|^2 \mathrm{d}x + \int F'_1(u_\epsilon) u_\epsilon \mathrm{d}x \le \int F'_2(u_\epsilon) u_\epsilon \mathrm{d}x,$$

which implies that $u^2 \log u^2 \in L^1(\mathbb{R}^3)$. This proves that u_ϵ is a critical point of $\mathcal{I}_\epsilon$ with $\phi = \phi_{u_\epsilon}$ for ϵ small enough. Finally, the last inequality together with Fatou's Lemma implies that

$$\mathcal{I}_\epsilon(u_\epsilon) \le C_\epsilon < 2m(c_0).$$

By Squassina–Szulkin [26], local estimates and standard bootstrap arguments show that $u_\epsilon \in C^2(\mathbb{R}^3, \mathbb{R})$. Moreover, by the Maximum Principle, we have that

$$u_\epsilon(x) > 0 \text{ for } x \in \mathbb{R}.$$

For each $\epsilon > 0$ small enough, let u_ϵ denote the positive solution obtained above. Setting $v_\epsilon = u_\epsilon(\frac{x}{\epsilon})$, then it shows that $(v_\epsilon, \phi_{v_\epsilon})$ gives rise to a pair of solutions of (1.1). □

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References

[1] C. O. Alves, Existence of a positive solution for a nonlinear elliptic equation with saddle-like potential and nonlinearity with exponential critical growth in $\mathbb{R}^2$, *Milan J. Math.* **84**(2016), 1–22. https://doi.org/10.1007/s00032-015-0247-9; MR3503193

[2] C. O. Alves, D. C. de Morais Filho, Existence of concentration of positive solutions for a Schrödinger logarithmic equation, *Z. Angew. Math. Phys.* **69**(2018), Paper No. 144 pp.https://doi.org/10.1007/s00033-018-1038-2; MR3869846

[3] C. O. Alves, C. Ji, Existence of a positive solution for a logarithmic Schrödinger equation with saddle-like potential, *Manuscripta Math.* **164**(2021), No. 3–4, 555–575.https://doi.org/10.1007/s00229-020-01197-z; MR4212205

[4] C. O. Alves, O. H. Miyagaki, A critical nonlinear fractional elliptic equation with saddle-like potential in $\mathbb{R}^N$, *J. Math. Phys.* **57**(2016), No. 8, 081501, 17 pp.https://doi.org/10.1063/1.4959221; MR3531162

[5] V. Ambrosio, Multiplicity and concentration results for a class of critical fractional Schrödinger–Poisson systems via penalization method, *Manuscripta Math.* **22**(2020), No. 1, 1850078, 45 pp.https://doi.org/10.1142/S0219199718500785; MR4064905

[6] D. Applebaum, *Lévy processes and stochastic calculus*, Cambridge Studies in Advanced Mathematics, Vol. 116, Cambridge University Press, Cambridge, second edition, 2009. MR2512800

[7] A. Bahrouni, V. D. Rădulescu, P. Winkert, Robin fractional problems with symmetric variable growth, *J. Math. Phys.* **61**(2020), No. 10, 101503, 14 pp.https://doi.org/10.1063/5.0014915; MR4157423

[8] X. Cabré, Y. Sire, Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates, *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **31**(2014), No. 1, 23–53.https://doi.org/10.1016/j.anihpc.2013.02.001

[9] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* **32**(2007), No. 7–9, 1245–1260.https://doi.org/10.1080/03605300600987306; MR2354493

[10] M. Chen, Q. Li, S. Peng, Bound states for fractional Schrödinger–Poisson system with critical exponent, *Discrete Contin. Dyn. Syst. Ser. S* **14**(2021), No. 6, 1819–1835.https://doi.org/10.3934/dcdss.2021038; MR4271198

[11] P. d'Avenia, E. Montefusco, M. Squassina, On the logarithmic Schrödinger equation, *Commun. Contemp. Math.* **16**(2014), Paper No. 1350032 pp. https://doi.org/10.1142/S0219199713500326

[12] M. del Pino, P. L. Felmer, O. H. Miyagaki, Existence of positive bound states of nonlinear Schrödinger equations with saddle-like potential, *Nonlinear Anal.* **34**(1998), No. 7, 979–989.https://doi.org/10.1016/S0362-546X(97)00593-2

[13] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136**(2012), No. 5, 521–573.https://doi.org/10.1016/j.bulsci.2011.12.004; MR2944369

[14] P. Felmer, A. Quaas, J. Tan, Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian, *Proc. Roy. Soc. Edinburgh Sect. A* **142**(2012), No. 6, 1237–1262.https://doi.org/10.1017/S0308210511000746; MR3002595

[15] C. Ji, Ground state sign-changing solutions for a class of nonlinear fractional Schrödinger–Poisson system in $\mathbb{R}^3$, *Ann. Mat. Pura Appl. (4)* **198**(2019), No. 5, 1563–1579. https://doi.org/10.1007/s10231-019-00831-2; MR4022109

[16] C. Ji, A. Szulkin, A logarithmic Schrödinger equation with asymptotic conditions on the potential, *J. Math. Anal. Appl.* **437**(2016), No. 1, 241–254. https://doi.org/10.1016/j.jmaa.2015.11.071; MR3451965

[17] E. H. Lieb, M. L. Loss, *Analysis*, Second Edition, Graduate Studies in Mathematics, Vol. 14, AMS, Providence, 2001. MR1817225

[18] P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. II, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1**(1984), No. 4, 223–283. MR778974

[19] Z. Liu, J. Zhang, Multiplicity and concentration of positive solutions for the fractional Schrödinger–Poisson systems with critical growth, *ESAIM Control Optim. Calc. Var.* **23**(2017), No. 4, 1515–1542. https://doi.org/10.1051/cocv/2016063; MR3716931

[20] Y. Meng, X. Zhang, X. He, Ground state solutions for a class of fractional Schrödinger–Poisson system with critical growth and vanishing potentials, *Adv. Nonlinear Anal.* **10**(2021), No. 1, 1328–1355. https://doi.org/10.1515/anona-2020-0179; MR4270477

[21] G. Molica Bisci, V. D. Rădulescu, R. Servadei, *Variational methods for nonlocal fractional problems*, of Encyclopedia of Mathematics and its Applications, Vol. 162, Cambridge University Press, Cambridge, 2016. https://doi.org/10.1017/CBO9781316282397; MR3445279

[22] E. G. Murcia, G. Siciliano, Positive semiclassical states for a fractional Schrödinger–Poisson system, *Differential Integral Equations* **30**(2017), No. 3–4, 231–258.MR3611500

[23] P. Pucci, M. Xiang, B. Zhang, Multiple solutions for nonhomogeneous Schrödinger–Kirchhoff type equations involving the fractional p-Laplacian in $\mathbb{R}^N$, *Calc. Var. Partial Differential Equations* **54**(2015), No. 3, 2785–2806. https://doi.org/10.1007/s00526-015-0883-5; MR3412392

[24] P. Pucci, M. Xiang, B. Zhang, Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations, *Adv. Nonlinear Anal.* **5**(2016), No. 1, 27–55. https://doi.org/10.1515/anona-2015-0102; MR3456737

[25] S. Qu, X. He, On the number of concentrating solutions of a fractional Schrödinger–Poisson system with doubly critical growth, *Anal. Math. Phys.* **12**(2022), No. 2, Paper No. 59, 49 pp. https://doi.org/10.1007/s13324-022-00675-9; MR4400605

[26] M. Squassina, A. Szulkin, Multiple solutions to logarithmic Schrödinger equations with periodic potential, *Calc. Var. Partial Differential Equations* **54**(2015), No. 1, 585–597. https://doi.org/10.1007/s00526-014-0796-8; MR3385171

[27] M. Squassina, A. Szulkin, Erratum to: Multiple solutions to logarithmic Schrödinger equations with periodic potential, *Calc. Var. Partial Differential Equations* **56**(2017), No. 3, 56.https://doi.org/10.1007/s00526-017-1127-7; MR3634793

[28] A. Szulkin, Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **3**(1986), No. 2, 77–109.https://doi.org/10.1016/S0294-1449(16)30389-4; MR837231

[29] K. Teng, Existence of ground state solutions for the nonlinear fractional Schrödinger–Poisson system with critical Sobolev exponent, *J. Differential Equations* **61**(2016), No. 6, 3061–3106. https://doi.org/10.1016/j.jde.2016.11.016; MR3582254

[30] M. Willem, *Minimax theorems*, Birkhäuser Boston, Inc., Boston, MA, 1996. https://doi.org/10.1007/978-1-4612-4146-1; MR1400007

[31] Z. Yang, Y. Yu, F. Zhao, Concentration behavior of ground state solutions for a fractional Schrödinger–Poisson system involving critical exponent, *Commun. Contemp. Math.* **21**(2019), No. 6, 1850027, 46 pp. https://doi.org/10.1142/S021919971850027X; MR3996970