Electronic Journal of Qualitative Theory of Differential Equations
2024, No. **20**, 1–25; https://doi.org/10.14232/ejqtde.2024.1.20 www.math.u-szeged.hu/ejqtde/

A variation of parameters formula for nonautonomous linear impulsive differential equations with piecewise constant arguments of generalized type

This paper is dedicated to the memory of Prof. Nicolás Yus Suárez

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Received 14 January 2024, appeared 29 April 2024
Communicated by Eduardo Liz

Abstract. In this work, we give a variation of parameters formula for nonautonomous linear impulsive differential equations with piecewise constant arguments of generalized type. We cover several cases of differential equations with deviated arguments investigated before as particular cases. We also give some examples showing the applicability of our results.

Keywords: variation of parameters formula, piecewise constant argument, linear functional differential equations, DEPCAG, IDEPCAG.

2020 Mathematics Subject Classification: 34A36, 34A37, 34A38, 34K34, 34K45.

1 Introduction

Occasionally, natural phenomena must be modeled using differential equations that may have discontinuous solutions, such as a piecewise constant, or the impulsive effect must be present. Some examples of such modeling can be found in the works of S. Busenberg and K. Cooke [7] (where the authors modeled vertical transmission diseases) and L. Dai and M. C. Singh [12] (oscillatory motion of spring-mass systems subject to piecewise constant forces such $Ax([t])$ or $A\cos([t])$). The last work studied the motion of mechanisms modeled by

$$mx''(t)+kx_1 = A\sin\left(\omega\left[\frac{t}{T}\right]\right),$$

where $[\cdot]$ is the greatest integer function. (See [11]).

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In the 70's, A. Myshkis [15] studied differential equations with deviating arguments ($h(t) \leq t$, such as $h(t) = [t]$ or $h(t) = [t-1]$). The Ukrainian mathematician M. Akhmet generalized those systems, introducing differential equations of the form

$$y'(t) = f(t, y(t), y(\gamma(t))), \tag{1.1}$$

where $\gamma(t)$ is a *piecewise constant argument of generalized type*. In order to define such γ, let $(t_n)_{n\in\mathbb{Z}}$ and $(\zeta_n)_{n\in\mathbb{Z}}$ such that $t_n < t_{n+1}, \forall n \in \mathbb{Z}$ with $\lim\limits_{n\to\infty} t_n = \infty$, $\lim\limits_{n\to-\infty} t_n = -\infty$ and $\zeta_n \in [t_n, t_{n+1}]$. Then, $\gamma(t) = \zeta_n$, if $t \in I_n = [t_n, t_{n+1})$. I.e., $\gamma(t)$ is a step function. An elementary example of such functions is $\gamma(t) = [t]$ which is constant in every interval $[n, n+1[$ with $n \in \mathbb{Z}$ (see (1.3)).

If a piecewise constant argument is used, the interval I_n is decomposed into an advanced and delayed subintervals $I_n = I_n^+ \cup I_n^-$, where $I_n^+ = [t_n, \zeta_n]$ and $I_n^- = [\zeta_n, t_{n+1}]$. This class of differential equations is known as *Differential Equations with Piecewise Constant Argument of Generalized Type* (*DEPCAG*). They have continuous solutions, even though γ is discontinuous. If we assume continuity of the solutions of (1.1), integrating from t_n to t_{n+1}, we define a finite-difference equation, so we are in the presence of a hybrid dynamic (see [3, 17]).

For example, taking $\gamma(t) = \left[\frac{t+l}{h}\right]h$ with $0 \leq l < h$, we have

$$\left[\frac{t+l}{h}\right]h = nh, \text{ when } t \in I_n = [nh - l, (n+1)h - l).$$

Then, we see that $\gamma(t) - t \geq 0 \Leftrightarrow t \leq nh$ and $\gamma(t) - t \leq 0 \Leftrightarrow t \geq nh$. Hence, we have

$$I_n^+ = [nh - l, nh], \quad I_n^- = [nh, (n+1)h - l].$$

Now, if an impulsive condition is defined at $\{t_n\}_{n\in\mathbb{Z}}$, we are in the presence of the *Impulsive differential equations with piecewise constant argument of generalized type* (*IDEPCAG*) (see [2]),

$$\begin{aligned} x'(t) &= f(t, x(t), x(\gamma(t))), \qquad t \neq t_n \\ \Delta x(t_n) := x(t_n) - x(t_n^-) &= J_n(x(t_n^-)), \qquad t = t_n, \quad n \in \mathbb{N}, \end{aligned} \tag{1.2}$$

where $x(t_n^-) = \lim_{t\to t_n^-} x(t)$, and J_n is the impulsive operator (see [18]).

When the piecewise constant argument used in a differential equation is explicit, it will be called DEPCA (IDEPCA if it has impulses).

An elementary and illustrative example of IDEPCA

Consider the scalar IDEPCA

$$\begin{aligned} x'(t) &= (\alpha - 1)x([t]), \qquad t \neq n \\ x(n) &= \beta x(n^-), \qquad\quad\; t = n, \quad n \in \mathbb{N}. \end{aligned} \tag{1.3}$$

where $\alpha, \beta \in \mathbb{R}$, $\beta \neq 1$.

If $t \in [n, n+1)$ for some $n \in \mathbb{Z}$, equation (1.3) can be written as

$$x'(t) = (\alpha - 1)x(n). \tag{1.4}$$

In the following, we will assume $t_0 = 0$. Now, integrating on $[n, n+1)$ from n to t we see that

$$x(t) = x(n)(1 + (\alpha - 1)(t - n)). \tag{1.5}$$

Next, assuming continuity at $t = n+1$, we have

$$x((n+1)^{-}) = \alpha x(n).$$

Applying the impulsive condition to the last expression, we get the following *finite-difference equation*

$$x((n+1)) = (\alpha\beta)x(n).$$

Its solution is

$$x(n) = (\alpha\beta)^{n} x(0). \tag{1.6}$$

Finally, applying (1.6) in (1.5) we have

$$x(t) = (\alpha\beta)^{[t]} \left(1 + (\alpha - 1)(t - [t])\right)x(0). \tag{1.7}$$

Remark 1.1. From (1.7), we can conclude that the underlying dynamic is of mixed type. The discrete and the continuous parts of the system are dependent. For example, A stable continuous part (associated with the coefficient α) can be unstabilized by the discrete part (associated with the parameter β). See [18].

In the next table, we describe some of the behavior of the solutions of (1.7):

Behavior of solutions	**Condition**
$\lvert x(t)\rvert \xrightarrow{t\to\infty} 0$ exponentially.	$\lvert\alpha\beta\rvert < 1$ and $\alpha\beta \neq 0$.
$x(t)$ is constant.	$\alpha\beta = 0$ or $\alpha = \beta = 1$
$x(t)$ is oscillatory.	$\alpha\beta < 0$
$x(t)$ is piecewise constant.	$\alpha = 1$
$\lvert x(t)\rvert$ is piecewise constant and $x(t) \xrightarrow{t\to\infty} +\infty$.	$\alpha = 1$ and $\lvert\beta\rvert > 1$
$x(t)$ is piecewise constant and $x(t) \xrightarrow{t\to\infty} 0$.	$\alpha = 1$ and $0 < \beta < 1$
$\lvert x(t)\rvert \xrightarrow{t\to\infty} +\infty$ exponentially.	$\lvert\alpha\beta\rvert > 1$.

Table 1.1: Behavior of solutions of (1.7)

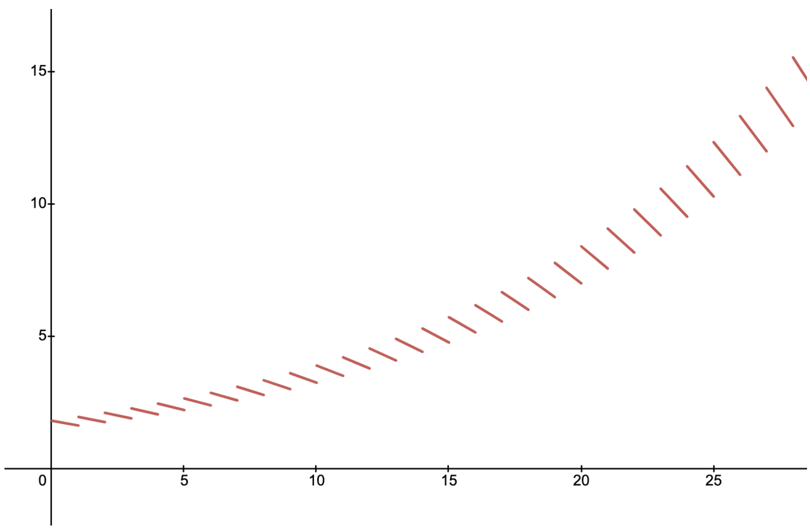

Figure 1.1: Solution of (1.3) with $\alpha = 0.9$, $\beta = 1.2$, $x_0 = 1.8$.

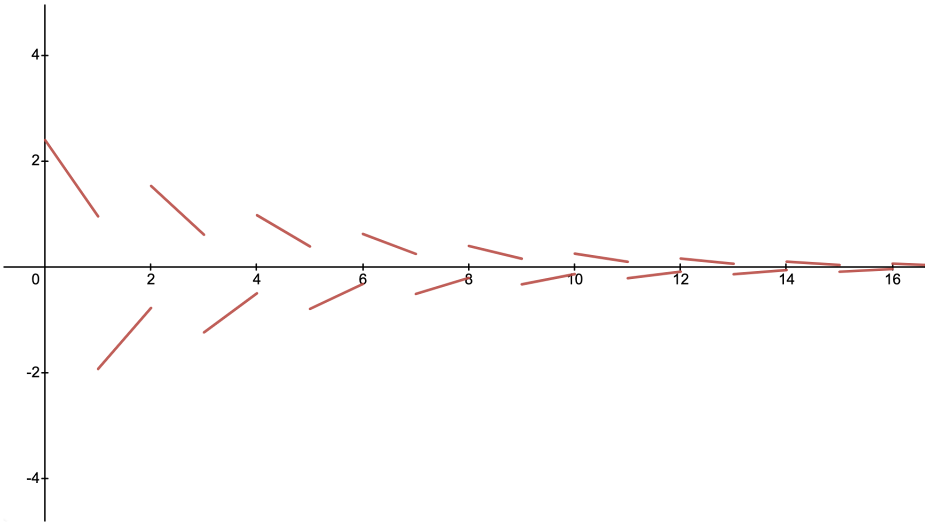

Figure 1.2: solution of (1.3) with $\alpha = 0.4$, $\beta = -2$, $x_0 = 2.4$.

1.1 Why study IDEPCAG?: impulses in action

Example 1.2. Let the following scalar linear DEPCA

$$x'(t) = a(t)(x(t) - x([t])), \qquad x(\tau) = x_0, \tag{1.8}$$

and the scalar linear IDEPCA

$$\begin{array}{ll} z'(t) = a(t)\left(z(t) - z([t])\right), & t \neq k \\ z(k) = c_k z(k^-), & t = k, \quad k \in \mathbb{Z}, \end{array} \tag{1.9}$$

where $a(t)$ is a continuous locally integrable function and $(c_k)_{k\in\mathbb{N}}$ a real sequence such that $c_k \notin \{0,1\}$, for all $k \in \mathbb{N}$. As $\gamma(t) = [t]$, we have $t_k = k = \zeta_k = k$ if $t \in [k, k+1)$, $k \in \mathbb{Z}$.

The solution of (1.8) is $x(t) = x_0$, $\forall t \geq \tau$. I.e., all the solutions are constant (see [17]).

On the other hand, as we will see, the solution of (1.9) is

$$z(t) = \left(\prod_{j=k(\tau)+1}^{k(t)} c_j\right) z(\tau), \qquad t \geq \tau,$$

where $k(t) = k$ is the only integer such that $t \in [k, k+1]$.

Hence, all the solutions are nonconstant if $c_j \neq 1$ and $c_j \neq 0$, for all $j \geq k(\tau)$. This example shows the differences between DEPCA and IDEPCA systems. The discrete part of the system can greatly impact the whole dynamic, determining the qualitative properties of the solutions.

1.2 Fundamental matrices and variation of parameters formulas: an overview

1.2.1 The fundamental matrix of a DEPCA system

In [9], K. L. Cooke and J. Wiener were the first to obtain a fundamental matrix for a scalar *DEPCA*'s using the delayed piecewise constant arguments $\gamma(t) = [t]$, $\gamma(t) = [t-1]$, $\gamma(t) = [t-n]$ and $\gamma(t) = t - n[t]$. Also, they considered the very interesting scalar DEPCA

$$x'(t) = a(t)x(t) + \sum_{i=0}^{n} a_i(t)x([t-i]), \qquad a_n \neq 0,$$

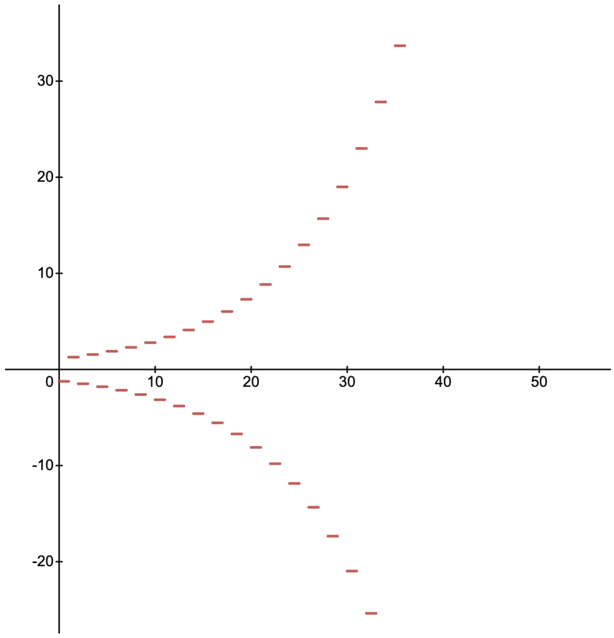

Figure 1.3: Solution of (1.9) with $c_k = -1.1$ and $z(0) = -1.2$

and

$$x'(t) = ax(t) + \sum_{i=1}^{n} a_i x(t - i[t]).$$

Also, in [19], S. M. Shah and J. Wiener studied the DEPCA

$$x'(t) = a(t)x(t) + \sum_{i=0}^{n} a_i(t)x([t+i]), \qquad a_n \neq 0, \quad n \geq 2.$$

Then, in [8], K. L. Cooke and J. Wiener studied the mixed-type piecewise constant argument $\gamma(t) = 2\left[\frac{t+1}{2}\right]$ and considered the DEPCA

$$z'(t) = az(t) + bz(2[(t+1)/2]).$$

Additionally, in [22], J. Wiener and A. R. Aftabizadeh considered the mixed-type piecewise constant argument $\gamma(t) = m\left[\frac{t+k}{m}\right]$ where $0 < k < m$, $k, m, n \in \mathbb{Z}^+$, and they studied the DEPCA

$$w'(t) = aw(t) + bw(m[(t+k)/m]).$$

1.2.2 Variation of parameters formula for a DEPCA

In [13] (1991), N. Jayasree and S. G. Deo were the first to consider the advanced and delayed parts of the solutions studying the equation

$$z'(t) = az(t) + bz(2[(t+1)/2]) + f(t),$$

obtaining a variation of parameters formula for this DEPCA, in terms of the homogeneous linear DEPCA associated:

$$
\begin{aligned}
z(t) = y(t) + & \sum_{j=0}^{[(t+1)/2]-1} \lambda^{-1}(1) \int_{2j}^{2j+1} \Psi(t,2j)\phi(2j+1,s)f(s)ds \\
& - \sum_{j=1}^{[(t+1)/2]} \lambda^{-1}(1) \int_{2j}^{2j-1} \Psi(t,2j)\phi(2j-1,s)f(s)ds \\
& + \int_{2[(t+1)/2]}^{t} \phi(t,s)f(s)ds,
\end{aligned}
$$

where

$$\lambda(t) = \exp(at)\left(1 + a^{-1}b\right) - a^{-1}b,$$

ϕ and Ψ are the fundamental solutions of $x'(t) = ax(t)$ and $y'(t) = ay(t) + by(2[(t+1)/2])$ respectively.

In [14] (2001), Q. Meng and J. Yan obtained a variation of parameters formula for the differential equation

$$x'(t) + a(t)x(t) + b(t)x(g(t)) = f(t) \quad \text{for } t > 0,$$

where $a(t), b(t)$ and $f(t)$ are locally integrable functions on $[0,\infty)$, $g(t)$ is a piecewise constant function defined by $g(t) = np$ for $t \in [np - l, (n+1)p - l)$ with $n \in \mathbb{N}$ and p, l positive constants such that $p > l$. The authors studied the oscillation and asymptotic stability properties of the solutions.

In [1] (2008), M. Akhmet considered the DEPCAG for systems

$$z'(t) = A(t)z(t) + B(t)z(\gamma(t)) + F(t), \tag{1.10}$$

$$w'(t) = A(t)w(t) + B(t)w(\gamma(t)) + g(t, w(t), w(\gamma(t))), \tag{1.11}$$

where $A(t), B(t) \in C(\mathbb{R})$ are $n \times n$ real valued uniformly bounded on $\mathbb{R}$ matrices, $g(t,x,y) \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ is an $n \times 1$ Lipschitz real valued function with $g(t,0,0) = 0$, $\gamma(t)$ is a piecewise constant argument of generalized type. The author found the following variation of parameters formula

$$
\begin{aligned}
w(t) = W(t,t_0)w_0 + W(t,t_0) & \int_{t_0}^{\zeta_i} X(t_0,s)g(s,w(s),w(\gamma(s)))ds \\
& + \sum_{k=i}^{j-1} W(t,t_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(t_{k+1},s)g(s,w(s),w(\gamma(s)))ds \\
& + \int_{\zeta_j}^{t} X(t,s)g(s,w(s),w(\gamma(s)))ds,
\end{aligned}
$$

where $j = j(t)$ is the only $j \in \mathbb{Z}$ such that $t_{j(t)} \le t \le t_{j(t)+1}$, $t_k \le \zeta_k \le t_{k+1}$, $t_i \le t_0 \le t_{i+1}$, X is the fundamental matrix of

$$x'(t) = A(t)x(t),$$

and W is the fundamental matrix of the homogeneous linear DEPCAG

$$y'(t) = A(t)y(t) + B(t)y(\gamma(t)).$$

Later, in [17] (2011), M. Pinto gave a new DEPCAG variation of parameters formula. This time, the author considered the delayed and advanced intervals defined by the general piecewise constant argument

$$
\begin{aligned}
z(t) = W(t,t_0)z_0 &+ \underbrace{W(t,t_0)\int_{t_0}^{\zeta_i} X(t_0,s)g(s,z(s),z(\gamma(s)))ds}_{I_k^+} \\
&+ \sum_{k=i+1}^{j} \underbrace{W(t,t_k)\int_{t_k}^{\zeta_k} X(t_k,s)g(s,z(s),z(\gamma(s)))ds}_{I_k^+} \\
&+ \sum_{k=i}^{j-1} \underbrace{W(t,t_{k+1})\int_{\zeta_k}^{t_{k+1}} X(t_{k+1},s)g(s,z(s),z(\gamma(s)))ds}_{I_k^-} \\
&+ \underbrace{\int_{\zeta_j}^{t} X(t,s)g(s,z(s),z(\gamma(s)))ds}_{I_k^-},
\end{aligned}
$$

where $t_i \le t_0 \le t_{i+1}$ and $t_{j(t)} \le t \le t_{j(t)+1}$.

In the DEPCAG theory, decomposing the interval I_n into the advanced and delayed subintervals is critical. As we will see, it is necessary for the forward or backward continuation of solutions.

1.2.3 Variation of parameters formula for an IDEPCA: the impulsive effect applied

For the IDEPCA case, In [16] (2012), G. Oztepe and H. Bereketoglu studied the scalar IDEPCA

$$
\begin{aligned}
&x'(t) = a(t)(x(t) - x([t+1])) + f(t), \qquad x(0) = x_0, \quad t \neq n \in \mathbb{N} \\
&\Delta x(n) = d_n, \qquad t = n, \quad n \in \mathbb{N}.
\end{aligned}
\tag{1.12}
$$

They proved the convergence of the solutions to a real constant when $t \to \infty$, and they showed the limit value in terms of x_0, using a suitable integral equation. They concluded the following expression for the solutions of (1.12)

$$
\begin{aligned}
x(t) = \ &\exp\left(\int_{[t]}^{t} a(u)du\right) x([t]) + \left(1 - \exp\left(\int_{[t]}^{t} a(u)du\right)\right) x([t+1]) \\
&+ \int_{[t]}^{t} \exp\left(\int_{s}^{t} a(u)du\right) f(s)ds,
\end{aligned}
$$

where

$$
x([t]) = x_0 + \sum_{j=0}^{[t]-1} \left(\int_j^{j+1} \exp\left(-\int_j^s a(u)du\right) f(s)ds + \exp\left(-\int_j^{j+1} a(u)du\right) d_{j+1} \right).
$$

For the IDEPCA case, in [6] (2023), K-S. Chiu and I. Berna considered the following impulsive differential equation with a piecewise constant argument

$$
\begin{aligned}
&y'(t) = a(t)y(t) + b(t)y\left(p\left[\frac{t+l}{p}\right]\right), \quad y(\tau) = c_0, \qquad t \neq kp - l \\
&\Delta y(kp - l) = d_k y(kp - l^-), \qquad t = kp - l, \quad k \in \mathbb{Z},
\end{aligned}
\tag{1.13}
$$

and

$$
\begin{aligned}
&y'(t) = a(t)y(t) + b(t)y\left(p\left[\frac{t+l}{p}\right]\right) + f(t), \qquad y(\tau) = c_0, \quad t \neq kp - l \\
&\Delta y(kp - l) = d_k y(kp - l^-), \qquad t = kp - l, \quad k \in \mathbb{Z},
\end{aligned} \tag{1.14}
$$

where $a(t) \neq 0$, $b(t)$ and $f(t)$ are real-valued continuous functions, $p < l$ and $d_k \in \mathbb{R} - \{1\}$. The authors obtained criteria for the existence and uniqueness, a variation of parameters formula, a Gronwall–Bellman inequality, stability and oscillation criteria for solutions for (1.13) and (1.14).

To our knowledge, there is no variation formula for impulsive differential equations with a generalized constant argument. As we have shown, some authors have studied just some particular cases before.

2 Aim of the work

We will get a variation of parameters formula associated with *IDEPCAG* system

$$
\begin{aligned}
x'(t) &= A(t)x(t) + B(t)x(\gamma(t)) + F(t), \qquad && t \neq t_k \\
\Delta x|_{t=t_k} &= C_k x(t_k^-) + D_k, && t = t_k,
\end{aligned} \tag{2.1}
$$

extending the particular case treated in [6] and the general results of the DEPCAG case studied in [17] to the IDEPCAG context.

3 Preliminaires

Let $\mathcal{PC}(X, Y)$ be the set of all functions $r : X \to Y$ which are continuous for $t \neq t_k$ and continuous from the left with discontinuities of the first kind at $t = t_k$. Similarly, let $\mathcal{PC}^1(X, Y)$ the set of functions $s : X \to Y$ such that $s' \in \mathcal{PC}(X, Y)$.

Definition 3.1 (DEPCAG solution)**.** A continuous function $x(t)$ is a solution of (1.1) if:

(i) $x'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception at the times t_k, $k \in \mathbb{Z}$, where the one side derivative exists.

(ii) $x(t)$ satisfies (1.1) on the intervals of the form (t_k, t_{k+1}), and it holds for the right derivative of $x(t)$ at t_k.

Definition 3.2 (IDEPCAG solution)**.** A piecewise continuous function $y(t)$ is a solution of (1.2) if:

(i) $y(t)$ is continuous on $I_k = [t_k, t_{k+1})$ with first kind discontinuities at t_k, $k \in \mathbb{Z}$, where $y'(t)$ exists at each $t \in \mathbb{R}$ with the possible exception at the times t_k, where lateral derivatives exist (i.e. $y(t) \in \mathcal{PC}^1([t_k, t_{k+1}), R^n)$).

(ii) The ordinary differential equation

$$
y'(t) = f(t, y(t), y(\zeta_k))
$$

holds on every interval I_k, where $\gamma(t) = \zeta_k$.

(iii) For $t = t_k$, the impulsive condition

$$\Delta y(t_k) = y(t_k) - y(t_k^-) = J_k(y(t_k^-))$$

holds. I.e., $y(t_k) = y(t_k^-) + J_k(y(t_k^-))$, where $y(t_k^-)$ denotes the left-hand limit of the function y at t_k.

Let the IDEPCAG system:

$$\begin{aligned} &x'(t) = f(t, x(t), x(\gamma(t))), && t \neq t_k \\ &x(t_k) - x\left(t_k^-\right) = J_k(x(t_k^-)), && t = t_k, \\ &x(\tau) = x_0, \end{aligned} \tag{3.1}$$

where $f \in C([\tau,\infty) \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $J_k \in C(\{t_k\}, \mathbb{R}^n)$ and $(\tau, x_0) \in \mathbb{R} \times \mathbb{R}^n$.

Let the following hypotheses hold:

(H1) Let $\eta_1, \eta_2 : \mathbb{R} \to [0,\infty)$ locally integrable functions and $\lambda_k \in \mathbb{R}^+$, $\forall k \in \mathbb{Z}$; such that

$$\begin{aligned} &\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq \eta_1(t)\, \|x_1 - x_2\| + \eta_2(t)\, \|y_1 - y_2\|, \\ &\left\|J_k(x_1(t_k^-)) - J_k(x_2(t_k^-))\right\| \leq \lambda_k \left\|x_1\left(t_k^-\right) - x_2\left(t_k^-\right)\right\|. \end{aligned}$$

where $\|\cdot\|$ is some matricial norm.

(H2)
$$\overline{\nu} = \sup_{k\in\mathbb{Z}} \left(\int_{t_k}^{t_{k+1}} (\eta_1(s) + \eta_2(s))\, ds \right) < 1.$$

In the following, we mention some useful results: an integral equation associated with (2.1) and two Gronwall–Bellman type inequalities necessary to prove the uniqueness and stability of solutions.

3.1 An integral equation associated to (3.1)

Theorem 3.3 ([4, Lemma 4.2]). *a function $x(t) = x(t, \tau, x_0)$, $\tau \in \mathbb{R}^+$ is a solution of (3.1) on $\mathbb{R}^+$ if and only if satisfies:*

$$x(t) = x_0 + \int_\tau^t f(s, x(s), x(\gamma(s)) ds + \sum_{\tau < t_k \leq t} J_k\left(x\left(t_k^-\right)\right),$$

where

$$\begin{aligned} \int_\tau^t f(s, x(s), x(\gamma(t))) ds = &\int_\tau^{t_1} f(s, x(s), x(\zeta_0))\, ds + \sum_{j=1}^{k(t)-1} \int_{t_j}^{t_{j+1}} f(s, x(s), x(\zeta_j)) ds \\ &+ \int_{t_{k(t)}}^t f\left(s, x(s), x\left(\zeta_{k(t)}\right)\right) ds, \end{aligned}$$

3.2 First IDEPCAG Gronwall–Bellman type inequality

Lemma 3.4 ([20], [4, Lemma 4.3]). *Let I an interval and $u, \eta_1, \eta_2 : I \to [0,\infty)$ such that u is continuous (with possible exception at $\{t_k\}_{k\in\mathbb{N}}$), η_1, η_2 are continuous and locally integrable functions, $\eta : \{t_k\} \to [0,\infty)$ and $\gamma(t)$ a piecewise constant argument of generalized type such that $\gamma(t) = \zeta_k$, $\forall t \in I_k = [t_k, t_{k+1})$ with $t_k \leq \zeta_k \leq t_{k+1}$ $\forall k \in \mathbb{N}$. Assume that $\forall t \geq \tau$*

$$u(t) \leq u(\tau) + \int_\tau^t (\eta_1(s)u(s) + \eta_2(s)u(\gamma(s)))\, ds + \sum_{\tau < t_k \leq t} \eta(t_k) u(t_k^-)$$

and

$$\widehat{\vartheta}_k = \int_{t_k}^{\zeta_k} (\eta_1(s) + \eta_2(s))\, ds \le \widehat{\vartheta} := \sup_{k\in\mathbb{N}} \widehat{\vartheta}_k < 1. \tag{3.2}$$

hold. Then, for $t \ge \tau$, *we have*

$$u(t) \le \left(\prod_{\tau<t_k\le t} (1+\eta(t_k))\right) \exp\left(\int_\tau^t \left(\eta_1(s) + \frac{\eta_2(s)}{1-\widehat{\vartheta}}\right) ds\right) u(\tau), \tag{3.3}$$

$$u(\zeta_k) \le (1-\vartheta) u(t_k) \tag{3.4}$$

$$u(\gamma(t)) \le (1-\vartheta)^{-1} \left(\prod_{\tau<t_k\le t} (1+\eta_3(t_j))\right) \exp\left(\int_\tau^t \left(\eta_1(s) + \frac{\eta_2(s)}{1-\widehat{\vartheta}}\right) ds\right) u(\tau). \tag{3.5}$$

3.3 Second IDEPCAG Gronwall–Bellman type inequality

Lemma 3.5 ([5, 20]). *Let* I *an interval and* $u, \eta_1, \eta_2 : I \to [0,\infty)$ *such that* u *is continuous (with possible exception at* $\{t_k\}_{k\in N}$*),* η_1, η_2 *are continuous and locally integrable functions,* $\eta : \{t_k\} \to [0,\infty)$ *and* $\gamma(t)$ *a piecewise constant argument of generalized type such that* $\gamma(t) = \zeta_k$*,* $\forall t \in I_k = [t_k, t_{k+1})$ *with* $t_k \le \zeta_k \le t_{k+1}$ $\forall k \in \mathbb{N}$*. Assume that* $\forall t \ge \tau$

$$u(t) \le u(\tau) + \int_\tau^t (\eta_1(s)u(s) + \eta_2(s)u(\gamma(s)))ds + \sum_{\tau<t_k\le t} \eta(t_k)u(t_k^-) \tag{3.6}$$

and

$$\varrho_k = \int_{t_k}^{\zeta_k} \left(\eta_2(s)e^{\int_s^{\zeta_k}\eta_1(r)dr}\right) ds \le \varrho := \sup_{k\in\mathbb{N}} \varrho_k < 1. \tag{3.7}$$

Then, for $t \ge \tau$, *we have*

$$\begin{aligned} u(t) \le{}& \left(\prod_{\tau<t_k\le t} (1+\eta(t_k))\right) \\ &\cdot \exp\left(\frac{1}{1-\vartheta}\sum_{j=k(\tau)+1}^{k(t)} \int_{t_{j-1}}^{t_j} \eta_2(s)\exp\left(\int_{t_{j-1}}^{\zeta_{j-1}} \eta_1(r)dr\right) ds \right. \\ &\left. + \frac{1}{1-\vartheta}\int_{t_{k(t)}}^t \eta_2(s)\exp\left(\int_{t_{k(t)}}^{\zeta_{k(t)}} \eta_1(r)dr\right) ds + \int_\tau^t \eta_1(s)ds\right) u(\tau). \end{aligned} \tag{3.8}$$

3.4 Existence and uniqueness for (3.1)

Theorem 3.6 (Uniqueness [4, Theorem 4.5]). *Consider the I.V.P for* (2.1) *with* $y(t,\tau,y(\tau))$*. Let (H1)–(H2) hold. Then, there exists a unique solution* y *for* (2.1) *on* $[\tau,\infty)$*. Moreover, every solution is stable.*

Lemma 3.7 (Existence of solutions in $[\tau, t_k)$ [4, Lemma 4.6]). *Consider the I.V.P for* (2.1) *with* $y(t,\tau,y(\tau))$*. Let (H1)–(H2) hold. Then, for each* $y_0 \in \mathbb{R}^n$ *and* $\zeta_k \in [t_{k-1}, t_k)$ *there exists a solution* $y(t) = y(t,\tau,y(\tau))$ *of* (2.1) *on* $[\tau, t_r)$ *such that* $y(\tau) = y_0$*.*

Theorem 3.8 (Existence of solutions in $[\tau,\infty$ [4, Theorem 4.7]). *Let (H1)–(H2) hold. Then, for each* $(\tau, y_0 \in \mathbb{R}_0^+ \times \mathbb{R}^n$*, there exists* $y(t) = y(t,\tau,y_0)$ *for* $t \ge \tau$*, a unique solution for* (2.1) *such that* $y(\tau) = y_0$*.*

4 Variation of parameters formula for IDEPCAG

In this section, we will construct a variation of parameters formula for the IDEPCAG system

$$\begin{aligned} y'(t) &= A(t)y(t) + B(t)y(\gamma(t)) + F(t), && t \neq t_k \\ \Delta y|_{t=t_k} &= C_k y(t_k^-) + D_k, && t = t_k \end{aligned} \tag{4.1}$$

where $y \in \mathbb{R}^{n\times 1}, t \in \mathbb{R}, F(t) \in \mathbb{R}^{n\times 1}$ is a real valued continuous matrix, $A(t), B(t) \in \mathbb{R}^{n\times n}$ are real valued continuous locally integrable matrices, $C_k, D_k \in \mathbb{R}^{n\times n}$, $(I + C_k)$ invertible $\forall k \in \mathbb{Z}$, where $I_{n\times n} = I$ is the identity matrix and $\gamma(t)$ is a generalized piecewise constant argument. This time, we will consider the advanced and the delayed intervals in our approach.

First, we will find the *fundamental matrix* for the linear IDEPCAG

$$\begin{aligned} w'(t) &= A(t)w(t) + B(t)w(\gamma(t)), && t \neq t_k \\ \Delta w|_{t=t_k} &= C_k w(t_k^-), && t = t_k. \end{aligned} \tag{4.2}$$

Then, we will give the variation of parameters formula for (4.1).

Let $\Phi(t,s)$, $t,s \in \mathbb{R}$, with $\Phi(t,t) = I$ the transition (Cauchy) matrix of the ordinary system

$$x'(t) = A(t)x(t), \qquad t \in I_k = [t_k, t_{k+1}). \tag{4.3}$$

We will assume the following hypothesis:

(H3) Let

$$\rho_{k^+}(A) = \exp\left(\int_{t_k}^{\zeta_k} \|A(u)\|\, du\right), \qquad \rho_{k^-}(A) = \exp\left(\int_{\zeta_k}^{t_{k+1}} \|A(u)\|\, du\right),$$
$$\rho_k(A) = \rho_{k^+}(A)\cdot\rho_{k^-}(A), \qquad \nu_k^{\pm}(B) = \rho_k^{\pm}(A)\ln\rho_k^{\pm}(B),$$

and assume that

$$\rho(A) = \sup_{k\in\mathbb{Z}} \rho_k(A) < \infty, \qquad \nu^{\pm}(B) = \sup_{k\in\mathbb{Z}} \nu_k^{\pm}(B) < \infty.$$

Consider the following matrices

$$J(t,\tau) = I + \int_{\tau}^{t} \Phi(\tau,s)B(s)ds, \qquad E(t,\tau) = \Phi(t,\tau)J(t,\tau), \tag{4.4}$$

where

$$\nu_k^{\pm}(B) < \nu^{\pm}(B) < 1. \tag{4.5}$$

Remark 4.1. It is important to notice the following facts:

a) As a consequence of (H3), $J(t_k,\zeta_k)$ and $J(t_{k+1},\zeta_k)$ are invertible $\forall k \in \mathbb{Z}$, and

$$\left\|J^{-1}(t_k,\zeta_k)\right\| \le \sum_{k=0}^{\infty}\left[\nu^{+}(B)\right]^k = \frac{1}{1-\nu^{+}(B)}, \qquad \|J(t_k,\zeta_k)\| \le 1 + \nu^{+}(B), \tag{4.6}$$

$$\left\|J^{-1}(t_{k+1},\zeta_k)\right\| \le \sum_{k=0}^{\infty}\left[\nu^{-}(B)\right]^k = \frac{1}{1-\nu^{-}(B)}, \qquad \|J(t_{k+1},\zeta_k)\| \le 1 + \nu^{-}(B). \tag{4.7}$$

Additionally, setting $t_0 := \tau$, we will assume that $J^{-1}(\tau,\gamma(\tau))$ exists.

b) Also, due to (H3) and the Gronwall inequality, we have

$$|\Phi(t)| \le \rho(A),$$

(see [17]).

4.1 The fundamental matrix of the linear homogeneous IDEPCAG

We adopt the following convention:

$$\overset{\leftarrow}{\prod_{k=j}^{j+p}} T_k = T_{j+p} \cdot T_{j+p-1} \cdot \ldots \cdot T_j.$$

Also, we will assume $\gamma(\tau) := \tau$ if $\gamma(\tau) < \tau$, where $k(\tau)$ is the only $k \in \mathbb{Z}$ such that $\tau \in I_{k(\tau)} = [t_{k(\tau)}, t_{k(\tau)+1})$. We will adopt the following notation:

$$\prod_{j=r+1}^{r} A_j = 1, \qquad \sum_{j=r+1}^{r} A_j = 0.$$

Let the system

$$\begin{array}{ll} w'(t) = A(t)w(t) + B(t)w(\gamma(t)), & t \neq t_k \\ w(t_k) = (I + C_k)\, w(t_k^-), & t = t_k \\ w_0 = w(\tau). & \end{array} \tag{4.8}$$

We will construct the fundamental matrix for system (4.8).
Let $t, \tau \in I_k = [t_k, t_{k+1})$ for some $k \in \mathbb{Z}$. In this interval, we are in the presence of the ordinary system

$$w'(t) = A(t)w(t) + B(t)w(\zeta_k).$$

So, the unique solution can be written as

$$w(t) = \Phi(t, \tau)w(\tau) + \int_\tau^t \Phi(t, s)B(s)w(\zeta_k)ds. \tag{4.9}$$

Keeping in mind $I_k^+ = [t_k, \zeta_k]$, evaluating the last expression at $t = \zeta_k$ we have

$$w(\zeta_k) = \Phi(\zeta_k, \tau)w(\tau) + \int_\tau^{\zeta_k} \Phi(\zeta_k, s)B(s)w(\zeta_k)ds. \tag{4.10}$$

Hence, we get

$$\left(I + \int_{\zeta_k}^{\tau} \Phi(\zeta_k, s)B(s)ds \right) w(\zeta_k) = \Phi(\zeta_k, \tau)w(\tau).$$

I.e.

$$w(\zeta_k) = J^{-1}(\tau, \zeta_k)\Phi(\zeta_k, \tau)w(\tau). \tag{4.11}$$

Then, by the definition of $E(t, \tau) = \Phi(t, \tau)J(t, \tau)$, we have

$$w(\zeta_k) = E^{-1}(\tau, \zeta_k)w(\tau). \tag{4.12}$$

Now, from (4.9) working on $I_k^- = [\zeta_k, t_{k+1})$, considering $\tau = \zeta_k$, we have

$$\begin{aligned} w(t) &= \Phi(t, \zeta_k)w(\zeta_k) + \int_{\zeta_k}^t \Phi(t, s)B(s)w(\zeta_k)ds \\ &= \Phi(t, \zeta_k) \left(I + \int_{\zeta_k}^t \Phi(\zeta_k, s)B(s)ds \right) w(\zeta_k). \end{aligned}$$

I.e.,

$$w(t) = E(t, \zeta_k)w(\zeta_k). \tag{4.13}$$

So, by (4.12), we can rewrite (4.13) as

$$w(t) = E(t,\zeta_k)E^{-1}(\tau,\zeta_k)w(\tau). \tag{4.14}$$

Then, setting

$$W(t,s) = E(t,\gamma(s))E^{-1}(s,\gamma(s)), \quad \text{if } t,s \in I_k = [t_k, t_{k+1}), \tag{4.15}$$

we have the solution for (4.8) for $t \in I_k$

$$w(t) = W(t,\tau)w(\tau). \tag{4.16}$$

Next, if we consider $\tau = t_k$ and assuming left side continuity of (4.16) at $t = t_{k+1}$, we have

$$w(t_{k+1}^-) = W(t_{k+1}, t_k)w(t_k).$$

Then, applying the impulsive condition to the last equation, we get

$$\begin{aligned} w(t_{k+1}) &= (I + C_{k+1})\, w(t_{k+1}^-) \\ &= (I + C_{k+1})\, W(t_{k+1}, t_k)w(t_k). \end{aligned}$$

This expression corresponds to a finite-difference equation. Then, by solving it, we get

$$w(t_{k(t)}) = \left(\overset{\leftarrow}{\prod_{j=k(\tau)+1}^{k(t)-1}} (I + C_{j+1})\, W(t_{j+1}, t_j) \right) w\left(t_{k(\tau)+1}\right). \tag{4.17}$$

Finally, by (4.16) and the impulsive condition, we have

$$w(t_{k(\tau)+1}) = (I + C_{k(\tau)+1})W(t_{k(\tau)+1}, \tau)w(\tau).$$

Hence, considering $\tau = t_k$ in (4.16) and applying (4.17) we get

$$\begin{aligned} w(t) &= W(t, t_{k(t)}) \left(\overset{\leftarrow}{\prod_{j=k(\tau)+1}^{k(t)-1}} (I + C_{j+1})\, W(t_{j+1}, t_j) \right) \left(I + C_{k(\tau)+1}\right) W(t_{k(\tau)+1}, \tau)w(\tau) \\ &= W(t,\tau)w(\tau), \quad \text{for } t \in I_{k(t)} \text{ and } \tau \in I_{k(\tau)}. \end{aligned} \tag{4.18}$$

The last equation is the solution of (4.8) on $[\tau, t)$.

We call the expression

$$W(t,\tau) = W(t, t_{k(t)}) \left(\overset{\leftarrow}{\prod_{j=k(\tau)+1}^{k(t)-1}} (I + C_{j+1})\, W(t_{j+1}, t_j) \right) \left(I + C_{k(\tau)+1}\right) W(t_{k(\tau)+1}, \tau) \tag{4.19}$$

the fundamental matrix for (4.8) for $t \in I_{k(t)}$ and $\tau \in I_{k(\tau)}$.

Remark 4.2. We use the decomposition of $I_k = I_k^+ \cup I_k^-$ to define W. In fact, we can rewrite (4.19) in terms of the advanced and delayed parts using (4.15):

$$\begin{aligned} W(t,\tau) = E(t,\zeta_{k(t)})E^{-1}(t_{k(t)},\zeta_{k(t)}) &\left(\overset{\leftarrow}{\prod_{j=k(\tau)+1}^{k(t)-1}} (I + C_{j+1})\, E(t_{j+1},\zeta_j)E^{-1}(t_j,\zeta_j) \right) \\ &\cdot \left(I + C_{k(\tau)+1}\right) E(t_{k(\tau)+1},\gamma(\tau))E^{-1}(\tau,\gamma(\tau)), \qquad \zeta_j = \gamma(t_j) \end{aligned}$$

for $t \in I_{k(t)}$ and $\tau \in I_{k(\tau)}$.

Remark 4.3.

a) Considering $B(t) = 0$, we recover the classical fundamental matrix of the impulsive linear differential equation (see [18]).

b) If $C_k = 0, \forall k \in \mathbb{Z}$, we recover the DEPCAG case studied by M. Pinto in [17].

c) If we consider $\gamma(t) = p\left[\frac{t+l}{p}\right]$, with $p < l$, we recover the IDEPCA case studied by K.-S. Chiu in [6].

4.2 The variation of parameter formula for IDEPCAG

Let the IDEPCAG

$$\begin{array}{ll} y'(t) = A(t)y(t) + B(t)y(\gamma(t)) + F(t), & t \neq t_k, \\ y(t_k) = (I + C_k)y(t_k^-) + D_k, & t = t_k, \\ y_0 = y(\tau). & \end{array} \tag{4.20}$$

If $\tau, t \in I_k = [t_k, t_{k+1})$, then the unique solution of (4.20) is

$$y(t) = \Phi(t,\tau)y(\tau) + \int_\tau^t \Phi(t,s)B(s)y(\zeta_k)ds + \int_\tau^t \Phi(t,s)f(s)ds.$$

Then, if $\tau = \zeta_k$, we have

$$\begin{aligned} y(t) &= \Phi(t,\zeta_k)y(\zeta_k) + \int_{\zeta_k}^t \Phi(t,s)B(s)y(\zeta_k)ds + \int_{\zeta_k}^t \Phi(t,s)f(s)ds \\ &= \Phi(t,\zeta_k)\left(I + \int_{\zeta_k}^t \Phi(\zeta_k,s)B(s)ds\right)y(\zeta_k) + \int_{\zeta_k}^t \Phi(t,s)f(s)ds \\ &= \Phi(t,\zeta_k)J\,(t,\zeta_k)\,y(\zeta_k) + \int_{\zeta_k}^t \Phi(t,s)f(s)ds, \end{aligned}$$

i.e.

$$y(t) = E(t,\zeta_k)y(\zeta_k) + \int_{\zeta_k}^t \Phi(t,s)f(s)ds. \tag{4.21}$$

Now, if we consider $t = \tau$ in (4.21) we have

$$y(\tau) = E(\tau,\zeta_k)y(\zeta_k) + \int_{\zeta_k}^\tau \Phi(\tau,s)f(s)ds,$$

and, by $(H3)$, we get the following estimation for $y(\zeta_k)$

$$y(\zeta_k) = E^{-1}(\tau,\zeta_k)\left(y(\tau) + \int_\tau^{\zeta_k} \Phi(\tau,s)f(s)ds\right). \tag{4.22}$$

Then, applying (4.22) in (4.21) we obtain

$$y(t) = E(t,\zeta_k)E^{-1}(\tau,\zeta_k)\left(y(\tau) + \int_\tau^{\zeta_k} \Phi(\tau,s)f(s)ds\right) + \int_{\zeta_k}^t \Phi(t,s)f(s)ds,$$

i.e.,

$$y(t) = W(t,\tau)y(\tau) + \int_\tau^{\zeta_k} W(t,\tau)\Phi(\tau,s)f(s)ds + \int_{\zeta_k}^t \Phi(t,s)f(s)ds, \qquad \tau, t \in I_k. \tag{4.23}$$

Next, taking the left-side limit to the last expression, we have

$$y(t_{k+1}^-) = W(t_{k+1},\tau)\left(y(\tau)+\int_{\tau}^{\zeta_k}\Phi(\tau,s)f(s)ds\right)+\int_{\zeta_k}^{t_{k+1}}\Phi(t_{k+1},s)f(s)ds. \tag{4.24}$$

Applying the impulsive condition, we get

$$y(t_{k+1}) = (I+C_{k+1})\,y(t_{k+1}^-)+D_{k+1},$$

or

$$\begin{aligned} y(t_{k+1}) = &\ (I+C_{k+1})\,W(t_{k+1},\tau)\left(y(\tau)+\int_{\tau}^{\zeta_k}\Phi(\tau,s)f(s)ds\right) \\ &+\int_{\zeta_k}^{t_{k+1}}(I+C_{k+1})\,\Phi(t_{k+1},s)f(s)ds+D_{k+1}. \end{aligned}$$

Therefore, considering $\tau = t_k$ in the last expression we have

$$\begin{aligned} y(t_{k+1}) = &\ (I+C_{k+1})\,W(t_{k+1},t_k)\left(y(t_k)+\int_{t_k}^{\zeta_k}\Phi(t_k,s)f(s)ds\right) \\ &+\int_{\zeta_k}^{t_{k+1}}(I+C_{k+1})\,\Phi(t_{k+1},s)f(s)ds+D_{k+1}, \end{aligned}$$

or

$$y(t_{k+1}) = W_k\left(y(t_k)+\alpha_k^+\right)+\alpha_k^-+\beta_k,$$

which corresponds to a non-homogeneous linear difference equation, where

$$\begin{aligned} W_k &= (I+C_{k+1})W(t_{k+1},t_k), \\ \alpha_k^+ &= \int_{t_k}^{\zeta_k}\Phi(t_k,s)f(s)ds, \\ \alpha_k^- &= \int_{\zeta_k}^{t_{k+1}}(I+C_{k+1})\,\Phi(t_{k+1},s)f(s)ds, \\ \beta_k &= D_{k+1}. \end{aligned}$$

Recalling that

$$W(t_{k(t)},\tau) = \left(\overset{\leftarrow}{\prod_{j=k(\tau)+1}^{k(t)-1}}(I+C_{j+1})\,W(t_{j+1},t_j)\right)\left(I+C_{k(\tau)+1}\right)W(t_{k(\tau)+1},\tau),$$

we get the discrete solution of (4.20):

$$\begin{aligned} y(t_{k(t)}) = &\left(\overset{\leftarrow}{\prod_{j=k(\tau)+1}^{k(t)-1}}(I+C_{j+1})\,W(t_{j+1},t_j)\right)(I+C_{k(\tau)+1})W(t_{k(\tau)+1},\tau)y(\tau) \\ &+\int_{\tau}^{\zeta_{k(\tau)}}W\left(t_{k(t)},\tau\right)\Phi(\tau,s)f(s)ds \\ &+\sum_{r=k(\tau)+1}^{k(t)-1}\left(\overset{\leftarrow}{\prod_{j=r}^{k(t)-1}}(I+C_{j+1})\,W(t_{j+1},t_j)\right)\int_{t_r}^{\zeta_r}\Phi(t_r,s)f(s)ds \\ &+\sum_{r=k(\tau)}^{k(t)-1}\left(\overset{\leftarrow}{\prod_{j=r+1}^{k(t)-1}}(I+C_{j+1})W(t_{j+1},t_j)\right)\int_{\zeta_r}^{t_{r+1}}(I+C_{r+1})\Phi(t_{r+1},s)f(s)ds \\ &+\sum_{r=k(\tau)}^{k(t)-1}\left(\overset{\leftarrow}{\prod_{j=r+1}^{k(t)-1}}(I+C_{j+1})W(t_{j+1},t_j)\right)D_{r+1}, \end{aligned}$$

or, written in terms of (4.19),

$$
\begin{aligned}
y(t_{k(t)}) = {} & W(t_{k(t)},\tau)y(\tau) + \int_{\tau}^{\zeta_{k(\tau)}} W\left(t_{k(t)},\tau\right)\Phi(\tau,s)f(s)ds \\
& + \sum_{r=k(\tau)+1}^{k(t)-1} \int_{t_r}^{\zeta_r} W(t_{k(t)},t_r)\Phi(t_r,s)f(s)ds \\
& + \sum_{r=k(\tau)}^{k(t)-1} \int_{\zeta_r}^{t_{r+1}} W(t_{k(t)},t_{r+1})(I+C_{r+1})\Phi(t_{r+1},s)f(s)ds \\
& + \sum_{r=k(\tau)}^{k(t)-1} W(t_{k(t)},t_{r+1})D_{r+1}.
\end{aligned}
\tag{4.25}
$$

Now, considering $\tau = t_k$ in (4.23) we have

$$
y(t) = W(t,t_{k(t)})y(t_{k(t)}) + \int_{t_{k(t)}}^{\zeta_{k(t)}} W(t,t_{k(t)})\Phi(t_{k(t)},s)f(s)ds + \int_{\zeta_{k(t)}}^{t} \Phi(t,s)f(s)ds.
$$

Finally, replacing $y(t_{k(t)})$ by (4.25) and rewriting in terms of (4.19), we get the variation of parameters formula for IDEPCAG (4.20):

$$
\begin{aligned}
y(t) = {} & W(t,\tau)y(\tau) \\
& + \int_{\tau}^{\zeta_{k(\tau)}} W(t,\tau)\Phi(\tau,s)f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} \int_{t_r}^{\zeta_r} W(t,t_r)\Phi(t_r,s)f(s)ds \\
& + \sum_{r=k(\tau)}^{k(t)-1} \int_{\zeta_r}^{t_{r+1}} W(t,t_{r+1})\left(I+C_{r+1}\right)\Phi(t_{r+1},s)f(s)ds \\
& + \int_{\zeta_{k(t)}}^{t} \Phi(t,s)f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} W(t,t_r)D_r, \quad \text{for } t \in [\tau, t_{k(t)+1}),
\end{aligned}
\tag{4.26}
$$

where W is the fundamental matrix of (4.8).

4.2.1 Green type matrix for IDEPCAG

If we define the following Green matrix type for IDEPCAG:

$$
\widetilde{W}(t,s) = \begin{cases} W^{+}(t,s), & \text{if } t_r \leq s \leq \gamma(s) \\ W^{-}(t,s), & \text{if } \gamma(s) < s \leq t_{r+1}, \end{cases}
\tag{4.27}
$$

where

$$
W^{+}(t,s) = W(t,t_r)\Phi(t_r,s), \quad \text{if } \; t_r \leq s \leq \gamma(s),\; s < t,
\tag{4.28}
$$

and

$$
W^{-}(t,s) = \begin{cases} W(t,t_{r+1})\left(I+C_{r+1}\right)\Phi(t_{r+1},s), & \text{if } \gamma(s) \leq s < t_{r+1},\; t > s,\; t \leq t_{k+1}, \\ \Phi(t,s), & \text{if } \gamma(t) < s \leq t < t_{r+1}. \end{cases}
\tag{4.29}
$$

Hence, we can see that

$$\int_{\tau}^{t} W^{+}(t,s)f(s)ds = \int_{\tau}^{\zeta_{k(\tau)}} W(t,\tau)\Phi(\tau,s)f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} \int_{t_r}^{\zeta_r} W(t,t_r)\Phi(t_r,s)f(s)ds,$$

$$\int_{\tau}^{t} W^{-}(t,s)f(s)ds = \sum_{r=k(\tau)}^{k(t)-1} \int_{\zeta_r}^{t_{r+1}} W(t,t_{r+1})\left(I+C_{r+1}\right)\Phi(t_{r+1},s)f(s)ds + \int_{\zeta_{k(t)}}^{t} \Phi(t,s)f(s)ds.$$

So, we have

$$\widetilde{W}(t,s) = W^{+}(t,s) + W^{-}(t,s).$$

In this way, (4.26) can be expressed as

$$y(t) = W(t,\tau)y(\tau) + \int_{\tau}^{t} \widetilde{W}(t,s)f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} W(t,t_r)D_r. \tag{4.30}$$

4.3 Some special cases of (4.20)

In the following, we present some r cases for (4.20).

1. Let $\gamma^{-}(t) = t_k$ and $\gamma^{+}(t) = t_{k+1}$, for all $t \in I_k = [t_k, t_{k+1})$. I.e., we are considering the completely delayed and advanced general piecewise constant arguments. Then, taking in account Remark 4.2, the solution of (4.20) for both cases $y_{-}(t)$ and $y_{+}(t)$ respectively are:

$$y_{-}(t) = W_{-}(t,\tau)y(\tau) + \sum_{r=k(\tau)}^{k(t)-1} \int_{t_r}^{t_{r+1}} W_{-}(t,t_{r+1})\left(I+C_{r+1}\right)\Phi(t_{r+1},s)f(s)ds + \int_{t_{k(t)}}^{t} \Phi(t,s)f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} W_{-}(t,t_r)D_r, \tag{4.31}$$

where

$$W_{-}(t,\tau) = E(t,t_{k(t)})\left(\overleftarrow{\prod_{j=k(\tau)+1}^{k(t)-1}}\left(I+C_{j+1}\right)E(t_{j+1},t_j)\right)\cdot\left(I+C_{k(\tau)+1}\right)E(t_{k(\tau)+1},\tau), \tag{4.32}$$

and

$$y_{+}(t) = W_{+}(t,\tau)y(\tau) + \int_{\tau}^{t_{k(\tau)+1}} W_{+}(t,\tau)\Phi(\tau,s)f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} \int_{t_r}^{t_{r+1}} W_{+}(t,t_r)\Phi(t_r,s)f(s)ds - \int_{t}^{t_{k(t)+1}} \Phi(t,s)f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} W_{+}(t,t_r)D_r, \tag{4.33}$$

where

$$W_+(t,\tau) = E(t,t_{k(t)+1})E^{-1}(t_{k(t)},t_{k(t)+1})\left(\leftarrow \prod_{j=k(\tau)+1}^{k(t)-1} \left(I+C_{j+1}\right) E^{-1}(t_j,t_{j+1})\right) \cdot \left(I+C_{k(\tau)+1}\right) E^{-1}(\tau,t_{k(\tau)+1}),$$

for $t \in I_{k(t)}$ and $\tau \in I_{k(\tau)}$, recalling that $\gamma(\tau) := \tau$ if $\gamma(\tau) < \tau$.

2. Let the IDEPCAG

$$\begin{aligned} w'(t) &= B(t)w(\gamma(t)), && t \neq t_k \\ w(t_k) &= (I+C_k)w(t_k^-), && t = t_k \\ w_0 &= w(\tau). \end{aligned} \tag{4.34}$$

We see that $\Phi(t,s) = I, E(t,s) = J(t,s)$ and $J(t,s) = I + \int_s^t B(u)du$, where I is the identity matrix. Hence the fundamental matrix for (4.34) is given by

$$W(t,\tau) = J(t,\zeta_{k(t)})J^{-1}(t_{k(t)},\zeta_{k(t)})\left(\leftarrow \prod_{j=k(\tau)+1}^{k(t)-1} \left(I+C_{j+1}\right) J(t_{j+1},\zeta_j)J^{-1}(t_j,\zeta_j)\right) \cdot \left(I+C_{k(\tau)+1}\right) J(t_{k(\tau)+1},\gamma(\tau))J^{-1}(\tau,\gamma(\tau)), \qquad \zeta_j = \gamma(t_j).$$

for $t \in I_{k(t)}$ and $\tau \in I_{k(\tau)}$.

This case is very important because it is used for the approximation of solutions of differential equations considering $\gamma(t) = \left[\frac{t}{h}\right] h$, with $h > 0$ fixed.

3. Let the IDEPCAG

$$\begin{aligned} w'(t) &= Aw(t) + Bw(\gamma(t)), && t \neq t_k \\ w(t_k) &= (I+C)w(t_k^-), && t = t_k \\ w_0 &= w(\tau), \end{aligned} \tag{4.35}$$

and

$$\begin{aligned} y'(t) &= Ay(t) + By(\gamma(t)) + f(t), && t \neq t_k \\ y(t_k) &= (I+C)y(t_k^-) + D_k, && t = t_k \\ y_0 &= y(\tau), \end{aligned} \tag{4.36}$$

where A^{-1} exist. By $(H3)$, we know that $J(t,\tau) = I + \int_\tau^t e^{A(\tau-s)}B\,ds$ is invertible, for $\tau, t \in I_k = [t_k, t_{k+1})$. Moreover, following [17], we see that

$$\begin{aligned} J(t,\tau) &= I + \int_\tau^t e^{A(\tau-s)}Bds \\ &= I + e^{A\tau}\left(\int_\tau^t (-A)e^{-As}ds\right)(-A^{-1})B \\ &= I + A^{-1}\left(I - e^{A(\tau-t)}\right)B. \end{aligned} \tag{4.37}$$

Then, as $E(t,\tau) = \Phi(t,\tau)J(t,\tau)$, we have

$$E(t,\tau) = e^{A(t-\tau)}\left(I + A^{-1}\left(I - e^{-A(t-\tau)}\right)B\right). \tag{4.38}$$

In light of the last calculations, we define

$$\begin{aligned}
&\widetilde{E}(t) = e^{At}\left(I + A^{-1}\left(I - e^{-At}\right)B\right)\\
&\eta_k^+ = \zeta_k - t_k, \qquad \eta_k^- = t_{k+1} - \zeta_k, \quad k \in \mathbb{Z},\\
&\eta(t) = t - \gamma(t).
\end{aligned}$$

Recalling that

$$\widehat{W}(t,s) = \widetilde{E}(t-\gamma(s))\widetilde{E}^{-1}(\eta(s)), \quad \text{if } t,s \in I_k = [t_k, t_{k+1}), \tag{4.39}$$

the solution of (4.35) is

$$w(t) = \widehat{W}(t,\tau)w(\tau),$$

where

$$\begin{aligned}
\widehat{W}(t,\tau) = \widetilde{E}(\eta(t))\widetilde{E}^{-1}(-\eta_{k(t)}^+)&\left(\overleftarrow{\prod_{j=k(\tau)+1}^{k(t)-1}}(I+C)\widetilde{E}(\eta_j^-)\widetilde{E}^{-1}(-\eta_j^+)\right) &(4.40)\\
&\cdot (I+C)\widetilde{E}(\eta_{k(\tau)+1}^-)\widetilde{E}^{-1}(\eta(\tau)), &(4.41)
\end{aligned}$$

is the fundamental matrix for (4.35) with $t \in I_{k(t)}$ and $\tau \in I_{k(\tau)}$.

The solution for (4.36) is given by

$$\begin{aligned}
y(t) = \widetilde{E}(\eta(t))\widetilde{E}^{-1}(-\eta_{k(t)}^+)&\left(\overleftarrow{\prod_{j=k(\tau)+1}^{k(t)-1}}(I+C)\widetilde{E}(\eta_j^-)\widetilde{E}^{-1}(-\eta_j^+)\right)\\
&\cdot (I+C)\widetilde{E}(\eta_{k(\tau)+1}^-)\widetilde{E}^{-1}(\eta(\tau))\left(y(\tau) + \int_\tau^{\zeta_{k(\tau)}} e^{A(\tau-s)}f(s)ds\right) \qquad (4.42)\\
&+ \widetilde{E}(\eta(t))\widetilde{E}^{-1}(-\eta_{k(t)}^+)\\
&\cdot\left\{\sum_{r=k(\tau)+1}^{k(t)}\left(\overleftarrow{\prod_{j=r}^{k(t)-1}}(I+C)\widetilde{E}(\eta_j^-)\widetilde{E}^{-1}(-\eta_j^+)\right)\int_{t_r}^{\zeta_r} e^{A(t_r-s)}f(s)ds\right.\\
&+\sum_{r=k(\tau)}^{k(t)-1}\left(\overleftarrow{\prod_{j=r+1}^{k(t)}}(I+C)\widetilde{E}(\eta_j^-)\widetilde{E}^{-1}(-\eta_j^+)\right)\int_{\zeta_r}^{t_{r+1}}(I+C)e^{A(t_{r+1}-s)}f(s)ds\\
&\left.+\sum_{r=k(\tau)}^{k(t)-1}\left(\overleftarrow{\prod_{j=r+1}^{k(t)}}(I+C)\widetilde{E}(\eta_j^-)\widetilde{E}^{-1}(-\eta_j^+)\right)D_r\right\}\\
&+\int_{\zeta_{k(t)}}^t e^{A(t-s)}f(s)ds.
\end{aligned}$$

Also, if $\eta = \eta_k^+ = \eta_k^-,\ k \in \mathbb{Z},\ \widehat{E} = (I+C)\widetilde{E}(\eta)\widetilde{E}^{-1}(-\eta)$, the solution of (4.35) is

$$w(t) = \widehat{W}(t,\tau)w(\tau),$$

where

$$\widehat{W}(t,\tau) = \widetilde{E}(\eta(t))\widetilde{E}^{-1}(-\eta_{k(t)}^+)\widehat{E}^{k(t)-k(\tau)-1}(I+C)\widetilde{E}(\eta)\widetilde{E}^{-1}(\eta(\tau)),$$

is the fundamental matrix for (4.35) with $t \in I_{k(t)}$ and $\tau \in I_{k(\tau)}$.
The solution for (4.36) is given by

$$\begin{aligned} y(t) &= \widetilde{E}(\eta(t))\widetilde{E}^{-1}(-\eta^{+}_{k(t)})\widehat{E}^{k(t)-k(\tau)-1}(I+C)\widetilde{E}(\eta^{-}_{k(\tau)+1})\widetilde{E}^{-1}(\eta(\tau)) \\ &\cdot \left(y(\tau) + \int_{\tau}^{\zeta_{k(\tau)}} e^{A(\tau-s)} f(s)ds \right) \\ &+ \widetilde{E}(\eta(t))\widetilde{E}^{-1}(-\eta^{+}_{k(t)}) \cdot \left\{ \sum_{r=k(\tau)+1}^{k(t)} \widehat{E}^{k(t)-r} \int_{t_r}^{\zeta_r} e^{A(t_r-s)} f(s)ds \right. \\ &+ \left. \sum_{r=k(\tau)}^{k(t)-1} \widehat{E}^{k(t)-r} \int_{\zeta_r}^{t_{r+1}} (I+C)e^{A(t_{r+1}-s)} f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} \widehat{E}^{k(t)-r} D_r \right\} \\ &+ \int_{\zeta_{k(t)}}^{t} e^{A(t-s)} f(s)ds. \end{aligned} \tag{4.43}$$

Remark 4.4.

1. We recover the variation of parameters concluded in [17] when $D_r = C_r = 0$.
2. Also, our result implies the variation of constant formulas given in section 1.2

5 Some examples of linear IDEPCAG systems

In [16], H. Bereketoglu and G. Oztepe studied the following linear IDEPCAG

$$\begin{array}{ll} z'(t) = A(t)\left(z(t) - z(\gamma(t))\right) + f(t), & t \neq t_k \\ z(t_k) = z(t_k^-) + D_k, & t = t_k. \\ z(\tau) = z_0 & \end{array} \tag{5.1}$$

where $\gamma(t)$ is some piecewise constant argument of generalized type, $A(t)$ is a continuous locally integrable matrix, $D : \mathbb{N} \to \mathbb{R}$ is such that $D_k \neq 0, \forall k \in \mathbb{N}$. The authors originally considered the cases $\gamma_1(t) = [t+1]$, and $\gamma_2(t) = [t-1]$. Hence, $t_k = k, \zeta_{1,k}k = k+1$ and $\zeta_{2,k} = k-1$, respectively.

Let $\Phi(t)$ be the fundamental matrix of the ordinary differential system

$$x'(t) = A(t)x(t). \tag{5.2}$$

It is well known that $\Phi^{-1}(t)$ is the fundamental matrix of the adjoint system associated with (5.2). So, it satisfies

$$\left(\Phi^{-1}\right)'(t) = -\Phi^{-1}(t)A(t).$$

Therefore, we have

$$\begin{aligned} J(t,t_k) &= I - \int_{t_k}^{t} \Phi(t_k,s)A(s)ds \\ &= I + \Phi(t_k)\left(\int_{t_k}^{t} -\Phi^{-1}(s)A(s)ds\right) \\ &= I + \Phi(t_k)\left(\Phi^{-1}(t) - \Phi^{-1}(t_k)\right) \\ &= \Phi(t_k,t) \\ &= \Phi^{-1}(t,t_k), \end{aligned}$$

$E(t,t_k) = \Phi(t,t_k)J(t,t_k) = \Phi(t,t_k)\Phi^{-1}(t,t_k) = I$, and, as a result of last estimations, for $t,t' \in I_k$, we have $W(t,t') = I$. Hence, the linear homogeneous IDEPCAG (is a DEPCAG because $C_k = 0$) system

$$\begin{aligned} &w'(t) = A(t)\left(w(t) - w(\gamma(t))\right), && t \neq t_k \\ &w(t_k) = w(t_k^-) && t = t_k. \\ &w(\tau) = w_0, && \end{aligned} \tag{5.3}$$

has the constant solution $w(t) = w(\tau)$.

Finally, for the variation of parameters formula (4.26), the solution for (5.1) is

$$\begin{aligned} y(t) = y(\tau) &+ \int_{\tau}^{\zeta_{k(\tau)}} \Phi(\tau,s)f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} \int_{t_r}^{\zeta_r} \Phi(t_r,s)f(s)ds \\ &+ \sum_{r=k(\tau)}^{k(t)-1} \int_{\zeta_r}^{t_{r+1}} \Phi(t_{r+1},s)f(s)ds + \int_{\zeta_{k(t)}}^{t} \Phi(t,s)f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} D_r, \end{aligned}$$

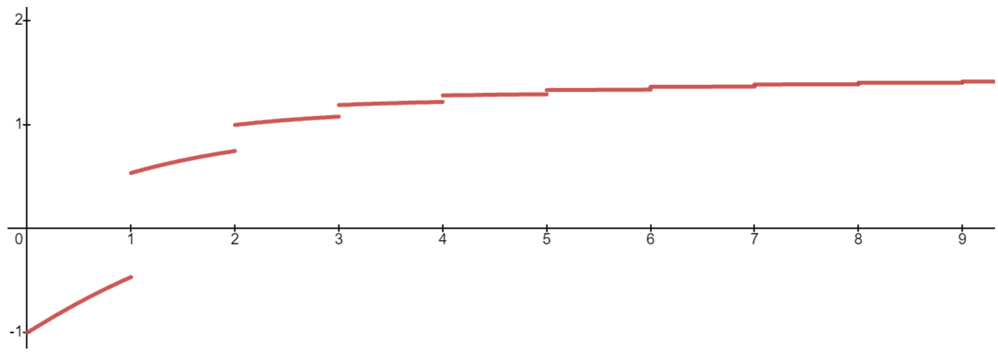

Figure 5.1: Solution of (5.1) with $\gamma(t) = [t] + 7/10$, $D_r = 1/r^2$, $A(t) = 1/(t+1)$, $f(t) = \exp(-t)$ and $y(0) = y_0 = -1$.

Remark 5.1. This is the IDEPCAG case for the well-known differential equation studied by K. L Cooke and J. A. Yorke in [10]. The authors investigated the following delay differential equation (DDE):

$$x'(t) = g(x(t)) - g(x(t-L)),$$

where $x(t)$ denotes the number of individuals in a population, the number of births is $g(x(t))$, and L is the constant life span of the individuals in the population. Then, the number of deaths $g(x(t-L))$. Since the difference $g(x(t)) - g(x(t-L)]$ means the change of the population. Therefore $x'(t)$ corresponds to the growth of the population at instant t.

In (5.3), we considered $g(x(t)) = A(t)x(t)$ and the constant delay in the Cooke–Yorke equation is regarded as a piecewise constant argument $\gamma(t)$. Notice that if D_r is summable and $f(t) = 0\ \forall t \geq \tau$, then the solution of (5.1) tends to the constant

$$y_\infty = y(\tau) + \sum_{t_r \geq t_{k(\tau)+1}} D_r, \quad \text{as } t \to \infty,$$

no matter what $\gamma(t)$ was used. For further about asymptotics in IDEPCAG, see [4].

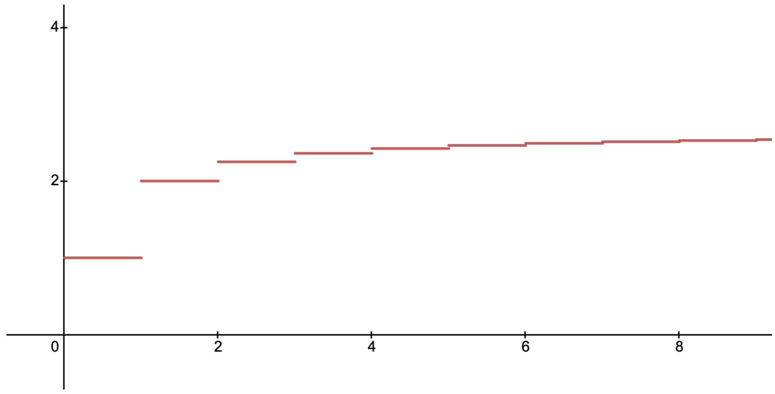

Figure 5.2: Solution of (5.1) with $D_k = 1/k^2,\ f(t) = 0$ and $z(0) = 1$.

Let us consider the following IDEPCA

$$\begin{aligned}
&z'(t) = \sin(2\pi t) z\left(\left[\tfrac{t}{h}\right] h + \beta h\right) + 1, && t \neq kh, \quad k \in \mathbb{N}, \\
&z(kh) = \left(-\frac{1}{2}\right) z(kh^-) + \frac{1}{2}, && t = kh, \\
&z(0) = z_0,
\end{aligned} \tag{5.4}$$

where $h > 0$, $0 \leq \beta \leq 1$.

It is easy to see that $t_k = kh,\ \zeta_k = (k+\beta)h,\ k \in \mathbb{N}$ and

$$I_k^+ = [kh, (k+\beta)h], \quad I_k^- = [(k+\beta)h, (k+1)h).$$

We see that

$$\begin{aligned}
&\nu_k^+(\sin(2\pi t)) \leq \beta h < 1, \quad \text{if } h \text{ is small enough}, \\
&\nu_k^-(\sin(2\pi t)) \leq (1-\beta) h < 1, \quad \text{if } h \text{ is small enough}, \\
&E(t,\tau) = 1 + \int_\tau^t \sin(2\pi s) ds.
\end{aligned}$$

The fundamental matrix of the homogeneous equation associated with (5.4) is

$$\begin{aligned}
W(t,0) = &\left(1 + \int_{[t/h]h+\beta h}^{t} \sin(2\pi s)\, ds\right) \left(1 + \int_{[t/h]h+\beta h}^{[t/h]h} \sin(2\pi s)\, ds\right)^{(-1)} \\
&\cdot \left(-\frac{1}{2}\right)^{[t/h]} \left(\prod_{j=0}^{[t/h]-1} \left(1 + \int_{(j+\beta)h}^{(j+1)h} \sin(2\pi s)\, ds\right) \left(1 + \int_{(j+\beta)h}^{jh} \sin(2\pi s)\, ds\right)^{(-1)} \right).
\end{aligned}$$

Hence, the solution of (5.4) is

$$\begin{aligned}
z(t) = W(t,0) z_0 + \left(-\frac{1}{2}\right) (1-\beta) h \sum_{r=0}^{[t/h]-1} W(t, (r+1)h) + (t - ([t/h]h + \beta h)) \\
+ W(t,0) \beta h + \beta h \sum_{r=1}^{[t/h]} W(t, rh) + \left(-\frac{1}{2}\right) \sum_{r=0}^{[t/h]-1} W(t, (r+1)h).
\end{aligned}$$

The piecewise constant used in this example was introduced in [21] to study the approximation of solutions of differential equations (under some stability assumptions and taking $h \to 0$.)

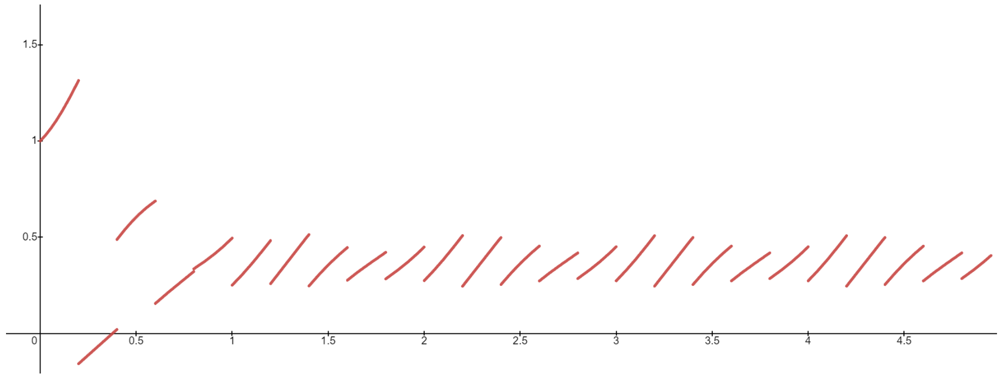

Figure 5.3: Solution of (5.4) with $h = \beta = 0, 2$.

6 Conclusions

In this work, we gave a variation of parameters formula for impulsive differential equations with piecewise constant arguments. We analyzed the constant coefficients case and gave several examples of formulas applied to some concrete piecewise constant arguments. We extended some cases treated before and showed the effect of the impulses in the dynamic.

Acknowledgments

Manuel Pinto thanks for the support of Fondecyt project 1170466.

Ricardo Torres thanks to DESMOS PBC for granting permission to use the images employed in this work. They were created with the DESMOS graphic calculator `https://www.desmos.com/calculator`.

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