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# Global bifurcation of positive solutions for a superlinear $p$-Laplacian system

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**Abstract.** We are concerned with the principal eigenvalue of

$$\begin{cases} -\Delta_p u = \lambda\theta_1\varphi_p(v), & x \in \Omega, \\ -\Delta_p v = \lambda\theta_2\varphi_p(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega \end{cases} \tag{$P$}$$

and the global structure of positive solutions for the system

$$\begin{cases} -\Delta_p u = \lambda f(v), & x \in \Omega, \\ -\Delta_p v = \lambda g(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{$Q$}$$

where $\varphi_p(s) = |s|^{p-2}s$, $\Delta_p s = \operatorname{div}(|\nabla s|^{p-2}\nabla s)$, $\lambda > 0$ is a parameter, $\Omega \subset \mathbb{R}^N$, $N > 2$, is a bounded domain with smooth boundary $\partial\Omega$, $f, g : \mathbb{R} \to (0, \infty)$ are continuous functions with $p$-superlinear growth at infinity. We obtain the principal eigenvalue of $(P)$ by using a nonlinear Krein–Rutman theorem and the unbounded branch of positive solutions for $(Q)$ via bifurcation technology.

**Keywords:** $p$-Laplacian, principal eigenvalue, positive solutions, bifurcation.

**2020 Mathematics Subject Classification:** 35B32, 35B40, 35J92.

## 1 Introduction

In this paper, we are concerned with the global structure of positive solutions for the system

$$\begin{cases} -\Delta_p u = \lambda f(v), & x \in \Omega, \\ -\Delta_p v = \lambda g(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Delta_p s = \text{div}(|\nabla s|^{p-2}\nabla s)$, $\lambda > 0$ is a parameter, $\Omega \subset \mathbb{R}^N$, $N > 2$, is a bounded domain with smooth boundary $\partial\Omega$, $f, g : \mathbb{R} \to (0, \infty)$ are continuous functions with $p$-superlinear growth at infinity.

The bifurcation behavior for $p$-Laplacian scalar equation has been investigated by many authors, see [11, 12, 22, 23, 30] for finite interval ($N = 1$) and [10, 18] for bounded domain ($N > 1$). For example, in [18], Fleckinger and Reichel established the global solution branches for the problem

$$\begin{cases} -\Delta_p u = \lambda(1 + u^q), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\lambda \geqslant 0$ and $q > p - 1$. Let $p^* := Np/(N - p)$ if $N > p$ and $p^* := \infty$ if $n \leqslant p$. They obtained, in supercritical case, that is, $q > p^* - 1$, then there exists an unbounded continuum $\mathscr{C}^+ \subset [0, +\infty) \times C_0^1(\bar{\Omega})$ of solutions of (1.2). Moreover, in subcritical case, that is, $q \in (p - 1, p^* - 1)$, then $\mathscr{C}^+$ is bounded in the $\lambda$-direction and becomes unbounded near $\lambda = 0$ under some additional conditions.

In [8], Chhetri and Girg studied global structure of the semilinear system

$$\begin{cases} -\Delta u = \lambda f(v), & x \in \Omega, \\ -\Delta v = \lambda g(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

they make the following assumptions:

(H1) $f, g \in C(\mathbb{R}, (0, \infty))$ are continuous and non-decreasing functions;

(C2) $f$ and $g$ satisfy

$$\lim_{s \to +\infty} \frac{f(s)}{s} = +\infty, \qquad \lim_{s \to +\infty} \frac{g(s)}{s} = +\infty.$$

Under (H1) and (C2), they obtained the global behavior of positive solutions set of (1.3) by using bifurcation technology. To be more precise, in supercritical case, they obtained a component of positive solutions for (1.3), emanating from the origin, which is bounded in positive $\lambda$-direction. If in addition, $\Omega$ is convex, and $f, g \in C^1$ satisfy certain subcriticality conditions, they showed that the component must bifurcate from infinity at $\lambda = 0$.

As for $p$-Laplacian system (1.1), the first result of which we are aware concerning is the one by Hai and Shivaji in [20], by means of sub- and supersolutions, it was proved that (1.1) has a large positive solution $(u, v)$ for $\lambda > \lambda_0$ with some $p$-sublinear conditions for $f$ and $g$. Quite recently, there are some authors concerned with the positive solutions for $p$-Laplacian systems, refer to [9, 15, 17, 26, 27, 29] and references therein. But all of them only obtain the positive solution and do not provide any information about the global structure of positive solutions set.

The global structure is very useful for computing the numerical solutions of differential equations as it can be used to guide numerical work. For example, it can be used to estimate the $u$-interval in advance in applying the finite difference method and when applying the shooting method, it can be used to restrict the range of initial values that need to be considered.

As we all know, if we want to get global structure of positive solutions for (1.1) by using bifurcation technology, it is necessary to investigate the eigenvalue of corresponding eigenvalue problem. However, to the best of our knowledge, the spectral theory of the corresponding $p$-Laplacian system has not yet been established. In fact, for $p = 2$, the principal eigenvalue $\nu_1$ of linear system corresponding to (1.3) can be directly expressed as

$$\nu_1 = \hat{\eta}_1 / \sqrt{\theta_1 \theta_2},$$

see [7, Prop. B.1], where $\hat{\eta}_1$ is the principal eigenvalue of scalar equation. However, this result do not hold for $p \neq 2$ because $p$-Laplacian operator is neither self-adjoint linear nor symmetric. In order to overcome this, we introduce a nonlinear version of Krein–Rutman theorem established by Arapostathis [2]. Let $\varphi_p(s) = |s|^{p-2}s$. By using the nonlinear Krein–Rutman theorem, we obtain that

$$\begin{cases} -\Delta_p u = \lambda \theta_1 \varphi_p(v), & x \in \Omega, \\ -\Delta_p v = \lambda \theta_2 \varphi_p(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

has a positive simple eigenvalue, expressed as $\mu_1$, having the smallest absolute value, that is to say, $\mu_1$ is the principal eigenvalue of (1.4).

The additional difficulty is the bifurcation results in dealing with semilinear boundary value problems [8] cannot be applied directly to quasilinear problems. Hence, we use the jumps of the index of the trivial solution to obtain a branch of nontrivial solutions. In addition to that, since $\Delta_p$ is asymmetric, the proof used in Theorem 1.1 of [8] do not applicable to (1.1). To this end, we adopt a new approach, with the help of sub- and supersolutions, to prove the nonexistence of solution for (1.1).

Let $X = C(\bar{\Omega}) \times C(\bar{\Omega})$, it is easy to know $X$ is a Banach space endowed with the norm $\|(u_1, u_2)\| = \|u_1\|_C + \|u_2\|_C$, where $\|u\|_C$ is equipped with the supremum norm. By a solution of (1.1), we mean a $(\lambda, (u, v))$ that solves (1.1) in the weak sense, that is, $(u, v) \in E$, where $E := W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$, and satisfies

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \omega \mathrm{d}x = \lambda \int_\Omega f(v) \omega \mathrm{d}x,$$
$$\int_\Omega |v|^{p-2} \nabla v \nabla \varpi \mathrm{d}x = \lambda \int_\Omega g(u) \varpi \mathrm{d}x$$

for all $(\omega, \varpi) \in E$. We denote $\Pi$ of the form

$$\Pi = \overline{\{(\lambda, (u, v)) \in \mathbb{R} \times E \mid (\lambda, (u, v)) \text{ solution of } (1.1)\}}.$$

If $(\lambda, (u, v)) \in \Pi$ and $u > 0$, $v > 0$, then we say that $(\lambda, (u, v))$ is a *positive solution* of (1.1). By a *continuum* of solutions of (1.1) we mean a subset $\mathscr{K} \subset \Pi$ which is closed and connected. By a *component* of solutions set $\Pi$ we mean a continuum which is maximal with respect to inclusion ordering. We say that $\lambda_\infty$ is a *bifurcation point* from infinity if the solution set $\Pi$ contains a sequence $(\lambda_n, (u_n, v_n))$ such that $\lambda_n \to \lambda_\infty$ and $\|(u_n, v_n)\| \to +\infty$ as $n \to +\infty$. We say that a continuum $\mathscr{C}$ bifurcates from infinity at $\lambda \in \mathbb{R}$ if there exists a sequence of solutions $(\lambda_n, (u_n, v_n)) \in \mathscr{C}$ such that $\lambda_n \to \lambda_\infty$ and $\|(u_n, v_n)\| \to +\infty$ as $n \to +\infty$.

Let $\eta_1 > 0$ be the principal eigenvalue of the problem

$$\begin{cases} -\Delta_p u = \lambda \varphi_p(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

by [1], $\eta_1$ is simple, isolated, and the unique positive eigenvalue having a nonnegative eigenfunction $\chi_1$.

Further we assume that:

(H2) $f$ and $g$ satisfy

$$\lim_{s\to+\infty}\frac{f(s)}{\varphi_p(s)}=+\infty, \qquad \lim_{s\to+\infty}\frac{g(s)}{\varphi_p(s)}=+\infty.$$

We first state a nonexistence result in Theorem 1.1, which holds under weaker assumptions than (H1)–(H2). Theorem 1.2 gives a existence result when $f$ and $g$ have supercritical growth at infinite. Specifically, there is an unbounded branch $\mathscr{C}$ of positive solutions for (1.1), bifurcating from infinity and going through trivial solution, which is bounded in positive $\lambda$-direction. Moreover, when $\Omega$ is convex with $C^2$ boundary and $p\in(1,2)$, $f$ and $g$ satisfy certain subcritical growth restrictions, Theorem 1.3 shows that $\lambda=0$ is the unique bifurcation point from infinity for the continuum $\mathscr{C}$ obtained in Theorem 1.2.

**Theorem 1.1.** *Let $m>0$, and suppose $f(s),g(s)\geqslant m\varphi_p(s)$ for all $s>0$, then (1.1) has no solution for $\lambda\geqslant\bar\lambda:=\eta_1/m$.*

**Theorem 1.2** (Supercritical case)**.** *Let (H1)–(H2) hold. Then there exists an unbounded component $\mathscr{C}\subset\Pi$ satisfying the following:*

(a) *$(\lambda,(u,v))\in\mathscr{C}$ is positive whenever $\lambda\in(0,\bar\lambda)$;*

(b) *(0,(0,0)) is the only element belonging to $\mathscr{C}$ with $\lambda=0$;*

(c) $\mathrm{Proj}_{\lambda\in[0,+\infty)}\mathscr{C}\overset{def}{=}\{\lambda\in[0,+\infty)\mid\exists(u,v)\in E \text{ with } (\lambda,(u,v))\in\mathscr{C}\}\subset[0,\bar\lambda)$;

(d) *any sequence $(\lambda_n,(u_n,v_n))\in\mathscr{C}$ such that $\|(u_n,v_n)\|\to+\infty$ as $n\to+\infty$ and $\lambda_n>0$ must satisfy $\lambda_n\to0^+$ as $n\to+\infty$.*

**Theorem 1.3** (Subcritical case)**.** *Assume that (H1)–(H2) hold. Let $p\in(1,2)$, $N>2$ and $\Omega$ be convex with $C^2$ boundary, $f,g$ satisfy*

$$\lim_{s\to+\infty}\frac{f(s)}{s^{q_1}}=C, \qquad \lim_{s\to+\infty}\frac{g(s)}{s^{q_2}}=D, \tag{1.6}$$

*for some positive constants $C,D$, and $q_1q_2>(p-1)^2$ satisfy*

$$\max\left\{\frac{p(q_1+p-1)}{q_1q_2-(p-1)^2}-\frac{N-p}{p-1},\ \frac{p(q_2+p-1)}{q_1q_2-(p-1)^2}-\frac{N-p}{p-1}\right\}\geqslant0.$$

*Then $\mu_1=0$ is the unique bifurcation point from infinity, for the continuum $\mathscr{C}\subset\Pi$ from Theorem 1.2.*

**Corollary 1.4.** *We may obtain the number of positive solutions of (1.1) from Theorem 1.3:*

*(i)* *(1.1) has no positive solution for $\lambda\geqslant\bar\lambda$;*

*(ii)* *there exists $\underline{\lambda}<\bar\lambda$ such that (1.1) has at least two positive solutions for each $\lambda\in(0,\underline{\lambda})$.*

**Remark 1.5.** It is worth remarking that Theorem 1.1 and Theorem 1.2 are direct generalizations of the results in [8]. However, Theorem 1.3 only holds for $p\in(1,2)$ because *a priori estimates* established in [4] is not available to $p=2$.

## 2 Preliminaries

Next we state some notations from [2].

An *ordered Banach space* is a real Banach space $W$ with a cone $K$. When the interior of $K$, denoted as int$K$, is nonempty, we call $W$ a *strongly ordered* Banach space. As usual, we write $x \preceq y$ if $y - x \in K$. A continuous map $T : W \to W$ is

- order-preserving or increasing if $x \preceq y \Rightarrow T(x) \preceq T(y)$;
- homogeneous of degree one, or 1-homogeneous, if $T(tx) = tT(x)$ for all $t \geqslant 0$.

**Lemma 2.1** ([4, Lemma 1.1])**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{1,\beta}$ for some $\beta \in (0,1)$ and $g \in L^\infty(\Omega)$. Then the problem*

$$\begin{cases} \int_\Omega |\nabla u|^{p-2}\nabla u \nabla \omega dx = \lambda \int_\Omega f(u)\omega dx, \quad \forall \omega \in C_c^\infty(\Omega), \\ u \in W_0^{1,p}(\Omega), \quad p > 1 \end{cases} \tag{2.1}$$

*has a unique solution $u \in C_0^1(\bar{\Omega})$. Moreover, if we define the operator $K : L^\infty(\Omega) \to C_0^1(\bar{\Omega}) : g \mapsto u$ where $u$ is the unique solution of (2.1), then $K$ is continuous, compact and order-preserving.*

Now we give a definition of weak sub- and supersolutions of (1.1), which is defined by [20].

**Definition 2.2.** We say that $(\alpha_u, \alpha_v)$ is a weak subsolution of problem (1.1) if $(\alpha_u, \alpha_v)$ satisfies

$$\begin{aligned} \int_\Omega |\nabla \alpha_u|^{p-2}\nabla \alpha_u \nabla \omega \mathrm{d}x \leqslant \lambda \int_\Omega f(\alpha_v)\omega \mathrm{d}x, \\ \int_\Omega |\nabla \alpha_v|^{p-2}\nabla \alpha_v \nabla \omega \mathrm{d}x \leqslant \lambda \int_\Omega g(\alpha_u)\omega \mathrm{d}x \end{aligned}$$

for all $\omega \in W_0^{1,p}(\Omega)$ with $\omega \geqslant 0$. Similarly, we say that $(\beta_u, \beta_v)$ is a weak supersolution of problem (1.1) if $(\beta_u, \beta_v)$ satisfies

$$\begin{aligned} \int_\Omega |\nabla \beta_u|^{p-2}\nabla \beta_u \nabla \omega \mathrm{d}x \geqslant \lambda \int_\Omega f(\beta_v)\omega \mathrm{d}x, \\ \int_\Omega |\nabla \beta_v|^{p-2}\nabla \beta_v \nabla \omega \mathrm{d}x \geqslant \lambda \int_\Omega f(\beta_u)\omega \mathrm{d}x \end{aligned}$$

for all $\omega \in W_0^{1,p}(\Omega)$ with $\omega \geqslant 0$.

**Proposition 2.3** ([34, Theorem 14.D])**.** *Let $Y$ be a Banach space with $Y \neq \{0\}$ and let $F : Y \to Y$ be compact. Then the solution component $\mathfrak{C} \subset \mathbb{R} \times Y$ of the equation*

$$x = \lambda F(x)$$

*which contains $(0,0) \in \mathbb{R} \times Y$ is unbounded as are both subsets*

$$\mathfrak{C}_\pm \overset{def}{=} \mathfrak{C} \cap (\mathbb{R}_\pm \times Y),$$

*where $\mathbb{R}_+ \overset{def}{=} [0,\infty)$ and $\mathbb{R}_- \overset{def}{=} (-\infty, 0]$.*

**Definition 2.4** ([33])**.** Let $Z$ be a Banach space and $\{C_n \mid n = 1, 2, \cdots\}$ be a certain infinite collection of subset of $Z$. Then the superior limit of $\mathcal{D}$ of $\{C_n\}$ is defined by

$$\mathcal{D} := \limsup_{n\to\infty} C_n = \{x \in Z \mid \exists \{n_i\} \subset \mathbb{N} \text{ and } x_{n_i} \in C_{n_i}, \text{ such that } x_{n_i} \to x\}.$$

**Lemma 2.5** ([33]). *Let $Z$ be a Banach space with the norm $\|\cdot\|_Z$, let $\{C_n\}$ be a family of closed subsets of $Z$. Assume that:*

(i) *there exist $z_n \in C_n$, $n = 1, 2, \dots$, and $z^* \in Z$, such that $z_n \to z^*$;*

(ii) *$d_n = \sup\{\|x\|_Z \mid x \in C_n\} = \infty$;*

(iii) *for every $R > 0$, $(\bigcup_{n=1}^{\infty} C_n) \cap B_R$ is a relatively compact set of $Z$, where*

$$B_R = \{x \in Z \mid \|x\|_Z \leqslant R\},$$

*then there exists an unbounded component $\mathcal{C}$ in $\mathcal{D}$ and $z^* \in \mathcal{C}$.*

The following nonlinear version of the Krein–Rutman theorem is firstly established by Mahadevan [25] and corrected by Arapostathis [2].

Let

$$\sigma_+(T) := \{\lambda > 0 : T(x) = \lambda x,\ x \in K \setminus \{0\}\}.$$

Consider the following hypotheses:

(B1) If $x \in \partial K \setminus \{0\}$, then $x - \beta T(x) \notin K$ for all $\beta > 0$.

(B2) If $x - y \in \partial K \setminus \{0\}$, then $x - y - \beta(T(x) - T(y)) \notin K$ for all $\beta > 0$.

**Lemma 2.6** ([2, Theorem 3]). *Let $W$ be strongly ordered, and $T : K \to K$ be an order-preserving, 1-homogeneous map with $\sigma_+(T) \neq \emptyset$.*

*(i) If (B1) holds, then $T(\operatorname{int} K) \subset \operatorname{int} K$, $\sigma_+(T)$ is a singleton, and all eigenvectors lie in* $\operatorname{int} K$.

*(ii) If (B2) holds, then the unique eigenvalue in $\sigma_+(T)$ is simple.*

## 3 Eigenvalue problems

Consider the eigenvalue of problem

$$\begin{cases} -\Delta_p u = \lambda \theta_1 \varphi_p(v), & x \in \Omega, \\ -\Delta_p v = \lambda \theta_2 \varphi_p(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{3.1}$$

where $\theta_1, \theta_2 > 0$, then (3.1) is equivalent to

$$-\Delta_p \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & \theta_1 \\ \theta_2 & 0 \end{pmatrix} \begin{pmatrix} \varphi_p(u) \\ \varphi_p(v) \end{pmatrix},$$

that is,

$$\begin{pmatrix} u \\ v \end{pmatrix} = -\Delta_p^{-1} \begin{pmatrix} 0 & \theta_1 \\ \theta_2 & 0 \end{pmatrix} \begin{pmatrix} \varphi_p(u) \\ \varphi_p(v) \end{pmatrix}.$$

It is equivalent to

$$U = -\Delta_p^{-1} A \varphi_p(U) =: HU,$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & \theta_1 \\ \theta_2 & 0 \end{pmatrix}.$$

Obviously, $H$ is 1-homogeneous by the property of $\varphi_p$ and $\varphi_p^{-1}$. Lemma 2.1 shows $H$ is a continuous, positively compact operator, and by the strictly increasing property of $\varphi_p$ and $(-\Delta_p)^{-1}$ we know that $H$ is strictly increasing.

Taking the cone to be

$$P = \left\{ w \in C_0^1(\overline{\Omega}) : w \geqslant 0 \text{ in } \Omega,\ \frac{\partial w}{\partial n} \leqslant 0 \text{ on } \partial\Omega \right\}.$$

**Lemma 3.1** ([32, Lemma 2.2.1])**.** *Let $u, v \in C^1(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$, $v|_{\partial\Omega} \geqslant 0$, $u > 0$ in $\Omega$, and $\partial_{\mathbf{n}} u|_{\partial\Omega} < 0$. Then there exists a positive constant $\varepsilon > 0$ such that $u + \varepsilon v > 0$ in $\Omega$.*

By Lemma 3.1, the interior of $P$ can be expressed as

$$\operatorname{int} P = \left\{ w \in C_0^1(\overline{\Omega}) : w > 0 \text{ in } \Omega,\ \frac{\partial w}{\partial n} < 0 \text{ on } \partial\Omega \right\}.$$

Let

$$K = P \times P.$$

Obviously, (B1) can be obtained by letting $y = 0$ in (B2), therefore, we only show that $H$ satisfies (B2). In fact, since $K$ is a closed set, if we choose $V_1, V_2 \in K$ such that $V_1 - V_2 \in \partial K \setminus \{0\}$, then $V_1 - V_2 \in K$, that is to say, $V_1 \succeq V_2$. The strong maximum principle shows $H(V_1), H(V_2) \in K$. In addition, by the property that $H$ is strictly increasing we have $H(V_1) \succeq H(V_2)$, this means $H(V_1) - H(V_2) \in K$ with $H(V_1) - H(V_2) \neq \{0\}$. On the other hand, $V_1 - V_2 \in \partial K \setminus \{0\}$, then any neighborhood of $V_1 - V_2$ contains some points that do not belong to $K$, this means

$$V_1 - V_2 - \beta(H(V_1) - H(V_2)) \notin K$$

for all $\beta > 0$.

**Lemma 3.2.** *System (3.1) has a positive and simple eigenvalue $\mu_1 > 0$ with a eigenfunction $U_1 = (\phi_1, \psi_1) > 0$.*

*Proof.* By Lemma 2.1, $H$ is compact, then $\sigma_+(T) \neq \emptyset$. By (i) of Lemma 2.6, $\sigma_+(T)$ is a singleton, denoted by $\lambda_1$, and all eigenvectors lie in $\operatorname{int} K$, that is, $\lambda_1$ is unique. Moreover, (B2) shows $\lambda_1$ is simple. Therefore, $H$ has a unique positive eigenvector (eigenfunction) $U_1 = (\phi_1, \psi_1)$, which satisfies

$$HU_1 = \lambda_1 U_1,$$

Subsequently, $\mu_1 = 1/\lambda_1$ is a simple, isolated and positive eigenvalue of (1.4). □

## 4 Auxiliary results

*Proof of Theorem 1.1.* Suppose that there is a sequence of $\{\lambda_n\}$ such that $\lambda_n \to \infty$ as $n \to \infty$, and (1.1) has a positive solution $(u_n, v_n)$ when $\lambda = \lambda_n$. Then there is a $\lambda_{n_0} > 0$ such that $\lambda_n > \frac{\eta_1}{m}$ for all $n > n_0$. We can choose a suitable $\epsilon_0 > 0$ such that $\lambda^* = \eta_1 + \epsilon_0$ and

$$\lambda_n m > \lambda^*, \qquad n > n_0.$$

Then

$$\begin{aligned}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla \omega \mathrm{d}x - \lambda^* \int_\Omega \varphi_p(v_n)\omega \mathrm{d}x\\ = \lambda_n \int_\Omega f(v_n)\omega \mathrm{d}x - \lambda^* \int_\Omega \varphi_p(v_n)\omega \mathrm{d}x\\ \geqslant \lambda_n \int_\Omega m\varphi_p(v_n)\omega \mathrm{d}x - \lambda^* \int_\Omega \varphi_p(v_n)\omega \mathrm{d}x\\ > 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla \omega \mathrm{d}x - \lambda^* \int_\Omega \varphi_p(u_n)\omega \mathrm{d}x\\ = \lambda_n \int_\Omega g(u_n)\omega \mathrm{d}x - \lambda^* \int_\Omega \varphi_p(u_n)\omega \mathrm{d}x\\ \geqslant \lambda_n \int_\Omega m\varphi_p(u_n)\omega \mathrm{d}x - \lambda^* \int_\Omega \varphi_p(u_n)\omega \mathrm{d}x\\ > 0.\end{aligned}$$

That is, $(u_n, v_n)$ is a weak supersolution of the problem

$$\begin{cases} -\Delta_p u = (\eta_1 + \epsilon_0)\varphi_p(v), & x \in \Omega,\\ -\Delta_p v = (\eta_1 + \epsilon_0)\varphi_p(u), & x \in \Omega,\\ u = 0 = v, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

On the other hand $t(\chi_1, \chi_1)$, $t > 0$, is a subsolution of (4.1). Letting $t > 0$ be such that $t(\chi_1, \chi_1) \leqslant (u_n, v_n)$, then by the method of sub- and super-solutions, for every $n > n_0$, (4.1) has a positive solution $(x_n, y_n)$. (The proof of the existence of $(x_n, y_n)$ have been showed, see [16, 20] for details. We omit it here.) On the other hand, since $\epsilon_0 > 0$ is arbitrary, this contradicts with the fact that $\eta_1$ is isolated. □

Now, consider an asymptotically positively homogeneous system of the form

$$\begin{cases} -\Delta_p u = \lambda\theta_1\varphi_p(v^+) + \lambda\widetilde{f}(v), & x \in \Omega,\\ -\Delta_p v = \lambda\theta_2\varphi_p(u^+) + \lambda\widetilde{g}(u), & x \in \Omega,\\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{4.2}$$

where $x^+ \overset{\text{def}}{=} \max\{0, x\}$ is the positive part of $x$, and $\theta_1, \theta_2$ are defined above. The nonlinear perturbations $\widetilde{f}, \widetilde{g} : \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions:

(A1) $\widetilde{f}$ and $\widetilde{g}$ are continuous, non-negative, and bounded functions;

(A2) $\theta_1\varphi_p(y^+) + \widetilde{f}(y) > 0$, $\theta_2\varphi_p(x^+) + \widetilde{g}(x) > 0$ for all $x, y \in \mathbb{R}$.

Let

$$\mathscr{T} = \{(\lambda, (u, v)) \in \mathbb{R} \times E \mid (\lambda, (u, v)) \text{ solution of } (4.2)\},$$

then we prove the following bifurcation result.

**Proposition 4.1.** *If $\mu_\infty$ is a bifurcation point from infinity for (4.2), then $\mu_\infty = \mu_1$. Moreover, for any sequence $(\lambda_j, (u_j, v_j)) \in \mathbb{R} \times E$ with $\lambda_j \to \mu_1$ and $\|(u_j, v_j)\| \to +\infty$ as $j \to +\infty$, there exists a subsequence $(\lambda_{j_k}, (u_{j_k}, v_{j_k}))$ such that*

$$\lim_{j_k \to +\infty} \frac{(u_{j_k}, v_{j_k})}{\|(u_{j_k}, v_{j_k})\|} = \frac{(\phi_1, \psi_1)}{\|(\phi_1, \psi_1)\|},$$

*where the convergence is in E.*

*Proof.* The operator equation corresponding to the system (4.2) is

$$\begin{pmatrix} u \\ v \end{pmatrix} = \lambda(-\Delta_p^{-1}) \begin{pmatrix} \theta_1 \varphi_p(v^+) + \widetilde{f}(v) \\ \theta_2 \varphi_p(u^+) + \widetilde{g}(u) \end{pmatrix}. \tag{4.3}$$

Let $(\lambda_j, (u_j, v_j)) \in \mathbb{R} \times E$ be a solution of (4.2) such that $\|(u_j, v_j)\| \to +\infty$ and $\lambda_j \to \mu_\infty$. Then $(\hat{u}_j, \hat{v}_j) = \frac{(u_j, v_j)}{\|(u_j, v_j)\|}$ satisfies

$$\hat{u}_j = \lambda_j(-\Delta_p^{-1}) \left( \theta_1 \varphi_p(\hat{v}_j^+) + \frac{\widetilde{f}(v_j)}{\|(u_j, v_j)\|} \right), \tag{4.4}$$

$$\hat{v}_j = \lambda_j(-\Delta_p^{-1}) \left( \theta_2 \varphi_p(\hat{u}_j^+) + \frac{\widetilde{g}(u_j)}{\|(u_j, v_j)\|} \right). \tag{4.5}$$

It then follows from (A1) that the right hand side of (4.4) and (4.5) are bounded in $X$ (independent of $j$). Hence $\|\hat{u}_j\|_{C^1}$ and $\|\hat{v}_j\|_{C^1}$ are bounded (independent of $j$), and there exists subsequence of $\hat{u}_j$ and $\hat{v}_j$ converging to $\hat{u}$ and $\hat{v}$ and satisfying

$$\begin{cases} -\Delta_p \hat{u} = \mu_\infty \theta_1 \varphi_p(\hat{v}^+), & x \in \Omega, \\ -\Delta_p \hat{v} = \mu_\infty \theta_2 \varphi_p(\hat{u}^+), & x \in \Omega, \\ \hat{u} = 0 = \hat{v}, & x \in \partial\Omega. \end{cases} \tag{4.6}$$

Suppose $\mu_\infty \leqslant 0$. Since $\hat{u}^+, \hat{v}^+ \geqslant 0$, it follows by applying the maximum principle to (4.6) that $\hat{u} \equiv 0$ and repeating the same argument we get $\hat{v} \equiv 0$ as well. This leads to a contradiction since $\|(\hat{u}, \hat{v})\| = 1$.

For $\mu_\infty > 0$, we distinguish two cases: the first case is $\hat{v}^+ \equiv 0$ and $\hat{u}^+ \equiv 0$, and the second is one of $\hat{v}^+ \not\equiv 0$ or $\hat{u}^+ \not\equiv 0$ holding. In the first case, we get $\hat{u} \equiv 0$, a contradiction as before. In the other case, we get $\hat{u} > 0$ from maximum principle and $\hat{v} > 0$ by repeating the same argument. Thus $\mu_\infty$ and $\hat{u}, \hat{v} > 0$ satisfies the eigenvalue problem (3.1).

However, we already discussed that (3.1) has precisely one eigenvalue $\mu_1$ with componentwise positive eigenfunction $(\phi_1, \psi_1)$. Therefore, it must be that $\mu_\infty = \mu_1$ and

$$(\hat{u}, \hat{v}) = \frac{(\phi_1, \psi_1)}{\|(\phi_1, \psi_1)\|}.$$

This concludes the proof of Proposition 4.1. □

**Lemma 4.2.** *Let (A1)–(A2) hold and $\Lambda$ be a compact interval with $\mu_1 \notin \Lambda$. Then there is a $M_\Lambda > 0$ such that all solutions $(\lambda, (u, v))$ of (4.2) with $\lambda \in \Lambda$ must satisfy $\|(u, v)\| < M_\Lambda$.*

*Proof.* Assume to the contrary that there exist sequences $\lambda_j \in \Lambda$ and $(u_j, v_j) \in E$ satisfying (4.2) with $\|(u_j, v_j)\| \to \infty$ as $j \to \infty$. Then there is a subsequence $(\lambda_{j_k}, (u_{j_k}, v_{j_k}))$ of $(\lambda_j, (u_j, v_j))$ satisfying $\lambda_{j_k} \to \tilde{\lambda}$ and $\|(u_{j_k}, v_{j_k})\| \to \infty$. Dividing (4.2) by $\|(u_{j_k}, v_{j_k})\|$, then the same argument as in the proof of Proposition 4.1, we obtain that $\tilde{\lambda} = \mu_1 \in \Lambda$, which contradicts $\mu_1 \notin \Lambda$. □

**Lemma 4.3.** *Let (A1)–(A2) hold, then for $\tau \in [0,1]$, the system*

$$\begin{cases} -\Delta_p u = \lambda\theta_1\varphi_p(v^+) + \lambda\widetilde{f}(v) + \tau\|(u,v)\|^p, & x \in \Omega, \\ -\Delta_p v = \lambda\theta_2\varphi_p(u^+) + \lambda\widetilde{g}(u) + \tau\|(u,v)\|^p, & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega \end{cases} \tag{4.7}$$

*has no solution for $\lambda > \nu := \mu_1 / \min\{\theta_1, \theta_2\}$.*

*Proof.* Suppose that there is a sequence of $\{\lambda_n\}$ such that $\lambda_n \to \infty$ as $n \to \infty$, and (4.7) has a positive solution $(u_n, v_n)$ when $\lambda = \lambda_n$. Then there is a $\lambda_{n_0} > 0$ such that $\lambda_n > \frac{\mu_1}{\min\{\theta_1,\theta_2\}}$ for all $n > n_0$. We can choose a suitable $\epsilon_0 > 0$ such that $\mu^* = \mu_1 + \epsilon_0$ and

$$\lambda_n \min\{\theta_1, \theta_2\} > \mu^*, n > n_0.$$

Since $(u_n, v_n)$ is the positive solution of (4.7), then $u_n^+ = u_n,\ v_n^+ = v_n$, and

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\omega \mathrm{d}x - \mu^* \int_\Omega \theta_1\varphi_p(v_n)\omega \mathrm{d}x \\ &\quad = \lambda_n \int_\Omega \theta_1\varphi_p(v_n^+)\omega \mathrm{d}x + \lambda_n \int_\Omega \widetilde{f}(v_n)\omega \mathrm{d}x + \int_\Omega \tau\|(u_n, v_n)\|^p\omega \mathrm{d}x - \mu^* \int_\Omega \varphi_p(v_n)\omega \mathrm{d}x \\ &\quad = \lambda_n \int_\Omega \theta_1\varphi_p(v_n)\omega \mathrm{d}x - \mu^* \int_\Omega \varphi_p(v_n)\omega \mathrm{d}x + \lambda_n \int_\Omega \widetilde{f}(v_n)\omega \mathrm{d}x + \int_\Omega \tau\|(u_n, v_n)\|^p\omega \mathrm{d}x \\ &\quad > \lambda_n \int_\Omega \widetilde{f}(v_n)\omega \mathrm{d}x + \int_\Omega \tau\|(u_n, v_n)\|^p\omega \mathrm{d}x \\ &\quad > 0. \end{aligned}$$

Similarly,

$$\begin{aligned} &\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla\omega \mathrm{d}x - \mu^* \int_\Omega \theta_2\varphi_p(u_n)\omega \mathrm{d}x \\ &\quad = \lambda_n \int_\Omega \theta_2\varphi_p(u_n^+)\omega \mathrm{d}x + \lambda_n \int_\Omega \widetilde{g}(u_n)\omega \mathrm{d}x + \int_\Omega \tau\|(u_n, v_n)\|^p\omega \mathrm{d}x - \mu^* \int_\Omega \theta_2\varphi_p(u_n)\omega \mathrm{d}x \\ &\quad = \lambda_n \int_\Omega \theta_2\varphi_p(u_n)\omega \mathrm{d}x - \mu^* \int_\Omega \theta_2\varphi_p(u_n)\omega \mathrm{d}x + \lambda_n \int_\Omega \widetilde{g}(u_n)\omega \mathrm{d}x + \int_\Omega \tau\|(u_n, v_n)\|^p\omega \mathrm{d}x \\ &\quad > \lambda_n \int_\Omega \widetilde{g}(v_n)\omega \mathrm{d}x + \int_\Omega \tau\|(u_n, v_n)\|^p\omega \mathrm{d}x \\ &\quad > 0. \end{aligned}$$

That is, $(u_n, v_n)$ is a weak supersolution of the problem

$$\begin{cases} -\Delta_p u = (\mu_1 + \epsilon_0)\theta_1\varphi_p(v), & x \in \Omega, \\ -\Delta_p v = (\mu_1 + \epsilon_0)\theta_2\varphi_p(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega. \end{cases} \tag{4.8}$$

On the other hand $t(\phi_1, \psi_1)$, $t > 0$, is a subsolution of (4.8). Letting $t > 0$ be such that $t(\phi_1, \psi_1) \leqslant (u_n, v_n)$, then the same argument with the proof of Theorem 1.1 we obtain a contradiction with that $\mu_1$ is isolated. □

Define the operator $L : X \to X$ by

$$Q_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \theta_1 \varphi_p(v^+) \\ \theta_2 \varphi_p(u^+) \end{pmatrix}$$

and

$$T_\lambda(u,v) := (-\Delta_p^{-1}) \circ \lambda Q_1(u,v).$$

It is well known that $T_\lambda$ is completely continuous in $X$. By Lemma 3.2, $\mu_1$ is a isolated eigenvalue of (3.1), therefore, there is no nontrivial solution of (3.1) in $\lambda \in (\mu_1 - \delta, \mu_1) \cup (\mu_1, \mu_1 + \delta)$ for some $\delta > 0$, that is, $T_\lambda$ has no fixed point in $\partial B_r(0)$ for arbitrary $r$-ball $B_r(0)$ when $\lambda \in (\mu_1 - \delta, \mu_1) \cup (\mu_1, \mu_1 + \delta)$. Subsequently, the Leray–Schauder degree $\deg(I - T_\lambda, B_r(0), (0,0))$ is well defined arbitrary $r$-ball $B_r(0)$ with $\lambda \in (\mu_1 - \delta) \cup (\mu_1, \mu_1 + \delta)$.

Define the operator $Q : X \to X$ by

$$Q \begin{pmatrix} u \\ v \end{pmatrix} = Q_1 \begin{pmatrix} u \\ v \end{pmatrix} + Q_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \theta_1 \varphi_p(v^+) \\ \theta_2 \varphi_p(u^+) \end{pmatrix} + \begin{pmatrix} \widetilde{f}(v) \\ \widetilde{g}(u) \end{pmatrix}.$$

Then it is clear that $Q$ is continuous operator and problem (4.2) can be equivalently written as

$$(u,v) = (-\Delta_p^{-1}) \circ \lambda Q(u,v) := \mathcal{F}_\lambda(u,v).$$

Since $-\Delta_p^{-1} : X \to X$ is compact, then $\mathcal{F}_\lambda : X \to X$ is completely continuous. Let

$$\Upsilon_\lambda^p(u,v) = (u,v) - (-\Delta_p^{-1}) \circ \mathcal{F}_\lambda(\tau, u, v) = (u,v) - (-\Delta_p^{-1}) \circ \lambda(Q_1(u,v) + Q_2(u,v)).$$

By (A1), $\widetilde{f}, \widetilde{g}$ are bounded, then we have $Q_2(u,v) \to (0,0)$ for $(u,v)$ large enough, that is, $\Upsilon_\lambda^p(u,v)$ close to $T_\lambda(u,v)$ when $(u,v) \to (+\infty, +\infty)$, and subsequently, $\Upsilon_\lambda^p(u,v)$ has no nontrivial solution when $\lambda \in (\mu_1 - \delta) \cup (\mu_1, \mu_1 + \delta)$. Let $(y,z) = (u,v)/\|(u,v)\|^p$, then $(u,v) \to (+\infty, +\infty)$ is equivalent to $(y,z) \to (0,0)$.

$$\begin{aligned} \Psi_\lambda^p(y,z) = \frac{\Upsilon_\lambda^p(u,v)}{\|(u,v)\|^p} &= (y,z) - \|(y,z)\|^p \mathcal{F}_\lambda\left(\frac{(y,z)}{\|(y,z)\|^p}\right) \\ &= (y,z) - \|(y,z)\|^p(-\Delta_p^{-1}) \circ \lambda \left[Q_1\left(\frac{(y,z)}{\|(y,z)\|^p}\right) + Q_2\left(\frac{(y,z)}{\|(y,z)\|^p}\right)\right]. \end{aligned} \tag{4.9}$$

Obviously, $\Psi_\lambda^p(y,z) = (0,0)$ has no nontrivial solution when $(y,z)$ small enough, a similar argument with $T_\lambda$ we have Leray–Schauder degree $\deg(\Psi_\lambda^p, B_r(0), (0,0))$ is well defined with $r$ small enough for $\lambda \in (\mu_1 - \delta, \mu_1) \cup (\mu_1, \mu_1 + \delta)$.

Next we will show $\mu_1$ is a bifurcation point, defined in Section 1, from infinity of (4.2). By Rabinowitz [28], it is equivalent to show that

$$\deg(\Psi_{r_1}^p, B_r(0), (0,0)) \neq \deg(\Psi_{r_2}^p, B_r(0), (0,0)),$$

where $r_1 \in [\mu_1 - \delta, \mu_1), r_2 \in (\mu_1, \mu_1 + \delta]$.

**Lemma 4.4.** *$\mu_1$ is a bifurcation point from infinity of problem (4.2).*

*Proof.* By (4.9), one has that $\mu_1$ is a bifurcation from infinity for (4.2) if and only if it is a bifurcation from the trivial solution for $\Psi^p_\lambda(u,v)=0$. Next we show

$$\deg(\Psi^p_\lambda, B_r(0),(0,0))=1, \qquad \lambda\in[\mu_1-\delta,\mu_1)$$

and

$$\deg(\Psi^p_\lambda, B_r(0),(0,0))=0, \qquad \lambda\in(\mu_1,\mu_1+\delta].$$

Suppose $(\mu_1,(0,0))$ is not a bifurcation point of (4.2), then there are $\delta,\rho_0>0$ such that for $|\lambda-\mu_1|\leqslant\delta$ and $0<\rho<\rho_0$, that is, equation

$$(y,z)-\mathscr{F}(\lambda,y,z)\neq 0$$

for $\|(y,z)\|_X=\rho$, where

$$\mathscr{F}(\lambda,y,z):=\|(y,z)\|^p\mathcal{F}_\lambda\left(\frac{(y,z)}{\|(y,z)\|^p}\right).$$

Consider the homotopy problem

$$(y,z)-\tau\mathscr{F}(\lambda,y,z)\neq 0,$$

where $\tau\in[0,1]$, then by the invariance of the degree of the compact, the same argument with [5, Corollary 3.2], we have

$$\deg(I-\mathscr{F}(\lambda,y,z),B_\rho(0),(0,0))=\deg(I,B_\rho(0),(0,0))=1 \tag{4.10}$$

for $\lambda\in[\mu_1-\delta,\mu_1)$.

Next we show, for $\lambda\in(\mu_1,\mu_1+\delta]$,

$$\deg(\Psi^p_\lambda(y,z),B_\rho(0),(0,0))=0.$$

Let $\check{\lambda}=\max\{\mu_1+\delta,\nu\}$. Define

$$\Phi(\lambda,(y,z))=\Psi^p_\lambda(y,z)$$

and consider the following one parameter family of operators

$$\Phi((1-\sigma)(\mu_1+\delta)+\sigma\check{\lambda},(y,z)), \qquad \sigma\in[0,1].$$

Obviously,

$$(1-\sigma)(\mu_1+\delta)+\sigma\check{\lambda}\in[\mu_1+\delta,\check{\lambda}].$$

By Lemma 4.2, the solutions of (4.2) satisfy

$$\|(u,v)\|<M(\varepsilon).$$

Since $(y,z)=(u,v)/\|(u,v)\|^p$, then all solutions of (4.2) satisfy

$$\|(y,z)\|=\frac{1}{\|(u,v)\|^{p-1}}>\frac{1}{[M(\varepsilon)]^{p-1}}.$$

Hence the problem (4.2) with $\lambda = (1-\sigma)(\mu_1+\delta)+\sigma\check{\lambda}$ does not have any solution on $\partial B_\rho$ with $0 < \rho < 1/[M(\varepsilon)]^{p-1}$. Then by the homotopy invariance of degree with respect to $\sigma \in [0,1]$, we have

$$\deg(\Phi(\mu_1+\delta,(y,z)), B_\rho,(0,0)) = \deg(\Phi(\check{\lambda},(y,z)), B_\rho,(0,0)). \tag{4.11}$$

Obviously, we wish to show that

$$\deg(\Phi(\check{\lambda},(y,z)), B_\rho,(0,0)) = 0.$$

This would be trivial if $\Phi(\check{\lambda},(y,z)) = (0,0)$ has no solution on $B_\rho$. Therefore, we construct an admissible homotopy connecting $\Phi(\check{\lambda},(y,z))$ to an operator which does not have any solution on $B_\rho$ for $0 < \rho < 1/[M(\varepsilon)]^{p-1}$. To this end, for $\tau \in [0,1]$, consider the operator

$$\Phi(\check{\lambda},(y,z)) - \tau\xi$$

with $(0,0) \neq \xi := (-\Delta_p)^{-1}(\chi_\Omega,\chi_\Omega)$, where $\chi_\Omega$ stands for the characteristic function of $\Omega$, that is,

$$\chi_\Omega(x) = \begin{cases} 1, & x \in \Omega, \\ 0, & x \notin \Omega. \end{cases}$$

First we show that, for all $\tau \in [0,1]$ and $0 < \rho < 1/[M(\varepsilon)]^{p-1}$,

$$\Phi(\check{\lambda},(y,z)) - \tau\xi = (0,0) \tag{4.12}$$

does not have any solution on $\partial B_\rho$. Indeed, assume to the contrary that there exists a solution $(y,z)$ of (4.12) with $\|(y,z)\| = \rho > 0$. Then $(u,v) = (y,z)/\|(y,z)\|^p$ must satisfy (4.7), which is absurd due to Lemma 4.3. Therefore, (4.2) does not have any nontrivial solution for all $\tau \in [0,1]$. Moreover, since $(0,0)$ is not a solution of (4.2), then

$$\deg(\Phi(\check{\lambda},(y,z)) - \tau\xi, B_\rho,(0,0)) = 0 \quad \text{for all } \tau \in [0,1].$$

Then homotopy invariance of degree with respect to $\tau \in [0,1]$ yields

$$\deg(\Phi(\check{\lambda},(y,z)), B_\rho,(0,0)) = \deg(\Phi(\check{\lambda},(y,z)) - \xi, B_\rho,(0,0)) = 0.$$

Since this holds for any $0 < \rho < 1/[M(\varepsilon)]^{p-1}$, it follows from (4.11) that

$$\deg(\Phi(\mu_1+\delta,(y,z)), B_\rho,(0,0)) = \deg(\Phi(\check{\lambda},(y,z)), B_\rho,(0,0)) = 0.$$

This combine (4.10) we have $\mu_1$ is a bifurcation point of (4.2) from infinity. □

**Lemma 4.5.** *Let (A1)–(A2) hold. Then $\mu_1$ is the unique bifurcation point from infinity for* (4.2)*. Moreover, there exists a continuum $\mathscr{D} \subset \mathscr{T}$ bifurcating from infinity at $\mu_1$ and satisfies the following:*

(i) *if $(\lambda,(u,v)) \in \mathscr{D}$ and $\lambda > 0$ then $u > 0$ and $v > 0$;*

(ii) *for $\lambda = 0$, $(u,v) = (0,0)$ is the only solution of* (4.2) *and $(0,(0,0)) \in \mathscr{D}$;*

(iii) *$\mathrm{Proj}_\lambda \mathscr{C} \overset{def}{=} \{\lambda \in \mathbb{R} \mid \exists\, (u,v) \in E \text{ with } (\lambda,(u,v)) \in \mathscr{D}\}$ is bounded from above and unbounded from below.*

*Proof.* By Lemma 4.4 and Proposition 4.1, $\mu_1$ is the unique bifurcation point from infinity for (4.2). It is easy to see that operator $\mathcal{F}_\lambda : X \to X$ satisfies the hypotheses of Proposition 2.3. Then there exist unbounded continua

$$\mathscr{D}_\pm \subset \widehat{\mathscr{S}} \stackrel{\text{def}}{=} \{(\lambda, (u, v)) \in \mathbb{R} \times E \mid (\lambda, (u, v)) \text{ solution of } (4.2)\}$$

containing $(0, (0, 0))$. By the nonexistence result of Theorem 1.1

$$\mathscr{D}_+ \subset ([0, \lambda^*) \times E),$$

and thus $\mathscr{D}_+$ must be unbounded in the Banach space $E$-direction. Then $\mathscr{D} \stackrel{\text{def}}{=} \mathscr{D}_+ + \mathscr{D}_-$ is a continuum containing (0,(0,0)). By Proposition 4.1, $\mu_1$ is the only bifurcation point from infinity for (4.2) and $\mathscr{D}_+$ is unbounded in the $E$-direction, hence $\mathscr{D}_+$ must bifurcate from infinity at $\mu_1$. To conclude the proof of Lemma 4.5, it remains to verify that $\mathscr{D}$ satisfies the properties (i)–(iii).

It follows from assumption (A2) and maximum principle that $u, v > 0$ whenever $(\lambda, (u, v)) \in \mathscr{D}$ and $\lambda > 0$, this implies part (i). For $\lambda = 0, (u, v) = (0, 0)$ is the only solution of (4.2) and $(0, (0, 0)) \in \mathscr{D}$, hence part (ii) holds. Applying Proposition 2.3, we see that $\mathscr{D}_-$ must be unbounded in $\mathbb{R} \times E$. However, by part (ii) and the fact that $\nu_1$ is the unique bifurcation point from infinity for (4.3), we see that $\mathscr{D}_-$ must be unbounded in the negative $\lambda$-direction, hence $(-\infty, \nu_1) \subset \mathrm{Proj}_\lambda \mathscr{D}$. This completes the proof of Lemma 4.5. □

## 5 Proof of Theorem 1.2

*Step 1. Approximation problems*

Fix $n \in \mathbb{N}$ and define $f_n(s), g_n(s) : \mathbb{R} \to (0, \infty)$ by

$$f_n(s) \stackrel{\text{def}}{=} \begin{cases} f(s); & s \leqslant n, \\ \frac{f(n)}{\varphi_p(n)} \varphi_p(s); & s > n, \end{cases}$$

$$g_n(s) \stackrel{\text{def}}{=} \begin{cases} g(s); & s \leqslant n, \\ \frac{g(n)}{\varphi_p(n)} \varphi_p(s); & s > n, \end{cases}$$

Then $f_n$ and $g_n$ are continuous functions. Note that, $f_n(s) = f(s)$ for $s \leqslant n$, $\lim_{s\to n^-} f_n(s) = f(n)$, hence $f_n$ is continuous. On the other hand, by assumption (H2),

$$\lim_{s\to\infty} \frac{f_n(s)}{\varphi_p(s)} = \lim_{s\to\infty} \frac{\frac{f(n)}{\varphi_p(n)} \varphi_p(s)}{\varphi_p(s)} = \frac{f(n)}{\varphi_p(n)} \to \infty$$

as $n \to \infty$, then $f_n$ approaches $f$. Similarly, $g_n$ approaches $g$ as $n \to \infty$.

For each $n$, we consider the following problem

$$\begin{cases} -\Delta_p u = \lambda f_n(v), & x \in \Omega, \\ -\Delta_p v = \lambda g_n(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{5.1}$$

by above argument, which approaches (1.1) as $n \to +\infty$. We will use Lemma 4.5 to treat (5.1) and thus we rewrite (5.1) in the form of system (4.2) as

$$\begin{cases} -\Delta_p u = \frac{f(n)}{\varphi_p(n)}\varphi_p(v^+) + \lambda \widetilde{f}_n(v), & x \in \Omega, \\ -\Delta_p v = \frac{g(n)}{\varphi_p(n)}\varphi_p(u^+) + \lambda \widetilde{g}_n(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{5.2}$$

where

$$\widetilde{f}_n(y) \overset{\text{def}}{=} f_n(y) - \frac{f(n)}{\varphi_p(n)}\varphi_p(y^+),$$
$$\widetilde{g}_n(y) \overset{\text{def}}{=} g_n(y) - \frac{g(n)}{\varphi_p(n)}\varphi_p(y^+).$$

We note that $\widetilde{f}_n(y)$ and $\widetilde{g}_n(y)$ are bounded in $\mathbb{R}$. Indeed, since $f_n(y)$ is nondecreasing and $f_n(x) = f(x) > 0$ for $s \leqslant n$, we get

$$|\widetilde{f}_n(x)| \leqslant \sup_{x\in\mathbb{R}} \left| f_n(x) - \frac{f(n)}{\varphi_p(n)}\varphi_p(x^+) \right| \leqslant \max_{x\in[0,n]} \left| f_n(x) - \frac{f(n)}{\varphi_p(n)}\varphi_p(x^+) \right| + f(0) = +\infty,$$

where the constant only depends on $n$. And a same argument we can get $\widetilde{g}_n$ is bounded.

Since $f_n, g_n > 0$, it is easy to see that (5.2) satisfies the hypotheses of Lemma 4.5 by taking $\theta_1 = \frac{f(n)}{\varphi_p(n)}$, $\theta_2 = \frac{g(n)}{\varphi_p(n)}$ Moreover, by Lemma 4.5, there is a $\mu_{1,n}$, such that $\nu_{1,n}$ is the unique bifurcation point from infinity for (5.2) and there exists a continuum $\mathscr{C}_n$ of positive solutions of (5.2), which bifurcates from infinity at $\mu_{1,n}$ and satisfies the properties (i)–(iii) of Lemma 4.5. In particular, $(0,(0,0)) \in \mathscr{C}_n$, $\mathscr{C}_n$ is bounded above by the hyperplane $\lambda = \bar{\lambda}$.

*Step 2. Passing to the limit*

Now we verify $\{\mathscr{C}_n\}$ satisfying the conditions of Lemma 2.5. By the definition of continuum, $\mathscr{C}_n$ is closed.

Since all of $\mathscr{C}_n$ contain $(0,(0,0))$, we can choose $z_n \in \mathscr{C}_n$ such that $z_n = (0,(0,0))$ for each $n = 1, 2, \ldots$. Naturally, $z_n \to z^* = (0,(0,0))$, the condition (i) of Lemma 2.5 is satisfied.

Obviously, because of the unboundedness of $\{\mathscr{C}_n\}$, then

$$d_n = \sup\{|\mu| + \|(u,v)\| \mid (\mu,(u,v)) \in \mathscr{C}_n\} = +\infty,$$

(ii) of Lemma 2.5 holds.

(iii) in Lemma 2.5 can be deduced directly from the Arzelà–Ascoli theorem and the definition of $f_n, g_n$.

Therefore, the superior limit of $\{\mathscr{C}_n\}$ contains a component $\mathscr{C} \subset \Pi$ joining $(0,(0,0))$ with infinity, and it follows from $u, v > 0$ for $\lambda > 0$ whenever $(\lambda,(u,v)) \in \mathscr{C}$, which establishes (a). Part (b) follows from $(0,(0,0)) \in \mathscr{C}$ and $f(0), g(0) > 0$. (c) in Theorem 1.2 can be deduced directly from the Theorem 1.1.

## 6 Proof of Theorem 1.3

Now we only show if the conditions of Theorem 1.3 are satisfied, then the unique point from infinity must be $\lambda = 0$.

In order to do this, we use a *rescaling* technology below, which is used by Ambrosetti et al. [3] to prove the scalar case as $p = 2$ and by Drábek et al. [6] to deal with the semipositone $p$-Laplacian system.

Let

$$F(s) := f(s) - C|s|^{q_1}, \qquad G(s) := g(s) - D|s|^{q_2},$$

where $C, D$ was defined in (1.6). Let

$$\lambda = \gamma^{\sigma}, \qquad w_1 = \gamma^{\kappa_1} u, \qquad w_2 = \gamma^{\kappa_2} v,$$

where $\sigma, \kappa_1, \kappa_2$ are parameters and provided by [6, Proof of Theorem 4.3]. Then (1.1) can be translated to the form

$$\begin{cases} -\Delta_p w_1 = \tilde{F}(\gamma, w_2), & x \in \Omega, \\ -\Delta_p w_2 = \tilde{G}(\gamma, w_1), & x \in \Omega, \\ w_1 = 0 = w_2, & x \in \partial\Omega \end{cases} \tag{6.1}$$

with

$$\begin{aligned} \tilde{F}(\gamma, s_2) &:= \gamma^{\kappa_2 q_1} F(s_2/\gamma^{\kappa_2}) + C|s_2|^{q_1}, \\ \tilde{G}(\gamma, s_1) &:= \gamma^{q_2} G(s_1/\gamma) + D|s_1|^{q_2}. \end{aligned}$$

Then, by a directly result of [4, Theorem 1.1], there is a $M > 0$ such that for all positive solutions $(w_1, w_2) \in C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$ of (6.1) we have

$$\|w_1\|_{C^1} + \|w_2\|_{C^1} \leqslant M. \tag{6.2}$$

Now let $\lambda_n \in \mathbb{R}$ be a decreasing sequence with $\lambda_1 < \bar{\lambda}$ such that $\lambda_n \to 0^+$ as $n \to \infty$. Then for each $n$, we can get $\gamma_n$ and the sequence $w_{1,n}, w_{2,n}$ of the solution for (6.1), such that

$$\|w_{1,n}\|_{C^1} + \|w_{2,n}\|_{C^1} \leqslant M_n. \tag{6.3}$$

Now let $n \to \infty$, then $\lambda_n \to 0^+$ and $\gamma_n = \lambda_n^{\sigma} \to 0^+$, then $(u_n, v_n) = (w_{1,n}/\gamma_n^{\kappa_1}, w_{2,n}/\gamma_n^{\kappa_2})$ gives the solution of (1.1) and $\|(u_n, v_n)\| \to \infty$ as $n \to \infty$.

Finally, we prove $\mathscr{C}$ must bifurcate from infinity at $\lambda \to 0^+$. Now let $(\lambda_n, (u_n, v_n)) \in \mathscr{C}$ with $\|(\mu_n, (u_n, v_n))\| \to +\infty$ as $n \to +\infty$ and $\lambda_n > 0$ for all $n \in \mathbb{N}$. Suppose to the contrary that $\lambda_n \to \lambda' > 0$ as $n \to +\infty$, then there exists a closed and bounded interval $I$ such that $\lambda' \in I$. By above proof,

$$\|(u_n, v_n)\| \leqslant M < +\infty$$

for all $\lambda'$, a contradiction to $\|(u_n, v_n)\| \to +\infty$ as $n \to +\infty$, which completes the proof of Theorem 1.3.

## Declarations

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