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New results concerning a Schrödinger equation involving logarithmic nonlinearity

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Abstract. In this paper, we investigate the existence of ground state solution to a class of Schrödinger equation involving logarithmic nonlinearity. To overcome the lack of smoothness, the corresponding functional J is first decomposed into the sum of a $\mathcal{C}^1$ functional and a convex lower semicontinuous functional by adapting to the approach of Squassina–Szulkin in [*Calc. Var. Partial Differential Equations* **54**(2015), 585–597]. Secondly, the existence of a ground state solution to the studied equation is proved by using the Mountain Pass Theorem under the weakened Ambrosetti–Rabinowitz conditions.

Keywords: Schrödinger equations, ground state solution, logarithmic nonlinearity, Mountain Pass Theorem.

Mathematics Subject Classification 35J35, 35J62, 35J75, 35D30.

1 Introduction

In this paper, we consider the following Schrödinger equation involving logarithmic nonlinearity

$$\begin{cases} -\Delta u + V(x)u = Q(x)u\log u^2 + f(x,u), & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 1$, the external potential $V(x)$, the term $Q(x)$ and $f(x,u)$ are continuous functions and satisfy certain properties given later.

The Schrödinger equation was first proposed by the Austrian physicist E. Schrödinger. As a more complex nonlinear Schrödinger equation, it is derived from the following classical model

$$i\partial_t\Psi + \Delta\Psi - (V(x)+w)\Psi + f(|\Psi|) = 0. \tag{1.2}$$

The solution Ψ is called the standing wave solution of (1.2). Standing wave phenomenon refers to the phenomenon that electromagnetic waves can stay in a fixed position in some media without propagating and forming a resident electromagnetic field. Its application is widely reflected in our daily lives, such as magnetic resonance imaging to scan and image the

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internal structure of the human body. Applied to optical filters that allow specific wavelengths of light to pass through for spectral analysis and filtering. It is used in the field of acoustic standing wave cancellers to reduce noise and quantum mechanics theory. The interference function f in (1.2) is a nonlinear term, it can also be used to describe a variety of nonlinear waves in quantum physics, such as laser beam propagation in the medium with refractive index and wave amplitude, ionic sound wave in plasma, etc, see [4, 23] and the references therein. At present, many scholars have studied the existence and multiplicity of solutions to Schrödinger equation, see [2, 5, 6, 8–11, 14, 18, 19, 21, 22, 25, 26, 28–31] and the references therein for an overview on the related topic.

The logarithmic Schrödinger equation

$$i\partial_t \Psi + \Delta\Psi + \Psi \log|\Psi|^2 = 0, \quad \Psi : [0, +\infty) \times \mathbb{R}^N \to \mathbb{C}, \quad N \geq 3, \tag{1.3}$$

possesses wide applications to quantum mechanics, nuclear physics, open quantum systems, effective quantum gravity, transport and diffusion phenomena, theory of superfluidity and Bose–Einstein condensation, see [32] and the references therein. We refer to [7, 12] for a study of the existence and uniqueness of the solutions of the associated Cauchy problem in some suitable condition as well as the global existence and blow-up of the solutions. On the other hand, many researchers are interested in the existence, multiplicity and qualitative properties of the standing waves solution of problem (1.3). Consider the following Schrödinger equation with logarithmic nonlinear terms

$$-\Delta u + V(x)u = Q(x)u \log u^2, \qquad x \in \mathbb{R}^N. \tag{1.4}$$

When $V(x) = Q(x) = 1$, Avenia–Montefusco–Squassina [13] proved that there are infinitely many solutions to equation (1.4) by introducing weak slope and using non-smooth critical point theory. Ji–Szulkin [16] proved the existence of infinitely many solutions to equation (1.4) by Fountain theorem in the case which $Q(x) = 1$ and potential function $V(x)$ satisfies the mandatory condition, that is $\lim_{|x|\to+\infty} V(x) = +\infty$. In addition, Shuai [20] used the constraint minimization method to obtain the existence of positive solutions and node solutions of equation (1.4) under different assumptions of potential function $V(x)$. When $V(x)$ and $Q(x)$ are 1-period, Squassina–Szulkin [22] investigated the existence of infinitely many different solutions to problem (1.4) by applying non-smooth critical point theory and $\mathbb{Z}_2$ index theory, and the existence of ground state solutions of problem (1.4) was proved .

Through the analysis of the above mentioned results, as far as we know, there is few corresponding result if $V(x) \neq 1$, $Q(x) \neq 1$ and $V(x)$, $Q(x)$ are not 1-period in equation (1.4). Therefore, **a natural question is whether the equation** (1.4) **plus the disturbance term** $f(x, u)$ **can also obtain the ground state solution through the Mountain Pass Theorem with the weakened Ambrosetti–Rabinowitz conditions?** There are two key difficulties in the proof process, one is that the energy functional is not well defined and not C^1 smooth, the other is to prove the existence of the ground state solution when the Ambrosetti–Rabinowitz condition (for short AR-condition) is not satisfied.

For convenience, in this paper, it is assumed that the potential function V and the disturbance term f satisfy the following conditions:

(V1) The potential function $V(x) \in \mathcal{C}(\mathbb{R}^N, [0, +\infty))$, and there exists a constant $a_0 > 0$ such that $|\{x \in \mathbb{R}^N : V(x) \leq a_0\}| < +\infty$;

(V2) $\text{int}V^{-1}(0) \neq \emptyset, Q(x) \in \mathcal{C}'(\mathbb{R}^N, [0, +\infty)), \min(V(x) + Q(x)) \geq 1.$

(f1) The function $f(x,t) \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and for any $\varepsilon > 0$ there exist constants $C(\varepsilon) > 0$ and $p \in (2, 2^*)$ such that $|f(x,t)| \leq \varepsilon |t| + C(\varepsilon)|t|^{p-1}$, where $2^* := 2N/(N-2)$ if $N \geq 3$, $2^* := \infty$ if $N = 1$ or 2;

(f2) $\lim_{|t|\to+\infty} \frac{F(x,t)}{t^2} = +\infty$, where $F(x,t) = \int_0^t f(x,\tau)d\tau$;

(f3) There exist constants $\alpha > 2$ and θ with $0 < \theta < \frac{S(\alpha-2)}{4}$ (S be given in (2.9) below), such that

$$\liminf_{|t|\to\infty} \frac{f(x,t)t - \alpha F(x,t)}{|t|^2} > -\theta, \quad \text{uniformly for a.e. } x \in \mathbb{R}^N.$$

Our main result states as follows.

Theorem 1.1. *Assume that (V1)–(V2) and (f1)–(f3) hold. Then problem (1.1) admits a ground state solution.*

Remark 1.2. *By (f3), there exist constants $M > 0, \alpha > 2$ such that $uf(x,u) \geq \alpha F(x,u)$ for $|u| \geq M, (x,u) \in \mathbb{R}^N \times \mathbb{R}$. Thus,*

$$uf(x,u) - aF(x,u) \geq 0 \geq -\theta|u|^2.$$

Obviously, we can show that the AR-condition implies (f3), while the inverse implication fails.

Remark 1.3. *There exists extensively the disturbance term $f(x,u)$ which satisfies conditions (f1)–(f3) of Theorem 1.1. Such as, taking $N = 3$, $2^* = 6$, then $p \in (2,6)$, if $p = 4$ for all $x \in \mathbb{R}^3$ and*

$$f(x,t) = \frac{S}{3}|\sin x| \left(|t|t + \frac{1}{2} t \sin 2t \right),$$

where S be given by formula (2.9) below, then

$$F(x,t) = \frac{S}{3}|\sin x| \left(\frac{1}{3}|t|^3 - \frac{1}{4} t \cos 2t + \frac{1}{8} \sin 2t \right).$$

Set $\alpha = 3$, $\theta = \frac{S}{5}$, then

$$\begin{aligned} f(x,t)t - \alpha F(x,t) &= \frac{S}{3}|\sin x| \left(|t|^3 + \frac{1}{2}t^2 \sin 2t - \frac{\alpha}{3}|t|^3 + \frac{\alpha}{4} t\cos 2t - \frac{\alpha}{8}\sin 2t \right) \\ &\geq \frac{S}{3}|\sin x| \left((1 - \frac{\alpha}{3})|t|^3 - \frac{1}{2}|t|^2 - \frac{\alpha}{4}t - \frac{\alpha}{8} \right) \\ &= -\frac{S}{3}|\sin x| \left(\frac{1}{2}|t|^2 + \frac{3}{4}t + \frac{3}{8} \right) \\ &\geq -\frac{S}{3} \left(\frac{1}{2}|t|^2 + \frac{3}{4}t + \frac{3}{8} \right), \end{aligned}$$

which implies

$$\liminf_{|t|\to\infty} \frac{f(x,t)t - 3F(x,t)}{|t|^2} = \lim_{|t|\to\infty} \frac{-\frac{S}{3}\left(\frac{1}{2}|t|^2 + \frac{3}{4}t + \frac{3}{8}\right)}{|t|^2} = -\frac{S}{6} > -\theta.$$

Obviously $f(x,t)$ satisfies all the conditions of Theorem 1.1.

This paper is organized as follows. In Section 2, we present some preliminary results that will be used later. Section 3 is devoted to proving the existence of ground state solutions to problem (1.1).

Notation. From now on, otherwise mentioned, we use the following notations:

- C, C_1, C_2, C_3, etc. will denote positive constants, whose exact values are not relevant.
- $\|\cdot\|_k$ denotes the usual norm of the Lebesgue space $L^k(\mathbb{R}^N)$, for $k \in [1, +\infty]$.
- $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$ as $n \to +\infty$.

2 Preliminaries

To prove Theorem 1.1, we need to present some notation and auxiliary lemmas, which will be crucial in dealing with ground state solutions of problem (1.1).

Towards problem (1.1), we define the space E as follows:

$$E := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (V(x) + Q(x))u^2 dx < +\infty \right\},$$

endowed with the following norm:

$$\|u\|_E := \left(\int_{\mathbb{R}^N} |\nabla u|^2 + (V(x) + Q(x))u^2 dx \right)^{\frac{1}{2}}, \qquad u \in E,$$

and the space E is a Hilbert space. Problem (1.1) has a variational structure and is properly associated with the energy functional $J : E \to \mathbb{R} \cup \{+\infty\}$ defined by

$$J(u) = \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^N} F(x, u)dx - \frac{1}{2} \int_{\mathbb{R}^N} Q(x)u^2 \log u^2 dx, \qquad u \in E. \tag{2.1}$$

One premise, the critical point of the energy functional J is the weak solution of the corresponding equation, is that the energy functional J can be well defined and smooth. But, in general, the logarithmic term may cause that the energy functional J fails to be finite and $\mathcal{C}^1$ smooth in $H^1(\mathbb{R}^N)$. Concretely, from [17], we have the following simple modification of the standard logarithmic Sobolev inequality:

$$\int_{\mathbb{R}^N} u^2 \log u^2 dx \le \frac{a^2}{\pi}\|\nabla u\|_2^2 + (\log \|u\|_2^2 - N(1 + \log a))\|u\|_2^2, \tag{2.2}$$

for any $u \in H^1(\mathbb{R}^N)$ and $a > 0$. It follows from (2.2) that $J(u) > -\infty$ for all $u \in H^1(\mathbb{R}^N)$, but there exists $u_* \in H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} u_*^2 \log u_*^2 dx = -\infty$.

Recently, many scholars have tried to find some different techniques and methods to overcome the above given difficulty. For instance, in [7], Cazenave's main idea was to find a suitable Banach space endowed with a Luxemburg type norm. On account of the definition of Luxemburg type norm and some special properties, the functional $J(u)$ is finally well defined and $\mathcal{C}^1$ smooth. Another way to overcome it comes from [15], the authors penalize the nonlinearity term around the origin and obtain a priori estimates to get a nontrivial solution at the limit.

In what follows, by *a solution* to problem (1.1) we shall usually indicate a function $u \in H^1(\mathbb{R}^N)$ such that $u^2 \log u^2 \in L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)dx = \int_{\mathbb{R}^N} Q(x)uv \log u^2 dx + \int_{\mathbb{R}^N} f(x,u)vdx, \quad \text{for any } v \in C_0^\infty(\mathbb{R}^N).$$

In this paper, we take the approach from [22], although $J(u)$ is not smooth, we split the functional $J(u)$ into the sum of a C^1 functional and a convex lower semicontinuous functional. In order to split $J(u)$, we define the following two functions for $\delta > 0$:

$$F_1(s) = \begin{cases} 0, & s = 0, \\ -\frac{1}{2}s^2 \log s^2, & 0 < |s| \le \delta, \\ -\frac{1}{2}s^2(\log \delta^2 + 3) + 2\delta|s| - \frac{1}{2}\delta^2, & |s| > \delta, \end{cases}$$

and

$$F_2(s) = \begin{cases} 0, & |s| \le \delta, \\ -\frac{1}{2}s^2 \log(s^2/\delta^2) + 2\delta|s| - \frac{3}{2}s^2 - \frac{1}{2}\delta^2, & |s| > \delta. \end{cases}$$

Then $F_2(s) - F_1(s) = \frac{1}{2}s^2 \log s^2$ for all $s \in \mathbb{R}$. Thus, the functional $J : E \to (-\infty, +\infty]$ can be rewritten as

$$J(u) = \tilde{\Phi}(u) + \Psi(u), u \in E$$

where

$$\tilde{\Phi}(u) = \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^N} Q(x)F_2(u)dx - \int_{\mathbb{R}^N} F(x,u)dx \tag{2.3}$$

and

$$\Psi(u) = \int_{\mathbb{R}^N} Q(x)F_1(u)dx. \tag{2.4}$$

In the sequel, we list some properties about $F_1(s), F_2(s)$ that shall be useful for our proofs later, which were proved in [16,22].

Proposition 2.1. *If $\delta > 0$ is sufficiently small, then the following results are true:*

(i) $F_1(s), F_2(s) \in C^1(\mathbb{R}, \mathbb{R})$.

(ii) The function $F_1(s)$ is convex, even, and

$$F_1(s) \ge 0, F_1'(s)s \ge 0, \qquad \forall s \in \mathbb{R}. \tag{2.5}$$

(iii) For each fixed $q \in (2, 2^*)$, there is a constant $C > 0$ such that

$$|F_2'(s)| \le C|s|^{q-1}, \qquad \forall s \in \mathbb{R}. \tag{2.6}$$

Hereafter, $\delta > 0$ is fixed and sufficiently small such that the above properties of F_1, F_2 hold.

Corollary 2.1. *Assume that (V1)–(V2) and (f1) hold, then $\tilde{\Phi} \in C^1(E, \mathbb{R})$.*

Proof. Let $\Phi(u) = \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^N} Q(x)F_2(u)dx$, according to Lemma 3.10 in [27] and (2.5), it is easy to show that $\Phi \in C^1(E, \mathbb{R})$. Since

$$\tilde{\Phi}(u) = \Phi(u) - \int_{\mathbb{R}^N} F(x,u)dx,$$

we just need to prove $\int_{\mathbb{R}^N} F(x,u)dx \in C^1(E, \mathbb{R})$. By condition (f1), there holds

$$|F(x,t)| \le \frac{1}{2}\varepsilon|t|^2 + \frac{1}{p}C(\varepsilon)|t|^p, \tag{2.7}$$

which implies $\int_{\mathbb{R}^N} F(x,u)dx \in C^1(E, \mathbb{R})$. Thus $\tilde{\Phi} \in C^1(E, \mathbb{R})$. □

Corollary 2.2. *The functional Ψ admits the following properties:*

(i) The functional $\Psi(u)$ is convex, $\Psi \geq 0$ for all $u \in E$, and $\Psi(u) = +\infty$ for certain $u \in E$. Furthermore, Ψ (and hence J) is lower semicontinuous.

(ii) If $\Omega \subset \mathbb{R}^N$ is a bounded domain, then the functional Ψ (and therefore J) is of class $\mathcal{C}^1$ in $H^1(\Omega)$.

Proof. (i) By the definition of the function $F_1(s)$, we get $\Psi \geq 0$. Since $Q(x) > 0$ and $F_1(s)$ is convex, then for all $\lambda \in [0,1]$ and u_1, $u_2 \in E$, there holds

$$\begin{aligned}\Psi(\lambda u_1 + (1-\lambda)u_2) &= \int_{\mathbb{R}^N} Q(x)F_1(\lambda u_1 + (1-\lambda)u_2)dx \\ &\leq \lambda \int_{\mathbb{R}^N} Q(x)F_1(u_1)dx + (1-\lambda)\int_{\mathbb{R}^N} Q(x)F_1(u_2)dx \\ &= \lambda\Psi(u_1) + (1-\lambda)\Psi(u_2).\end{aligned}$$

This implies that Ψ is convex, and $\Psi(u) = +\infty$ for certain $u \in E$. Moreover, there exists $s_0 \in E$, for all sequence $\{s_n\}$ with $s_n \to s_0$ as $n \to +\infty$. It follows naturally from the Fatou's lemma that

$$\begin{aligned}\lim_{n\to+\infty} \inf \Psi(s_n) &= \lim_{n\to+\infty} \inf \left(\int_{\mathbb{R}^N} Q(x)F_1(s_n)\mathrm{d}x\right) \\ &\geq \int_{\mathbb{R}^N} \lim_{n\to+\infty} \inf Q(x)F_1(s_n)dx = \Psi(s_0).\end{aligned}$$

Thus Ψ is lower semicontinuous, and the functional J is also lower semicontinuous.

(ii) By the definition of the function $F_1(s)$, we have $|F_1'(s)| \leq C(1 + |s|^{p-1})$ for $p \in (2, 2^*)$. Then it follows from Lemma 2.16 of [27] that the conclusion is true in $H_0^1(\Omega)$ but the argument remains valid in $H^1(\Omega)$. □

According to some arguments in [24] and by Corollary 2.1 and Corollary 2.2, we also give the following definition:

Definition 2.3. *Let E be a Banach space, E' be the dual space of E, and $\langle \cdot, \cdot \rangle$ be the duality pairing between E' and E. Let $J : E \to \mathbb{R}$ be a functional of the form $J(u) = \tilde{\Phi}(u) + \Psi(u)$, where $\tilde{\Phi} \in \mathcal{C}^1(E, \mathbb{R})$ and $\Psi(u)$ is convex and lower semicontinuous. Let us list some definitions:*

(1) The sub-differential $\partial J(u)$ of the functional J at a point $u \in E$ is the following set

$$\{w \in E' : \langle \tilde{\Phi}'(u), v-u \rangle + \Psi(v) - \Psi(u) \geq \langle w, v-u \rangle,\ \forall v \in E\}.$$

(2) A critical point of J is a point $u \in E$ such that $J(u) < +\infty$ and $0 \in \partial J(u)$, i.e.

$$\langle \tilde{\Phi}'(u), v-u \rangle + \Psi(v) - \Psi(u) \geq 0, \qquad \forall v \in E.$$

(3) A Palais–Smale sequence at level c for J is a sequence $\{u_n\} \subset E$ such that $J(u_n) \to c$ and there is a numerical sequence $\tau_n \to 0^+$ with

$$\langle \tilde{\Phi}'(u_n), v-u_n \rangle + \Psi(v) - \Psi(u_n) \geq -\tau_n \|v-u_n\|_E, \qquad \forall v \in E.$$

(4) The functional J satisfies the Palais–Smale condition at level c ($(PS)_c$ condition, for short) if all Palais–Smale sequences at level c have a convergent subsequence.

(5) The effective domain of J is the set $D(J) = \{u \in E : J(u) < +\infty\}$.

In the sequel, for each $u \in D(J)$, we set the following functional $J'(u) : H_c^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\langle J'(u), z\rangle = \langle \tilde{\Phi}'(u), z\rangle + \int_{\mathbb{R}^N} Q(x)F_1'(u)z dx \quad \text{for any } z \in H_c^1(\mathbb{R}^N),$$

where $H_c^1(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : \ u \text{ has compact support}\}$, and define

$$\|J'(u)\| = \sup\{\langle J'(u), z\rangle : z \in H_c^1(\mathbb{R}^N) \text{ and } \|z\|_E \le 1\}.$$

Hence, $J'(u)$ can be extended to a bounded operator in E when $\|J'(u)\|$ is finite, and it may be seen as an element of E'.

In order to prove Theorem 1.1, we will use the following Lemma 2.4, whose proof can be found in Lemma 2.4 of [22] .

Lemma 2.4. *If $u \in D(J)$, then $\partial J(u) \neq \emptyset$, i.e. there exists $w \in E'$ such that*

$$\langle \Phi'(u), v-u\rangle + \Psi(v) - \Psi(u) \ge \langle w, v-u\rangle, \quad \text{for all } v \in E.$$

Moreover, this w is unique and satisfies

$$\langle \Phi'(u), z\rangle + \int_{\mathbb{R}^N} Q(x)F_1'(u)z dx = \langle w, z\rangle, \quad \text{for all } z \in E \text{ such that } F_1'(u)z \in L^1(\mathbb{R}^N).$$

As an immediate consequence, we know that the unique element $w \in E'$ introduced in Lemma 2.4 will be denoted by $J'(u)$. Moreover, there holds

$$\langle J'(u), u\rangle = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx - \int_{\mathbb{R}^N} Q(x)u^2 \log u^2 dx - \int_{\mathbb{R}^N} f(x,u)u dx, \tag{2.8}$$

for each $u \in D(J)$ with $\|J'(u)\| < +\infty$. Thus solution of problem (1.1) is equivalent to a nontrivial critical point of the functional J.

Theorem 2.5 ([27, Theorem 1.8], Sobolev imbedding theorem)**.** *The following imbeddings are continuous:*

$$\begin{aligned} H^1(\mathbb{R}^N) &\hookrightarrow L^p(\mathbb{R}^N), & 2 \le p < \infty, N = 1,2, \\ H^1(\mathbb{R}^N) &\hookrightarrow L^p(\mathbb{R}^N), & 2 \le p \le 2^*, N \ge 3, \\ D^{1,2}(\mathbb{R}^N) &\hookrightarrow L^{2^*}(\mathbb{R}^N), & N \ge 3. \end{aligned}$$

According to the norm in the space E and $L^2(\mathbb{R}^N)$ respectively, we have $\|u\|_2^2 \le \|u\|_E^2$. In particular, the best constant for the Sobolev embedding $E \hookrightarrow L^2(\mathbb{R}^N)$ is given by

$$S := \inf_{u \in E\setminus\{0\}} \frac{\|u\|_E^2}{\|u\|_2^2}. \tag{2.9}$$

3 Proof of Theorem 1.1

In what follows, we will show that the functional J satisfies the Mountain pass geometry. The following two conclusions can be found in Theorem 3.1 and Corollary 3.1 of [1], which are crucial in our approach.

Theorem 3.1 (Mountain Pass Theorem without (PS) condition)**.** *Let X be a real Banach space and $J : X \to \mathbb{R}$ be a functional such that:*

(i) $J(u) = \tilde{\Phi}(u) + \Psi_{(}u), u \in X$ *with* $\tilde{\Phi}(u) \in \mathcal{C}^1(X, \mathbb{R})$, *and* $\Psi : X \to \mathbb{R}$ *is convex,* $\Psi \not\equiv +\infty$ *and is lower semicontinuous (l.s.c);*

(ii) $J(0) = 0$ *and* $J|_{\partial B_\rho} \geq \alpha_0$, *for real constants* $\rho, \alpha_0 > 0$*;*

(iii) $J(e) \leq 0$, *for some* $e \notin \overline{B_\rho}(0)$.

If

$$c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)) \geq \alpha_0 > 0, \qquad \Gamma = \{\gamma \in \mathcal{C}([0,1], X) : \gamma(0) = 0, J(\gamma(1)) < 0\}.$$

Then, for a given $\epsilon > 0$, *there is* $u_\epsilon \in X$ *such that*

$$\langle \tilde{\Phi}'(u_\epsilon), v - u_\epsilon \rangle + \Psi(v) - \Psi(u_\epsilon) \geq -3\epsilon \|v - u_\epsilon\|, \qquad \forall v \in X, \tag{3.1}$$

and $J(u_\epsilon) \in [c - \epsilon, c + \epsilon]$.

Corollary 3.2. *Under the conditions of Theorem 3.1, there is a* $(PS)_c$ *sequence* $\{u_n\}$ *of the functional* $J(u)$, *that is,* $J(u_n) \to c$ *and*

$$\langle \tilde{\Phi}'(u_n), v - u_n \rangle + \Psi(v) - \Psi(u_n) \geq -\tau_n \|v - u_n\|, \qquad \forall v \in X,$$

with $\tau_n \to 0^+$.

Lemma 3.3. *Assume that (V1)–(V2) and (f1)–(f2) hold. Then the functional* $J(u)$, *defined with* $\tilde{\Phi}(u)$ *and* $\Psi(u)$ *in (2.3) and (2.4), has the Mountain pass geometry.*

Proof. Now we will show that the functional J satisfies (i), (ii) and (iii) of the Theorem 3.1.

(i) For each $u \in D(J)$, the functional $J(u)$ defined with $\tilde{\Phi}(u)$ and $\Psi(u)$ in (2.3)–(2.4), respectively. According to Corollary 2.1–2.2, then $\tilde{\Phi}(u) \in \mathcal{C}^1$, $\Psi(u)$ is a convex lower semicontinuous function and $\Psi \not\equiv +\infty$.

(ii) Obviously, $J(0) = 0$. It follows from (2.5)–(2.7), and embedding theorem that there holds

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^N} F(x,u)dx - \int_{\mathbb{R}^N} Q(x)F_2(u)dx + \int_{\mathbb{R}^N} Q(x)F_1(u)dx \\
&\geq \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^N} F(x,u)dx - \int_{\mathbb{R}^N} Q(x)F_2(u)dx \\
&\geq \frac{1}{2}\|u\|_E^2 - \frac{1}{2}\varepsilon \int_{R^N} |u|^2 dx - \frac{1}{p}C(\varepsilon) \int_{R^N} |u|^p dx - \int_{R^N} Q(x)F_2(u)dx. \\
&\geq \frac{1}{2}\|u\|_E^2 - \frac{1}{2}\varepsilon\|u\|_2^2 - \frac{1}{p}C(\varepsilon)\|u\|_p^p - CQ_\infty\|u\|_q^q \\
&\geq \frac{1}{2}\|u\|_E^2 - \frac{\varepsilon}{2S}\|u\|_E^2 - \frac{C(\varepsilon)}{pS^{\frac{p}{2}}}\|u\|_E^p - C_1 Q_\infty \|u\|_E^q \\
&= \left(\frac{1}{2} - \frac{\varepsilon}{2S} - \frac{C(\varepsilon)}{pS^{\frac{p}{2}}}\|u\|_E^{p-2} - C_1 Q_\infty \|u\|_E^{q-2}\right)\|u\|_E^2,
\end{aligned}$$

where $Q_\infty := |Q(x)|_\infty$ and C_1 is a positive constant. We may choose $\varepsilon = \frac{S}{2}$ and ρ sufficiently small (i.e. ρ is such that $\frac{1}{4} - \frac{C(\varepsilon)}{pS^{\frac{p}{2}}}\rho^{p-2} - C_1 Q_\infty \rho^{q-2} > 0$), thus

$$J(u) \geq \left(\frac{1}{4} - \frac{C(\varepsilon)}{pS^{\frac{p}{2}}}\rho^{p-2} - C_1 Q_\infty \rho^{q-2}\right)\rho^2 : = \alpha_0 > 0, \quad \text{for any } u \in \partial B_\rho.$$

(iii) We may choose $u_* \in D(J)$ with $u_* \geq 0$ and $\mathrm{supp}(\phi) \subset B_R(0)$ for some $R > 0$. By the condition (f2) we know that there exist constants $C_2, C_3 > 0$, such that $|F(x,u)| \geq C_2|u|^2 - C_3$ for any $u \in \mathbb{R}^+$. Then let $e := tu_*$ for any $t > 0$, there holds

$$\begin{aligned} J(e) = J(tu_*) &= \frac{1}{2}\|tu_*\|_E^2 - \frac{1}{2}\int_{\mathbb{R}^N} Q(x)t^2u_*^2 \log(tu_*)^2 dx - \int_{\mathbb{R}^N} F(x,tu_*)dx \\ &\leq t^2\left(I(u_*) - \frac{1}{2}\int_{\mathbb{R}^N} Q(x)u_*^2 \log t^2 dx\right) - C_2t^2\left(\int_{B_R(0)} u_*^2 dx + \int_{\mathbb{R}^N\setminus B_R(0)} u_*^2 dx\right) + C_4 \\ &\leq t^2\left(I(u_*) - \log t\int_{\mathbb{R}^N} Q(x)u_*^2 dx - C_2\int_{B_R(0)} u_*^2 dx\right) + C_4 \\ &\to -\infty \quad \text{as } t \to +\infty, \end{aligned}$$

where $I(u_*) = \frac{1}{2}\|u_*\|_E^2 - \frac{1}{2}\int_{\mathbb{R}^N} Q(x)u_*^2 \log u_*^2 dx$ is the energy functional of (1.4). Therefore there exists enough large $t_0 > 0$ with $\|e\|_E = \|t_0u_*\|_E > \rho$, i.e. $e \notin \bar{B}_\rho(0)$ such that

$$J(e) = J(t_0u_*) \leq 0.$$

So the proof of Lemma 3.3 is now completed. □

By Theorem 3.1 and Lemma 3.3, $J(u)$ admits a $(PS)_c$ sequence, where c is the Mountain level of $J(u)$.

Lemma 3.4. *Assume that (V1)–(V2), and (f1)–(f3) hold, then all $(PS)_c$ sequence $\{u_n\}$ are bounded in E.*

Proof. If $\{u_n\}$ is unbounded in E, then we can take, passing to a subsequence if necessary, that $\|u_n\|_E > 1$. Since $\{u_n\} \subset E$ is a $(PS)_c$ sequence, then $\{J(u_n)\}$ is bounded above and $\langle J'(u_n), u_n\rangle \to 0$ as $n \to +\infty$. Thus

$$\langle J'(u_n), z\rangle = o_n(1)\|z\|_E, \qquad \forall z \in E.$$

Since $Q(x) > 0$, then take $\sqrt{Q(x)}u$ instead of u in (2.2), there yields

$$\begin{aligned} &\int_{\mathbb{R}^N} Q(x)u^2 \log(Q(x)u^2)dx \\ &\qquad \leq \frac{a^2}{\pi}\|\nabla(\sqrt{Q(x)}u)\|_2^2 + \left(\log\|\sqrt{Q(x)}u\|_2^2 - N(1+\log a)\right)\|\sqrt{Q(x)}u\|_2^2. \end{aligned} \tag{3.2}$$

From Lemma 2.2 of [3], there exists a positive constant C_5 such that

$$N(1+\log a)\|u\|_2^2 \leq (\log\|u\|_2^2)\|u\|_2^2 + C_5\|u\|_2^2. \tag{3.3}$$

Then taking $a > 0$ enough small, there exists a positive constant C_6 such that

$$\int_{\mathbb{R}^N} Q(x)u^2 \log u^2 dx \leq \frac{1}{2}\|\nabla u\|_2^2 + C_6(\log\|u\|_2^2 + 1)\|u\|_2^2. \tag{3.4}$$

We have exploited the fact that the function $t \to \log t$ $(t > 0)$ is increasing. Then for $r \in (0,1)$, there is a constant $C_7 > 0$ satisfying

$$|t\log t| \leq C_7(1 + |t|^{1+r}), \quad \text{for any } t > 0.$$

Therefore there exists a positive constant C_8 such that

$$\|u_n\|_2^2 \log(\|u_n\|_2^2) \leq C_7(1 + (\|u_n\|_2^2)^{1+r}) \leq C_8(1 + \|u_n\|_E)^{1+r}. \tag{3.5}$$

From (3.3)–(3.5) and Theorem 2.5, there holds

$$\int_{\mathbb{R}^N} Q(x)u^2 \log u^2 dx \leq \frac{1}{2}\|\nabla u\|_2^2 + C_9(1 + \|u_n\|_E)^{1+r}, \tag{3.6}$$

where C_9 is a positive constant. By the condition (f3), there exists a constant $M_0 > 0$ with $|u| > M_0$ such that $f(x,t)t - \alpha F(x,t) \geq -\theta|t|^2$ for all $|t| > M_0$, $x \in \mathbb{R}^N$. Let $\Omega_n = \{x \in \mathbb{R}^N : |u| > M_0\}$. It follows from (2.1), (2.8), (3.2) and (3.6) that

$$\begin{aligned}
c + \|u_n\|_E &\geq J(u_n) - \frac{1}{\alpha}\langle J'(u_n), u_n\rangle \\
&= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + (V(x) + Q(x))u_n^2\right) dx - \frac{1}{\alpha}\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 dx + V(x)u_n^2\right) dx \\
&\quad + \left(\frac{1}{\alpha} - \frac{1}{2}\right)\int_{\mathbb{R}^N} Q(x)u_n^2 \log u_n^2 dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} \left(f(x,u_n)u_n - \alpha F(x,u_n)\right) dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\alpha}\right) ||u_n||_E^2 + \left(\frac{1}{\alpha} - \frac{1}{2}\right)\left(\frac{1}{2}\|\nabla u_n\|_2^2 + C_9(1 + \|u_n\|_E)^{1+r}\right) \\
&\quad - \frac{\theta}{\alpha}\left(\int_{\Omega_n} + \int_{\mathbb{R}^N\setminus\Omega_n}\right)|u_n|^2 dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|u_n\|_E^2 + \left(\frac{1}{\alpha} - \frac{1}{2}\right)\left(\frac{1}{2}\|u_n\|_E^2 + C_9\left(1 + \|u_n\|_E\right)^{1+r}\right) - \frac{\theta}{\alpha}\int_{\Omega_n}|u_n|^2 dx - C_{10} \\
&\geq \left(\frac{1}{4} - \frac{1}{2\alpha} - \frac{\theta}{S\alpha}\right)\|u_n\|_E^2 + \left(\frac{1}{\alpha} - \frac{1}{2}\right) C_9\left(1 + \|u_n\|_E\right)^{1+r} - C_{10},
\end{aligned}$$

where C_{10} is a positive constant. Divide both sides of this inequality by the norm $\|u_n\|_E^2$, then this leads to the following contradiction:

$$0 > \frac{S(\alpha - 2) - 4\theta}{4S\alpha} > 0,$$

and the proof of Lemma 3.4 is completed. □

Since the sequence $\{u_n\}$ is bounded in E, it has a weakly convergent subsequence in E. Without loss of generality we can assume that there exist $u \in E$ and a subsequence of $\{u_n\}$, still denoted by itself, such that

$$\begin{cases}
u_n \rightharpoonup u & \text{in } E, \\
u_n \to u & \text{in } L^p_{\text{loc}}(\mathbb{R}^N),\ p \in (2, 2^*), \\
u_n \to u & \text{a.e. } x \in \mathbb{R}^N,
\end{cases}$$

as $n \to \infty$.

Lemma 3.5. *Assume that (f1) satisfies, then there hold*

$$\begin{aligned}
\lim_{n\to+\infty}\int_{\mathbb{R}^N} F_2'(u_n)u_n dx &= \int_{\mathbb{R}^N} F_2'(u)u dx, \\
\lim_{n\to+\infty}\int_{\mathbb{R}^N} f(x,u_n)u_n dx &= \int_{\mathbb{R}^N} f(x,u)u dx.
\end{aligned}$$

Proof. It follows from (2.6) that we know $|F_2'(s)| \leq C|s|^{q-1}$, then

$$\left| \int_{\mathbb{R}^N} F_2'(u_n) u_n dx - \int_{\mathbb{R}^N} F_2'(u) u dx \right| \leq \int_{\mathbb{R}^N} C \left| |u_n|^q - |u|^q \right| dx \to 0, \quad \text{as } n \to +\infty.$$

By the condition (f1), there holds

$$\begin{aligned} \left| \int_{\mathbb{R}^N} (f(x,u_n)u_n - f(x,u)u)\, dx \right| &\leq \left| \int_{\mathbb{R}^N} \left(\varepsilon|u_n|^2 + C(\varepsilon)|u_n|^p - \varepsilon|u|^2 - C(\varepsilon)|u|^p\right) dx \right| \\ &\leq \varepsilon \int_{\mathbb{R}^N} \left||u_n|^2 - |u|^2\right| dx + C(\varepsilon) \int_{\mathbb{R}^N} \left||u_n|^p - |u|^p\right| dx \to 0, \end{aligned}$$

as $n \to +\infty$. The proof of Lemma 3.5 is completed. □

Lemma 3.6. *Let $\{u_n\}$ be a $(PS)_c$ sequence of J in E, then $u_n \to u$ in E.*

Proof. By Lemma 3.4, the sequence $\{u_n\}$ is bounded in E. Then without loss of generality we can assume that $u_n \rightharpoonup u$ in E, recalling that $\langle J'(u_n), u_n \rangle = o_n(1)\|u_n\|_E$ yields

$$\begin{aligned} &\int_{\mathbb{R}^N} \left[|\nabla u_n|^2 + (V(x) + Q(x))u_n^2\right] dx + \int_{\mathbb{R}^N} Q(x) F_1'(u_n) u_n dx \\ &\quad = \int_{\mathbb{R}^N} f(x,u_n) u_n dx + \int_{\mathbb{R}^N} Q(x) F_2'(u_n) u_n dx + o_n(1). \end{aligned} \tag{3.7}$$

Moreover, $\lim_{n\to\infty} \langle J'(u_n), u_n \rangle = 0$, i.e.

$$\begin{aligned} &\int_{\mathbb{R}^N} \left[|\nabla u|^2 + (V(x) + Q(x))u^2\right] dx + \int_{\mathbb{R}^N} Q(x) F_1'(u) u dx \\ &\quad = \int_{\mathbb{R}^N} f(x,u) u dx + \int_{\mathbb{R}^N} Q(x) F_2'(u) u dx. \end{aligned} \tag{3.8}$$

By the Lemma 3.5, the right-hand side of (3.7) and (3.8) are equal. Therefore, there holds

$$\|u_n\|_E^2 + \int_{\mathbb{R}^N} Q(x) F_1'(u_n) u_n dx + o_n(1) = \|u\|_E^2 + \int_{\mathbb{R}^N} Q(x) F_1'(u) u dx.$$

Without loss of generality we have $\int_{\mathbb{R}^N} F_1'(u_n) u_n dx \to \int_{\mathbb{R}^N} F_1'(u) u dx$, and

$$\|u_n\|_E^2 \to \|u\|_E^2,$$

as $n \to +\infty$. Thus we can conclude that $u_n \to u$ in E. □

Because $\{u_n\} \subset E$ is the $(PS)_c$ sequence of the functional $J(u)$, and by (3) of Definition 2.3, then there exists a function $v \in C_0^\infty(\mathbb{R}^N)$, for $\tau_n \to 0^+$ such that

$$\begin{aligned} -\tau_n \|v - u_n\|_E \leq &\int_{\mathbb{R}^N} \left[\nabla u_n \nabla (v - u_n) + (V(x) + Q(x)) u_n (v - u_n)\right] dx - \int_{\mathbb{R}^N} f(x,u_n)(v - u_n) dx \\ &- \int_{\mathbb{R}^N} Q(x) F_2'(u_n)(v - u_n) dx + \int_{\mathbb{R}^N} Q(x) F_1(v) dx - \int_{\mathbb{R}^N} Q(x) F_1(u_n) dx. \end{aligned}$$

Since Ψ is lower semicontinuous, then

$$\Psi(u_n) \geq \liminf_{n\to\infty} \Psi(u_n) \geq \Psi(u).$$

It follows from Lemmas 3.5–3.6 that there holds

$$\begin{aligned}\lim_{n\to\infty}(-\tau_n\|v-u\|_E) \leq & \int_{\mathbb{R}^N}[\nabla u\nabla(v-u)+(V(x)+Q(x))u(v-u)]\,dx \\ & -\int_{\mathbb{R}^N}f(x,u)(v-u)dx-\int_{\mathbb{R}^N}Q(x)F_2'(u)(v-u)dx \\ & +\int_{\mathbb{R}^N}Q(x)F_1(v)dx-\int_{\mathbb{R}^N}Q(x)F_1(u)dx.\end{aligned} \tag{3.9}$$

The above formula (3.9) is equivalent to

$$\langle\Phi'(u),v-u\rangle+\Psi(v)-\Psi(u)\geq 0.$$

It can be seen that it satisfies (2) of Definition 2.3. This implies that u is the critical point of the functional $J(u)$. Therefore, u is the solution of problem (1.1).

In what follows, we mainly prove that the solution u is nontrivial and reachable.

Lemma 3.7. *Assume that (V1)–(V2) and (f1)–(f3) hold, then the functional $J(u)$ satisfies $(PS)_c$ condition, and u is a critical point of J. Furthermore, u is nontrivial solution of equation* (1.1).

Proof. If $u=0$, then $u_n\rightharpoonup 0$ in E. One of the following two cases is always true.

(i) $u_n\to 0$, as $n\to+\infty$.

(ii) $\liminf_{n\to+\infty}\sup_{y\in\mathbb{R}^N\setminus\{0\}}\int_{B_r(y)}|u_n|^2dx>0$.

Now we next will prove that neither (i) nor (ii) is true:

If (i) is true, then $J(u)\to 0$, but $J(u)\to c$ $(c>0)$, which leads to a contradiction. Therefore (ii) is true. Let $\beta:=\liminf_{n\to+\infty}\sup_{y\in\mathbb{R}^N\setminus\{0\}}\int_{B_r(y)}|u_n|^2dx>0$. Choose $A_r:=\{x\in\mathbb{R}^N\setminus B_r(0): V(x)<a_0\}$, as $r\to+\infty$. It follows from (V1) that $\text{meas}(A_r)\to 0$. There exists a constant r^* such that if $r>r^*$ and $q^*\in(2,2^*)$, and according to the Hölder inequality and Sobolev imbedding inequality, we have

$$\begin{aligned}\int_{A_r}|u_n|^2dx &\leq\left(\int_{A_r}|u_n|^{q^*}dx\right)^{\frac{2}{q^*}}\left(\int_{A_r}1dx\right)^{\frac{q^*-2}{q^*}} \\ &\leq C\|u_n\|_E^2\,(\text{meas}(A_r))^{\frac{q^*-2}{q^*}}\leq\frac{\beta}{4}.\end{aligned} \tag{3.10}$$

Take $D_r:=\{x\in\mathbb{R}^N\setminus B_r(0): V(x)\geq a_0\}$, as a_0 is enough large. Then there holds

$$\int_{D_r}|u_n|^2dx\leq\frac{1}{1+a_0}\int_{\mathbb{R}^N}(1+V(x))|u_n|^2dx\leq\frac{C}{1+a_0}\leq\frac{\beta}{4}. \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\begin{aligned}\beta &=\liminf_{n\to+\infty}\sup_{y\in\mathbb{R}^N\setminus\{0\}}\int_{B_r(y)}|u_n|^2dx \\ &\leq\liminf_{n\to+\infty}\int_{\mathbb{R}^N\setminus B_r(0)}|u_n|^2dx \\ &=\liminf_{n\to+\infty}\left(\int_{D_r}|u_n|^2dx+\int_{A_r}|u_n|^2dx\right)\leq\frac{\beta}{2},\end{aligned}$$

which leads to a contradiction. Thus we have $u\neq 0$. The proof of Lemma 3.7 is completed. □

Proof of Theorem 1.1. By the statement in Section 3, we admit that the sequence $\{u_n\} \subset E$ is the $(PS)_c$ sequence of the functional $J(u)$. From Lemma 3.7, the weak limit of $(PS)_c$ sequence is nontrivial, we easily infer that the weak limit is the desired ground state. On the one hand, since $u_n \rightharpoonup u$ $(n \to \infty)$, the norm $\|u\|_E$ and $\Psi(u)$ are lower semicontinuous, it follows from (2.1), (2.3), (2.4) that

$$\begin{aligned}
J(u) &= \tilde{\Phi}(u) + \Psi(u) \\
&= \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^N} F(x,u)dx - \int_{\mathbb{R}^N} Q(x)F_2(u)dx + \int_{\mathbb{R}^N} Q(x)F_1(u)dx \\
&\leq \frac{1}{2}\|u\|_E^2 - \inf \int_{\mathbb{R}^N} F(x,u)dx - \inf \int_{\mathbb{R}^N} Q(x)F_2(u)dx + \int_{\mathbb{R}^N} Q(x)F_1(u)dx \\
&\leq \liminf_{n\to+\infty} \frac{1}{2}\|u_n\|_E^2 - \liminf_{n\to+\infty} \int_{\mathbb{R}^N} F(x,u_n)dx - \liminf_{n\to+\infty} \int_{\mathbb{R}^N} Q(x)F_2(u_n)dx \\
&\quad + \liminf_{n\to+\infty} \int_{\mathbb{R}^N} Q(x)F_1(u_n)dx \\
&= \liminf_{n\to+\infty} J(u_n) = c.
\end{aligned}$$

Then we can get $J(u) \leq c$. On the other hand, by the definition of c, we have $J(u) \geq c$. Hence $J(u) = c$. In a word, we deduce that c is attained and the corresponding minimizer is a ground state solution of problem (1.1). The proof of Theorem 1.1 is completed. □

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References

[1] C. O. Alves, D. C. De Morais Filho, Existence and concentration of positive solutions for a Schrödinger logarithmic equation, *Z. Angew. Math. Phys.* **69**(2018), No. 144, 1–22. https://doi.org/10.1007/s00033-018-1038-2; Zbl 1411.35090

[2] C. O. Alves, I. S. Da Silva, Existence of multiple solutions for a Schrödinger logarithmic equation via Lusternik–Schnirelmann category, *Anal. Appl. (Singap.)* **21**(2023), No. 6, 1477–1516. https://doi.org/10.1142/S0219530523500240; Zbl 1528.35035

[3] A. H. Ardila, Orbital stability of Gausson solutions to logarithmic Schrödinger equations, *Electron. J. Differential Equations* **2016**, No. 335, 1–9. https://doi.org/10.48550/arXiv.1607.01479.

[4] L. Bergé, Wave collapse in physics: principles and applications to light and plasma waves, *Phys. Rep.* **303**(1998), No, 5–6, 259–370. https://doi.org/10.1016/S0370-1573(97)00092-6;

[5] Y. Cai, Y. Zhao, Ground state solution for the logarithmic Schrödinger–Poisson system with critical growth, *Qual. Theory Dyn. Syst.* **24**(2025), Art. No. 16, 21 pp. https://doi.org/10.1007/s12346-024-01174-x.

[6] D. Cassani, L. Du, Z. Liu, Positive solutions to the planar logarithmic Choquard equation with exponential nonlinearity, *Nonlinear Anal.* **241**(2024), No. 113479, pp. 1–19. https://doi.org/10.1016/j.na.2023.113479; Zbl 1533.35171

[7] T. Cazenave, Stable solutions of the logarithmic Schrödinger equation, *Nonlinear Anal.* **7**(1983), 1127–1140. https://doi.org/10.1016/0362-546X(83)90022-6; Zbl 0529.35068

[8] X. Chang, V. D. Rădulescu, R. Wang, D. Yan, Convergence of least energy sign-changing solutions for logarithmic Schrödinger equations on locally finite graphs, *Commun. Nonlinear Sci. Numer. Simul.* **125**(2023), Art. ID 107418, 19 pp. https://doi.org/10.1016/j.cnsns.2023.107418; Zbl 1522.35010

[9] X. Chang, R. Wang, D. Yan, Ground states for logarithmic Schrödinger equations on locally finite graphs, *J. Geom. Anal.* **33**(2023), No. 211, 1–26. https://doi.org/10.1007/s12220-023-01267-0; Zbl 1514.35133

[10] J. Chao, X. Ying, Existence and concentration of positive solution for a fractional logarithmic Schrödinger equation, *Differential Integral Equations* **35**(2022), No. 11–12, 677–704. https://doi.org/10.57262/die035-1112-677; Zbl 1524.35028

[11] Z. Chen, 1/2-Laplacian problem with logarithmic and exponential nonlinearities, *Electron. J. Qual. Theory Differ. Equ.* **2023**, No. 37, 1–19. https://doi.org/10.14232/ejqtde.2023.1.37; Zbl 1538.35423

[12] H. Chen, P. Luo, G. W. Liu, Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity, *J. Math. Anal. Appl.* **422**(2015), No. 1, 84–98. https://doi.org/10.1016/j.jmaa.2014.08.030; Zbl 1302.35071

[13] P. D'Avenia, E. Montefusco, M. Squassina, On the logarithmic Schrödinger equation, *Commun. Contemp. Math.* **16**(2014), Art. No. 1350032, 10 pp. https://doi.org/10.1142/S0219199713500326

[14] W. Feng, X. Tang, L. Zhang, Existence of a positive bound state solution for logarithmic Schrödinger equation, *J. Math. Anal. Appl.* **531**(2024), No. 127861, 16 pp. https://doi.org/10.1016/j.jmaa.2023.127861; Zbl 1540.35138

[15] P. Guerrero, J. L. López, J. Nieto, Global H^1 solvability of the 3D logarithmic Schrödinger equation, *Nonlinear Anal. Real World Appl.* **11**(2010), No. 1, 79–87. https://doi.org/10.1016/j.nonrwa.2008.10.017; Zbl 1180.81071

[16] C. Ji, A. Szulkin, A logarithmic Schrödinger equation with asymptotic conditions on the potential, *J. Math. Anal. Appl.* **437**(2016), 241–254. https://doi.org/10.1016/j.jmaa.2015.11.071; Zbl 1333.35010

[17] E. H. Lieb, M. Loss, *Analysis*, 2nd ed., Graduate Studies in Mathematics, Vol. 14, American Mathematical Society, Providence, RI, 2001. Zbl 0873.26002

[18] S. B. Liu, On superlinear Schrödinger equations with periodic potential, *Calc. Var. Partial Differential Equations* **45**(2013), No. 1–2, pp. 1–9. https://doi.org/10.1007/s00526-011-0447-2; Zbl 1247.35149

[19] X. Peng, G. Jia, Existence and concentration behavior of solutions for the logarithmic Schrödinger–Poisson system with steep potential, *Z. Angew. Math. Phys.* **74**(2022), Paper No. 29, 21 pp. https://doi.org/10.1007/s00033-022-01922-6; Zbl 1505.35149

[20] W. Shuai, Multiple solutions for logarithmic Schrödinger equations, *Nonlinearity* **32**(2019), No. 6, 2201–2225. https://doi.org/10.1088/1361-6544/ab08f4; Zbl 1418.35102

[21] W. Shuai, Two sequences of solutions for the semilinear elliptic equations with logarithmic nonlinearities, *J. Differential Equations* **343**(2023), 263–284. https://doi.org/10.1016/j.jde.2022.10.014; Zbl 1505.35185

[22] M. Squassina, A. Szulkin, Multiple solutions to logarithmic Schrödinger equations with periodic potential, *Cal. Var. Partial Differential Equations* **54**(2015), 585–597. https://doi.org/10.1007/s00526-014-0796-8; Zbl 1326.35358

[23] C. Sulem, P. L. Sulem, *The nonlinear Schrödinger equation: self-focusing and wave collapse*, Applied Mathematical Sciences, Vol. 139, Springer, New York, 1999. https://doi.org/10.1007/b98958; Zbl 0928.35157

[24] A. Szulkin, Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems, *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **3**(1986), 77–109. https://doi.org/10.1016/S0294-1449(16)30389-4; Zbl 0612.58011

[25] X. Tang, S. Chen, X. Lin, J. Yu, Ground state solutions of Nehari–Pankov type for Schrödinger equations with local super-quadratic conditions, *J. Differential Equations* **268**(2020), No. 8, 4663–4690. https://doi.org/10.1016/j.jde.2019.10.041; Zbl 1437.35224

[26] X. Tang, X. Lin, J. Yu, Nontrivial solutions for Schrödinger equation with local super-quadratic conditions, *J. Dynam. Differential Equations* **31**(2019), No. 1, 369–383. https://doi.org/10.1007/s10884-018-9662-2; Zbl 1414.35062

[27] M. Willem, *Minimax theorems*, Birkhäuser, Boston, 1996. https://doi.org/10.1007/978-1-4612-4146-1; Zbl 0856.49001

[28] M. Yang, Ground state Solutions for a periodic Schrödinger equation with superlinear nonlinearities, *Nonlinear Anal.* **72**(2010), 2620–2627. https://doi.org/10.1016/j.na.2009.11.009; Zbl 1184.35139

[29] Y. Yu, Y. Zhao, Caffarelli–Kohn–Nirenberg inequality for biharmonic equations with inhomogeneous term and Rellich potential, *Electron. J. Qual. Theory Differ. Equ.* **2023**, No. 26, 1–11. https://doi.org/10.14232/ejqtde.2023.1.26; Zbl 1538.35143

[30] Y. Yu, Y. Zhao, C. Luo, New results concerning a singular biharmonic equations with p-Laplacian and Hardy potential, *Anal. Appl. (Singap.)* **103**(2024), No. 18, 3295–3312. https://doi.org/10.1080/00036811.2024.2351469.

[31] C. Zhang, L. Zhang, Qualitative analysis on logarithmic Schrödinger equation with general potential, *J. Fixed Point Theory Appl.* **24**(2022), No. 74, pp. 1–33. https://doi.org/10.1007/s11784-022-00989-y; Zbl 1500.35178

[32] K. G. Zloshchastiev, Logarithmic nonlinearity in theories of quantum gravity: origin of time and observational consequences, *Gravit. Cosmol.* **16**(2010), No. 4, 288–297. https://doi.org/10.1134/S0202289310040067; Zbl 1232.83044