Existence of Positive Solutions for Boundary Value Problems of Second-order Functional Differential Equations

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Abstract

We use a fixed point index theorem in cones to study the existence of positive solutions for boundary value problems of second-order functional differential equations of the form

$$\begin{cases} y''(x) + r(x)f(y(w(x))) = 0, & 0 < x < 1, \\ \alpha y(x) - \beta y'(x) = \xi(x), & a \le x \le 0, \\ \gamma y(x) + \delta y'(x) = \eta(x), & 1 \le x \le b; \end{cases}$$

where $w(x)$ is a continuous function defined on $[0,1]$ and $r(x)$ is allowed to have singularities on $[0,1]$. The result here is the generalization of a corresponding result for ordinary differential equations.

Keywords: functional differential equation, boundary value problem, positive solution, superlinear and sublinear

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1 Introduction

Boundary value problems (abbr. as BVP) associated with second order diffferential equations have a long history and many different methods and techniques have been used and developed in order to obtain various qualitative properties of the solutions (see [1-5, 8, 11, 15, 17]). In recent years, accompanied by the development of theory of functional differential equations (abbr. as FDE), many authors have paid attention to BVP of second order FDE such as

$$[p(t)x'(t)]' = f(t, x_t, x(t)),$$

or

$$x''(t) = f(t, x(t), x(\sigma(t)), x'(t), x'(\tau(t)))$$

(for example see [6-7, 10, 12-14, 16]). As pointed out by the authors of [6], the background for these problems lies in many areas of physics, applied mathematics and variational problems of control theory.

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In this paper, we investigate the existence of positive solutions for BVP of second-order FDE with the form:

$$\begin{cases} y''(x) + r(x)f(y(w(x))) = 0, & 0 < x < 1, \\ \alpha y(x) - \beta y'(x) = \xi(x), & a \le x \le 0, \\ \gamma y(x) + \delta y'(x) = \eta(x), & 1 \le x \le b. \end{cases} \tag{1.1}$$

Here we assume that

(P_1) $w(x)$ is a continuous function defined on $[0,1]$ satisfying

$$c = \inf\{w(x);\ 0 \le x \le 1\} < 1, \qquad d = \sup\{w(x);\ 0 \le x \le 1\} > 0.$$

Thus $E := \{x \in [0,1];\ 0 \le w(x) \le 1\}$ is a compact set and $mes\ E > 0$;

(P_2) $\xi(x)$ and $\eta(x)$ are continuous functions defined on $[a,0]$ and $[1,b]$ respectively, where $a := \min\{0, c\}$, $b := \max\{1, d\}$; furthermore, $\xi(0) = \eta(1) = 0$; $\xi(x) \ge 0$ as $\beta = 0$; $\int_x^0 e^{-\frac{\alpha}{\beta}s}\xi(s)ds \ge 0$ as $\beta > 0$; $\eta(x) \ge 0$ as $\delta = 0$; $\int_1^x e^{\frac{\gamma}{\delta}s}\eta(s)ds \ge 0$ as $\delta > 0$.

For the case of $w(x) \equiv x$, BVP (1.1) is related to two point BVP of ODE for which Erbe and Wang[4] have got the following theorem:

Theorem A.[4] Assume that $w(x) \equiv x,\ \xi(x) \equiv 0,\ \eta(x) \equiv 0$ and

(A_1) $f \in C([0,+\infty),[0,+\infty))$;

(A_2) $r \in C([0,1],[0,+\infty))$ and $r(x) \not\equiv 0$ for any subinterval of $[0,1]$;

(A_3) $\alpha,\ \beta,\ \gamma,\ \delta \ge 0,\ \rho := \gamma\beta + \alpha\gamma + \alpha\delta > 0$.

Then any one of the following is a sufficient condition for the existence of at least one positive solution of BVP (1.1):

(1) $\lim\limits_{v\downarrow 0} \frac{f(v)}{v} = +\infty$ and $\lim\limits_{v\uparrow +\infty} \frac{f(v)}{v} = 0$ (sublinear case);

(2) $\lim\limits_{v\downarrow 0} \frac{f(v)}{v} = 0$ and $\lim\limits_{v\uparrow +\infty} \frac{f(v)}{v} = +\infty$ (superlinear case).

Motivated by [4], in this paper we shall extend the results of [4] to BVP (1.1). Firstly, we have the following hypotheses:

(H_1) $\alpha,\ \beta,\ \gamma,\ \delta \ge 0,\ \rho := \gamma\beta + \alpha\gamma + \alpha\delta > 0$;

(H_2) $r(x)$ is a measurable function defined on $[0,1]$, and

$$0 < \int_E h(x)r(x)dx \le \int_0^1 h(x)r(x)dx < +\infty,$$

$$0 < \int_E \phi_1(x)r(x)dx \le \int_0^1 \phi_1(x)r(x)dx < +\infty,$$

where E is defined as in (P_1); $h(x) : [0,1] \to [0,1]$ is defined by

$$h(x) = \begin{cases} 1, & \delta\beta > 0, \\ x, & \beta = 0,\ \delta > 0, \\ 1 - x, & \beta > 0,\ \delta = 0, \\ x(1-x), & \beta = \delta = 0 \end{cases}$$

and $\phi_1(x)$ $(\phi_1(x) > 0,\ x \in (0,1))$ is the eigenfunction related to the smallest eigenvalue λ_1 $(\lambda_1 > 0)$ of the eigenvalue problem

$$-\phi'' = \lambda\phi, \qquad \alpha\phi(0) - \beta\phi'(0) = 0, \qquad \gamma\phi(1) + \delta\phi'(1) = 0;$$

(H_3) $f(y)$ is a nonnegative continuous function defined on $[0,+\infty)$.

Under the assumptions $(H_1)-(H_3)$, we allow that $r(x)\equiv 0$ on some subset of E, and $r(x)$ has some kind of singularities on $[0,1]$. For example, if $\beta=\delta=0$, then $\phi_1(x)=\sin\pi x$ and

$$r(x)=x^{-m}(1-x)^{-n},\qquad 0<m<2,\ 0<n<2,$$

satisfies (H_2).

To the best of the authors' knowledge, there has not been much work done about the positive solutions for the singular boundary value problems with deviating arguments, although they have importance in applications.

2 Main Theorem

First, we give the following definitions of solution and positive solution of BVP (1.1).

Definition. $y(x)$ is said to be a solution of BVP (1.1) if it satisfies the following:

1. $y(x)$ is nonnegative and continuous on $[a,b]$;
2. $y(x)=y(a;x)$ as $x\in[a,0]$, where $y(a;x):[a,0]\to[0,+\infty)$ is defined by
$$y(a;x)=\begin{cases} e^{\frac{\alpha}{\beta}x}\left(\frac{1}{\beta}\int_x^0 e^{-\frac{\alpha}{\beta}s}\xi(s)ds+y(0)\right), & \beta>0,\\ \frac{1}{\alpha}\xi(x), & \beta=0;\end{cases}\tag{2.1}$$
3. $y(x)=y(b;x)$ as $x\in[1,b]$, where $y(b;x):[1,b]\to[0,+\infty)$ is defined by
$$y(b;x)=\begin{cases} e^{-\frac{\gamma}{\delta}x}\left(\frac{1}{\delta}\int_1^x e^{\frac{\gamma}{\delta}s}\eta(s)ds+e^{\frac{\gamma}{\delta}}y(1)\right), & \delta>0,\\ \frac{1}{\gamma}\eta(x), & \delta=0;\end{cases}\tag{2.2}$$
4. while $\delta\beta>0$, $y'(x)$ exists and is absolutely continuous on $[0,1]$; while $\beta=0,\ \delta>0$, $y'(x)$ exists and is locally absolutely continuous on $(0,1]$; while $\beta>0,\ \delta=0$, $y'(x)$ exists and is locally absolutely continuous on $[0,1)$; while $\beta=\delta=0$, $y'(x)$ exists and is locally absolutely continuous on $(0,1)$;
5. $y''(x)=-r(x)f(y(w(x)))$ for $x\in(0,1)$ almost everywhere.

Furthermore, a solution $y(x)$ of (1.1) is called a positive solution if $y(x)>0$ for $x\in(0,1)$.

Suppose that $y(x)$ is a solution of BVP (1.1), then it could be expressed as

$$y(x)=\begin{cases} y(a;x), & a\le x\le 0,\\ \int_0^1 G(x,t)r(t)f(y(w(t)))dt, & 0\le x\le 1,\\ y(b;x), & 1\le x\le b,\end{cases}\tag{2.3}$$

and *Green's* function

$$G(x,t):=\begin{cases} \frac{1}{\rho}(\delta+\gamma-\gamma x)(\beta+\alpha t), & 0\le t\le x\le 1,\\ \frac{1}{\rho}(\delta+\gamma-\gamma t)(\beta+\alpha x), & 0\le x\le t\le 1,\end{cases}$$

where ρ is given in (H_1). It is obvious that $0 < G(x,t) \leq G(t,t)$ for $(x,t) \in (0,1) \times (0,1)$.

By an elementary calculation, one can find constants λ and B such that

$$\lambda B h(x) h(t) \leq G(x,t) \leq B h(t), \quad (x,t) \in [0,1] \times [0,1], \tag{2.4}$$

where $h(x)$ is decided by (H_2).

By using (2.3) and (2.4), we know that for every solution $y(x)$ of BVP (1.1), one has

$$\begin{cases} \|y\|_{[0,1]} \leq B \int_0^1 h(t) r(t) f(y(w(t))) dt, \\ y(x) \geq \lambda \|y\|_{[0,1]} h(x), \qquad x \in [0,1], \end{cases} \tag{2.5}$$

where $\|y\|_{[0,1]} := \sup\{|y(x)|; \ \ 0 \leq x \leq 1\}$.

Choose $\sigma \in (0, \frac{1}{4})$ such that

$$\int_{E_\sigma} h(s) r(s) ds > 0, \quad \int_{E_\sigma} \phi_1(s) r(s) ds > 0, \tag{2.6}$$

where $E_\sigma := \{x \in E; \ \sigma \leq w(x) \leq 1 - \sigma\}$. In this paper, we always assume that σ satisfies (2.6). Then we have from (2.5) that

$$y(x) \geq \lambda \|y\|_{[0,1]} h(x) \geq \lambda \sigma_o \|y\|_{[0,1]}, \qquad x \in [\sigma, 1-\sigma],$$

here

$$\sigma_o = \begin{cases} 1, & \delta\beta > 0, \\ \sigma, & \beta > 0, \delta = 0 \quad or \quad \beta = 0, \delta > 0, \\ \sigma(1-\sigma), & \delta = \beta = 0. \end{cases} \tag{2.7}$$

The following theorem is our main result.

Theorem 1. If $(P_1), (P_2), (H_1) - (H_3)$ are satisfied, then any one of the following is a sufficient condition for the existence of at least one positive solution of BVP(1.1):

(H_4) $\liminf\limits_{v \to 0+} \frac{f(v)}{v} > k\lambda_1, \quad \limsup\limits_{v \to +\infty} \frac{f(v)}{v} < q\lambda_1;$

(H_5) $\liminf\limits_{v \to +\infty} \frac{f(v)}{v} > k\lambda_1, \quad \limsup\limits_{v \to 0+} \frac{f(v)}{v} < q\lambda_1, \quad \xi(x) \equiv 0, \ \eta(x) \equiv 0;$

where $k > 0$ is large enough such that

$$k\lambda\sigma_o \int_{E_\sigma} r(x)\phi_1(x)dx \geq \int_0^1 \phi_1(x)dx,$$

and $q > 0$ is small enough such that

$$q \int_0^1 r(x)\phi_1(x)dx \leq \lambda\sigma_o \int_\sigma^{1-\sigma} \phi_1(x)dx$$

(λ and σ_o are defined as in (2.4) and (2.7) respectively).

Corollary. Using the following (H_6) or (H_7) instead of (H_4) or (H_5), the conclusion of Theorem 1 is true.

(H_6) $\lim\limits_{v \downarrow 0} \frac{f(v)}{v} = +\infty, \quad \lim\limits_{v \uparrow +\infty} \frac{f(v)}{v} = 0$ (sublinear);

(H_7) $\lim\limits_{v \downarrow 0} \frac{f(v)}{v} = 0, \quad \lim\limits_{v \uparrow +\infty} \frac{f(v)}{v} = +\infty$ (superlinear), $\quad \xi(x) \equiv 0, \ \eta(x) \equiv 0.$

It is obvious that our corollary is an extension of Theorem A, and Theorem 1 is an improvement of Theorem A even for the case $w(x) = x$. We remark here that only if $\delta\beta > 0$ and $r(x)$ is continuous on $[0, 1]$, every positive solution of BVP (1.1) belongs to $C^1[a, b] \bigcap C^2[0, 1]$.

3 Proof of Theorem

In this section, we shall show the conclusion of Theorem 1 only for the situation $\beta > 0,\ \delta = 0$. The arguments for the other three cases are similar. First we give a lemma which will be used later (see [5] or [11]).

Lemma 1. Assume that X is a Banach space, and $K \subset X$ is a cone in X. Let $K_p = \{u \in K;\ \|u\| \le p\}$. Furthermore, assume that $\Phi : K \to K$ is a compact map, and $\Phi u \neq u$ for $u \in \partial K_p = \{u \in K;\ \|u\| = p\}$. Then one has the following conclusions:

1. if $\|u\| \le \|\Phi u\|$ for $u \in \partial K_p$, then $i(\Phi, K_p, K) = 0$;
2. if $\|u\| \ge \|\Phi u\|$ for $u \in \partial K_p$, then $i(\Phi, K_p, K) = 1$.

If $u_o(x)$ is a solution of BVP (1.1) for $f \equiv 0$, then it can be expressed as

$$u_o(x) = \begin{cases} \frac{1}{\beta} e^{\frac{\alpha}{\beta}x} \int_x^0 e^{-\frac{\alpha}{\beta}s} \xi(s) ds, & a \le x \le 0, \\ 0, & 0 \le x \le 1, \\ \frac{1}{\gamma}\eta(x), & 1 \le x \le b. \end{cases} \tag{3.1}$$

If $y(x)$ is a solution of BVP(1.1), let $u(x) = y(x) - u_o(x)$, noting that $u(x) \equiv y(x)$ as $0 \le x \le 1$, then by using (3.1), we have

$$u(x) = \begin{cases} e^{\frac{\alpha}{\beta}x} u(0), & a \le x \le 0, \\ \int_0^1 G(x,t) r(t) f(u(w(t)) + u_o(w(t))) dt, & 0 \le x \le 1, \\ 0, & 1 \le x \le b. \end{cases} \tag{3.2}$$

Let K be a cone in the Banach space $X = C[a, b]$ which is defined as

$$K = \{u \in C[a, b];\ \ u(x) \ge \lambda\sigma_o \|u\|,\ x \in [\sigma, 1 - \sigma]\},$$

where $\|u\| := \sup\{|u(x)|;\ \ a \le x \le b\}$ (noting that $\|u\|_{[0,1]}$ is defined as in (2.5)).

Define an operator $\Phi : K \to K$ by

$$(\Phi u)(x) = \begin{cases} e^{\frac{\alpha}{\beta}x} \int_0^1 G(0,t) r(t) f(u(w(t)) + u_o(w(t))) dt, & a \le x \le 0, \\ \int_0^1 G(x,t) r(t) f(u(w(t)) + u_o(w(t))) dt, & 0 \le x \le 1, \\ 0, & 1 \le x \le b. \end{cases} \tag{3.3}$$

Then we have the following four lemmas.

Lemma 2. $\Phi(K) \subset K$.

Proof. It is obvious that $0 \le (\Phi u)(x) \le (\Phi u)(0)$ as $a \le x \le 0$, then one has $\|\Phi u\| = \|\Phi u\|_{[0,1]}$. We have from (2.4) and (3.3) that (noting that $h(x) = 1 - x$)

$$\|\Phi u\|_{[0,1]} \le B \int_0^1 (1 - t) r(t) f(u(w(t)) + u_o(w(t))) dt,$$

thus we have

$$\begin{aligned}(\Phi u)(x) &\geq \lambda B \int_0^1 (1-x)(1-t)r(t)f(u(w(t))+u_o(w(t)))dt,\\ &\geq \lambda(1-x)\|\Phi u\|_{[0,1]},\\ &= \lambda(1-x)\|\Phi u\|, \qquad x\in[0,1].\end{aligned}$$

Thus $(\Phi u)(x) \geq \lambda\sigma_o\|\Phi u\|, \quad \sigma \leq x \leq 1-\sigma$, i.e. $\Phi(K)\subset K$.

Lemma 3. $\Phi : K \to K$ is completely continuous.

Proof. We can obtain the continuity of Φ from the continuity of f. In fact, if $u_n,\ u \in K$ and $\|u_n - u\| \to 0$ as $n\to\infty$, then we have from (3.3) that for $a \leq x \leq b$,

$$\begin{aligned}&|(\Phi u_n)(x) - (\Phi u)(x)|\\ \leq\ &\max_{0\leq x\leq 1} |f(u_n(w(x))+u_o(w(x))) - f(u(w(x))+u_o(w(x)))|\ B\int_0^1(1-t)r(t)dt,\end{aligned}$$

which implies that $\|\Phi u_n - \Phi u\| \to 0$ as $n\to\infty$.

Suppose that $A\subset K$ is a bounded set, and there exists a constant $M_1>0$ such that $\|u\|\leq M_1$ for $u\in A$. Let $\|u_o\| = M_2$, then $\|u+u_o\| \leq M_1+M_2 = M$ for $u \in A$. We have from (3.3) that

$$\|\Phi u\| \leq B\max_{0\leq v\leq M} f(v)\int_0^1 (1-t)r(t)dt. \tag{3.4}$$

Thus $\Phi(A)$ is bounded in K. Now it is easy to see that $\Phi u \in C^1[a,1)\cap C[a,b]$, and

$$\begin{aligned}(\Phi u)'(x) =\ & \tfrac{-1}{\beta+\alpha}\int_0^x(\beta+\alpha t)r(t)f(u(w(t))+u_o(w(t)))dt\\ &+\tfrac{\alpha}{\beta+\alpha}\int_x^1(1-t)r(t)f(u(w(t))+u_o(w(t)))dt, \quad 0\leq x<1,\\ (\Phi u)'(x) =\ & \tfrac{\alpha}{\beta}e^{\frac{\alpha}{\beta}x}\int_0^1 G(0,t)r(t)f(u(w(t))+u_o(w(t)))dt, \quad a\leq x\leq 0.\end{aligned}\tag{3.5}$$

For $u\in A,\ 0\leq x\leq 1$, we have

$$(\Phi u)(x) \leq F(x) := \max_{0\leq v\leq M} f(v)\int_0^1 G(x,t)r(t)dt, \quad 0\leq x\leq 1. \tag{3.6}$$

Noting the facts that $F(1)=0$ and the continuity of $F(x)$ on $[0,1]$, we have from (3.6) that for any $\epsilon>0$, one can find a $\delta_1>0$ (independent with u) such that, $0<\delta_1<\frac{1}{4}$ and

$$(\Phi u)(x) < \frac{\epsilon}{2}, \quad 1-2\delta_1 < x < 1. \tag{3.7}$$

On the other hand, for $x\in[0,1-\delta_1]$ one has

$$\begin{aligned}|(\Phi u)'(x)| &\leq \max_{0\leq v\leq M} f(v)\left\{\int_0^{1-\delta_1} r(t)dt + \int_0^1(1-t)r(t)dt\right\}\\ &\leq \tfrac{1+\delta_1}{\delta_1}\max_{0\leq v\leq M} f(v)\int_0^1(1-t)r(t)dt = L_1.\end{aligned}$$

For $x\in[a,0]$, one has from (3.5) and (3.4) that

$$|(\Phi u)'(x)| \leq \frac{\alpha}{\beta}|(\Phi u)(x)| \leq \frac{\alpha}{\beta}B\max_{0\leq v\leq M} f(v)\int_0^1 r(t)(1-t)dt = L_2.$$

Let $\delta_2 = \frac{\epsilon}{\max\{L_1, L_2\}}$, then for $x_1,\ x_2 \in [a, 1-\delta_1],\ |x_1 - x_2| < \delta_2$, we have

$$|(\Phi u)(x_1) - (\Phi u)(x_2)| \le \max\{L_1, L_2\}|x_1 - x_2| < \epsilon. \tag{3.8}$$

Define $\delta_o = \min\{\delta_1, \delta_2\}$, then by using (3.7) – (3.8) and the fact that $(\Phi u)(x) \equiv 0$ for $x \in [1, b]$, we obtain that

$$|(\Phi u)(x_1) - (\Phi u)(x_2)| < \epsilon,$$

for $x_1,\ x_2 \in [a, b],\ |x_1 - x_2| < \delta_o$, which implies that $\Phi(A)$ is equicontinuous. In view of the Arzela-Ascoli lemma, we know that $\bar{\Phi}(A)$ is a compact set; thus $\Phi : K \to K$ is completely continuous.

Lemma 4. (H_6) implies that there exists $r_o,\ R_o:\ 0 < r_o < R_o$ such that

$$i(\Phi, K_r, K) = 0, \quad 0 < r \le r_o; \qquad i(\Phi, K_R, K) = 1, \quad R \ge R_o.$$

Proof. By using the first equality of (H_6) we can choose $r_o > 0$ such that

$$f(v) \ge Mv, \qquad 0 \le v \le r_o,$$

where M satisfies $\lambda^2 T B \sigma_o M > 2$ and

$$T = \int_{E_\sigma} (1-s)r(s)ds.$$

If $u \in \partial K_r$ $(0 < r \le r_o)$, one has

$$u(x) \ge \lambda\sigma_o\|u\| = \lambda\sigma_o r, \qquad x \in [\sigma, 1-\sigma]. \tag{3.9}$$

Then we obtain (noting that $u_o(x) \equiv 0$ as $x \in [0, 1]$)

$$\begin{aligned}(\Phi u)(\tfrac{1}{2}) &= \textstyle\int_0^1 G(\frac{1}{2}, s)r(s)f(u(w(s)) + u_o(w(s)))ds \\ &\ge \tfrac{1}{2}\lambda B \textstyle\int_{E_\sigma}(1-s)r(s)f(u(w(s)))ds \\ &\ge \tfrac{1}{2}\lambda^2 B\sigma_o r T M \\ &> r = \|u\|.\end{aligned}$$

This leads to

$$\|\Phi u\| > \|u\|, \qquad \forall u \in \partial K_r.$$

Thus we have from Lemma 1 $i(\Phi, K_r, K) = 0$, for $0 < r \le r_o$.

On the other hand, the second equality of (H_6) leads to: for $\forall\epsilon > 0$, there is a $R' > r_o + \|u_o\|$ such that

$$f(v) \le \epsilon v, \qquad v > R', \tag{3.10}$$

where ϵ satisfies

$$\epsilon B(1 + \|u_o\|)\int_0^1 (1-s)r(s)ds < \frac{1}{2}. \tag{3.11}$$

Choose

$$R_o > 1 + 2B\max\{f(v);\ 0 \le v \le R' + \|u_o\|\}\int_0^1 (1-s)r(s)ds. \tag{3.12}$$

Thus, if $u \in \partial K_R$ and $R \geq R_o$, then we have from (3.10)-(3.12) that

$$\begin{aligned}
(\Phi u)(x) &\leq B \int_0^1 (1-s)r(s)f(u(w(s))+u_o(w(s)))ds \\
&= B \int_{u(w(s))>R'} (1-s)r(s)f(u(w(s))+u_o(w(s)))ds \\
&\quad + B \int_{0 \leq u(w(s)) \leq R'} (1-s)r(s)f(u(w(s))+u_o(w(s)))ds \\
&\leq \epsilon B(\|u\|+\|u_o\|) \int_0^1 (1-s)r(s)ds \\
&\quad + B \max\{f(v);\ 0 \leq v \leq R'+\|u_o\|\} \int_0^1 (1-s)r(s)ds \\
&< \tfrac{1}{2}\|u\| + \tfrac{1}{2} + B \max\{f(v);\ 0 \leq v \leq R'+\|u_o\|\} \int_0^1 (1-s)r(s)ds \\
&< \tfrac{1}{2}\|u\| + \tfrac{1}{2}R, \qquad 0 \leq x \leq 1.
\end{aligned}$$

That is

$$\|\Phi u\| = \|\Phi u\|_{[0,1]} < \|u\|, \qquad \forall u \in \partial K_R, \quad .$$

Thus $i(\Phi, K_R, K) = 1$ for $R \geq R_o$.

Lemma 5. (H_7) implies that there exists $r_o,\ R_o:\ 0 < r_o < R_o$ such that

$$i(\Phi, K_r, K) = 1, \quad 0 < r \leq r_o; \qquad i(\Phi, K_R, K) = 0, \quad R \geq R_o.$$

Proof. Since $\xi(x) \equiv 0,\ \eta(x) \equiv 0$, we have $u_o(x) \equiv 0$. By the first equality of (H_7), one can choose a $r_o > 0$ such that

$$f(v) \leq \epsilon v, \qquad 0 \leq v \leq r_o, \tag{3.13}$$

where $\epsilon > 0$ satisfies

$$0 < \epsilon B \int_0^1 (1-s)r(s)ds < \frac{1}{2}. \tag{3.14}$$

If $u \in \partial K_r,\ 0 < r \leq r_o$, then we have from (3.3), (3.13) and (3.14) that

$$\begin{aligned}
0 \leq (\Phi u)(x) &\leq B \int_0^1 (1-s)r(s)f(u(w(s)))ds \\
&\leq \epsilon B \int_0^1 (1-s)r(s)u(w(s))ds \\
&\leq \epsilon B\|u\| \int_0^1 (1-s)r(s)ds \\
&< \|u\|, \qquad 0 \leq x \leq 1,
\end{aligned}$$

That is

$$\|\Phi u\| = \|\Phi u\|_{[0,1]} < \|u\|, \qquad \forall u \in \partial K_r.$$

So we have the conclusion that $i(\Phi, K_r, K) = 1,\ 0 < r \leq r_o$.

On the other hand, the second equality of (H_7) implies that $\forall M > 0$, there is an $R_o > r_o$ such that

$$f(v) \geq Mv, \qquad v > \lambda \sigma_o R_o; \tag{3.15}$$

here we choose $M > 0$ such that $\lambda^2 TB\sigma_o M > 2$. For $u \in \partial K_R,\ R \geq R_o$, we have from the definition of K_R that

$$u(x) \geq \lambda \sigma_o \|u\| = \lambda \sigma_o R, \qquad x \in [\sigma, 1-\sigma]. \tag{3.16}$$

Thus we have from (3.3),(3.15)-(3.16) that

$$\begin{aligned}
(\Phi u)(\tfrac{1}{2}) &= \int_0^1 G(\tfrac{1}{2}, s)r(s)f(u(w(s)))ds \\
&\geq \tfrac{1}{2}\lambda B \int_{E_\sigma} (1-s)r(s)f(u(w(s)))ds \\
&\geq \tfrac{1}{2}\lambda^2 B\sigma_o RM \int_{E_\sigma} (1-s)r(s)ds \\
&= \tfrac{1}{2}\lambda^2 B\sigma_o RTM \\
&> R = \|u\|,
\end{aligned}$$

which leads to

$$\|\Phi u\| > \|u\| \qquad \forall u \in \partial K_R.$$

Thus $i(\Phi, K_R, K) = 0$ for $R \geq R_o$.

Now by using the above lemmas, we can show the conclusions of Theorem 1.

Proof of Theorem 1. For $0 < m < 1 < n$, define $f_1(u) = u^m,\ f_2(u) = u^n, \quad u \geq 0$, so that $f_1(u)$ satisfies (H_6) and $f_2(u)$ satisfies (H_7). Define $\Phi_i : K \to K\ (i = 1, 2)$ as follows:

$$(\Phi_i u)(x) = \begin{cases} e^{\frac{\alpha}{\beta}x} \int_0^1 G(0,t)r(t)f_i(u(w(t)) + u_o(w(t)))dt, & a \leq x \leq 0, \\ \int_0^1 G(x,t)r(t)f_i(u(w(t)) + u_o(w(t)))dt, & 0 \leq x \leq 1, \\ 0, & 1 \leq x \leq b. \end{cases} \tag{3.17}$$

Then $\Phi_i u\ (i = 1, 2)$ are completely continuous operators. One has from Lemma 4-5 that

$$i(\Phi_1, K_r, K) = 0, \quad 0 < r \leq r_o; \qquad i(\Phi_1, K_R, K) = 1, \quad R \geq R_o, \tag{3.18}$$

and

$$i(\Phi_2, K_r, K) = 1, \quad 0 < r \leq r_o; \qquad i(\Phi_2, K_R, K) = 0, \quad R \geq R_o, \tag{3.19}$$

Define $H_i(s, u) = (1 - s)\Phi u + s\Phi_i u\ (i = 1, 2)$ so that for any $s \in [0, 1]$, H_i is a completely continuous operator. Furthermore, for any $\omega > 0$ and $i = 1, 2$, we have

$$|H_i(s_1, u) - H_i(s_2, u)| \leq |s_1 - s_2|[\|\Phi_i u\| + \|\Phi u\|]$$

as $s_1, s_2 \in [0, 1],\ u \in K_\omega$. Note that $\|\Phi_i u\| + \|\Phi u\|$ is uniformly bounded in K_ω. Thus $H_i(s, u)$ is continuous on $u \in K_\omega$ uniformly for $s \in [0, 1]$. According to Lemma 7.2.3 in [18], we conclude that $H_i(s, u)$ is a completely continuous operater on $[0, 1] \times K_\omega$.

Suppose that (H_4) holds. By using the first inequality of (H_4) and the definition of f_1, one can find $\epsilon > 0$ and $r_1 :\ 0 < r_1 \leq r_o$ such that

$$\begin{cases} f(u) \geq (k\lambda_1 + \epsilon)u, & 0 \leq u \leq r_1, \\ f_1(u) \geq (k\lambda_1 + \epsilon)u, & 0 \leq u \leq r_1. \end{cases} \tag{3.20}$$

In what follows, we shall show that $H_1(s, u) \neq u$ for $u \in \partial K_{r_1}$ and $s \in [0, 1]$. If this is not true, then there exist $s_1 : 0 \leq s_1 \leq 1$ and $u_1 \in \partial K_{r_1}$ such that $H_1(s_1, u_1) = u_1$. Note that $u_1(x)$ satisfies

$$\begin{aligned} -u_1''(x) = \ & (1 - s_1)r(x)f(u_1(w(x)) + u_o(w(x))) \\ & +s_1 r(x)f_1(u_1(w(x)) + u_o(w(x))), \qquad 0 < x < 1; \end{aligned} \tag{3.21}$$

and

$$\begin{cases} \alpha u_1(x) - \beta u_1'(x) = 0, & a \leq x \leq 0, \\ u_1(x) = 0, & 1 \leq x \leq b. \end{cases} \tag{3.22}$$

Multiplying both sides of (3.21) by $\phi_1(x)$ and then integrating it from 0 to 1, after two times of integrating by parts, we get from (3.22) that

$$\begin{aligned} \lambda_1 \int_0^1 u_1(x)\phi_1(x)dx \ & = \int_0^1 \phi_1(x)[(1 - s_1)r(x)f(u_1(w(x)) + u_o(w(x))) \\ & +s_1 r(x)f_1(u_1(w(x)) + u_o(w(x)))]dx. \end{aligned} \tag{3.23}$$

Noting that $u_o(w(x)) \equiv 0$ as $x \in E_\sigma$, we obtain from (3.20) and (H_4) that

$$\begin{array}{ll} \lambda_1 \int_0^1 u_1(x)\phi_1(x)dx & \geq \int_{E_\sigma} \phi_1(x)r(x)[(1-s_1)f(u_1(w(x)))) + s_1 f_1(u_1(w(x)))]dx \\ & \geq \int_{E_\sigma} \phi_1(x)r(x)[(1-s_1)(k\lambda_1+\epsilon)u_1(w(x)) \\ & \quad + s_1(k\lambda_1+\epsilon)u_1(w(x))]dx \\ & \geq (\lambda_1 + \frac{\epsilon}{k})k\lambda\sigma_o\|u_1\| \int_{E_\sigma} \phi_1(x)r(x)dx \\ & \geq (\lambda_1 + \frac{\epsilon}{k})\|u_1\| \int_0^1 \phi_1(x)dx. \end{array} \tag{3.24}$$

We also have

$$\lambda_1 \int_0^1 u_1(x)\phi_1(x)dx \leq \lambda_1\|u_1\| \int_0^1 \phi_1(x)dx, \tag{3.25}$$

which together with (3.24) leads to

$$\lambda_1 \geq \lambda_1 + \frac{\epsilon}{k}.$$

This is impossible. Thus $H_1(s,u) \neq u$ for $u \in \partial K_{r_1}$ and $s \in [0,1]$. In view of the homotopic invariant property of topological degree (see [9] or [18]) and (3.18) we know that

$$\begin{array}{ll} i(\Phi, K_{r_1}, K) & = i(H_1(0,\cdot), K_{r_1}, K) \\ & = i(H_1(1,\cdot), K_{r_1}, K) = i(\Phi_1, K_{r_1}, K) = 0. \end{array} \tag{3.26}$$

On the other hand, according to the second inequality of (H_4), there exist $\epsilon > 0$ and $R' > R_o$ such that

$$f(u) \leq (q\lambda_1 - \epsilon)u, \quad u \geq R'.$$

If $C = \max_{0 \leq u \leq R'} |f(u) - (q\lambda_1 - \epsilon)u| + 1$, then we deduce that

$$f(u) \leq (q\lambda_1 - \epsilon)u + C, \quad \forall u \geq 0. \tag{3.27}$$

Define $H(s,u) = s\Phi u,\ s \in [0,1]$. We shall show that there exists a $R_1 \geq R'$ such that

$$H(s,u) \neq u, \quad \forall s \in [0,1],\ u \in K,\ \|u\| \geq R_1. \tag{3.28}$$

If $\exists s_1 \in [0,1],\ u_1 \in K$ such that $H(s_1,u_1) = u_1$, then it is similar to the argument of (3.24)-(3.25) that

$$\begin{array}{ll} & \lambda_1 \int_0^1 u_1(x)\phi_1(x)dx \\ = & s_1 \int_0^1 r(x)\phi_1(x)f(u_1(w(x)) + u_o(w(x)))dx \\ \leq & q(\lambda_1 - \frac{\epsilon}{q})\|u_1 + u_o\| \int_0^1 r(x)\phi_1(x)dx + C\int_0^1 r(x)\phi_1(x)dx \\ \leq & q(\lambda_1 - \frac{\epsilon}{q})\|u_1\| \int_0^1 r(x)\phi_1(x)dx + C_1 \int_0^1 r(x)\phi_1(x)dx, \end{array} \tag{3.29}$$

and

$$\begin{array}{ll} \lambda_1 \int_0^1 u_1(x)\phi_1(x)dx & \geq \lambda_1\lambda\sigma_o\|u_1\| \int_\sigma^{1-\sigma} \phi_1(x)dx \\ & \geq \lambda_1 q\|u_1\| \int_0^1 r(x)\phi_1(x)dx, \end{array} \tag{3.30}$$

where $C_1 = q(\lambda_1 - \frac{\epsilon}{q})\|u_o\| + C$. Combining (3.29) with (3.30), we have

$$\|u_1\| \leq \frac{C_1}{\epsilon} = \bar{R}.$$

Define $R_1 = \max\{R', \bar{R}\}$, then (3.28) is true. By the homotopic invariant property of topological degree, one has

$$\begin{aligned} i(\Phi, K_{R_1}, K) &= i(H(1,\cdot), K_{R_1}, K) \\ &= i(H(0,\cdot), K_{R_1}, K) = i(\theta, K_{R_1}, K) = 1, \end{aligned} \tag{3.31}$$

where θ is zero operator. In view of (3.26),(3.31), we obtain

$$i(\Phi, K_{R_1} \backslash K_{r_1}, K) = 1.$$

Thus Φ has a fixed point in $K_{R_1} \backslash K_{r_1}$.

Now assume that (H_5) is true. The first inequality and the definition of f_2 lead to: $\exists \epsilon > 0$ and $R' > R_o$ such that

$$\begin{cases} f(u) \geq (k\lambda_1 + \epsilon)u, & u > R', \\ f_2(u) \geq (k\lambda_1 + \epsilon)u, & u > R'. \end{cases}$$

Let

$$C = \max_{0 \leq u \leq R'} |f(u) - (k\lambda_1 + \epsilon)u| + \max_{0 \leq u \leq R'} |f_2(u) - (k\lambda_1 + \epsilon)u| + 1,$$

then we have

$$f(u) \geq (k\lambda_1 + \epsilon)u - C, \quad f_2(u) \geq (k\lambda_1 + \epsilon)u - C, \quad \forall u \geq 0. \tag{3.32}$$

We want to show that $\exists R_1 \geq R'$ such

$$H_2(s,u) \neq u, \quad \forall s \in [0,1],\ u \in K,\ \|u\| \geq R_1. \tag{3.33}$$

In fact, if there are $s_1 \in [0,1],\ u_1 \in K$ such that $H_2(s_1, u_1) = u_1$, then using (3.32), it is analogous to the argument of (3.24)-(3.25) that

$$\begin{aligned} \lambda_1 \int_0^1 u_1(x)\phi_1(x)dx &\geq \int_{E_\sigma} \phi_1(x)r(x)\{(1-s_1)[(k\lambda_1+\epsilon)u_1(w(x)) - C] \\ &\quad + s_1[(k\lambda_1+\epsilon)u_1(w(x)) - C]\}dx \\ &\geq (\lambda_1 + \tfrac{\epsilon}{k})k\lambda\sigma_o\|u_1\| \int_{E_\sigma} \phi_1(x)r(x)dx - \int_{E_\sigma} C\phi_1(x)r(x)dx; \end{aligned} \tag{3.34}$$

$$\begin{aligned} \lambda_1 \int_0^1 u_1(x)\phi_1(x)dx &\leq \lambda_1\|u_1\| \int_0^1 \phi_1(x)dx \\ &\leq \lambda_1 k\lambda\sigma_o\|u_1\| \int_{E_\sigma} r(x)\phi_1(x)dx. \end{aligned} \tag{3.35}$$

(3.34)-(3.35) lead to $\|u_1\| \leq \frac{C}{\lambda\sigma_o\epsilon} = \bar{R}$. Let $R_1 = \max\{R', \bar{R}\}$. We obtain (3.33) and then we have

$$i(\Phi, K_{R_1}, K) = i(\Phi_2, K_{R_1}, K) = 0. \tag{3.36}$$

On the other hand, noting that $\xi(x) \equiv 0,\ \mu(x) \equiv 0$, one has $u_o(x) \equiv 0$ for $x \in [a,b]$. Define $H(s,u)$ as above. By the second inequality of (H_5), there exist $\epsilon > 0$ and $r_1 :\ 0 < r_1 \leq r_o$ such that

$$f(u) \leq (q\lambda_1 - \epsilon)u, \quad 0 \leq u \leq r_1. \tag{3.37}$$

We could also show that

$$H(s,u) \neq u, \quad \forall s \in [0,1],\ u \in \partial K_{r_1}.$$

But we omit the details. Thus we obtain

$$i(\Phi, K_{r_1}, K) = i(\theta, K_{r_1}, K) = 1. \tag{3.38}$$

In view of (3.36),(3.38), we obtain

$$i(\Phi, K_{R_1} \backslash K_{r_1}, K) = -1.$$

Thus Φ has a fixed point in $K_{R_1} \backslash K_{r_1}$.

Suppose that u is the fixed point of Φ in $K_{R_1} \backslash K_{r_1}$. Let $y(x) = u(x) + u_o(x)$. Since $y(x) = u(x)$ for $x \in [0, 1]$ and $0 < r_1 \le \|u\| = \|u\|_{[0,1]} = \|y\|_{[0,1]} \le R_1$, we have from (2.5) that $y(x)$ is the positive solution of BVP(1.1).

Thus we complete the proof.

Example. Let us introduce an example to illustrate the usage of our theorem. Consider the BVP:

$$\begin{cases} y''(x) + r(x) y^{\frac{1}{2}}(x - \frac{1}{3}) = 0, & 0 < x < 1, \\ y(x) = -\sin \pi x, & -\frac{1}{3} \le x \le 0, \\ y(1) = 0; & \end{cases} \tag{3.39}$$

where

$$r(x) = \begin{cases} \pi^2 (\sin \pi x) \left| \sin \pi (x - \frac{1}{3}) \right|^{-\frac{1}{2}}, & x \in [0, \frac{1}{3}) \cup (\frac{1}{3}, 1], \\ 0, & x = \frac{1}{3}. \end{cases}$$

Then $w(x) = x - \frac{1}{3}$, $a = -\frac{1}{3}$, $b = 1$, $f(v) = v^{\frac{1}{2}}$, $\alpha = \gamma = 1$, $\beta = \delta = 0, E = [\frac{1}{3}, 1]$. Since

$$\frac{f(v)}{v} = \frac{v^{\frac{1}{2}}}{v} = v^{-\frac{1}{2}}$$

we have $\lim\limits_{v \to +\infty} \frac{f(v)}{v} = 0$, $\lim\limits_{v \to 0+} \frac{f(v)}{v} = +\infty$. Thus (P_1), (P_2), $(H_1) - (H_3)$, (H_6) are satisfied and (3.39) has at least one positive solution $y(x)$. In fact,

$$y(x) = \begin{cases} -\sin \pi x, & -\frac{1}{3} \le x \le 0, \\ \sin \pi x, & 0 < x \le 1, \end{cases}$$

is a positive solution of (3.39).

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