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Slope restrictions and monotonicity in nonlinear systems: almost linear behavior

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Abstract

The problems of absolute stability and existence of forced oscillations for systems with sector restricted nonlinearities are considered. An overview of the results for systems described by ordinary differential equations with constant and periodic coefficients is presented. A result is obtained on exponential absolute stability for systems described by ordinary differential equations. Next, a theorem of Persidskii type, showing that uniform asymptotic stability implies exponential stability, is obtained for a rather general class of nonlinear time varying functional differential equations. The paper ends with some problems awaiting solutions.

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Motto: Happiness=linearity (R. Bellman)

1 What is almost linear behavior?

"Almost linear behavior" was coined in some papers of I. Barbălat and A. Halanay [1, 2, 3] on forced oscillations and dissipativeness, being concerned with systems containing the so called *sector restricted nonlinear functions*. We shall not insist here on the various motivations arising from electrical and control engineering or from flight dynamics, but focus from the very beginning on the mathematical aspects.

A. Consider the following linear system with time varying coefficients

$$\dot{x} = A(t)x + f(t) \tag{1}$$

for which the following properties are known: let $A(t)$ define an exponentially stable evolution i.e. the Cauchy (state transition) matrix $X_A(t,\tau)$ defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}X_A(t,\tau) = A(t)X_A(t,\tau)\ ,\ X_A(\tau,\tau) = I \tag{2}$$

satisfies $|X_A(t,\tau)| \leq \beta_0 \exp\{-\alpha(t-\tau)\}$ for some $\alpha > 0$, $\beta_0 > 0$. Then $x(t) \equiv 0$ is the unique equilibrium of the free (for $f(t) \equiv 0$)

$$\dot{x} = A(t)x, \tag{3}$$

it is exponentially stable (globally) and the forced system (1) has a unique "global" solution (defined on $\mathbb{R}$) which reads

$$x_\infty(t) = \int_{-\infty}^{t} X_A(t,\theta)f(\theta)\mathrm{d}\theta \tag{4}$$

and has the following properties: i) it is bounded on $\mathbb{R}$; ii) it is exponentially stable (globally); iii) it "reproduces" some properties of the forcing term f - if $A(t)$ and $f(t)$ are both T periodic or have rationally dependent periods T_1 and T_2, then $x_\infty(t)$ is periodic; if these periods are rationally independent then $x_\infty(t)$ is quasi-periodic; quasi-periodicity is ensured also if both $A(t)$ and $f(t)$ are quasi-periodic; if $A(t)$ and $f(t)$ are almost periodic then $x_\infty(t)$ is also almost periodic.

B. Consider now the following nonlinear system with time varying parameters

$$\dot{x} = A(t)x - b\phi(t, c^*(t)x) + f(t) \tag{5}$$

where the nonlinear function $\phi : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is subject to the following

$$\phi(t, 0) \equiv 0 \ , \ 0 \leq \frac{\phi(t, \nu_1) - \phi(t, \nu_2)}{\nu_1 - \nu_2} \leq L \ , \ \forall \nu_1 \neq \nu_2. \tag{6}$$

If $f(t) \equiv 0$ i.e. the system is free, then $x(t) \equiv 0$ is its unique equilibrium. For this free system i.e. for

$$\dot{x} = A(t)x - b\phi(t, c^*(t)x) \tag{7}$$

the following is stated.

Problem 1. *(***Absolute stability problem***): find conditions on* (A, b, c, L) *for the equilibrium* $x(t) \equiv 0$ *to be globally asymptotically stable for all functions (linear or nonlinear) subject to (6).*

For the forced system (5) the following is stated.

Problem 2. *(***Forced oscillations problem***): find conditions on* (A, b, c, L) *- under the assumptions of absolute stability for the free system (7) - in order that (5) should have a unique "global" solution* $x_\infty(t)$ *(defined on* $\mathbb{R}$*) with the following properties: i) it is bounded on* $\mathbb{R}$*; ii) it is asymptotically stable (globally); iii) it "reproduces" some properties of the forcing term* f*: if system's coefficients are all* T*-periodic or have rationally dependent periods, then* $x_\infty(t)$ *is periodic; if these periods are rationally independent then* $x_\infty(t)$ *is quasi-periodic; quasi-periodicity is ensured also if all these functions are quasi-periodic; if these functions are almost periodic then* $x_\infty(t)$ *is also almost periodic.*

It becomes now clear that by almost linear behavior it is understood the following gathering of qualitative properties: i) existence of a unique equilibrium (e.g. at the origin) which is globally asymptotically stable; ii) existence and global asymptotic stability of a unique limit regime of the forced system, the corresponding solution being "of the same type" as the forcing term, in the sense described above.

Summarizing this presentation, it appears that systems with sector restricted nonlinearities are very suitable to be checked for an almost linear behavior. This requires solving the two problems stated previously - the absolute stability and forced nonlinear oscillations problems.

2 Almost linear behavior for ordinary differential equations

The main mathematical object of this section will be the system

$$\dot{x} = Ax - b\phi(c^*x) + f(t) \ , \ |f(t)| \leq M \tag{8}$$

for which we define $\chi : \mathbb{C} \mapsto \mathbb{C}$ by $\chi(\sigma) = c^*(\sigma I - A)^{-1}b$, the transfer function of its linear part.

A. We give below some of the most known results concerning its almost linear behavior.

Theorem 1. *Consider system (8) with $f(t) \equiv 0$ under the following assumptions: i) A is a Hurwitz matrix; ii) the nonlinear function $\phi : \mathbb{R} \mapsto \mathbb{R}$ is subject to the following sector restriction*

$$0 \leq \phi(\nu)/\nu \leq L \ , \ \phi(0) = 0. \tag{9}$$

If there exists some $\theta \in \mathbb{R}$ in order that the following frequency domain inequality holds

$$\frac{1}{L} + \Re e(1 + \imath\omega\theta)\chi(\imath\omega) \geq 0 \ \ \omega \in \mathbb{R} \tag{10}$$

and the following alternative is true: either (10) is strict (including for $\omega \to \infty$) and (9) are non-strict or (10) is non-strict and (9) are strict, then the free system (8) is absolutely stable i.e. its equilibrium at $x = 0$ is globally asymptotically stable for all functions ϕ subject to (9).

This theorem is but too well known and we shall not insist on it. Concerning the forced system (8) there is known the following result [4].

Theorem 2. *Consider system (8) under the assumptions of Theorem 1 in a relaxed form i.e. (9) hold only for "large deviations" of the form $|\nu| \geq \lambda_0$ but with the additional condition*

$$\liminf_{|\nu| \to \infty} \frac{\theta}{\nu^2} \left[\int_0^\nu \phi(\lambda) \mathrm{d}\lambda - \frac{1}{2}\nu\phi(\nu) \right] \geq 0 \tag{11}$$

Then (8) is uniformly dissipative in the sense of N. Levinson (uniformly ultimately bounded) and, therefore, if f is T-periodic, (8) has a T-periodic solution.

One could be tempted to think that, were (9) valid for all $\nu \in \mathbb{R}$, Theorem 2 would express some kind of almost linear behavior while restricted to periodic regimes only. This assertion is not true because the periodic solution whose existence is proved might not be unique and by no means is it a limit regime: there exist solutions of (8) such that $\lim_{t\to\infty} |x(t) - x_\infty(t)| \neq 0$ where $x_\infty(t)$ denotes as previously the periodic solution and $x(t)$ some perturbed solution in the sense of Liapunov.

The stronger result, which is indeed a theorem on almost linear behavior, reads as follows (in a slightly modified form with respect to the original one [4])

Theorem 3. *Consider system (8) under the following assumptions: i) A is a Hurwitz matrix; ii) the nonlinear function $\phi : \mathbb{R} \mapsto \mathbb{R}$ is subject to the following conditions*

$$\phi(0) = 0 \ , \ 0 \leq \frac{\phi(\nu_1) - \phi(\nu_2)}{\nu_1 - \nu_2} \leq L \ , \ \forall \nu_1 \neq \nu_2 \tag{12}$$

iii) the frequency domain inequality holds

$$\frac{1}{L} + \Re e \chi(\imath\omega) > 0 \ , \ \omega \geq 0 \tag{13}$$

Then system (8) displays an almost linear behavior - the free system (with $f(t) \equiv 0$) is absolutely stable in the class of nonlinear functions defined by (9) and the forced system has a limit regime $x_\infty(t)$ which is bounded on $\mathbb{R}$, T-periodic if f is such and almost periodic if f is such; moreover $x_\infty(t)$ is globally exponentially stable.

This theorem has a counterpart in the class of the systems with periodic coefficients [5] which was obtained from the more general results on systems with periodic coefficients [6].

Theorem 4. *Consider system (5) with A, b, c being continuous and T-periodic. Let $\phi(t, \nu)$ be subject to (6) and T-periodic for all $\nu \in \mathbb{R}$; suppose $|f(t)| \leq M$ for all $t \in \mathbb{R}$. Let also the following assumptions hold: i) the multipliers of $A(t)$ are inside the unit disk i.e. $A(t)$ defines an exponentially stable evolution; ii) the following linear Hamiltonian system*

$$\begin{aligned} \dot{x} &= (A(t) - \frac{1}{2L} b(t)c^*(t))x + \frac{1}{L} b(t)b^*(t)p \\ \dot{p} &= -\frac{4}{L} c(t)c^*(t)x - (A(t) - \frac{1}{2L} b(t)c^*(t))^* p \end{aligned} \tag{14}$$

is exponentially dichotomic and strongly disconjugate. Then system (5) displays an almost linear behavior (in the sense of Theorem 3).

Since the significance of the exponential dichotomy is well known, we have to explain strong disconjugacy. The multipliers of a periodic Hamiltonian being located symmetrically with respect to the unit circle, system (14) has n linearly independent real solutions $(x_j(t)^* \ p_j(t)^*)^*$, $j = \overline{1,n}$, that tend exponentially to 0 for $t \to \infty$. Introducing the matrix

$$X(t) = (x_1(t) \dots x_n(t)) \tag{15}$$

the fulfilment of the condition

$$\det\ X(t) \neq 0, \forall t \in [0,T] \tag{16}$$

is called strong disconjugacy (non-oscillatory behavior) of the Hamiltonian.

Theorems 3 and 4 correspond to a frequency domain inequality without free parameters or with multiplier of the type **P**; if the Yakubovich Kalman Popov lemma is taken into account, it appears that the main mathematical tool for the study of the almost linear behavior is a quadratic Liapunov function of the form

$$V(t,x) = x^* H(t) x \tag{17}$$

where $H(t)$ is T-periodic if the coefficients of (5) are such and $H(t) \equiv H$ for the case of (8).

If we are concerned with Theorems 1 and 2 (which account for an "almost" almost linear behavior)), the frequency domain condition is of V.M. Popov type with the stability multiplier $1+\theta\sigma$, where σ is the complex variable and θ is the free parameter, also called **PD** multiplier; if again the Yakubovich Kalman Popov lemma is taken into account, a Liapunov function of the type "quadratic form plus the integral of the nonlinearity" is obtained

$$V(x) = x^* H x + \theta \int_0^{c^* x} \phi(\lambda) \mathrm{d}\lambda \tag{18}$$

where $\theta \in \mathbb{R}$ is the free parameter of the frequency domain inequality (10).

B. The further development of the absolute stability theory was directed to the relaxation of the stability conditions: since all of them are only sufficient conditions, this trend may be considered as aiming to reduce the gap between sufficient and necessary conditions i.e. the "degree of conservatism" as is it called in the engineering world. The most natural way in this direction was that of making additional assumptions on the nonlinear functions (monotone, slope restricted) and to obtain, as a consequence, new frequency domain inequalities based on new stability multipliers.

Monotonicity, expressed by the first inequality in (12), possibly combined with the global Lipschitz condition or with the even stronger slope restrictions $0 \leq \phi'(\nu) \leq \bar{\nu}$ are the additional restrictions that generated most of the stability multipliers other than the **PD** - Popov multiplier. In [7] there is to be found an account concerning these multipliers; worth mentioning that they are in close connection with the *systems with augmented dynamics* [8].

Here we shall discuss only the multiplier introduced by Yakubovich in [9, 10] which has the form

$$\zeta(\sigma) = 1 + \theta\sigma - \beta\sigma^2 \ , \ \theta \in \mathbb{R} \ , \ \beta > 0. \tag{19}$$

This special attention is due to the fact that, according to our knowledge, this is the only one of these newer multipliers for which a dissipativeness result exists [3]. It has the form below (adapted to the case considered in this paper).

Theorem 5. *Consider the system (8) under the following assumptions: i) A is a Hurwitz matrix; ii) the nonlinear function $\phi : \mathbb{R} \mapsto \mathbb{R}$ is subject to the sector restriction (9) and to the slope restriction*

$$0 \leq \phi'(\nu) \leq \bar{\nu}. \tag{20}$$

If there exist some $\theta \in \mathbb{R}$, $\beta > 0$ in order that the following frequency domain inequality holds

$$\left(\frac{1}{L} + \frac{\beta}{\bar{\nu}}\omega^2\right) + \Re e(1 + \imath\omega\theta + \beta\omega^2)\chi(\imath\omega) \geq 0 \ , \ \omega \in \mathbb{R} \tag{21}$$

and the following alternative is true: either the frequency domain inequality is strict (including for $\omega \to \infty$) and the sector and slope restrictions are non-strict or the frequency domain inequality is non-strict and the sector and slope restrictions are strict then the free system (8) is absolutely stable for all functions ϕ subject to (9) and (20).

If (9) and (20) hold only for "large deviations" of the form $|\nu| \geq \lambda_0$ but with the additional condition (11), then (8) is dissipative in the sense of Levinson and, therefore, if f is T-periodic then (8) has a T-periodic solution.

We just mention, with reference to Theorem 4, that Theorem 5 might also have its counterpart for systems with periodic coefficients. However, a more interesting idea of extending the above results arises from [11]: it was shown in this paper that, under the assumption (12) on ϕ, if f is T-periodic and a T-periodic solution of (8) *does exist*, then fulfilment of the frequency domain inequality

$$\Re e\left(\frac{1}{L}+\chi(\imath\omega)\right)\zeta(\imath\omega)>0\ ,\ \omega\in\mathbb{R}, \tag{22}$$

where $\zeta:\mathbb{C}\mapsto\mathbb{C}$ is the Zames Falb multiplier, implies input output stability of the periodic solution. The Zames Falb multiplier is defined by

$$\zeta(\sigma)=1-\sum_{1}^{\infty}\theta_n \mathrm{e}^{n\sigma T}\ ,\ \theta_n\geq 0\ ,\ \sum_{1}^{\infty}\theta_n<1 \tag{23}$$

Since (22) is already a sufficient condition of absolute stability, one may ask whether it is not also a sufficient condition for dissipativeness in the sense of Levinson and, therefore, for the existence of a periodic solution. In this way the frequency domain inequality with the Zames Falb multiplier might be a condition for almost linear behavior.

3 A result of the Persidskii type

A classical result due to K. P. Persidskii states that *in the case of the linear systems uniform asymptotic stability is equivalent to exponential stability.* A generalization of this result to nonlinear systems is due to A. Halanay (1960) [12, 13] and we reproduce it for the sake of completeness

Lemma 1. *Consider the nonlinear system*

$$\dot{x}=f(t,x)\ ,\ \dim\ x=n \tag{24}$$

under the following assumptions: i) f is continuous with respect to both arguments and $f(t,0)\equiv 0$; ii)$|f(t,x)|\leq L(\rho)|x|$ for $|x|\leq\rho$. Let the equilibrium at the origin be uniformly asymptotically stable and satisfy the estimate

$$|x(t;t_0,x_0)|\leq k_0|x_0|\psi(t-t_0) \tag{25}$$

where $\psi \in \mathcal{L}$ (the class of the Kamke-Massera functions that are continuous, monotone decreasing and such that $\lim_{t\to\infty}\psi(t) = 0$). Then the stability of the equilibrium is exponential.

In the following we shall show that *the absolute stability obtained from the frequency domain inequality of V.M. Popov is exponential.* For simplicity we shall consider the stable case only. The result is as follows.

Proposition 1. *Consider system (8) with $f(t) \equiv 0$ under the assumptions of Theorem 1 with the special mention that L may be finite only. Then the free system (8) is exponentially absolutely stable i.e. its equilibrium at $x = 0$ is globally exponentially stable.*

Outline of proof We shall make use of some intermediate results of the proof of Theorem 1. From the frequency domain inequality (10) and applying the positiveness theorem of V.M. Popov [14] i.e. the Yakubovich-Kalman-Popov lemma, we deduce existence of a Hermitian matrix $P_0 = P_0^*$, a vector w_0 and a scalar γ_0 such that the following holds

$$\begin{aligned}\mathcal{V}^{\mathrm{o}}(x(t)) = \mathcal{V}^{\mathrm{o}}(x_0) - \int_0^t |-\gamma_0\phi(c^*x(\lambda)) + w_0^*x(\lambda)|^2\mathrm{d}\lambda -\\ - \int_0^t \phi(c^*x(\lambda))(c^*x(\lambda) - \phi(c^*x(\lambda))/L)\mathrm{d}\lambda\end{aligned} \tag{26}$$

where the candidate Liapunov function $\mathcal{V}^{\mathrm{o}}(x)$ has the usual structure "quadratic form plus integral of the nonlinear function"

$$\mathcal{V}^{\mathrm{o}}(x) = x^*P_0x + \theta\int_0^{c^*x}\phi(\lambda)\mathrm{d}\lambda \tag{27}$$

and $\theta \in \mathbb{R}$ is the parameter for which (10) holds. If the strict frequency domain inequality holds then for some $\varepsilon > 0$ the Yakubovich-Kalman-Popov lemma allows existence of a Hermitian matrix $P_\varepsilon = P_\varepsilon^*$, a vector w_ε and a scalar γ_ε such that the following holds

$$\begin{aligned}\mathcal{V}^{\varepsilon}(x(t)) = \mathcal{V}^{\varepsilon}(x_0) - \int_0^t |-\gamma_\varepsilon\phi(c^*x(\lambda)) + w_\varepsilon^*x(\lambda)|^2\mathrm{d}\lambda -\\ - \int_0^t \phi(c^*x(\lambda))(c^*x(\lambda) - \phi(c^*x(\lambda))/L)\mathrm{d}\lambda - \varepsilon\int_0^t |x(\lambda)|^2\mathrm{d}\lambda\end{aligned} \tag{28}$$

where

$$\mathcal{V}^\varepsilon(x) = x^* P_\varepsilon x + \theta \int_0^{c^* x} \phi(\lambda)\mathrm{d}\lambda \tag{29}$$

Worth mentioning that $\lim_{\varepsilon\to 0} P_\varepsilon = P_0$, $\lim_{\varepsilon\to 0} w_\varepsilon = w_0$, $\lim_{\varepsilon\to 0}\gamma_\varepsilon = \gamma_0$ from continuity reasons.

Equalities (26) and (28) give some information about the derivative along system's solutions. If (10) is strict we deduce from (28) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}^\varepsilon(x(t)) \leq -\varepsilon|x(t)|^2 < 0 \tag{30}$$

hence the derivative is negative definite. If (10) is non-strict but the sector conditions are strict then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}^{\mathrm{o}}(x(t)) \leq -\phi(c^*x(t))(c^*x(t) - \phi(c^*x(t))/L) \leq 0 \tag{31}$$

and the derivative vanishes on the set $c^*x = 0$ only. Since A is a Hurwitz matrix, the pair (c^*, A) is detectable and the largest invariant set contained in $\{x \in \mathbb{R} | c^*x = 0\}$ is the singleton $\{0\}$.

We need now some sign information for the Liapunov function itself. At this point we consider the linear functions of the sector i.e. $\phi(\nu) = h\nu$, $0 \leq h \leq L$. The free system (8) becomes

$$\dot{x} = (A - bhc^*)x \tag{32}$$

and it is associated with the quadratic Liapunov function

$$\mathcal{V}(x) = x^*(P + \frac{1}{2}\theta hcc^*)x \tag{33}$$

with $\mathcal{V}$ accounting for both $\mathcal{V}^\varepsilon$ and $\mathcal{V}^{\mathrm{o}}$ and P for both P_ε and P_0. The frequency domain inequality (10) and the sector inequality (9) ensure exponential stability of the linear system for all $h \in (0, L]$. Consider first the case of the strict inequality (10) and suppose there exists some $\tilde{h} \in (0, L]$ and some $\omega_0 \geq 0$ such that

$$\det(\imath\omega_0 I - A + b\tilde{h}c^*) = 0 = \det(\imath\omega_0 I - A)(1 + \tilde{h}c^*(\imath\omega_0 I - A)^{-1}b)$$

Since A is a Hurwitz matrix, the second factor should be 0 hence $c^*(\imath\omega_0 I - A)^{-1}b = -1/\tilde{h}$. Substituting in (10) it follows that

$$\frac{1}{L} + \Re e(1 + \imath\omega_0\theta)(-\frac{1}{\tilde{h}}) = \frac{1}{L} - \frac{1}{\tilde{h}} > 0$$

what contradicts $0 < \tilde{h} \leq L$. Let now (10) be non-strict and (9) strict hence $0 < \tilde{h} < L$; proceeding as previously we obtain from (10) that $1/L - 1/\tilde{h} \geq 0$ again a contradiction.

In the first case this exponential stability of the linear system allows to obtain from (30) and the properties of the Liapunov matrix equation that the matrix $P^\varepsilon + (1/2)\theta hcc^*$ is positive definite for all $\tilde{h} \in [0, L]$ and the $\theta \in \mathbb{R}$ ensuring (10). In particular $P^\varepsilon > 0$.

In the second case the following Liapunov matrix inequality is obtained

$$(P^{\mathrm{o}} + \frac{1}{2}\theta hcc^*)(A - bhc^*) + (A - bhc^*)^*(P^{\mathrm{o}} + \frac{1}{2}\theta hcc^*) \leq -h(1 - h/L)cc^* \tag{34}$$

Since (c^*, A) is detectable, $(c^*, A - bhc^*)$ it is such; it follows from the fact that $A - bhc^*$ is a Hurwitz that $P^{\mathrm{o}} + (1/2)\theta hcc^*$ is positive definite for all $h \in [0, L)$ and the $\theta \in \mathbb{R}$ ensuring (10). In particular $P^{\mathrm{o}} > 0$. We deduce in both cases

$$P + \frac{1}{2}\theta hcc^* \geq \delta_0 I$$

with $\delta_0 > 0$ being independent of $h \in [0, L]$.

We turn now to the nonlinear case and consider the Liapunov function

$$\mathcal{V}(x) = x^*Px + \theta \int_0^{c^*x} \phi(\lambda)\mathrm{d}\lambda$$

If $\theta \geq 0$ then $\theta hcc^* \geq 0$ and since $\delta_0 > 0$ is independent of h we deduce $\mathcal{V}(x) \geq \delta_0|x|^2$, taking into account also that the integral is nonnegative. If $\theta < 0$ then we may write

$$\mathcal{V}(x) = x^*(P + \frac{1}{2}\theta L(cc^*))x - \theta \int_0^{c^*x} (L\lambda - \phi(\lambda))\mathrm{d}\lambda \geq \delta_0|x|^2 \tag{35}$$

A quadratic upper estimate for $\mathcal{V}(x)$ is even easier to obtain since L is finite. We shall have

$$\mathcal{V}(x) \leq |x^*Px| + \frac{1}{2}|\theta|L(c^*x)^2 \leq \Lambda_0|x|^2\ ,\ \Lambda_0 > 0 \tag{36}$$

We are now in position to obtain exponential stability. If (10) is strict then we use (30) and (36) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(x(t)) \leq -\frac{\varepsilon}{\Lambda_0}\mathcal{V}(x(t))$$

hence

$$\mathcal{V}(x(t)) \leq \mathcal{V}(x_0)\exp\{-\frac{\varepsilon}{\Lambda_0}t\}$$

and finally, from the quadratic bounds of $\mathcal{V}$ it follows that

$$|x(t)| \leq \sqrt{\frac{\Lambda_0}{\delta_0}}\, \mathrm{e}^{-\frac{\varepsilon}{2\Lambda_0}t}|x_0| \tag{37}$$

In the second case all we know is that $\lim_{t\to\infty} x(t) = 0$ hence

$$\lim_{t\to\infty} \mathcal{V}(x(t)) = 0$$

from the continuity of $\mathcal{V}$. There will exist $\psi \in \mathcal{L}$ - a Kamke-Massera function mentioned in Lemma 1 such that $\mathcal{V}(x(t)) \leq \mathcal{V}(x_0)\ \psi(t)$ hence

$$|x(t)| \leq \sqrt{\frac{\Lambda_0}{\delta_0}}\ |x_0|\ \psi(t)$$

Application of Lemma 1 ends the proof.

4 Exponential stability for time lag systems

In the previous section we obtained a result of exponential stability for nonlinear time lag systems. This result represents a step ahead in the analysis of the almost linear behavior of the time lag systems with sector restricted nonlinearities. In the previous section we reproduced (Lemma 1) a generalization of the classical Persidskii theorem stating that "for linear systems uniform asymptotic stability is always exponential". The theorem of K. P. Persidskii as well as its generalization due to A. Halanay [12] concern ordinary differential equations. For linear time delay systems the Persidskii type result is also due to A. Halanay [12].

In the following we shall obtain the same Persidskii type theorem for nonlinear functional differential equations of delayed type.

Theorem 6. *Consider the system*

$$\dot{x}(t) = f(t, x_t)\ ,\ \ f(t, 0) \equiv 0 \tag{38}$$

with the standard Krasovskii-Halanay-Hale notations. Here the vector functional $f : \mathbb{R} \times \mathcal{C}(-\tau, 0; \mathbb{R}^n) \mapsto \mathbb{R}^n$ is continuous in both arguments and locally Lipschitz, with constant L, with respect to the second one.

Let the equilibrium at $x \equiv 0$ be uniformly asymptotically stable i.e. be subject to

$$|x(t; t_0; \phi)| \leq \chi(\|\phi\|)\psi(t - t_0) \ , \ \|\phi\| \leq \rho_0 \tag{39}$$

where $\| \cdot \|$ denotes the sup *norm on the Banach space $\mathcal{C}(-\tau, 0; \mathbb{R}^n)$ - the system's state space. In (39) $\chi(\cdot)$ is a $\mathcal{K}$ function i.e. a strictly increasing Kamke-Massera function while $\psi(\cdot)$ is a $\mathcal{L}$ function i.e. a strictly decreasing Kamke-Massera function approaching 0 asymptotically at ∞. If χ is linear i.e. $\chi(\rho) = k_0\rho$, $k_0 > 0$ then the stability is exponential.*

Proof From (39) we deduce, using the fact that ψ is decreasing

$$\|x_t(t_0, \phi)\| = \sup_{-\tau \leq \theta \leq 0} |x(t + \theta; t_0, \phi)| \leq k_0\|\phi\|\psi(t - \tau - t_0) \tag{40}$$

Consider now the Liapunov functional $\mathcal{V} : \mathbb{R} \times \mathcal{C}(-\tau, 0; \mathbb{R}^n) \to \mathbb{R}_+$ defined by

$$\mathcal{V}(t, \phi) = \int_0^T \|x_{t+\lambda}(t, \phi)\|^2 \mathrm{d}\lambda + \sup_{\lambda \geq 0} \|x_{t+\lambda}(t, \phi)\|^2 \tag{41}$$

for some fixed $T > 0$ which is subject to choice. Taking into account (40) and (41) the following inequalities are straightforward

$$\|\phi\|^2 \leq \mathcal{V}(t, \phi) \leq (1 + T)k_0^2\psi(0)^2\|\phi\|^2 \tag{42}$$

Moreover, we may show that this Liapunov functional is Lipschitz in its second argument. We shall have

$$|\mathcal{V}(t, \phi_1) - \mathcal{V}(t, \phi_2)| \leq \left| \int_0^T \|x_{t+\lambda}(t, \phi_1)\|^2 \mathrm{d}\lambda - \int_0^T \|x_{t+\lambda}(t, \phi_2)\|^2 \mathrm{d}\lambda \right| +$$

$$+ \left| \sup_{\lambda \geq 0} \|x_{t+\lambda}(t, \phi_1)\|^2 - \sup_{\lambda \geq 0} \|x_{t+\lambda}(t, \phi_2)\|^2 \right| \leq$$

$$\leq \int_0^T (\|x_{t+\lambda}(t, \phi_1)\| + \|x_{t+\lambda}(t, \phi_2)\|)\|x_{t+\lambda}(t, \phi_1) - x_{t+\lambda}(t, \phi_2)\| \mathrm{d}\lambda +$$

$$+ \sup_{\lambda \geq 0}(\|x_{t+\lambda}(t, \phi_1)\| + \|x_{t+\lambda}(t, \phi_2)\|) \sup_{\lambda \geq 0} \|x_{t+\lambda}(t, \phi_1) - x_{t+\lambda}(t, \phi_2)\|$$

Making again use of (39) and (40) we find

$$\|x_{t+\lambda}(t,\phi_1)\| + \|x_{t+\lambda}(t,\phi_2)\| \leq k_0\psi(0)(\|\phi_1\| + \|\phi_2\|)$$

Using the Lipschitz assumption on $f(t,\cdot)$ the following estimate is obtained

$$\|x_{t+\lambda}(t,\phi_1) - x_{t+\lambda}(t,\phi_2)\| \leq \mathrm{e}^{\int_0^\lambda L(\mu)\mathrm{d}\mu}\|\phi_1 - \phi_2\|$$

Consequently the following is obtained

$$\begin{aligned}
&\int_0^T (\|x_{t+\lambda}(t,\phi_1)\| + \|x_{t+\lambda}(t,\phi_2)\|)\|x_{t+\lambda}(t,\phi_1) - x_{t+\lambda}(t,\phi_2)\|\mathrm{d}\lambda \leq \\
&\leq k_0\psi(0)\left(\int_0^T \mathrm{e}^{\int_0^\lambda L(\mu)\mathrm{d}\mu}\mathrm{d}\lambda\right)(\|\phi_1\| + \|\phi_2\|)\|\phi_1 - \phi_2\|
\end{aligned} \tag{43}$$

It has been shown in [12], Section 4.2, that $\sup_{\lambda\geq 0}\|x_{\lambda+\cdot}(t,\phi)\|$ is monotone decreasing. In the same way (and quite straightforward) it can be proved that $\sup_{\lambda\geq 0}(\|x_{\lambda+\cdot}(t,\phi_1)\| + \|x_{\lambda+\cdot}(t,\phi_2)\|)$ and $\sup_{\lambda\geq 0}\|x_{\lambda+\cdot}(t,\phi_1) - x_{\lambda+\cdot}(t,\phi_2)\|$ are also monotone decreasing. Therefore

$$\begin{aligned}
&\sup_{\lambda\geq 0}(\|x_{t+\lambda}(t,\phi_1)\| + \|x_{t+\lambda}(t,\phi_2)\|)\sup_{\lambda\geq 0}\|x_{t+\lambda}(t,\phi_1) - x_{t+\lambda}(t,\phi_2)\| \leq \\
&\leq k_0\psi(0)(\|\phi_1\| + \|\phi_2\|)\|\phi_1 - \phi_2\|
\end{aligned} \tag{44}$$

Combining (43) and (44) we summarize

$$|\mathcal{V}(t,\phi_1) - \mathcal{V}(t,\phi_2)| \leq k_0\psi(0)\left(1 + \int_0^T \mathrm{e}^{\int_0^\lambda L(\mu)\mathrm{d}\mu}\mathrm{d}\lambda\right)(\|\phi_1\| + \|\phi_2\|)\|\phi_1 - \phi_2\| \tag{45}$$

i.e. the Lipschitz property.

We discuss now the behavior of the Liapunov functional (41) along the solutions of (38). We shall have

$$\begin{aligned}
\mathcal{V}(t, x_t(t_0,\phi)) &= \int_0^T \|x_{t+\lambda}(t, x_t(t_0,\phi))\|^2\mathrm{d}\lambda + \sup_{\lambda\geq 0}\|x_{t+\lambda}(t, x_t(t_0,\phi))\|^2 = \\
&= \int_t^{t+T} \|x_\lambda(t_0,\phi)\|^2\mathrm{d}\lambda + \sup_{\lambda\geq 0}\|x_{t+\lambda}(t_0,\phi)\|^2
\end{aligned}$$

and, further

$$\frac{1}{h}(\mathcal{V}(t+h, x_{t+h}(t_0,\phi)) - \mathcal{V}(t, x_t(t_0,\phi))) =$$

$$= \frac{1}{h}\int_t^{t+h} (\|x_{\lambda+T}(t_0,\phi)\|^2 - \|x_\lambda(t_0,\phi)\|^2)\mathrm{d}\lambda +$$

$$+\frac{1}{h}(\sup_{\lambda\geq 0}\|x_{t+h+\lambda}(t_0,\phi)\|^2 - \sup_{\lambda\geq 0}\|x_{t+\lambda}(t_0,\phi)\|^2) \leq$$

$$\leq \frac{1}{h}\int_t^{t+h} (\|x_{\lambda+T}(t_0,\phi)\|^2 - \|x_\lambda(t_0,\phi)\|^2)\mathrm{d}\lambda$$

(We used again monotonicity of $\sup_{\lambda\geq 0}\|x_{\lambda+\cdot}(t_0,\phi)\|^2$). Further

$$\|x_{\lambda+T}(t_0,\phi)\|^2 - \|x_\lambda(t_0,\phi)\|^2 =$$

$$= (\|x_{\lambda+T}(t_0,\phi)\| + \|x_\lambda(t_0,\phi)\|)(\|x_{\lambda+T}(t_0,\phi)\| - \|x_\lambda(t_0,\phi)\|) \leq$$

$$\leq \|x_\lambda(t_0,\phi)\|(\|x_{\lambda+T}(\lambda, x_\lambda(t_0,\phi)\| - \|x_\lambda(t_0,\phi)\|) \leq$$

$$\leq (k_0\psi(T) - 1)\|x_\lambda(t_0,\phi)\|^2 \leq -\frac{1}{2}\|x_\lambda(t_0,\phi)\|^2$$

provided $T > 0$ is chosen sufficiently large in order to have $k_0\psi(T) < 1/2$. We then deduce from (42) that

$$-\frac{1}{2}\|x_\lambda(t_0,\phi)\|^2 \leq -\frac{1}{2(1+T)k_0^2\psi(0)^2}\mathcal{V}(\lambda, x_\lambda(t_0,\phi))$$

Denoting $\mathcal{V}^\star(\lambda) := \mathcal{V}(\lambda, x_\lambda(t_0,\phi))$ we obtain

$$\frac{1}{h}(\mathcal{V}^\star(t+h) - \mathcal{V}^\star(t)) \leq -\frac{1}{2(1+T)k_0^2\psi(0)^2}\left(\frac{1}{h}\int_t^{t+h}\mathcal{V}^\star(\lambda)\mathrm{d}\lambda\right)$$

Since $\mathcal{V}^\star(t)$ is at least integrable, all its definition points are Lebesgue points and we shall have

$$\limsup_{h\to 0_+}\frac{1}{h}(\mathcal{V}^\star(t+h) - \mathcal{V}^\star(t)) \leq -\frac{1}{2(1+T)k_0^2\psi(0)^2}\mathcal{V}^\star(t)$$

hence

$$\mathcal{V}^\star(t) \leq \mathcal{V}^\star(t_0)\exp\left\{-\frac{1}{2(1+T)k_0^2\psi(0)^2}(t-t_0)\right\} \tag{46}$$

Making use again of (42) we obtain finally

$$\|x_t(t_0,\phi)\| \leq k_0\psi(0)\sqrt{1+T}\exp\left\{-\frac{1}{4(1+T)k_0^2\psi(0)^2}(t-t_0)\right\}\|\phi\| \tag{47}$$

and the proof is complete. This theorem offers the possibility to obtain exponential absolute stability for time delay systems provided the necessary estimates of the solutions may be obtained. However absolute stability of time delay systems is obtained from Liapunov functionals as well as from frequency domain inequalities applied to nonlinear integral equations; in this last case the estimates of the solutions must be considered with additional care since the method lacks, according to our opinion, the sharpness of the Liapunov approach - at least in the case of the systems with time lags.

5 Conclusions and perspectives

We have discussed throughout the paper the concept of almost linear behavior and its implications for systems with sector restricted nonlinearities. In the case of the systems described by ordinary differential equations with constant coefficients we obtained a result of Persidskii type i.e. the exponential absolute stability. For rather general time varying nonlinear systems with time delay we were able, by slightly modifying a Liapunov functional from [12] to obtain exponential stability from uniform asymptotic stability. This opens the way to the exponential absolute stability of the time delay systems with sector restricted nonlinearities provided "good" estimates of the solutions are available. With respect to this we already mentioned the "competition" between the Liapunov method and the frequency domain (input/output) method - see [15].

The same choice between the two methods appears also in the problem of the forced oscillations. It had been mentioned in [15] that the Liapunov method appeared as more suitable for the problem of the forced oscillations than the input/output method. This assertion turned to be particularly true in the problem of the existence of forced periodic oscillations: only when equivalence of the two methods was taken into account, it was possible to obtain frequency domain inequalities with free parameters ensuring this existence. For time delay systems the frequency domain inequalities without free parameters only allowed to obtain both existence and exponential stability of forced oscillations.

The problem of these criteria containing free parameters has been approached quite recently in [11] where stability of the forced periodic oscillations was obtained from a frequency domain inequality of the Zames Falb type [16]; an existence theorem for periodic oscillations based on the Zames Falb criterion is still missing.

To end this discussion we mention that all problems discussed throughout the paper are significant for systems described by neutral functional differential equations also. And this sends to hyperbolic systems of the propagation and of the conservation laws.

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