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An existence result for (p,q)-Laplacian BVP with falling zeros

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Abstract. We show the existence of a positive solution to the (p,q)-Laplacian problem

$$\begin{cases} -\Delta_p u - a\Delta_q u = \lambda f(u) - h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for λ large, where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega, a$ is a nonnegative constant, $h \in L^\infty(\Omega)$, $p > q > 1$, and f satisfies $f(0) = f(r) = 0$ with $f > 0$ on $(0,r)$ for some $r > 0$.

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1 Introduction

Consider the (p,q) Laplacian problem

$$\begin{cases} -\Delta_p u - a\Delta_q u = \lambda f(u) - h(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $p > q > 1$, $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u)$, $f : [0,\infty) \to \mathbb{R}$, $h : \Omega \to \mathbb{R}$, a is a nonnegative constant, and λ is a positive parameter.

In contrast to the p-Laplacian, the (p,q) -Laplacian is not homogenous and occurs in applied areas such as chemical reactions and quantum physics (see e.g. [2, 6]) and has been studied extensively in recent years. The existence of a positive solution to (1.1) for λ large when f is p-sublinear at ∞ was studied in [1]. We are interested here in the case when f has falling zeroes and are motivated by a result in [9, Theorem 1.1], where the existence of a positive solution to (1.1) was established for λ large when $a = 0$ (the p-Laplacian equation), $h \equiv \varepsilon$ is small, and f satisfies the following condition:

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(H) There exists a constant $r > 0$ such that $f : [0, r] \to \mathbb{R}$ is continuous with $f(0) = f(r) = 0$ and $f > 0$ on $(0, r)$.

This result extended a previous work in [4] where $p = 2$ and $f(u) = u - u^3$. Note that under the assumption (H), the function $g(u) = \lambda f(u) - \varepsilon$ has at least two zeroes for λ large as $g(0) = g(r) < 0$ and $g(r/2) = \lambda f(r/2) - \varepsilon > 0$ for λ large. The purpose of this note is to extend the result in [9] to the general (p, q)-Laplacian. In fact, we show that for any $h \in L^\infty(\Omega)$, (1.1) has a positive solution provided that λ is large enough. This extension is nontrivial since the lack of homogeneity of the operator makes it difficult to create a positive subsolution.

Our main result is

Theorem 1.1. *Let (H) hold and $c_0 > 0$. Suppose $h \in L^\infty(\Omega)$ with $0 \le h \le c_0$ in Ω. Then there exists a constant $\lambda_0 > 0$ depending on c_0 such that (1.1) has a positive solution for $\lambda > \lambda_0$.*

We shall denote by $\|\cdot\|_p, |\cdot|_1$, and $|\cdot|_{1,\nu}$ the norms in $L^p(\Omega), C^1(\bar{\Omega})$, and $C^{1,\nu}(\bar{\Omega})$ respectively.

Lemma 1.2. *Let $f \in L^\infty(\Omega)$ with $\|f\|_\infty \le M$. Then the problem*

$$\begin{cases} -\Delta_p u - a\Delta_q u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

has a unique solution $u \in C^{1,\nu}(\bar{\Omega})$ for some $\nu \in (0, 1)$. Furthermore $|u|_{1,\nu} \le C$, where $C > 0$ is a constant depending on M (but not on a and f).

Proof. Let $E = W_0^{1,p}(\Omega)$ with norm $\|u\| = \left(\int_\Omega |\nabla u|^p\right)^{1/p}$. Define

$$\langle Au, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v + a\int_\Omega |\nabla u|^{q-2}\nabla u \cdot \nabla v$$

and

$$F(v) = \int_\Omega fv$$

for $u, v \in E$. Then it is easily seen that $A : E \to E^*$ is continuous with

$$\frac{\langle Au, u\rangle}{\|u\|} \ge \|u\|^{p-1} \to \infty \quad \text{as } \|u\| \to \infty$$

and

$$\langle Au - Av, u - v\rangle \ge \int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \cdot \nabla u - \nabla v\right) > 0 \quad \text{for } u \ne v.$$

Hence by the Minty–Browder Theorem (see [3]), there exists a unique $u \in E$ such that $Au = F$ in E^* i.e. u is the unique weak solution of (1.2). To show that $u \in C^{1,\nu}(\bar{\Omega})$ for some $\nu \in (0, 1)$, we need Lieberman's regularity result in [8]. By the weak comparison principle [10, Theorem 10.1], $|u| \le \tilde{u}$ in Ω, where $\tilde{u}$ satisfies

$$\begin{cases} -\Delta_p \tilde{u} - a\Delta_q \tilde{u} = M & \text{in } B(0, R), \\ \tilde{u} = 0 & \text{on } \partial B(0, R), \end{cases}$$

where $R > 0$ is such that $\Omega \subset B(0,R)$ and $B(0,R)$ denotes the open ball centered at 0 with radius R in $\mathbb{R}^n$. Note that $\tilde{u}$ is unique, radial, and

$$\tilde{u}(x) = \int_{|x|}^{R} \phi^{-1}\left(\frac{Ms}{n}\right) ds \le \int_0^R \left(\frac{Ms}{n}\right)^{\frac{1}{p-1}} ds = \left(\frac{M}{n}\right)^{\frac{1}{p-1}} R^{\frac{p}{p-1}} \equiv M_0 \qquad \forall x \in B(0,R),$$

where $\phi(t) = |t|^{p-2}t + a|t|^{q-2}t$.

Next, let $w \in C^{1,\nu}(\bar{\Omega})$ satisfy $\Delta w = f$ in $\Omega, w = 0$ on $\partial\Omega$. Then the equation in (1.2) becomes

$$\operatorname{div} A(x,u,\nabla u) = 0 \quad \text{in } \Omega,$$

where $A(x,z,\mu) = |\mu|^{p-2}\mu + a|\mu|^{q-2}\mu + \nabla w(x)$. Since $A(x,z,\mu)$ satisfies assumptions (1.10a)–(1.10d) in [8, p. 320] and $|u| \le M_0$ in Ω, it follows from the remark after Theorem 1.7 in [8] that $u \in C^{1,\nu}(\bar{\Omega})$ for some $\nu \in (0,1)$ and $|u|_{1,\nu} \le C$, where C depends on M. □

Lemma 1.3. *Let $f, g \in L^\infty(\Omega)$ and $u, v \in W_0^{1,p}(\Omega)$ satisfy*

$$\begin{cases} -\Delta_p u - a\Delta_q u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta_p v - a\Delta_q v = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $|u - v|_1 \to 0$ as $\|f - g\|_1 \to 0$.

Proof. By Lemma 1.2, $u, v \in C^{1,\nu}(\bar{\Omega})$ for some $\nu \in (0,1)$ and $|u|_{1,\nu}, |v|_{1,\nu} \le C$, where C depends on an upper bound of $\|f\|_\infty, \|g\|_\infty$.

Multiplying the equation

$$-(\Delta_p u - \Delta_p v) - a(\Delta_q u - \Delta_q v) = f - g \quad \text{in } \Omega$$

by $u - v$ and integrating, we get

$$\begin{aligned} \int_\Omega |\nabla(u-v)|^p + a\int_\Omega |\nabla(u-v)|^q &= \int_\Omega (f-g)(u-v) \\ &\le 2C\|f-g\|_1 \to 0 \end{aligned}$$

as $\|f - g\|_1 \to 0$. From this and the interpolation inequality [7, Corollary 1.3],

$$|w|_1 \le c|w|_{1,\beta}^{1-\theta}\|w\|_{W^{1,p}}^{\theta} \qquad \forall w \in C^{1,\beta}(\bar{\Omega})$$

for some $c > 0$ and $\theta \in (0,1)$, we obtain $|u - v|_1 \to 0$ as $\|f - g\|_1 \to 0$, which completes the proof. □

Lemma 1.4. *Let $m > 0$ and u_m be the solution of*

$$\begin{cases} -\Delta_p u - a\Delta_q u = m & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

(i) $\|u_m\|_\infty \to \infty$ *as* $m \to \infty$.

(ii) $\|u_m\|_\infty \to 0$ *as* $m \to 0$.

Proof. *(i)* A calculation shows that $u_m = m^{\frac{1}{p-1}} v_m$, where v_m satisfies

$$\begin{cases} -\Delta_p v_m - a m^{\frac{q-p}{p-1}} \Delta_q v_m = 1 & \text{in } \Omega, \\ v_m = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Suppose $\|u_m\|_\infty \not\longrightarrow \infty$ as $m \to \infty$. Then by going to a subsequence if necessary, we can assume that $\|u_m\|_\infty \leq M \; \forall m > 0$ for some $M > 0$.

This implies $|v_m| \leq M m^{-\frac{1}{p-1}} \leq M$ in Ω for $m > 1$. By Lemma 1.2, $|v_m|_{1,\nu} \leq C$, where $C > 0$ is independent of m. Hence there exists $v_0 \in C^1(\bar{\Omega})$ and a subsequence of (v_m), which we still denote by (v_m), such that $v_m \to v_0$ in $C^1(\bar{\Omega})$. Since

$$\int_\Omega |\nabla v_m|^{p-2} \nabla v_m \cdot \nabla \psi + a m^{\frac{q-p}{p-1}} \int_\Omega |\nabla v_m|^{q-2} \nabla v_m \cdot \nabla \psi = \int_\Omega \psi \qquad \forall \psi \in W_0^{1,p}(\Omega),$$

it follows by letting $m \to \infty$ that

$$\int_\Omega |\nabla v_0|^{p-2} \nabla v_0 \cdot \nabla \psi = \int_\Omega \psi \qquad \forall \psi \in W_0^{1,p}(\Omega),$$

i.e v_0 satisfies $-\Delta_p v_0 = 1$ in Ω, $v_0 = 0$ on $\partial\Omega$. Consequently,

$$\|u_m\|_\infty = m^{\frac{1}{p-1}} \|v_m\|_\infty \to \infty \qquad \text{as } m \to \infty$$

a contradiction which proves (i).

(ii) Using Lemma 1.3 with $f = m$ and $g = 0$, we obtain the result. □

Proof of Theorem 1.1. Let u_m be defined by Lemma 1.4. By Lemma 1.3, the map $m \mapsto \|u_m\|_\infty$ is continuous. This, together with Lemma 1.4, implies the existence of an $m > 0$ such that $\|u_m\|_\infty = r$. By [10, Corollary 8.4], $u_m > 0$ in Ω and $\frac{\partial u_m}{\partial n} < 0$ on $\partial\Omega$, where n denotes the outward unit normal on $\partial\Omega$. Let $0 < \alpha < \beta < r$ and $z_{\alpha,\beta} \in C^{1,\beta}(\bar{\Omega})$ be the solution of

$$-\Delta_p z - a\Delta_q z = \begin{cases} m & \text{if } u_m \in [\alpha, \beta], \\ -c_0 & \text{otherwise} \end{cases} \equiv h_{\alpha,\beta}, \qquad z = 0 \quad \text{on } \partial\Omega.$$

Note that the existence of $z_{\alpha,\beta}$ follows from Lemma 1.2. Since $-\Delta_p u_m - a\Delta_q u_m = m$ in Ω and

$$\|h_{\alpha,\beta} - m\|_1 = (m + c_0)|B| \to 0$$

as $\alpha \to 0$ and $\beta \to r$, where $|B|$ denotes the Lebesgue measure of

$$B = \{x : u_m(x) < \alpha\} \cup \{x : \beta < u_m(x) \leq r\},$$

it follows from Lemma 1.3 that $|z_{\alpha,\beta} - u_m|_1 \to 0$ as $\alpha \to 0$ and $\beta \to r$. Hence there exist α, β such that $z_{\alpha,\beta} \equiv z_0$ such that

$$\frac{u_m}{2} \leq z_0 \leq u_m \quad \text{in } \Omega. \tag{1.4}$$

Note that the right side inequality in (1.4) follows from the weak comparison principle in [10, Theorem 10.1]. In particular, $\frac{\alpha}{2} \leq z_0 \leq \beta$ when $u_m \in [\alpha, \beta]$, which implies $f(z_0) \geq \inf_{[\alpha/2,\beta]} f \equiv \gamma > 0$ and therefore

$$-\Delta_p z_0 - a\Delta_q z_0 = m \leq \lambda\gamma - c_0 \leq \lambda f(z_0) - h(x) \tag{1.5}$$

for $u_m \in [\alpha, \beta]$ and $\lambda > \frac{m+c_0}{\gamma}$. For such λ and $u_m \notin [\alpha, \beta]$,

$$-\Delta_p z_0 - a\Delta_q z_0 = -c_0 \leq -h(x) \leq \lambda f(z_0) - h(x) \tag{1.6}$$

since $f(z_0) \geq 0$ in view of (1.4). Combining (1.5) and (1.6), we see that z_0 is a subsolution of (1.1). Clearly, $z_1 \equiv r$ is a supersolution of (1.1) with $z_0 \leq z_1$ in Ω. Hence (1.1) has a solution z with $z_0 \leq z \leq z_1$ in Ω by [5, Corollary 1], which completes the proof. □

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