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Two regularity criteria of the 3D magneto-micropolar equations in Vishik spaces

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Abstract. In this paper, we utilize the Littlewood–Paley decomposition theory to establish two regularity criteria for the 3D magneto-micropolar equations in Vishik spaces, specifically focusing on the gradient of the velocity field.

Keywords: magneto-micropolar equations, regularity criteria, Vishik spaces.

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1 Introduction

In this paper, we study the magneto-micropolar system in the whole space $\mathbb{R}^3$:

$$\begin{cases} \partial_t u - (\mu+\chi)\Delta u + (u\cdot\nabla)u - (b\cdot\nabla)b - \chi\nabla\times\omega + \nabla p = 0, \\ \partial_t \omega - \gamma\Delta\omega - \kappa\nabla\nabla\cdot\omega + 2\chi\omega + (u\cdot\nabla)\omega - \chi\nabla\times u = 0, \\ \partial_t b - \nu\Delta b + (u\cdot\nabla)b - (b\cdot\nabla)u = 0, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(x,0) = u_0(x),\ \omega(x,0) = \omega_0(x), b(x,0) = b_0(x), \end{cases} \tag{1.1}$$

where $(x,t)\in\mathbb{R}^3\times\mathbb{R}^+$, $u(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))$, $\omega(x,t)=(\omega_1(x,t),\omega_2(x,t),\omega_3(x,t))$, $b(x,t) = (b_1(x,t), b_2(x,t), b_3(x,t))$ and p denote the fluid velocity, micro-rotational velocity, magnetic field and scalar pressure, respectively. Here, μ is the kinematic viscosity, χ is the vortex viscosity, $\frac{1}{\nu}$ is the magnetic Reynolds number, while κ and γ denote angular viscosities. This model has been used to study microelectrode fluid motion in the presence of a magnetic field. It was first proposed by Galdi and Rionero [8] to address microscopic physical phenomena, such as the motion of animal blood, liquid crystals, and dilute aqueous polymer solutions, which cannot be accurately described by the classical Navier–Stokes equations for incompressible viscous fluids. These fluids are characterized by asymmetric stress tensors, which is why they are referred to as asymmetric fluids. Due to the complex physical background and the richness of the phenomena involved, incompressible micropolar fluids have been extensively studied (see [1,2,4,5,12,17,19,23,24] and references therein).

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Function spaces are essential tools for studying and solving systems of fluid mechanics equations. By means of Sobolev spaces [13], researchers can effectively characterize solutions to fluid dynamics problems and investigate their existence, uniqueness, and regularity. To further refine the analysis of local behavior and regularity, especially when dealing with nonlinear partial differential equations, Morrey-type spaces are employed (see [6, 16, 21, 27]). These spaces better capture the local integrability and smoothness of solutions, aiding in the study of conditions for the emergence of local singularities and vortex structures, while also providing more precise integral estimates for nonlinear terms. The concept of weak solutions relies on the weak formulation within function spaces. By examining the well-posedness of fluid mechanics equations in various function spaces, researchers can not only ensure the reasonableness and solvability of these systems but also gain a deeper understanding of solution regularity, nonlinear characteristics, stability, and the feasibility of numerical computations. For more details, please refer to [9, 10, 14].

Rojas-Medar and Boldrini [18] used the Galerkin method to prove, for the first time, the existence of weak solutions to the 2D and 3D magnetic differential equations (1.1). Yuan [25] showed that if $\nabla u \in L^1(0, T; \dot{B}^0_{\infty,\infty}(\mathbb{R}^3))$, then the weak solution is smooth on $\mathbb{R} \times [0, T]$. Subsequently, Gala [7], Zhang et al. [28] and Xu [22] extended the regularity criterion to Morrey–Campanato spaces, Triebel–Lizorkin spaces and Besov spaces, respectively. Yuan and Li [26] further refined the results of Xu [22]. Recently, Wu [20] established the regularity of the weak solution to this system by imposing specific conditions on the partial derivatives of the velocity and magnetic field components. Additionally, Qin and Zhang [15] obtained optimal decay estimates for the higher-order derivatives of the strong solution to the system (1.1).

The aim of this paper is to study the regularity criterion for the solution of the magneto-micropolar equations (1.1). Understanding this criterion is crucial for comprehending the physical laws governing magneto-micropolar motion. Notably, $\dot{B}^0_{\infty,\infty}(\mathbb{R}^3) \subset \dot{V}^0_{\infty,\infty,\theta}(\mathbb{R}^3)$, where the Vishik spaces $\dot{V}^s_{p,r,\theta}(\mathbb{R}^3)$ are introduced as a class of Banach spaces (see Definition 2.3). Consequently, we anticipate that weak solution exhibit corresponding smoothness in such Banach spaces. In this paper, we demonstrate that to ensure the regularity of weak solution to (1.1), it is sufficient to impose certain conditions on the fluid's velocity field. This finding also indirectly suggests that, in the study of weak solution regularity, the fluid velocity u plays a more significant role than both the microscopic rotational velocity ω of the particles and the magnetic field b.

Our main result of the paper is stated as follows:

Theorem 1.1. *Let $(u_0, \omega_0, b_0) \in H^1(\mathbb{R}^3)$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Assume that (u, ω, b) is a weak solution to the system (1.1) on the interval $[0, T]$. If the velocity gradient ∇u satisfies one of the following conditions:*

$$\nabla u \in L^1\Big((0, T; \dot{V}^{\frac{3}{p}}_{p,r,1}(\mathbb{R}^3)\Big), \qquad p \geq 1, \tag{1.2}$$

$$\nabla u \in L^{\frac{2p}{2p-3}}(0, T; \dot{V}^0_{p,r,1}(\mathbb{R}^3)), \qquad p \geq \frac{3}{2}, \tag{1.3}$$

then the weak solution (u, ω, b) is smooth on $[0, T]$.

Remark 1.2. Notice that for $\theta \in [1, \infty]$, we have $\dot{B}^0_{\infty,\infty}(\mathbb{R}^3) \subset \dot{V}^0_{\infty,\infty,\theta}(\mathbb{R}^3)$. Therefore, Theorem 1.1 can be viewed as a further improvement of [24].

The rest of this paper is organized as follows. Section 2 reviews some preliminaries. Section 3 is devoted to the proof of Theorem 1.1.

2 Preliminaries

Let us begin with a brief review of the definition of Littlewood–Paley decomposition, as detailed in [3]. Let χ be a smooth, radially non-increasing function that takes values in $[0,1]$ and is supported within the ball $|\xi| \le \frac{4}{3}$. Define φ in terms of χ by setting $\varphi(\xi) := \chi(\frac{\xi}{2}) - \chi(\xi)$, so that φ is supported in the annulus $\{\frac{3}{4} \le |\xi| \le \frac{8}{3}\}$. These functions satisfy the following partition of unity:

$$\chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^3; \qquad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \neq 0.$$

Let $h = \check{\varphi}, \tilde{h} = \check{\chi}$, where $\check{\varphi}$ and $\check{\chi}$ denote the inverse Fourier transforms of φ and χ, respectively. The dyadic blocks $\dot{\Delta}_j u$ and low-frequency cut-off $\dot{S}_j u$ can then be defined as:

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{3j} \int_{\mathbb{R}^3} h(2^j y) u(x-y) \mathrm{d}y,$$

$$\dot{S}_j u = \sum_{k\le j-1} \Delta_j u = 2^{3j} \int_{\mathbb{R}^3} \tilde{h}(2^j y) u(x-y) \mathrm{d}y, \qquad j \in \mathbb{Z}.$$

According to the Bony decomposition, any distribution $u \in \mathcal{S}'(\mathbb{R}^3) \setminus \mathcal{P}(\mathbb{R}^3)$ can be expressed as:

$$u = \sum_{j=-\infty}^{+\infty} \dot{\Delta}_j u, \qquad u \in \mathcal{S}'(\mathbb{R}^3) \setminus \mathcal{P}(\mathbb{R}^3),$$

where $\mathcal{P}(\mathbb{R}^3)$ denotes the set of polynomials.

Recall the definition of the homogeneous Besov spaces [3], which are based on the Littlewood–Paley decomposition.

Definition 2.1. Let $p, r \in [1, \infty]$ and $s \in \mathbb{R}$. The homogeneous Besov spaces $\dot{B}^s_{p,r}(\mathbb{R}^3)$ are defined as

$$\dot{B}^s_{p,r}(\mathbb{R}^3) := \left\{ f \in \mathcal{S}'(\mathbb{R}^3) \setminus \mathcal{P}(\mathbb{R}^3) : \ \|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^3)} < \infty \right\},$$

where

$$\|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^3)} := \begin{cases} \left(\sum_{j=1}^{\infty} 2^{jrs} \|\dot{\Delta}_j f\|^r_{L^p} \right)^{\frac{1}{r}}, & r \neq \infty, \\ \sup_{j\in\mathbb{Z}} \|\dot{\Delta}_j f\|_{L^p}, & r = \infty. \end{cases}$$

We also recall the Bernstein inequality, which plays a key role in the proof of the main result, see [3].

Lemma 2.2. *Let $k \ge 0$ and $1 \le a, b \le \infty$. Then the following inequality holds*

$$\sum_{|\alpha|=k} \|\partial^\alpha \dot{\Delta}_j u\|_{L^b} \le C 2^{kj + 3j(\frac{1}{a} - \frac{1}{b})} \|\dot{\Delta}_j u\|_{L^a},$$

where $C > 0$ is a constant depending only on k, a, b.

Next, we introduce a class of Banach spaces, known as Vishik spaces [11], which generalize the homogeneous Besov spaces.

Definition 2.3. Let $p, r \in [1,\infty]$, $s \in \mathbb{R}$ and $\theta \in [1, r]$. The Vishik spaces $\dot{V}^s_{p,r,\theta}(\mathbb{R}^3)$ are defined as

$$\dot{V}^s_{p,r,\theta}(\mathbb{R}^3) := \left\{ f \in \mathcal{S}'(\mathbb{R}^3) \setminus \mathcal{P}(\mathbb{R}^3) : \|f\|_{\dot{V}^s_{p,r,\theta}(\mathbb{R}^3)} < \infty \right\},$$

where

$$\|f\|_{\dot{V}^s_{p,r,\theta}(\mathbb{R}^3)} := \begin{cases} \sup_{N\in\mathbb{N}^*} \frac{(\sum_{j=-N}^{N} 2^{j\theta s}\|\dot{\Delta}_j f\|^\theta_{L^p})^{\frac{1}{\theta}}}{N^{\frac{1}{\theta}-\frac{1}{r}}}, & \theta \neq \infty, \\ \|f\|_{B^0_{p,\infty}(\mathbb{R}^3)}, & \theta = \infty. \end{cases}$$

3 The proof of Theorem 1.1

Proof. By taking the L^2 inner product of the first equation, the second equation and the third equation of (1.1) with u, ω and b, respectively, summing the results, and then integrating with respect to t, we obtain

$$\|u\|^2_{L^2} + \|\omega\|^2_{L^2} + \|b\|^2_{L^2} + 2\int_0^T (\mu\|\nabla u\|^2_{L^2} + \gamma\|\nabla\omega\|^2_{L^2} + \nu\|\nabla b\|^2_{L^2})\mathrm{d}t \le C(u_0, \omega_0, b_0).$$

The first equation, as well as the second and third equations in (1.1), are multiplied by $-\Delta u$, $-\Delta\omega$ and $-\Delta b$, respectively, and then integrated over $\mathbb{R}^3$ with respect to x, which yields

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|^2_{L^2} + \|\nabla\omega\|^2_{L^2} + \|\nabla b\|^2_{L^2}) + (\mu+\chi)\|\Delta u\|^2_{L^2} + \gamma\|\Delta\omega\|^2_{L^2} + \nu\|\Delta b\|^2_{L^2} \\ &\qquad + 2\chi\|\nabla\omega\|^2_{L^2} + \kappa\|\nabla\nabla\cdot\omega\|^2_{L^2} \\ &\quad = \int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\mathrm{d}x - \int_{\mathbb{R}^3}(b\cdot\nabla)b\cdot\Delta u\mathrm{d}x - \chi\int_{\mathbb{R}^3}\nabla\times\omega\cdot\Delta u\mathrm{d}x - \chi\int_{\mathbb{R}^3}\nabla\times u\cdot\Delta\omega\mathrm{d}x \\ &\qquad + \int_{\mathbb{R}^3}(u\cdot\nabla)\omega\cdot\Delta\omega\mathrm{d}x + \int_{\mathbb{R}^3}(u\cdot\nabla)b\cdot\Delta b\mathrm{d}x - \int_{\mathbb{R}^3}(b\cdot\nabla)u\cdot\Delta b\mathrm{d}x \\ &\quad =: I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + I_6(t) + I_7(t). \end{aligned} \tag{3.1}$$

According to the Littlewood–Paley decomposition theory, it can be obtained that

$$\nabla u = \sum_{j<-N}\dot{\Delta}_j\nabla u + \sum_{j=-N}^{N}\dot{\Delta}_j\nabla u + \sum_{j>N}\dot{\Delta}_j\nabla u, \tag{3.2}$$

where N is to be determined. Without loss of generality, we first estimate $I_5(t)$. Using integration by parts and (3.2), we have that

$$\begin{aligned} I_5(t) &= \int_{\mathbb{R}^3}(u\cdot\nabla)\omega\cdot\Delta\omega\mathrm{d}x \\ &= -\int_{\mathbb{R}^3}\partial_k u_i\partial_i\omega_j\partial_k\omega_j\mathrm{d}x - \int_{\mathbb{R}^3}u_i\partial_k\partial_i\omega_j\partial_k\omega_j\mathrm{d}x \\ &\le \int_{\mathbb{R}^3}|\nabla\omega|^2|\nabla u|\mathrm{d}x \\ &\le \sum_{j<-N}\int_{\mathbb{R}^3}|\nabla\omega|^2|\dot{\Delta}_j\nabla u|\mathrm{d}x + \sum_{j=-N}^{N}\int_{\mathbb{R}^3}|\nabla\omega|^2|\dot{\Delta}_j\nabla u|\mathrm{d}x + \sum_{j>N}\int_{\mathbb{R}^3}|\nabla\omega|^2|\dot{\Delta}_j\nabla u|\mathrm{d}x \\ &=: I_{51} + I_{52} + I_{53}. \end{aligned} \tag{3.3}$$

Below we estimate I_{51}–I_{53} separately. For I_{51}, by means of the Hölder inequality and the Bernstein inequality, it follows that

$$
\begin{aligned}
I_{51}(t) &= \sum_{j<-N} \int_{\mathbb{R}^3} |\nabla \omega|^2 |\dot{\Delta}_j \nabla u| \mathrm{d}x \\
&\le \sum_{j<-N} \|\dot{\Delta}_j \nabla u\|_{L^\infty} \|\nabla \omega\|_{L^2}^2 \\
&\le C \sum_{j<-N} 2^{\frac{3j}{2}} \|\dot{\Delta}_j \nabla u\|_{L^2} \|\nabla \omega\|_{L^2}^2 \\
&\le C 2^{-\frac{3N}{2}} \|\nabla u\|_{L^2} \|\nabla \omega\|_{L^2}^2. \qquad (3.4)
\end{aligned}
$$

For I_{52}. Let $p \ge 1$. By the Hölder inequality, the Bernstein inequality and the definition of Vishik spaces, we have

$$
\begin{aligned}
I_{52}(t) &= \sum_{j=-N}^{N} \int_{\mathbb{R}^3} |\nabla \omega|^2 |\dot{\Delta}_j \nabla u| \mathrm{d}x \\
&\le \sum_{j=-N}^{N} \|\dot{\Delta}_j \nabla u\|_{L^\infty} \|\nabla \omega\|_{L^2}^2 \\
&\le C \sum_{j=-N}^{N} 2^{\frac{3j}{p}} \|\dot{\Delta}_j \nabla u\|_{L^p} \|\nabla \omega\|_{L^2}^2 \\
&\le C N^{1-\frac{1}{r}} \sup_{N \in \mathbb{N}^*} \frac{\sum_{j=-N}^{N} 2^{\frac{3j}{p}} \|\dot{\Delta}_j \nabla u\|_{L^p}}{N^{1-\frac{1}{r}}} \|\nabla \omega\|_{L^2}^2 \\
&\le C N^{1-\frac{1}{r}} \|\nabla u\|_{V_{p,r,1}^{\frac{3}{p}}} \|\nabla \omega\|_{L^2}^2. \qquad (3.5)
\end{aligned}
$$

For I_{53}. From the Hölder inequality, the Bernstein inequality and space embedding relation, it follows that

$$
\begin{aligned}
I_{53}(t) &= \sum_{j>N} \int_{\mathbb{R}^3} |\nabla \omega|^2 |\dot{\Delta}_j \nabla u| \mathrm{d}x \\
&\le C \sum_{j>N} \|\dot{\Delta}_j \nabla u\|_{L^3} \|\nabla \omega\|_{L^2} \|\nabla \omega\|_{L^6} \\
&\le C \sum_{j>N} 2^{\frac{j}{2}} \|\dot{\Delta}_j \nabla u\|_{L^2} \|\nabla \omega\|_{L^2} \|\Delta \omega\|_{L^2} \\
&\le C \Big(\sum_{j>N} 2^{-j} \Big)^{\frac{1}{2}} \Big(\sum_{j>N} 2^{2j} \|\dot{\Delta}_j \nabla u\|_{L^2}^2 \Big)^{\frac{1}{2}} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \\
&\le C 2^{-\frac{N}{2}} \|\nabla u\|_{B_{2,2}^1} \|\nabla \omega\|_{L^2} \|\Delta \omega\|_{L^2} \\
&\le C 2^{-\frac{N}{2}} \|\Delta u\|_{L^2} \|\nabla \omega\|_{L^2} \|\Delta \omega\|_{L^2}. \qquad (3.6)
\end{aligned}
$$

Combining (3.4)–(3.6), there are

$$
\begin{aligned}
I_5(t) &\le C 2^{-\frac{3N}{2}} \|\nabla u\|_{L^2} \|\nabla \omega\|_{L^2}^2 + C N^{1-\frac{1}{\sigma}} \|\nabla u\|_{V_{p,r,1}^{\frac{3}{p}}} \|\nabla \omega\|_{L^2}^2 + C 2^{-\frac{N}{2}} \|\nabla \omega\|_{L^2} \|\Delta u\|_{L^2} \|\Delta \omega\|_{L^2} \\
&\le C 2^{-\frac{3N}{2}} (\|\nabla u\|_{L^2}^3 + \|\nabla \omega\|_{L^2}^3) + C N^{1-\frac{1}{r}} \|\nabla u\|_{V_{p,r,1}^{\frac{3}{p}}} \|\nabla \omega\|_{L^2}^2 \\
&\quad + C 2^{-\frac{N}{2}} \|\nabla \omega\|_{L^2} (\|\Delta u\|_{L^2}^2 + \|\Delta \omega\|_{L^2}^2). \qquad (3.7)
\end{aligned}
$$

Similarly, we have

$$I_1(t) \leq C2^{-\frac{3N}{2}} \|\nabla u\|_{L^2}^3 + CN^{1-\frac{1}{\sigma}} \|\nabla u\|_{V^{\frac{3}{p}}_{p,r,1}} \|\nabla u\|_{L^2}^2 + C2^{-\frac{N}{2}} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2}^2. \tag{3.8}$$

Using the Hölder inequality and the Young inequality, one obtains that

$$\begin{aligned} I_3(t) + I_4(t) = & -\chi \int_{\mathbb{R}^3} \nabla \times \omega \cdot \Delta u \mathrm{d}x - \chi \int_{\mathbb{R}^3} \nabla \times u \cdot \Delta \omega \mathrm{d}x \\ & \leq \chi (\|\nabla \omega\|_{L^2} \|\Delta u\|_{L^2} + \|\nabla u\|_{L^2} \|\Delta \omega\|_{L^2}) \\ & \leq \frac{\mu + \chi}{4} \|\Delta u\|_{L^2}^2 + \frac{\gamma}{4} \|\Delta \omega\|_{L^2}^2 + C\|\nabla u\|_{L^2} + C\|\nabla \omega\|_{L^2}. \end{aligned} \tag{3.9}$$

Similar to I_5, we have

$$\begin{aligned} I_2(t) + I_6(t) + I_7(t) \leq & \int_{\mathbb{R}^3} |\nabla b|^2 |\nabla u| \mathrm{d}x \\ & \leq C2^{-\frac{3N}{2}} (\|\nabla u\|_{L^2}^3 + \|\nabla b\|_{L^2}^3) + CN^{1-\frac{1}{\sigma}} \|\nabla u\|_{V^{\frac{3}{p}}_{p,r,1}} \|\nabla b\|_{L^2}^2 \\ & \quad + C2^{-\frac{N}{2}} \|\nabla b\|_{L^2} (\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2). \end{aligned} \tag{3.10}$$

Combining (3.2) and (3.7)–(3.10), one has

$$\begin{aligned} & \frac{1}{2} \frac{\mathrm{d}t}{\mathrm{d}t} (\|\nabla u\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \frac{\mu + \chi}{2} \|\Delta u\|_{L^2}^2 + \frac{\gamma}{2} \|\Delta \omega\|_{L^2}^2 + \nu \|\Delta b\|_{L^2}^2 \\ & \qquad + 2\chi \|\nabla \omega\|_{L^2}^2 + \kappa \|\nabla \nabla \cdot \omega\|_{L^2}^2 \\ & \quad \leq C2^{-\frac{3N}{2}} (\|\nabla u\|_{L^2}^3 + \|\nabla \omega\|_{L^2}^3 + \|\nabla b\|_{L^2}^3) + CN^{1-\frac{1}{r}} \|\nabla u\|_{V^{\frac{3}{p}}_{p,r,1}} (\|\nabla u\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \\ & \qquad + C2^{-\frac{N}{2}} (\|\nabla u\|_{L^2} + \|\nabla \omega\|_{L^2} + \|\nabla b\|_{L^2}) (\|\Delta u\|_{L^2}^2 + \|\Delta \omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) \\ & \qquad + C(\|\nabla u\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2) \\ & \quad \leq C2^{-\frac{3N}{2}} (\|\nabla u\|_{L^2} + \|\nabla \omega\|_{L^2} + \|\nabla b\|_{L^2}) (\|\nabla u\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \\ & \qquad + CN^{1-\frac{1}{r}} \|\nabla u\|_{V^{\frac{3}{p}}_{p,r,1}} (\|\nabla u\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \\ & \qquad + C2^{-\frac{N}{2}} (\|\nabla u\|_{L^2} + \|\nabla \omega\|_{L^2} + \|\nabla b\|_{L^2}) (\|\Delta u\|_{L^2}^2 + \|\Delta \omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2). \end{aligned} \tag{3.11}$$

We fix a large enough N, which obeys

$$C2^{-\frac{N}{2}} (\|\nabla u\|_{L^2} + \|\nabla \omega\|_{L^2} + \|\nabla b\|_{L^2}) \leq \frac{1}{4} \min\{\mu + \chi, \gamma, 2\nu\},$$

i.e.

$$N \geq 4 + \frac{2lnC + 2ln(\|\nabla u\|_{L^2} + \|\nabla \omega\|_{L^2} + \|\nabla b\|_{L^2}) - 2\min\{\mu + \chi, \gamma, 2\nu\}}{ln2}.$$

Taking

$$N = \left[4 + \frac{2lnC + 2ln(\|\nabla u\|_{L^2} + \|\nabla \omega\|_{L^2} + \|\nabla b\|_{L^2}) - 2\min\{\mu + \chi, \gamma, 2\nu\}}{ln2} \right] + 1,$$

it follows from (3.11) that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}t}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\left(\frac{\mu+\chi}{2}-\frac{1}{4}\min\{\mu+\chi,\gamma,2\nu\}\right)\|\Delta u\|_{L^2}^2+2\chi\|\nabla\omega\|_{L^2}^2\\
&\quad+\left(\frac{\gamma}{2}-\frac{1}{4}\min\{\mu+\chi,\gamma,2\nu\}\right)\|\Delta\omega\|_{L^2}^2\\
&\quad+\left(\nu-\frac{1}{4}\min\{\mu+\chi,\gamma,2\nu\}\right)\|\Delta b\|_{L^2}^2+\kappa\|\nabla\nabla\cdot\omega\|_{L^2}^2\\
&\le C\left(1+\|\nabla u\|_{V^{\frac{3}{p}}_{p,r,1}}\right)(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+\|\nabla b\|_{L^2}^2). \qquad (3.12)
\end{aligned}$$

Using the Gronwall inequality yields

$$\begin{aligned}
&\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+C\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta\omega\|_{L^2}^2+\|\Delta b\|_{L^2}^2)(t)\,\mathrm{d}t\\
&\quad\le\ \exp\left(CT+C\int_0^T\|\nabla u\|_{V^{\frac{3}{p}}_{p,r,1}}\mathrm{d}t\right)(\|\nabla u_0\|_{L^2}^2+\|\nabla\omega_0\|_{L^2}^2+\|\nabla b_0\|_{L^2}^2).
\end{aligned}$$

Using the hypothetical condition (1.2), we get

$$u,\omega,b\in L^\infty(0,T;H^1(\mathbb{R}^3))\cap L^2(0,T;H^2(\mathbb{R}^3)).$$

For the regularity criterion (1.3), we focus our analysis on I_5. Similarly, by applying the theory of Littlewood–Paley decompositions, it follows that

$$\begin{aligned}
I_5(t)&\le\sum_{j<-N}\int_{\mathbb{R}^3}|\nabla u|^2|\dot{\Delta}_j\nabla u|\mathrm{d}x+\sum_{j=-N}^{N}\int_{\mathbb{R}^3}|\nabla u|^2|\dot{\Delta}_j\nabla u|\mathrm{d}x+\sum_{j>N}\int_{\mathbb{R}^3}|\nabla u|^2|\dot{\Delta}_j\nabla u|\mathrm{d}x\\
&=:J_{51}+J_{52}+J_{53}. \qquad (3.13)
\end{aligned}$$

Using $I_{51}(t)$ and $I_{53}(t)$, one obtains that

$$J_{51}(t)+J_{53}(t)\le C2^{-\frac{3N}{2}}\|\nabla u\|_{L^2}^3+C2^{-\frac{N}{2}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^2. \qquad (3.14)$$

Let $p\ge\frac{3}{2}$. From the Hölder inequality, the definition of Vishik spaces, the Gagliardo–Nirenberg inequality and the Young inequality, we have

$$\begin{aligned}
J_{52}(t)&=\sum_{j=-N}^{N}\int_{\mathbb{R}^3}|\nabla u|^2|\dot{\Delta}_j\nabla u|\mathrm{d}x\\
&\le\sum_{j=-N}^{N}\|\dot{\Delta}_j\nabla u\|_{L^p}\|\nabla u\|^2_{L^{\frac{2p}{p-1}}}\\
&\le CN^{1-\frac{1}{r}}\sup_{N\in\mathbb{N}^*}\frac{\sum_{j=-N}^{N}\|\dot{\Delta}_j\nabla u\|_{L^p}}{N^{1-\frac{1}{r}}}\|\nabla u\|_{L^2}^{2-\frac{3}{p}}\|\Delta u\|_{L^2}^{\frac{3}{p}}\\
&\le CN^{1-\frac{1}{r}}\|\nabla u\|_{V^0_{p,r,1}}\|\nabla u\|_{L^2}^{2-\frac{3}{p}}\|\Delta u\|_{L^2}^{\frac{3}{p}}\\
&\le\frac{\mu+\chi}{4}\|\Delta u\|_{L^2}^2+CN^{\frac{(r-1)2p}{(2p-3)r}}\|\nabla u\|_{V^0_{p,r,1}}^{\frac{2p}{2p-3}}\|\Delta u\|_{L^2}^2. \qquad (3.15)
\end{aligned}$$

Combining (3.13)–(3.15), yields

$$I_5(t) \le C2^{-\frac{3N}{2}}\|\nabla u\|_{L^2}^3 + \frac{\mu+\chi}{4}\|\Delta u\|_{L^2}^2 + CN^{\frac{(r-1)2p}{(2p-3)r}}\|\nabla u\|_{V^0_{p,r,1}}^{\frac{2p}{2p-3}}\|\Delta u\|_{L^2}^2 + C2^{-\frac{N}{2}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^2.$$

The analysis that follows is similar to that of the regularity criterion (1.3), except for a slight difference in the choice of N, which is omitted here. We leave the details to the interested reader.

Based on the above analysis, we complete the proof of Theorem 1.1. □

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References

[1] R. Agarwal, A. Alghamdi, S. Gala, M. Ragusa, On the regularity criterion on one velocity component for the micropolar fluid equations, *Math. Model. Anal.* **28**(2023), No. 2, 271–284. https://doi.org/10.3846/mma.2023.15261; MR4568298; Zbl 1514.35342

[2] D. Bresch, B. Desjardins, C. Lin, On some compressible fluid models: Korteweg, lubrication, and shallow water systems, *Comm. Partial Differential Equations* **28**(2003), No. 3–4, 843–868. https://doi.org/10.1081/pde-120020499; MR1978317; Zbl 1106.76436

[3] J. Chemin, *Perfect incompressible fluids*, Oxford University Press, New York, 1998. MR1688875

[4] F. Cruz, M. Novais, On the strong solutions of the 3D magneto-micropolar equations, *Appl. Anal.* **101**(2022), No. 6, 1963–1970. https://doi.org/10.1080/00036811.2020.1791831; MR4413094; Zbl 1487.35284

[5] L. Deng, H. Shang, Global well-posedness for n-dimensional magneto-micropolar equations with hyperdisspation, *Appl. Math. Lett.* **140**(2021), 1–8. https://doi.org/10.1016/j.aml.2020.106610; MR4121518; Zbl 1451.35129

[6] H. Ding, F. Wu, Liouville-type theorems for 3D stationary tropical climate model in mixed local Morrey spaces, *Bull. Malays. Math. Sci. Soc.* **46**(2023), 1–34. https://doi.org/10.1016/j.aml.2022.108533; MR4530222; Zbl 1507.35292

[7] S. Gala, Regularity criteria for the 3D magneto-micropolar fluid equations in the Morrey–Campanato space, *NoDEA Nonlinear Differential Equations Appl.* **17**(2010), No. 2, 181–194. https://doi.org/10.1007/s00030-009-0047-4; MR2639150; Zbl 1191.35214

[8] G. Galdi, S. Rionero, A note on the existence and uniqueness of solutions of the micropolar fluid equations, *Internat. J. Engrg. Sci.* **15**(1977), No. 2, 105–108. https://doi.org/10.1016/0020-7225(77)90025-8; MR0467030; Zbl 0351.76006

[9] E. Guariglia, Fractional calculus, zeta functions and Shannon entropy, *Open Math.* **19**(2021), No. 1, 87–100. https://doi.org/10.1515/math-2021-0010; MR4248257; Zbl 1475.11151

[10] E. Guariglia, S. Silvestrov, Fractional-wavelet analysis of positive definite distributions and wavelets on $\mathcal{D}'(\mathbb{C})$, in: *Engineering mathematics II*, Springer Proc. Math. Stat., Vol. 179, Springer, Berlin, 2016, pp. 337–353. https://doi.org/10.1007/978-3-319-42105-6_16; MR3630586; Zbl 1365.65294

[11] R. Kanamaru, Optimality of logarithmic interpolation inequalities and extension criteria to the Navier–Stokes and Euler equations in Vishik spaces, *J. Evol. Equ.* **20**(2020), No. 4, 1381–1397. https://doi.org/10.1007/s00028-020-00559-0; MR4181952; Zbl 1466.35287

[12] C. Niche, C. Perusato, Sharp decay estimates and asymptotic behaviour for 3D magneto-micropolar fluids, *Z. Angew. Math. Phys.* **73**(2022), No. 2, 1–20. https://doi.org/10.1007/s00033-022-01683-2; MR4379088; Zbl 1507.35195

[13] S. Polidoro, M. A. Ragusa, Sobolev–Morrey spaces related to an ultraparabolic equation, *Manuscripta Math.* **96**(1998), 371–392. https://doi.org/10.1007/s002290050072; MR1638177; Zbl 0910.35037

[14] S. Polidoro, M. A. Ragusa, Harnack inequality for hypoelliptic ultraparabolic equations with a singular lower order term, *Rev. Mat. Iberoam.* **24**(2008), No. 3, 1011–1046. https://doi.org/10.4171/rmi/565; MR2490208; Zbl 1175.35081

[15] L. Qin, Y. Zhang, Optimal decay rates for higher-order derivatives of solutions to the 3D magneto-micropolar fluid equations, *Appl. Math. Lett.* **133**(2022), 1–7. https://doi.org/10.1016/j.aml.2022.108286; MR4447676; Zbl 1504.35372

[16] M. A. Ragusa, Commutators of fractional integral operators on vanishing-Morrey spaces, *J. Global Optim.* **40**(2008), 361–368. https://doi.org/10.1007/s10898-007-9176-7; MR2373563; Zbl 1143.42020

[17] M. A. Ragusa, F. Wu, Eigenvalue regularity criteria of the three-dimensional micropolar fluid equations, *Bull. Malays. Math. Sci. Soc.* **47**(2024), No. 3, 1–11. https://doi.org/10.1007/s40840-024-01679-3; MR4718588; Zbl 07850937

[18] M. Rojas-Medar, J. Boldrini, Magneto-micropolar fluid motion: existence of weak solutions, *Rev. Mat. Complut.* **11**(1998), No. 2, 443–460. https://doi.org/10.5209/rev_rema.1998.v11.n2.17276; MR1484679; Zbl 0918.35114

[19] Y. Wang, L. Gu, Global regularity of 3D magneto-micropolar fluid equations, *Appl. Math. Lett.* **101**(2020), 1–9. https://doi.org/10.1016/j.aml.2019.07.011; MR3990123; Zbl 1428.35402

[20] F. Wu, A refined regularity criteria of weak solutions to the magneto-micropolar fluid equations, *J. Evol. Equ.* **21**(2021), 725–734. https://doi.org/10.1007/s00028-020-00598-7; MR4238222; Zbl 1464.35261

[21] B. Xu, Bilinear θ-type Calderón–Zygmund operators and its commutators on generalized variable exponent Morrey spaces, *AIMS Math.* **7**(2022), No. 7, 12123–12143. https://doi.org/10.3934/math.2022674; MR4431773

[22] F. Xu, Regularity criterion of weak solution for the 3D magneto-micropolar fluid equations in Besov spaces, *Commun. Nonlinear Sci. Numer. Simul.* **17**(2012), No. 6, 2426–2433. https://doi.org/10.1016/j.cnsns.2011.09.038; MR2877688; Zbl 1335.35206

[23] Z. Ye, Remark on exponential decay-in-time of global strong solutions to 3D inhomogeneous incompressible micropolar equations, *Discrete Contin. Dyn. Syst. Ser. B* **25**(2019), No. 12, 6725–6735. https://doi.org/10.3934/dcdsb.2019164; MR4026901; Zbl 1428.35407

[24] J. Yuan, Existence theorem and blow-up criterion of the strong solutions to the magneto-micropolar fluid equations, *Math. Method Appl. Sci.* **31**(2008), No. 9, 1113–1130. https://doi.org/10.1002/mma.967; MR2419091; Zbl 1137.76071

[25] B. Yuan, Regularity of weak solutions to magneto-micropolar fluid equations, *Acta Math. Sci. Ser. B* **30**(2010), No. 5, 1469–1480. https://doi.org/10.1016/s0252-9602(10)60139-7; MR2778615; Zbl 1240.35421

[26] B. Yuan, X. Li, Regularity of weak solutions to the 3D magneto-micropolar equations in Besov spaces, *Acta Appl. Math.* **163**(2019), 207–223. https://doi.org/10.1007/s10440-018-0220-z; MR4008703; Zbl 1428.35409

[27] B. Yuan, F. Wang, The Liouville theorems for 3D stationary tropical climate model in local Morrey spaces, *Appl. Math. Lett.* **138**(2023), 1–7. https://doi.org/10.1016/j.aml.2022.108533; MR4522953; Zbl 1502.86005

[28] Z. Zhang, Z. Yao, X. Wang, A regularity criterion for the 3D magneto-micropolar fluid equations in Triebel–Lizorkin spaces, *Nonlinear Anal.* **74**(2011), No. 6, 2220–2225. https://doi.org/10.1016/j.na.2010.11.026; MR2781751; Zbl 1209.35105