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Existence of two infinite families of solutions to a singular superlinear equation on exterior domains

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Abstract. We are concerned with the radial solutions of the Dirichlet problem $-\Delta u = K(|x|)f(u)$ on the exterior of the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$ with $N \geq 3$ where f is superlinear at ∞ and has a singularity at 0 with $f(u) \sim \frac{1}{|u|^{q-1}u}$ and $0 < q < 1$ for small u. We prove that if $K(|x|) \sim |x|^{-\alpha}$ with $\alpha > 2(N-1)$ then there exist two infinite families of sign-changing radial solutions.

Keywords: exterior domains, singular, superlinear, radial solution.

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1 Introduction

In this paper we study the radial solutions of

$$-\Delta u = K(|x|)f(u) \text{ on } \mathbb{R}^N \backslash B_R(0) \tag{1.1}$$

$$u(x) = 0 \text{ on } \partial B_R(0), \quad \lim_{|x|\to\infty} u(x) = 0 \tag{1.2}$$

where $\Delta : C^k(\mathbb{R}^N) \to C^{k-2}(\mathbb{R}^N)$ denotes the N-dimensional Laplacian, $B_R(0)$ denotes the unit ball centered at the origin, $|x|$ denotes the Euclidean distance of x, and $u : \mathbb{R}^N \to \mathbb{R}$ with $N \geq 3$.

Numerous papers have proved the existence of *positive* solutions of these equations with $K(|x|) = 1$. See for example [4,5,10]. In [10], Miyamoto and Naito studied the problem in the domain $B_R(0) \setminus \{0\}$. Some other papers have dealt with the *positive* solutions of these equations with various nonlinearities $f(u)$ and $K(|x|) \sim |x|^{-\alpha}$ with $\alpha > 0$. (See [1,9,11]).

We prove the existence of sign-changing solutions of (1.1)–(1.2) and analyze their properties. The papers [2,3,7,8] examined the case where the non-linear function $f(u)$ in (1.1) has a unique positive zero. We choose a superlinear function $f(u)$ that has no positive zeros.

Our study of the solutions of (1.1)–(1.2) is based on the following assumptions:

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(H1) $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is odd, locally Lipschitz, and $f > 0$ on $(0, \infty)$. (So, by the symmetry of f about the origin, $f < 0$ on $(-\infty, 0)$),

(H2) $f(u) = |u|^{p-1}u + g(u)$ with $p > 1$ for large u and $\lim_{u\to\infty} \frac{|g(u)|}{|u|^p} = 0$,

(H3) there exists a locally Lipschitz function $g_1 : \mathbb{R} \to \mathbb{R}$ such that $f(u) = \frac{1}{|u|^{q-1}u} + g_1(u)$ with $0 < q < 1$ and $g_1(0) = 0$,

(H4) $K(r)$, $K'(r)$ are continuous on $[R, \infty)$ with $K(r) > 0$ such that $2(N-1) + \frac{rK'}{K} < 0$ on $[R, \infty)$,

(H5) there exist a constant $k_0 > 0$ and $\alpha > 2(N-1)$ such that $\frac{k_0}{r^\alpha} \le K(r)$ on $[R, \infty)$.

Let $F(u) = \int_0^u f(t)\, dt$. From $(H3)$ it follows that f is integrable at 0 and therefore F is continuous with $F(0) = 0$. Also, since f is odd and $f > 0$ on $(0, \infty)$, it follows that F is even and $F(u) > 0$ for $u \neq 0$.

Since we are studying the radial solutions of (1.1)–(1.2), we let $u(x) = u(|x|) = u(r)$ where $r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$. Denoting $\frac{\partial u}{\partial r}$ by u' and $\frac{\partial^2 u}{\partial r^2}$ by u'' then (1.1)–(1.2) becomes:

$$u''(r) + \frac{N-1}{r}u'(r) + K(r)f(u) = 0 \quad \text{for } R < r < \infty, \tag{1.3}$$

$$u(R) = 0, \quad \lim_{r\to\infty} u(r) = 0. \tag{1.4}$$

In this paper we prove the following:

Theorem 1.1. *Assume (H1)–(H5) hold and $N \ge 3$. There exist two infinite families of non-trivial radial solutions of (1.3)–(1.4). In addition, $\exists n_0 \ge 0$ such that for every $n \ge n_0$ then there are at least two solutions of (1.3)–(1.4) with exactly n zeros on (R, ∞).*

2 Preliminaries and behavior for large a

We prove the existence of a solution of (1.3)–(1.4) with

$$u(R) = 0, \quad u'(R) = a > 0 \tag{2.1}$$

on $[R, R+\epsilon)$ for some $\epsilon > 0$. We denote $u(r)$ by $u_a(r)$ to emphasize the dependence of u on the initial parameter a. We begin first by making the following change of variables

$$u_a(r) = v_a(r^{2-N}).$$

Let $r^{2-N} = t$ and denote R^{2-N} by R^*. We observe then that solving (1.3), (2.1) is equivalent to solving the following initial value problem

$$v_a'' + h(t)f(v_a) = 0 \quad \text{on } (0, R^*) \tag{2.2}$$

$$v_a(R^*) = 0, \quad v_a'(R^*) = -\frac{aR^{N-1}}{N-2} < 0 \tag{2.3}$$

where $h(t) = \frac{t^{\frac{2(N-1)}{2-N}} K(t^{\frac{1}{2-N}})}{(N-2)^2}$. We will then try to find values of a such that $v_a(0) = 0$. From $(H4)$, $(H5)$, and the definition of $h(t)$ it follows that

$$\begin{aligned} &h(t) > 0,\ h'(t) > 0 \quad \text{on } (0, R^*] \\ \text{and } \exists h_1 > 0 &\text{ such that } h_1 t^{\tilde{\alpha}} \le h(t) \text{ on } (0, R^*] \text{ where } \tilde{\alpha} = \frac{\alpha - 2(N-1)}{N-2} > 0. \end{aligned} \tag{2.4}$$

We first prove the existence of a solution for (2.2)–(2.3) on $[R^* - \epsilon, R^*]$ for some $\epsilon > 0$. To do this, we transform this equation into an integral equation and use the contraction mapping principle to solve it. Let $t > 0$ and let v_a be a solution of (2.2)–(2.3). By integrating (2.2) over (t, R^*) and using (2.3) we obtain

$$v_a'(t) = -\frac{aR^{N-1}}{N-2} + \int_t^{R^*} h(x) f\left(v_a(x)\right) dx. \tag{2.5}$$

Now integrate (2.5) over (t, R^*) and use (2.3). This gives

$$v_a(t) = \frac{aR^{N-1}}{N-2}(R^* - t) - \int_t^{R^*} \left(\int_s^{R^*} h(x) f(v_a(x))\, dx \right) ds. \tag{2.6}$$

Letting $v_a(t) = (R^* - t) y(t)$ and $y(R^*) \equiv \lim_{t \to R^{*-}} \frac{v_a(t)}{R^* - t} = -v_a'(R^*) = \frac{aR^{N-1}}{N-2}$, we can rewrite the equation (2.6) in terms of $y(t)$ as

$$y(t) = \frac{aR^{N-1}}{N-2} - \frac{1}{R^* - t} \int_t^{R^*} \left(\int_s^{R^*} h(x) f\left((R^* - x) y(x)\right)\ dx \right) ds. \tag{2.7}$$

We now solve (2.7) by defining an operator on an appropriate space and showing that it has a fixed point. For this, let $a > 0$ and consider the Banach space

$$X = \left\{ y \in C[R^* - \epsilon, R^*] : y(R^*) = \frac{aR^{N-1}}{N-2}, \left| y(t) - \frac{aR^{N-1}}{N-2} \right| \le \frac{aR^{N-1}}{2(N-2)} \text{ on } [R^* - \epsilon, R^*] \right\}$$

equipped with the supremum norm defined by

$$\|y\| = \sup_{x \in [R^* - \epsilon, R^*]} |y(x)|.$$

We define a map $T : X \to C[R^* - \epsilon, R^*]$ by

$$(Ty)(t) = \frac{aR^{N-1}}{N-2} - \frac{1}{R^* - t} \int_t^{R^*} \left(\int_s^{R^*} h(x) f\left((R^* - x) y(x)\right) dx \right) ds \quad \text{for } R^* - \epsilon \le t < R^* \tag{2.8}$$

and $T(R^*) = \frac{aR^{N-1}}{N-2}$. Since $f = \frac{1}{|u|^{q-1} u} + g_1(u)$ by $(H3)$, we have from (2.8) that

$$(Ty)(t) = \frac{aR^{N-1}}{N-2} - \frac{1}{R^* - t} \int_t^{R^*} \left(\int_s^{R^*} h(x) \left(\frac{1}{(R^* - x)^q y^q(x)} + g_1\left((R^* - x) y(x)\right) \right) dx \right) ds. \tag{2.9}$$

Since $0 < q < 1$ by $(H3)$, it follows that $\frac{1}{(R^* - x)^q}$ is integrable on $[0, R^*]$. Using this fact together with that g_1 is locally Lipschitz, it can be shown that T is a contraction mapping from X into itself for sufficiently small ϵ (the details are carried out in [3]). Thus by the contraction mapping principle [6], there exists a unique element $y \in X$ such that $Ty = y$ on

$[R^* - \epsilon, R^*]$. Hence, we obtain a solution $v_a(t) = (R^* - t)y(t)$ of (2.2)–(2.3) on $[R^* - \epsilon, R^*]$ if $a > 0$ and $\epsilon > 0$ is sufficiently small.

Next let $(R_1, R^*]$ be the maximal half-open interval of existence of the solution to (2.2)–(2.3). Now we define the energy of the solution

$$E_a = \frac{1}{2}\frac{v_a'^2}{h(t)} + F(v_a) \quad \text{for } R_1 < t \le R^*. \tag{2.10}$$

Then it follows from (2.2) and (2.4) that

$$E_a' = -\frac{v_a'^2 h'}{2h^2} \le 0 \quad \text{on } (R_1, R^*]. \tag{2.11}$$

Thus, E_a is non-increasing on $(R_1, R^*]$ and hence for $R_1 < t \le R^*$ we have

$$0 < \frac{1}{2}\frac{a^2 R^{2(N-1)}}{(N-2)^2 h(R^*)} = \frac{1}{2}\frac{v_a'^2(R^*)}{h(R^*)} = E_a(R^*) \le E_a = \frac{1}{2}\frac{v_a'^2}{h(t)} + F(v_a) \quad \text{on } (R_1, R^*]. \tag{2.12}$$

So $E_a > 0$ on $(R_1, R^*]$.

We next claim that the solution of (2.2)–(2.3) exists on $[0, R^*]$ and analyze the properties of the solution in several lemmas.

Lemma 2.1. *Assume (H1)–(H5) hold, $N \ge 3$ and $a > 0$. Let v_a be the solution of (2.2)–(2.3). Then v_a can be extended to the maximal interval $[0, R^*]$.*

Proof. Let v_a be the unique solution of (2.2)–(2.3) on the maximal half-open interval of existence $(R_1, R^*]$. We show that $R_1 = 0$. Suppose on the contrary that $R_1 > 0$. Using (2.2), (2.4) and that $F(v_a) \ge 0$ we obtain

$$\left(\frac{1}{2}v_a'^2 + h(t)F(v_a)\right)' = h'(t)F(v_a) \ge 0 \quad \text{on } (R_1, R^*]. \tag{2.13}$$

Let $0 < t < R_1$. Now by integrating (2.13) over (t, R^*), using (2.3) and that $h(t) > 0$, $F(v_a) \ge 0$ we obtain

$$\frac{1}{2}v_a'^2 \le \frac{1}{2}v_a'^2 + h(t)F(v_a) \le \frac{1}{2}\frac{a^2 R^{2(N-1)}}{(N-2)^2} \quad \text{on } (R_1, R^*]. \tag{2.14}$$

Therefore,

$$|v_a'| \le \frac{aR^{N-1}}{N-2} \quad \text{on } (R_1, R^*]. \tag{2.15}$$

Also, we have

$$|v_a| = \left|\int_t^{R^*} v_a'\, ds\right| \le \int_t^{R^*} |v_a'|\, ds \le \frac{aR^{N-1}}{N-2}(R^* - t) \le \frac{aR^{N-1}}{N-2}R^* = \frac{aR}{N-2} \quad \text{on } (R_1, R^*]. \tag{2.16}$$

Now let $(t_n) \subset (R_1, R^*]$ such that $t_n \to R_1^+$. Then by the mean value theorem and (2.15) we obtain

$$|v_a(t_n) - v_a(t_m)| = |v_a'(c_{n,m})||t_n - t_m| \le \frac{aR^{N-1}}{N-2}|t_n - t_m| \to 0 \quad \text{as } m, n \to \infty.$$

This shows that $(v_a(t_n))$ is a Cauchy sequence on $(R_1, R^*]$ and so $\exists L \in \mathbb{R}$ such that $\lim_{t\to R_1^+} v_a(t) = L$. Also since $h(t)F(v_a)$ and $h'(t)F(v_a)$ are continuous on $(R_1, R^*]$, integrating (2.13) on (t, R^*) we see that $\lim_{t\to R_1^+} v_a'(t) = L_1$ exists. From (2.12) we see $0 < E_a \le \frac{1}{2}\frac{L_1^2}{h(R^*)} + F(L)$ on $(R_1, R^*]$ which shows that L and L_1 cannot both be zero. Now if $L = 0$ then $L_1 \neq 0$ and we can use the contraction mapping principle as we did earlier to extend our solution to $(R_1 - \delta, R^*]$ for some $\delta > 0$. On the other hand, if $L \neq 0$, then we can use the standard existence theorem for ordinary differential equations to obtain a solution on $(R_1 - \delta, R^*]$ for some $\delta > 0$. Therefore in both cases the solution of (2.2)-(2.3) can be extended to $(R_1 - \delta, R^*]$ for some $\delta > 0$, contradicting the maximality of $(R_1, R^*]$. Hence $R_1 = 0$. It then follows from (2.15) and (2.16) that v_a and v_a' are bounded on $(0, R^*]$ and so in a similar way to earlier we see that the limits $\lim_{t\to 0^+} v_a(t)$ and $\lim_{t\to 0^+} v_a'(t)$ exist. Thus v_a and v_a' are defined and continuous $[0, R^*]$. □

Remark 2.2. If v_a solves (2.2)–(2.3) and $z \in (0, R^*)$ is such that $v_a(z) = 0$ then by (2.12), $0 < E_a(z_a) = \frac{1}{2}\frac{v_a'^2(z)}{h(z)}$ and hence $v_a'(z) \neq 0$. Thus the zeros of v_a on $(0, R^*)$ are simple. Also, since $\lim_{u\to 0} |f(u)| = \infty$, by (H3) it follows that the solution to (2.2)–(2.3) is twice differentiable except at points where $v_a(t_0) = 0$. Therefore, by a solution v_a of (2.2)–(2.3) we mean a continuously differentiable function v_a on $[0, R^*]$ that satisfies the equation (2.6) with (2.3).

Lemma 2.3. *Assume (H1)–(H5) hold, $N \ge 3$ and $a > 0$. Let v_a solve (2.2)–(2.3) on $[0, R^*]$. Then v_a depends continuously on the initial parameter a on $[0, R^*]$.*

Proof. Let $0 < a_1 < a < a_2$. Then from (2.15) we have

$$|v_a'| \le \frac{aR^{N-1}}{N-2} \le a_2 c_1 \quad \text{for all } a \text{ such that } 0 < a_1 \le a \le a_2 \tag{2.17}$$

where $c_1 = \frac{R^{N-1}}{N-2}$. And from (2.16) we have

$$|v_a| = \frac{aR}{N-2} \le a_2 c_2 \quad \text{for all } a \text{ such that } 0 < a_1 \le a \le a_2 \tag{2.18}$$

where $c_2 = \frac{R}{N-2}$. Thus, (2.17) and (2.18) show that the upper bounds for $|v_a|, |v_a'|$ can be chosen to be independent of a on $[0, R^*]$ for all a such that $0 < a_1 \le a \le a_2$.

Now let $\tilde{a} > 0$ and suppose $a \to \tilde{a}$. Then, we want to show that $v_a \to v_{\tilde{a}}$ uniformly on $[0, R^*]$. Suppose on the contrary, that there is a subsequence $(a_j) \subset \mathbb{R}$ such that $a_j \to \tilde{a}$ as $j \to \infty$ and $\epsilon_0 > 0$ such that

$$|v_{a_j}(t_j) - v_{\tilde{a}}(t_j)| \ge \epsilon_0 \quad \text{for some sequence } t_j \in [0, R^*]. \tag{2.19}$$

Since $a_j \to \tilde{a}$, there exists $N_0 \in \mathbb{N}$ such that for all $j \ge N_0$ $|a_j| \le \tilde{a} + 1$. From (2.15) and (2.16) we know that v_a and v_a' are uniformly bounded on the compact domain $[0, R^*]$. Hence, by the Arzelà–Ascoli theorem, there exists a subsequence $(v_{a_{j_k}}) \subset (v_{a_j})$ such that $v_{a_{j_k}} \to v_{\tilde{a}}$ uniformly on $[0, R^*]$ as $k \to \infty$. Therefore, as $k \to \infty$ from (2.19) we obtain

$$0 \leftarrow |v_{a_{j_k}}(t_{j_k}) - v_{\tilde{a}}(t_{j_k})| \ge \epsilon_0$$

which is a contradiction. Thus, $v_a \to v_{\tilde{a}}$ uniformly on $[0, R^*]$ and this completes the proof of the lemma. □

Lemma 2.4. *Assume (H1)–(H5) hold and $N \geq 3$. If $a > 0$ and v_a is a solution of (2.2)–(2.3), then v_a has at most finitely many zeros on $(0, R^*)$.*

Proof. Suppose on the contrary that $\exists$ a sequence $(z_{k,a}) \subset (0, R^*)$ with $0 < \cdots < z_{2,a} < z_{1,a}$ such that $v_a(z_{k,a}) = 0$. Then $z_{k,a}$ converges to some z_a^* on $[0, R^*]$. Since v_a has infinitely many zeros, $z_{k,a}$, and $v_a'(z_{k,a}) \neq 0$ by the Remark 2.2, it follows that v_a has infinitely many local extrema, $\{M_{k,a}\}_{k=1}^{\infty}$, with $z_{k+1,a} < M_{k,a} < z_{k,a}$ and so $\lim_{k\to\infty} M_{k,a} = z_a^*$. Since $E_a(t) > 0$ on $(0, R^*]$ and E is non-increasing by (2.12) we have $F(v_a(M_{k,a})) = E_a(M_{k,a}) \geq \frac{1}{2}\frac{a^2R^{2(N-1)}}{(N-2)^2h(R^*)} > 0$. So $\exists \beta_a > 0$ such that $|v_a(M_{k,a})| \geq \beta_a$ for all k. Now by the mean value theorem and (2.15) $\exists t_{k,a} \in (M_{k,a}, z_{k,a})$ such that

$$0 < \beta_a \leq |v_a(M_{k,a})| = |v_a(M_{k,a}) - v_a(z_{k,a})| = |v_a'(t_{k,a})||M_{k,a} - z_{k,a}| \leq \frac{aR^{N-1}}{(N-2)}|M_{k,a} - z_{k,a}|. \quad (2.20)$$

Since $M_{k,a} \to z_a^*$ and $z_{k,a} \to z_a^*$ as $k \to \infty$, the right-hand side of (2.20) goes to 0 as $k \to \infty$ which gives a contradiction. Therefore v_a has at most finitely many zeros on $(0, R^*)$ for $a > 0$. □

Lemma 2.5. *Assume (H1)–(H5) hold, $N \geq 3$ and let v_a solve (2.5). Then for $a > 0$ sufficiently large v_a has a local maximum, M_a. In addition, $v_a(M_a) \to \infty$ and $M_a \to R^*$ as $a \to \infty$.*

Proof. First we show for any $0 \leq t_0 < R^*$ that $\max_{[t_0,R^*)} |v_a(t)| \to \infty$ as $a \to \infty$.

If v_a has a local maximum $M_a \in [t_0, R^*)$, then $v_a'(M_a) = 0$. So, by letting $t = M_a$ in (2.12) we obtain

$$F(v_a(M_a)) \geq \frac{1}{2}\frac{a^2R^{2(N-1)}}{h(R^*)(N-2)^2}. \quad (2.21)$$

Since $h(R^*) > 0$, it follows that the right-hand side of (2.21) approaches infinity as $a \to \infty$ and hence from the definition of F we see that

$$v_a(M_a) \to \infty \quad \text{as } a \to \infty. \quad (2.22)$$

On the other hand, if v_a has no local maximum on (t_0, R^*) then v_a is decreasing on (t_0, R^*). We want to show that $\max_{[t_0,R^*)} |v_a(t)| \to \infty$ as $a \to \infty$. Suppose on the contrary that this is false. Then there exists a constant $c_3 > 0$ independent of a such that $|v_a(t)| \leq c_3$ on $[t_0, R^*]$. Then by the continuity of F there exists $c_4 > 0$ such that $F(v_a(t)) \leq c_4$. Using this and (2.3), it follows from (2.12) that

$$\frac{1}{2}\frac{v_a'^2(t)}{h(t)} + c_4 \geq \frac{1}{2}\frac{v_a'^2(t)}{h(t)} + F(v_a(t)) \geq \frac{1}{2}\frac{v_a'^2(R^*)}{h(R^*)} = \frac{1}{2}a^2c_5^2 \quad \text{on } [t_0, R^*] \quad (2.23)$$

where $c_5 = \frac{R^{N-1}}{(N-2)\sqrt{h(R^*)}}$. Rewriting (2.23) we obtain

$$|v_a'(t)| \geq \sqrt{a^2c_5^2 - 2c_4}\,\sqrt{h(t)}. \quad (2.24)$$

By (2.4) there exists $h_1 > 0$ such that $h(t) \geq h_1 t^{\tilde{\alpha}}$ on $[t_0, R^*]$. By using this and choosing a sufficiently large we can ensure that

$$|v_a'(t)| \geq \frac{ac_5}{2}\sqrt{h(t)} \geq \frac{ac_5}{2}\sqrt{h_1}t^{\frac{\tilde{\alpha}}{2}}. \quad (2.25)$$

Since v_a is decreasing, then by (2.25) we have $v_a' < 0$ on $[t_0, R^*]$. Now integrating (2.25) over (t_0, R^*) yields

$$c_3 \geq v_a(t_0) = \int_{t_0}^{R^*} -v_a'(t)\, dt \geq \frac{ac_5}{2}\sqrt{h_1}\int_{t_0}^{R^*} t^{\frac{\tilde{\alpha}}{2}}\, dt = \frac{ac_5}{2}\sqrt{h_1}\left(\frac{(R^*)^{\frac{\tilde{\alpha}}{2}+1} - t_0^{\frac{\tilde{\alpha}}{2}+1}}{\tilde{\alpha}+2}\right). \tag{2.26}$$

The left hand side of (2.26) is a constant while the right-hand side approaches ∞ as $a \to \infty$ which is a contradiction. Thus we conclude that for any $t_0 \in [0, R^*)$

$$\max_{[t_0, R^*)} |v_a(t)| \to \infty \quad \text{as } a \to \infty. \tag{2.27}$$

We claim next that v_a has a local max, M_a, and $\frac{1}{2}R^* < M_a < R^*$ if a is sufficiently large. Suppose on the contrary that v_a is decreasing on $[\frac{1}{2}R^*, R^*]$. Let

$$C_a = \frac{1}{2}\min_{[\frac{1}{2}R^*, \frac{3}{4}R^*]} \frac{h(t)f(v_a)}{v_a}. \tag{2.28}$$

By letting $t_0 = \frac{3}{4}R^*$ in (2.27), we obtain $v_a(\frac{3}{4}R^*) \to \infty$ as $a \to \infty$. Since v_a is decreasing on the interval $[\frac{1}{2}R^*, \frac{3}{4}R^*]$ we see that $v_a \to \infty$ uniformly as $a \to \infty$ on the interval $[\frac{1}{2}R^*, \frac{3}{4}R^*]$. By (2.4) $h_1 t^{\tilde{\alpha}} \leq h(t)$ on $(0, R^*]$ for some constant $h_1 > 0$ from which it follows that $h(t)$ is bounded from below on $[\frac{1}{2}R^*, \frac{3}{4}R^*]$. Also we have $f(v_a) = |v_a|^{p-1}v_a + g(v_a)$ by $(H2)$ and so it follows that if v_a is large then $f(v_a) \geq \frac{1}{2}v_a^p$. It then follows from this that $\frac{f(v_a)}{v_a} \geq \frac{1}{2}v_a^{p-1}(t) \geq \frac{1}{2}v_a^{p-1}\left(\frac{3}{4}R^*\right)$ on $[\frac{1}{2}R^*, \frac{3}{4}R^*]$. Since $p-1 > 0$ and $v_a(\frac{3}{4}R^*) \to \infty$ as $a \to \infty$, then we see $\frac{f(v_a)}{v_a} \to \infty$ on $[\frac{1}{2}R^*, \frac{3}{4}R^*]$ as $a \to \infty$. And since h is bounded from below on $[\frac{1}{2}R^*, \frac{3}{4}R^*]$, it follows from this and (2.28) that

$$C_a \to \infty \quad \text{as } a \to \infty.$$

Now we consider the differential equation

$$w_a'' + C_a w_a = 0 \tag{2.29}$$

with

$$\begin{aligned} w_a\left(\frac{3}{4}R^*\right) &= v_a\left(\frac{3}{4}R^*\right) > 0, \\ w_a'\left(\frac{3}{4}R^*\right) &= v_a'\left(\frac{3}{4}R^*\right) < 0. \end{aligned} \tag{2.30}$$

Clearly, $\{\cos\sqrt{C_a}(t - \frac{3}{4}R^*), \sin\sqrt{C_a}(t - \frac{3}{4}R^*)\}$ is a fundamental set of solutions of (2.29). So, $w_a = \alpha_1\cos\sqrt{C_a}(t - \frac{3}{4}R^*) + \alpha_2\sin\sqrt{C_a}(t - \frac{3}{4}R^*)$ for some constants α_1 and α_2. We also know that the distance between two consecutive zeros of w_a is $\frac{\pi}{\sqrt{C_a}} \to 0$ as $a \to \infty$. So, for $a > 0$ sufficiently large we have $\frac{1}{2}R^* < \frac{3}{4}R^* - \frac{\pi}{\sqrt{C_a}}$. Therefore, for $a > 0$ sufficiently large w_a has a zero on $[\frac{1}{2}R^*, \frac{3}{4}R^*]$ and hence has a local maximum $\tilde{M}$ on this interval with $w_a' < 0$ on $(\tilde{M}, \frac{3}{4}R^*]$.

Next, we rewrite equation (2.2) and consider

$$v_a'' + \left(\frac{h(t)f(v_a)}{v_a}\right)v_a = 0. \tag{2.31}$$

Multiplying (2.29) by v_a, (2.31) by w_a, and subtracting we obtain

$$(w_a' v_a - w_a v_a')' + \left(C_a - \frac{h(t)f(v_a)}{v_a}\right) w_a v_a = 0.$$

Integrating this on $(\tilde{M}, \frac{3}{4}R^*)$ and using (2.30) gives

$$w_a(\tilde{M})v_a'(\tilde{M}) = \int_{\tilde{M}}^{\frac{3}{4}R^*} \left(\frac{h(t)f(v_a)}{v_a} - C_a\right) w_a v_a \, dt. \tag{2.32}$$

Since $w_a(\tilde{M}) > 0$, $C_a < \frac{h(t)f(v_a)}{v_a}$ on $[0, \frac{3}{4}R^*]$, and w_a, v_a stay positive on $[\tilde{M}, \frac{3}{4}R^*]$ it follows from (2.32) that $v_a'(\tilde{M}) > 0$, contradicting our assumption that v_a is decreasing on $[\frac{1}{2}R^*, R^*]$. Thus v_a has a local maximum, M_a, and $\frac{1}{2}R^* < M_a < R^*$ with v_a decreasing on $[M_a, R^*]$ for $a > 0$ sufficiently large. It also follows immediately from (2.22) that $v_a(M_a) \to \infty$ as $a \to \infty$.

Next we show that $M_a \to R^*$ as $a \to \infty$. Since v_a is decreasing on $[M_a, R^*)$ and $v_a(R^*) = 0$ so we see $v_a > 0$ on $[M_a, R^*)$. But then from (2.2) we know $v_a'' = -h(t)f(v_a) < 0$ on $[M_a, R^*)$ and so v_a is concave down on $[M_a, R^*)$. This implies

$$v_a\left(\lambda M_a + (1-\lambda)R^*\right) \geq \lambda v_a(M_a) + (1-\lambda)v_a(R^*) \quad \text{for } 0 \leq \lambda \leq 1.$$

So by letting $\lambda = \frac{1}{2}$ we obtain

$$v_a\left(\frac{M_a + R^*}{2}\right) \geq \frac{v_a(M_a) + v_a(R^*)}{2} = \frac{v_a(M_a)}{2} \to \infty \quad \text{as } a \to \infty. \tag{2.33}$$

By the superlinearity of f it follows that $f(v_a(t)) \geq \frac{1}{2}v_a^p(t)$ on $[M_a, \frac{M_a+R^*}{2}]$ if a is sufficiently large. By using this in (2.2) we obtain

$$v_a'' = -h(t)f(v_a(t)) \leq -\frac{1}{2}v_a^p(t).$$

Now integrating this on $[M_a, t]$ where $M_a \leq t \leq \frac{M_a+R^*}{2}$ and recalling that M_a is a local maximum of v_a with v_a decreasing on $[M_a, R^*]$ yields

$$v_a'(t) \leq -\frac{1}{2}\int_{M_a}^t v_a^p(x)\,dx \leq -\frac{1}{2}v_a^p(t)\int_{M_a}^t h(x)\,dx.$$

Rewriting the above gives

$$\frac{-v_a'}{v_a^p} \geq \frac{1}{2}\int_{M_a}^t h(x)\,dx.$$

Integrating again on (M_a, t) gives,

$$\frac{1}{(p-1)v_a^{p-1}(t)} \geq \frac{1}{p-1}[v_a^{1-p}(t) - v_a^{1-p}(M_a)] \geq \frac{1}{2}\int_{M_a}^t \int_{M_a}^s h(x)\,dx\,ds.$$

Evaluating at $t = \frac{M_a+R^*}{2}$ we obtain

$$\frac{1}{(p-1)v_a^{p-1}(\frac{M_a+R^*}{2})} \geq \frac{1}{2}\int_{M_a}^{\frac{M_a+R^*}{2}} \int_{M_a}^s h(x)\,dx\,ds. \tag{2.34}$$

Since $p - 1 > 0$, it follows from (2.33) that the left-hand side of (2.34) goes to zero as $a \to \infty$. Thus, since $h(x) > 0$ and h is continuous on $[M_a, R^*]$, it follows from (2.34) that $M_a \to R^*$ as $a \to \infty$. This completes the lemma. □

Lemma 2.6. *Assume (H1)–(H5) hold, $N \geq 3$ and let v_a solve (2.5). Then for $a > 0$ sufficiently large v_a has a zero, z_a, with $0 < z_a < M_a < R^*$ where $z_a \to R^*$ and $|v_a'(z_a)| \to \infty$ as $a \to \infty$. In addition, if a is sufficiently large and $n \geq 1$, then v_a has n zeros on $(0, R^*)$.*

Proof. First we show that $\exists z_a \in (0, M_a)$ such that $v_a(z_a) = 0$. Suppose on the contrary that v_a stays positive on $(0, M_a)$. We note that v_a cannot have a positive critical point on $(0, M_a)$. If it has a positive critical point c_a with $v_a' > 0$ on (c_a, M_a), then $v_a(c_a) > 0$ and $v_a''(c_a) \geq 0$. So by (2.2) $f(v_a(c_a)) \leq 0$ but then $v_a(c_a) \leq 0$ contradicting that $v_a > 0$ on $(0, M_a)$. Thus v_a is increasing on $(0, R^*)$. Next recall from (2.11) that $E_a' \leq 0$ on $(0, R^*]$. So we have

$$\frac{1}{2}\frac{v_a'^2}{h(t)} + F(v_a) \geq F(v_a(M_a)) \quad \text{on } (0, M_a]. \tag{2.35}$$

Rewriting (2.35) and integrating on $(0, M_a)$ by making the change of variable $s = v_a(t)$ gives

$$\begin{aligned}\int_0^{M_a} \sqrt{2h(t)}\, dt \leq \int_0^{M_a} \frac{v_a'(t)\, dt}{\sqrt{F(V_a(M_a) - F(v_a(t))}} &= \int_{v_a(0)}^{v_a(M_a)} \frac{ds}{\sqrt{F(v_a(M_a)) - F(s)}} \\ &\leq \int_0^{v_a(M_a)} \frac{ds}{\sqrt{F(v_a(M_a)) - F(s)}}.\end{aligned} \tag{2.36}$$

We now estimate the integral on the right-hand side of (2.36). Letting $s = v_a(M_a)x$, we obtain

$$\int_0^{v_a(M_a)} \frac{ds}{\sqrt{F(v_a(M_a)) - F(s)}} = \frac{v_a(M_a)}{\sqrt{F(v_a(M_a))}} \int_0^1 \frac{dx}{\sqrt{1 - \frac{F(v_a(M_a)x)}{F(v_a(M_a))}}}. \tag{2.37}$$

Let $G(u) = \int_0^u g(s)\, ds$. Then by $(H2)$ it follows that

$$\begin{aligned}\frac{F(v_a(M_a)x)}{F(v_a(M_a))} &= \frac{v_a^{p+1}(M_a)x^{p+1} + G(v_a(M_a)x)}{v_a^{p+1}(M_a) + G(v_a(M_a))} \\ &= \frac{x^{p+1} + \frac{G(v_a(M_a)x)}{v_a^{p+1}(M_a)}}{1 + \frac{G(v_a(M_a))}{v_a^{p+1}(M_a)}}.\end{aligned} \tag{2.38}$$

By $(H2)$ and L'Hôpital's rule it follows that $\frac{|G(u)|}{|u^{p+1}|} \to 0$ as $u \to \infty$. This implies that given $\epsilon > 0$ there exists U such that $|G(u)| \leq \epsilon |u|^{p+1}$ for $|u| \geq U$. Also the continuity of G implies that there exists $c_6 > 0$ such that $|G(u)| \leq c_6$ for $|u| \leq U$. Therefore

$$|G(u)| \leq c_6 + \epsilon |u|^{p+1} \quad \text{for all } u.$$

Letting $u = v_a(M_a)x$ in the above inequality and using (2.22) we obtain

$$\begin{aligned}\frac{|G(v_a(M_a)x)|}{v_a^{p+1}(M_a)} &\leq \frac{c_6}{v_a^{p+1}(M_a)} + \epsilon x^{p+1} \\ &\leq \frac{c_6}{v_a^{p+1}(M_a)} + \epsilon (R^*)^{p+1} \\ &\leq 2(R^*)^{p+1}\epsilon \quad \text{for } a \text{ sufficiently large.}\end{aligned}$$

Therefore $\lim_{a\to\infty} \frac{G(v_a(M_a)x)}{v_a^{p+1}(M_a)} = 0$ uniformly on $[0, 1]$. In particular it follows that $\lim_{a\to\infty} \frac{G(v_a(M_a))}{v_a^{p+1}(M_a)} = 0$. Thus it follows from (2.38) that $\frac{F(v_a(M_a)x)}{F(v_a(M_a))} \to x^{p+1}$ uniformly as $a \to \infty$.

Also we know that $\int_0^1 \frac{dx}{\sqrt{1-x^{p+1}}} < \infty$ since $p > 1$. So it follows from this and the fact that f is superlinear that $\frac{v_a(M_a)}{\sqrt{F(v_a(M_a)}} \to 0$ as $a \to \infty$. Therefore it follows from (2.37) that

$$\lim_{a\to\infty} \int_0^{v_a(M_a)} \frac{ds}{\sqrt{F(v_a(M_a)) - F(s)}} = 0.$$

Hence, the right-hand side of (2.36) goes to 0 as $a \to \infty$. However, we know $h(t) > 0$ on $(0, R^*)$ and $M_a \to R^*$ as $a \to \infty$ (by Lemma 2.4), so the integral on the left-hand side of (2.36) goes to $\int_0^{R^*} \sqrt{2h(t)}\, dt > 0$ which gives a contradiction. Therefore v_a has a zero, z_a, with $0 < z_a < M_a < R^*$. Now we show that $z_a \to R^*$ as $a \to \infty$. Rewriting (2.35) and integrating on (z_a, M_a) by letting $x = v_a(t)$ we obtain

$$\int_0^{v_a(M_a)} \frac{dx}{\sqrt{F(v_a(M_a)) - F(x)}} \geq \int_{z_a}^{M_a} \sqrt{2h(t)}\, dt. \tag{2.39}$$

As we have just proved above that the left-hand side of (2.39) goes to 0 as $a \to \infty$. Thus since $h > 0$ is continuous we must have $(M_a - z_a) \to 0$ as $a \to \infty$. Since we know from Lemma 2.4 that $M_a \to R^*$ as $a \to \infty$, it follows that $z_a \to R^*$ as $a \to \infty$.

Next we show that $|v_a'(z_a)| \to \infty$ as $a \to \infty$. Since $0 < z_a < M_a$ and E_a is non-increasing we have

$$\frac{1}{2}\frac{v_a'^2(z_a)}{h(z_a)} = E_a(z_a) \geq E_a(M_a) = F(v_a(M_a)).$$

So by rewriting this we obtain

$$2h(z_a)F(v_a(M_a)) \leq v_a'^2(z_a). \tag{2.40}$$

Since $z_a \to R^*$ as $a \to \infty$ and h is continuous then $h(z_a) \to h(R^*) > 0$ as $a \to \infty$. Also, in Lemma 2.4 we saw that $v_a(M_a) \to \infty$ as $a \to \infty$ and thus since F is continuous, it follows that $F(v_a(M_a)) \to \infty$ as $a \to \infty$. Thus, from (2.40) we see that $v_a'^2(z_a) \to \infty$ as $a \to \infty$ which then implies $|v_a'(z_a)| \to \infty$ as $a \to \infty$.

Finally, we denote the largest zero of v_a on $(0, R^*)$ as $z_{1,a}$. Using a similar argument as in Lemma 2.5, it can be shown that v_a has a local minimum, $m_a \in (0, z_{1,a})$ if a is sufficiently large. And by following a similar argument as above we can show that there exists a second zero, $z_{2,a} \in (0, m_a)$ of v_a, $z_{2,a} \to R^*$ as $a \to \infty$, and $|v_a'(z_{2,a})| \to \infty$ as $a \to \infty$. Continuing in this way if a is sufficiently large and n is a given non-negative integer, then v_a has n zeros on $(0, R^*)$ if a is sufficiently large. □

3 Behavior for small $a > 0$

Lemma 3.1. *Assume (H1)–(H5) hold and let v_a solve (2.2)–(2.3). Suppose a is sufficiently small. Then v_a has a zero, z_a, and a local maximum, M_a, with $0 < z_a < M_a < R^*$. In addition, $z_a \to R^*$, $M_a \to R^*$, $|v_a'(z_a)| \to 0$, and $v_a(M_a) \to 0$ as $a \to 0^+$. Furthermore, given $n \geq 1$, if a is sufficiently small then v_a has n zeros on $(0, R^*)$.*

Proof. First we want to show that v_a has a zero on $(0, R^*)$ if a is sufficiently small. Suppose on the contrary that $v_a > 0$ on $(0, R^*)$ for all $a > 0$. By (2.6) we have

$$v_a(t) = \frac{aR^{N-1}}{N-2}(R^* - t) - \int_t^{R^*} \left(\int_s^{R^*} h(x) f(v_a(x))\, dx \right) ds. \tag{3.1}$$

Since $v_a > 0$ near R^* it follows from (2.2) that $v_a'' < 0$ near R^* so by integrating this inequality twice we obtain

$$0 < v_a < \frac{aR^{N-1}}{N-2}(R^* - t). \tag{3.2}$$

From $(H1)$ and $(H3)$ there exists $f_1 > 0$ such that $f(v_a) \geq f_1 v_a^{-q}$. Substituting this into (3.1) gives

$$v_a(t) \leq ac_7(R^* - t) - f_1 \int_t^{R^*} \left(\int_s^{R^*} h(x) v_a^{-q}(x)\, dx \right) ds \tag{3.3}$$

where $c_7 = \frac{R^{N-1}}{N-2}$. Since h is increasing on $[0, R^*]$ then from (3.2) and (3.3) we obtain

$$v_a(t) \leq ac_7(R^* - t) - f_1 h(t) \int_t^{R^*} \left(\int_s^{R^*} v_a^{-q}(x)\, dx \right) ds = ac_7(R^* - t) - \frac{f_1 h(t)(R^* - t)^{2-q}}{a^q c_7^q (1-q)(2-q)}. \tag{3.4}$$

Therefore if $v_a > 0$ on $[\frac{R^*}{2}, R^*]$, then from (3.4) we obtain

$$\frac{f_1 h(t)(R^* - t)^{1-q}}{c_7^{q+1}(1-q)(2-q)} \leq a^{q+1}. \tag{3.5}$$

Letting $t = \frac{R^*}{2}$ in (3.5) we obtain

$$\frac{f_1 h(\frac{R^*}{2})(R^*)^{1-q}}{c_7^{q+1} 2^{1-q}(1-q)(2-q)} \leq a^{q+1}. \tag{3.6}$$

The left-hand side of (3.6) is a positive constant but the right-hand side goes to 0 as $a \to 0^+$. Thus we obtain a contradiction if a is sufficiently small. Hence v_a has a zero, z_a, on $[\frac{R^*}{2}, R^*]$ if $a > 0$ is sufficiently small and $v_a > 0$ on (z_a, R^*). Since $v_a(z_a) = 0 = v_a(R^*)$ and $v_a'(R^*) < 0$, it follows that v_a has a local maximum, M_a, with $0 < z_a < M_a < R^*$.

Next by letting $t = z_a$ in (3.5) we obtain

$$\frac{f_1 h(z_a)(R^* - z_a)^{1-q}}{c_7^{q+1}(1-q)(2-q)} \leq a^{q+1}. \tag{3.7}$$

Since the right-hand side of (3.7) goes to 0 as $a \to 0^+$ it follows that $z_a \to R^*$ as $a \to 0^+$. Since $z_a < M_a < R^*$ it then follows that $M_a \to R^*$ as $a \to 0^+$.

Next we know that $\frac{1}{2}v_a'^2 + h(t)F(v_a)$ is increasing by (2.13). So it follows that

$$\frac{1}{2}v_a'^2(z_a) = \frac{1}{2}v_a'^2(z_a) + h(z_a)F(v_a(z_a)) \leq \frac{1}{2}v_a'^2(R^*) + h(R^*)F(v_a(R^*)) = \frac{1}{2}\frac{a^2 R^{2(N-1)}}{(N-2)^2}. \tag{3.8}$$

The right-hand side of (3.8) goes to 0 as $a \to 0^+$ which implies that $|v_a'(z_a)| \to 0$ as $a \to 0^+$.

Now we show that $v_a(M_a) \to 0$ as $a \to 0^+$. From (2.16) we have $|v_a| \leq \frac{aR}{N-2}$ on $(0, R^*)$. Since $v_a(M_a) \geq 0$ it then follows that

$$0 \leq v_a(M_a) \leq \frac{aR}{N-2} \to 0 \quad \text{as } a \to 0^+.$$

Now if we denote the largest zero of v_a on $(0, R^*)$ as $z_{1,a}$ then by using a similar argument as above we can show that v_a has a local minimum, m_a, on $(0, z_{1,a})$ if a is sufficiently small. Also, it can be shown that there exists a zero, $z_{2,a} \in (0, m_a)$ of v_a and $z_{2,a} \to R^*$ as $a \to 0^+$. Continuing in this way, given $n \geq 1$ then v_a has n zeros on $(0, R^*)$ if a is sufficiently small . □

4 Proof of Theorem 1.1

Let $n \geq 0$ and consider the set

$$S_n = \{a > 0 \mid v_a \text{ solves (2.2)–(2.3) and } v_a \text{ has exactly } n \text{ zeros on } (0, R^*)\}.$$

By Lemma 2.4 we observe that if $a > 0$ then $S_n \neq \emptyset$ for some n. Let $n_0 \geq 0$ be the least integer n such that $S_n \neq \emptyset$ (i.e, $S_{n_0} \neq \emptyset$ and $S_n = \emptyset$ for all $0 \leq n < n_0$). Also it follows from Lemma 2.6 that S_{n_0} is bounded from above. So let

$$a_{n_0}^+ = \sup S_{n_0}.$$

Lemma 4.1. *$v_{a_n^+}$ has exactly n zeros, $v_{a_n^+}(0) = 0$, and $v'_{a_n^+}(0) \neq 0$ for all $n \geq n_0$.*

Proof. It follows from the definition of S_{n_0} that $v_{a_{n_0}^+}$ has at least n_0 zeros on $(0, R^*)$. Suppose that $v_{a_{n_0}^+}$ has an (n_0+1)st zero. Then by the continuous dependence of v_a on a it follows that v_a has an (n_0+1)st zero if a is sufficiently close to a_{n_0}. But if we choose $a \in S_{n_0}$ such that $a < a_{n_0}$ and a is sufficiently close to a_{n_0}, then v_a has only n_0 zeros on $(0, R^*)$ which gives a contradiction. Thus $v_{a_{n_0}^+}$ has exactly n_0 zeros on $(0, R^*)$. Now we want to show that $v_{a_{n_0}^+}(0) = 0$. Assume without the loss of generality that $v_{a_{n_0}^+} > 0$ on $(0, z_{a_{n_0}})$. Then by the continuity of $v_{a_{n_0}^+}$ we have $v_{a_{n_0}^+}(0) \geq 0$. Suppose $v_{a_{n_0}^+}(0) > 0$. Since the zeros of v_a are simple and $v_a(0) > 0$ it follows that v_a has exactly n_0 zeros on $(0, R^*)$ if a is close to a_{n_0}. But if $a > a_{n_0}$ then v_a has at least $n_0 + 1$ zeros on $(0, R^*)$ which is a contradiction. Therefore, we must have $v_{a_{n_0}^+}(0) = 0$.

Next we want to show that $v'_{a_{n_0}^+}(0) \neq 0$. Assume without loss of generality that $v_{a_{n_0}^+} > 0$ on $(0, z_{n_0})$ where z_{n_0} is the n_0^{th} zero of $a_{n_0}^+$ on $(0, R^*)$. Since $v_{a_{n_0}^+}$ solves (2.2) we have

$$v''_{a_{n_0}^+} + h(t) f(v_{a_{n_0}^+}) = 0.$$

From the above equation it follows that

$$(t v'_{a_{n_0}^+} - v_{a_{n_0}^+})' = t v''_{a_{n_0}^+} = -t h(t) f(v_{a_{n_0}^+}) < 0.$$

Thus, $t v'_{a_{n_0}^+} - v_{a_{n_0}^+}$ is decreasing. Also, since $\lim_{t \to 0^+}(t v'_{a_{n_0}^+} - v_{a_{n_0}^+})$=0 we have that $(t v'_{a_{n_0}^+} - v_{a_{n_0}^+}) \leq 0$ on $(0, z_{n_0})$. It then follows that

$$\left(\frac{v_{a_{n_0}^+}}{t} \right)' \leq 0. \tag{4.1}$$

Since $v_{a_{n_0}^+} > 0$ on $(0, z_{a_{n_0}})$, we see from (4.1) that $\lim_{t \to 0^+} \frac{v_{a_{n_0}^+}}{t}$ exists. Integrating (4.1) on (t, t_0) we obtain

$$0 < \frac{v_{a_{n_0}^+}(t_0)}{t_0} \leq \lim_{t \to 0^+} \frac{v_{a_{n_0}^+}(t)}{t} = v'_{a_{n_0}^+}(0).$$

Therefore, $v'_{a_{n_0}^+}(0) > 0$. □

Next let

$$S_{n_0+1} = \{a > 0 \mid v_a \text{ solves (2.2)–(2.3) and } v_a \text{ has exactly } (n_0+1) \text{ zeros on } (0, R^*)\}.$$

If a is sufficiently close to $a^+_{n_0}$ with $a > a^+_{n_0}$, then by the definition of $a^+_{n_0}$ it follows that v_a has an (n_0+1)st zero, $z_{a_{n_0}+1} \in (0, R^*)$. By integrating (2.13) on (t, R^*) we obtain

$$\frac{1}{2}v_a'^2 = \frac{1}{2}\frac{a^2 R^{2(N-1)}}{(N-2)^2} - \int_t^{R^*} h' F(v_a). \tag{4.2}$$

Similarly, we have

$$\frac{1}{2}v_{a^+_{n_0}}'^2 = \frac{1}{2}\frac{{a^+_{n_0}}^2 R^{2(N-1)}}{(N-2)^2} - \int_t^{R^*} h' F(v_{a^+_{n_0}}). \tag{4.3}$$

Since $v_a \to v_{a^+_{n_0}}$ uniformly as $a \to a^+_{n_0}$ it follows from (4.2) and (4.3) that

$$\lim_{a\to a^+_{n_0}} v_a'^2 = v_{a^+_{n_0}}'^2 \text{ uniformly on } [0, t_0] \text{ for } t_0 > 0. \tag{4.4}$$

Since $v_{a^+_{n_0}}'^2(0) > 0$ it follows from (4.4) that $v_a'(t) \neq 0$ if $a > a^+_{n_0}$ and a close to $a^+_{n_0}$ and t is close to 0. Hence, v_a has at most (n_0+1) zeros and therefore v_a has exactly (n_0+1) zeros if a is sufficiently close to $a^+_{n_0}$ and $a > a^+_{n_0}$. Thus, $S_{n_0+1} \neq \emptyset$. Also it follows from Lemma 2.6 that S_{n_0+1} is bounded above.

Now let

$$a^+_{n_0+1} = \sup S_{n_0+1}.$$

Then by using a similar argument as above we can show that $v_{a^+_{n_0}+1}$ has exactly (n_0+1) zeros on $(0, R^*)$ and that $v_{a^+_{n_0}+1}(0) = 0$. Continuation of this process will generate an infinite family of solutions $\{v_{a^+_n}\}_{n\geq n_0}$ of (2.2)–(2.3) where $v_{a^+_n}$ has exactly n zeros on $(0, R^*)$ and $v_{a^+_{n_0}}(0) = 0$.

To complete the proof we again consider the set S_{n_0} as above which is non-empty. By Lemma 3.1 it follows that S_{n_0} is bounded from below by a positive real number. So we define

$$a^-_{n_0} = \inf S_{n_0}.$$

Then by using the continuous dependence of the solution v_a on a as above we can show that $v_{a^-_{n_0}}$ has exactly n_0 zeros and $v_{a^-_{n_0}}(0) = 0$ and $v'_{a^-_{n_0}}(0) \neq 0$. Now it may be possible that S_{n_0} is a singleton set. Then we have $a^-_{n_0} = a^+_{n_0}$. In this case there is only one solution with n_0 zeros. But we know that if $a > a^+_{n_0}$ then $S_{n_0+1} \neq \emptyset$. Also if $a < a^-_{n_0} = a^+_{n_0}$ and a is close to $a^-_{n_0}$, then v_a has exactly (n_0+1) zeros. Thus S_{n_0+1} has at least two points. Next let

$$a^-_{n_0+1} = \inf S_{n_0+1}.$$

Then $a^-_{n_0+1} < a^+_{n_0+1}$ and we can also show that $v_{a^-_{n_0}+1}$ has exactly (n_0+1) solutions and $v_{a^-_{n_0}+1}(0) = 0$. Thus, $v_{a^+_{n_0}+1}$ and $v_{a^-_{n_0}+1}$ are two solutions with exactly (n_0+1) zeros on $(0, R^*)$. Continuation of this process will generate a second infinite family of solutions $\{v_{a^-_n}\}_{n\geq n_0}$ of (2.2)–(2.3) where $v_{a^-_n}$ has exactly n zeros on $(0, R^*)$ and $v_{a^-_{n_0}}(0) = 0$.

Finally, by letting $u^+_n(t) = v^+_{a_n}(t^{\frac{1}{2-N}})$ and $u^-_n(t) = v^-_{a_n}(t^{\frac{1}{2-N}})$ we obtain two infinite families of solutions of (1.3)–(1.4) with prescribed number of zeros. This ends the proof of Theorem 1.1. $\square$

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