Homoclinic solution to zero of a non-autonomous, nonlinear, second order differential equation with quadratic growth on the derivative

[ID] **Luiz Fernando de Oliveira Faria**[1] and
[ID] **Pablo dos Santos Corrêa Junior**[✉1, 2]

[1]Universidade Federal de Juiz de Fora, Via Local, 880 – São Pedro, Juiz de Fora, 36036-900, Brazil
[2]Universidade de São Paulo, Avenida Trabalhador São-carlense,
400 - Centro, São Carlos, 13566-590, Brazil

Abstract. This work aims to obtain a positive, smooth, even, and homoclinic to zero (i.e. zero at infinity) solution to a non-autonomous, second-order, nonlinear differential equation involving quadratic growth on the derivative. We apply Galerkin's method combined with Strauss' approximation on the term involving the first derivative to obtain weak solutions. We also study the regularity of the solutions and the dependence on their existence with a parameter.

Keywords: Galerkin method, homoclinic solution, quadratic growth on the derivative, differential equation.

2020 Mathematics Subject Classification: 34A34, 34C37, 34D10, 34K12, 34L30.

1 Introduction

The existence of homoclinic solutions for autonomous and non-autonomous differential equations and Hamiltonian systems is a crucial subject in qualitative theory (see [19]).

In this work, the second-order equation in the real line considered is

$$\begin{cases} -(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + g(|u'(t)|), & \text{in } \mathbb{R} \\ u(t) > 0, & \text{in } \mathbb{R} \\ \lim_{t \to \pm\infty} u(t) = 0, \end{cases} \quad (1.1)$$

with

(H_1) $1 < q < 2 < p < +\infty$ and $a_1 \in L^{\mathfrak{s}}(\mathbb{R}) \cap C(\mathbb{R})$, $\mathfrak{s} = \frac{2}{2-q}$, a positive even function;

✉Corresponding author. Email: p.correa@usp.br

(H_2) $A : \mathbb{R} \to \mathbb{R}$ a Lipschitz, smooth (at least $C^1(\mathbb{R})$), non-decreasing function satisfying:

$$\exists \, \gamma \in (0,1) \quad \text{such that} \quad 0 < \gamma \leq A(t) \quad \forall t \in \mathbb{R};$$

(H_3) $g : \mathbb{R} \to \mathbb{R}$ a continuous function satisfying:

$$0 \leq sg(s) \leq |s|^{\theta} \quad \text{for all } s \in \mathbb{R}, \text{ where } 2 < \theta \leq 3. \tag{1.2}$$

The equation in (1.1) arises in several real phenomena, for instance, as the study of traveling wavefronts for parabolic reaction-diffusion equations with a local reaction term, chemical models, and others, as mentioned in [13, 14], and generalizes several classical equations such as Duffing-type equations [3, 10, 16] or Liénard-like systems [18].

Now, we state our main result.

Theorem 1.1. *There exists $\lambda^* > 0$ such that, for all $\lambda \in (0, \lambda^*]$, problem (1.1) has an even, positive and $C^2(\mathbb{R})$ homoclinic solution to the origin. Moreover, as $\lambda \to 0$, this solution goes to 0 in $C^0(\mathbb{R})$.*

We also find an additional result with respect to appropriate ranges of λ in order to guarantee the existence of solutions.

Proposition 1.2. *Assume the hypotheses of Theorem 1.1. If $\lambda > 0$ is sufficiently large, then equation (1.1) has no (positive) solution in $H^1(\mathbb{R})$.*

The idea to consider problem (1.1) came from article [3], where the authors considered a similar equation but with a different set of hypotheses; namely, their formulation was focused on the study of the equation

$$\begin{cases} -(A(u)u')' + u(t) = h(t, u(t)) + g(t, u'(t)) \text{ in } \mathbb{R} \\ u(\pm\infty) = u'(\pm\infty) = 0, \end{cases}$$

with

$(\widetilde{H}_1)$ $h, g : \mathbb{R}^2 \to \mathbb{R}$ locally Hölder continuous, even in the first variable and $h(t,0) = g(t,0) = 0$;

$(\widetilde{H}_2)$ there exist constants $0 < r_1, r_2 < 1$ and smooth functions $b \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $b(t) > 0$ for all $t \in \mathbb{R}$, $a_1 \in L^2(\mathbb{R})$ and $a_2 \in L^{\frac{2}{1-r_2}}(\mathbb{R})$, satisfying

$$b(t)|\mu|^{r_1} \leq h(t, \mu) \leq a_1(t) + a_2(t)|\mu|^{r_2}, \quad \forall (t, \mu) \in \mathbb{R}^2;$$

$(\widetilde{H}_3)$ there exist a constant $0 < r_3 < 1$ and smooth functions $a_3 \in L^{\frac{2}{1-r_3}}(\mathbb{R})$ and $a_4 \in L^2(\mathbb{R})$ satisfying

$$0 \leq g(t, \eta) \leq a_4(t) + a_3(t)|\eta|^{r_3} \quad \forall (t, \eta) \in \mathbb{R}^2;$$

$(\widetilde{H}_4)$ the function A is smooth, nondecreasing and there exists $\gamma \in (0,1)$ satisfying

$$0 < \gamma \leq A(t) \quad \forall t \in \mathbb{R}.$$

By comparison with our work, we considered sup-linear growth on u and u', terms involving this type of growth are not covered in [3]. Another aspect that we would like to emphasize is the weakening of the hypothesis over g: comparing with [3], we asked only for continuity over g, instead of Hölder continuity. Although the formulation presented here is not an immediate consequence of [3], some techniques therein proved to be quite solid and very useful in the study of this type of problem, transcending the circumstances framed by the authors.

Our formulation presented some interesting challenges, for instance, the problem is not variational. Among the non-variational techniques, we chose the Galerkin method as a tool to gather information about the existence of weak solutions. Although proving itself beneficial, the Galerkin method presented us with other types of challenges to circumvent. For example, the nonlinear term $g(|u'(t)|)$ with $0 \leq sg(s) \leq |s|^{\theta}$ and $2 < \theta \leq 3$ enables us to take $g(s) \equiv \text{sign}(s)|s|^2$. Thus estimations involving $\int_{\Omega} |u'|^2$ become essential to the calculations but, at the same time, we cannot say much about it *a priori*: this is due to the lack of information about u', since the embedding theorems of $H_0^1(\Omega)$ do not provide substation information about u' as they do for u.

We consider the case $\theta = 3$ as the *critical* one and treat it separately in our estimations. For $\theta > 3$ we would get expressions involving $\int |u'|^{\theta-1}$ that we could not control, because $\theta - 1 > 2$ and we only know that $u' \in L^2$; for this reason we limited $\theta \leq 3$, and $\theta > 2$ was required because we wanted to focus on the sup-linear case.

There is some literature about equations on domains in $\mathbb{R}^n$ involving the term $|\nabla u|^2$ in the nonlinearity (see [1,2,9,15]), some authors call this type of growth: "*critical growth on the gradient*". Simple changes on how this term appears in the equation can have dramatic effects on the outcome. For instance, a simple change in the sign of $|\nabla u|^2$ can lead to a total failure to obtain a solution (even in the weak sense), see the article [2] for more information. We also would like to emphasize article [9] for its results and broader discussion about PDE with quadratic growth on the gradient: the model problem studied by the authors is

$$- \text{div}(A(x)\nabla u) = c_0(x)u + \mu(x)|\nabla u|^2 + f(x),$$

with suitable hypothesis. In this context, our problem (1.1) presents a similar structure that was not covered before, thus we believe it contributes to the discussion previously mentioned.

The methods applied in our work require certain symmetry, which is due mainly to a lack of a comparison principle (known to the authors) to guarantee that some limit-functions are not zero almost everywhere (a.e.), (see Proposition 2.24). To overcome this obstacle, we founded this work focusing on the set $\mathbb{E}_0^1(I) = \{u \in H_0^1(I); u(t) = u(-t) \text{ a.e.}\}$, $I \subset \mathbb{R}$ a symmetric interval, which is the subset of $H_0^1(I)$ consisting of even (or radial symmetric) functions.

In order to develop our study, in Section 2, we started by analyzing our equation on a bounded interval:

$$\begin{cases} -(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + g(|u'(t)|), & \text{in } (-n, n) \\ u(n) = u(-n) = 0. \end{cases} \quad (P_n)$$

This restriction was essential to realize our estimations and to obtain upper bound constants that were crucial to construct the solution in $\mathbb{R}$ to the problem (1.1). The process developed in Section 2 consisted mainly of two steps:

First Approximate g by a sequence of Lipschitz functions (f_k) using the *Strauss Approximation*; this sequence received this name after its first appearance in the article [17].

This approximation was useful because it helped us to work with the necessary estimations without extra hypotheses over g. We followed [5] in the definition and presentation of the properties of the sequence (f_k). In this article, the authors used this approximation to avoid the usage of the Ambrosetti–Rabinowitz condition and were able to obtain a positive solution to the equation

$$\begin{cases} -\Delta u = \lambda u^{q(r)-1} + f(r,u) & \text{in } B(0,1) \\ u > 0 & \text{in } B(0,1) \\ u = 0 & \text{on } \partial B, \end{cases}$$

see [5] for more information.

We would like to emphasize that, in [5], the authors used this approximation in a term involving u; namely they used it to approximate $f(r,u)$. In our work, we used it in u'.

Second We used the sequence (f_k) to define an approximate problem in $(-n,n)$ and used the Galerkin method to obtain a weak solution. Then, using the work done by Gary M. Liberman [12], we obtained the *a priori estimation* summarized in Proposition 2.21. Thus we obtained a strong solution to this problem. Afterward, we were able to construct a strong solution to the problem (P_n).

In Section 4 we used the pieces of information gathered in Section 2 to construct a solution to the problem (1.1), thus proving Theorem 1.1. We also prove Proposition 1.2 in Section 4. We would like to point out the role of Section 3: there we study the asymptotic behavior, in respect to λ, of the solution to the problem (P_n) – the arguments presented were inspired by the article [8]. This was useful to tackle the last assertion of Theorem 1.1.

2 Solution in a bounded interval

First, we will obtain a solution to a problem related to (1.1); namely, we will study

$$\begin{cases} -(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + g(|u'(t)|), & \text{in } (-n,n) \\ u(n) = u(-n) = 0, \end{cases} \qquad (P_n)$$

with the same set of hypothesis $(H_1), (H_2)$ and (H_3). The motivation for this approach is to construct a solution to the problem (1.1) using the solutions of (P_n). Although the analysis of (P_n) is easier since it is defined over $(-n,n)$, rather than $\mathbb{R}$, the lack of hypothesis over g creates a difficult situation for our estimations. To overcome this matter, we will utilize the *Strauss Approximation* on g at the same time that we approximate the problem (P_n). Let us define the sequence of functions that will approximate g.

Define $G(s) = \int_0^s g(t)dt$ so that G is differentiable and $G'(s) = g(s)$. By means of G we shall construct a sequence of approximations of g by Lipschitz functions $f_k : \mathbb{R} \longrightarrow \mathbb{R}$. Let

$$f_k(s) = \begin{cases} -k[G(-k-\frac{1}{k}) - G(-k)], & \text{if } s \leq -k \\ -k[G(s-\frac{1}{k}) - G(s)], & \text{if } -k \leq s \leq \frac{-1}{k} \\ k^2 s[G(\frac{-2}{k}) - G(\frac{-1}{k})], & \text{if } \frac{-1}{k} \leq s \leq 0 \\ k^2 s[G(\frac{2}{k}) - G(\frac{1}{k})], & \text{if } 0 \leq s \leq \frac{1}{k} \\ k[G(s+\frac{1}{k}) - G(s)], & \text{if } \frac{1}{k} \leq s \leq k \\ k[G(k+\frac{1}{k}) - G(k)], & \text{if } s \geq k. \end{cases} \qquad (2.1)$$

Remark 2.1. The construction of the sequence (f_k) is due to [17].

The advantage of this sequence lies in the properties that one can obtain from it:

Theorem 2.2 ([5, Lemma 1]). *The sequence f_k as defined above satisfies:*

1. *$sf_k(s) \geq 0$ for all $s \in \mathbb{R}$;*

2. *for all $k \in \mathbb{N}$ there is a constant $c(k)$ such that $|f_k(\xi) - f_k(\eta)| \leq c(k)|\xi - \eta|$, for all $\xi, \eta \in \mathbb{R}$;*

3. *f_k converges uniformly to g in bounded sets.*

Remark 2.3. From the definition of the sequence f_k, and the fact that $\text{sign}(g(s)) = \text{sign}(s)$ for all $s \in \mathbb{R}$, it follows without difficulties that *1* is true. In [5, p. 6, Prop. 5] one can find a detailed proof of *2*, so we will only prove *3* by an alternative argumentation.

Proof. Given a bounded set $J \subset \mathbb{R}$, there exists $m_0 \in \mathbb{N}$ such that $J \subset (-m_0, m_0)$; so to prove *3* we only need to prove that it holds in intervals such as $(-m, m)$, $m \in \mathbb{N}$. We may also assume that $k > m$. Given $\epsilon > 0$, for $s \in (-m, m)$ there are four possible cases:

Case I. $-m < s \leq \frac{-1}{k}$ Here we have that

$$|f_k(s) - g(s)| = \left| -k \left[G\left(s - \frac{1}{k}\right) - G(s) \right] - g(s) \right| = \left| \frac{\left[G(s - \frac{1}{k}) - G(s) \right]}{\frac{-1}{k}} - g(s) \right|.$$

Then, since $G'(s) = g(s)$, there exists $\delta(s) > 0$ such that $0 < |h| < \delta(s)$ implies

$$\left| \frac{[G(s+h) - G(s)]}{h} - g(s) \right| < \epsilon$$

From the family of open sets $\{(s - \delta(s), s + \delta(s)); s \in [-m, 0]\}$ we extract a finite subcover $\{(s_i - \delta(s_i), s_i + \delta(s_i)); i = 1 \ldots l\}$ of the compact set $[-m, 0]$ and take $\delta = \min\{\delta(s_i); i = 1, \ldots l\}$. Thus, for $k > \frac{1}{\delta}$ we get $|f_k(s) - g(s)| < \epsilon$.

Case II. $\frac{-1}{k} \leq s \leq 0$

Since $g(0) = 0$ and g is continuous, for the given $\epsilon > 0$ there exists $\delta > 0$ such that $|t| < \delta$ implies $|g(t)| < \epsilon/2$. Let $k_0 \in \mathbb{N}$ be such that $k_0 > m$ and $k_0 > 2/\delta$. Then, for $k > k_0$

$$|f_k(s) - g(s)| = \left| k^2 s \left[G\left(\frac{-2}{k}\right) - G\left(\frac{-1}{k}\right) \right] - g(s) \right|$$

$$\leq k^2 |s| \left| \int_{-1/k}^{-2/k} |g(t)| dt \right| + |g(s)|$$

$$\leq k^2 \left(\frac{1}{k^2}\right) \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall s \in \left[\frac{-1}{k}, 0\right].$$

The cases $0 \leq s \leq \frac{1}{k}$ and $\frac{1}{k} \leq s < m$ can be analyzed in a similar fashion. Thus we see that, for $\epsilon > 0$, we can take $k \in \mathbb{N}$ big enough such that $|f_k(s) - g(s)| < \epsilon$ independently of $s \in (-m, m)$. □

Lemma 2.4 ([5, Lemma 2]). *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (1.2). Then the sequence f_k of Theorem 2.2 satisfies*

1. *For all $k \in \mathbb{N}$, $0 \leq sf_k(s) \leq C_1|s|^\theta$ for every $|s| \geq \frac{1}{k}$;*

2. *for all $k \in \mathbb{N}$, $0 \leq sf_k(s) \leq C_1|s|^2$ for every $|s| \leq \frac{1}{k}$;*

where C_1 is a constant independent of k.

Proof. See [5, Page 8, Lemma 2].   □

Now we are in condition – using (f_k) – to define a problem that approximates problem (P_n). Let $\psi \in L^2(-n,n) \cap C(-n,n)$ be a positive, even function. We define our *approximate problem* by:

$$\begin{cases} -(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + f_k(|u'(t)|) + \frac{\psi}{k}, \text{ in } (-n,n) \\ u(n) = u(-n) = 0. \end{cases} \quad (P_n^k)$$

In the next subsection we will utilize the *Galerkin method* to obtain a solution to (P_n^k); afterward, we will let k vary and thus, as $k \to \infty$, obtain a solution to (P_n). Before jumping into the next subsection, let us define what we understand as *weak solution* to problem (P_n):

Definition 2.5. We will call $w \in H_0^1(-n,n)$ a *weak solution* of (P_n) if

$$\int_{-n}^n A(w)w'v' + \int_{-n}^n wv = \int_{-n}^n \lambda a_1|w|^{q-1}v + \int_{-n}^n |w|^{p-1}v + \int_{-n}^n g(|w'|)v$$

for all $v \in H_0^1(-n,n)$.

Remark 2.6. We will use, for the sake of clarity, the notation $\|\cdot\|_{W^{1,2}}$ for the usual norm of H_0^1 and for $(\|u\|_{L^2} + \|u'\|_{L^2})$ or $(\|u\|_{L^2}^2 + \|u'\|_{L^2}^2)^{1/2}$. Since these norms are equivalent the results will not change but the constants may. In most of the cases $\|u\|_{W^{1,2}} = \|u\|_{L^2} + \|u'\|_{L^2}$. We also emphasize that, when the context is clear, we will omit the domain in norms such as those from the spaces $L^p(-n,n)$.

Remark 2.7. The integrals in the definition above are well defined, see for instance the estimations of Proposition 2.16. The same is true for the definitions given in the next subsection.

2.1 Solution to the approximate problem

Our main goal in this subsection is to prove the following theorem:

Theorem 2.8. *There exist $\lambda^* > 0$, $\beta \in (0,1)$ and $k^* \in \mathbb{N}$ for which the problem (P_n^k) admits a nontrivial, even, non-negative $C^{1,\beta}[-n,n] \cap C^2(-n,n)$ solution for every $\lambda \in (0,\lambda^*)$ and $k \geq k^*$.*

As mentioned, we will utilize the *Galerkin method*; thus we will start by presenting the foundations that this method requires. The next lemma is a well-known result, but it plays a central role in all arguments involving the Galerkin method.

Lemma 2.9. *Let $\mathfrak{F} : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function such that $\langle \mathfrak{F}(x), x \rangle \geq 0$ for all $x \in \mathbb{R}^N$ with $\|x\|_{\mathbb{R}^N} = r$. Then there exists x_0 in the closed ball $B[0,r]$ such that $\mathfrak{F}(x_0) = 0$.*

Proof. See [11, Chap. 5, Theorem 5.2.5].   □

Now we will define an entity called *E-weak solution*. It is well known that the main focus of the Galerkin method is to obtain a weak solution, but we will utilize it to obtain an E-weak solution first.

Definition 2.10. A function $w \in H_0^1(-n, n)$ is called an *E-weak solution* of (P_n^k) if w is an *even function* satisfying

$$\int_{-n}^{n} A(w)w'\varphi' + \int_{-n}^{n} w\varphi = \int_{-n}^{n} \lambda a_1 |w|^{q-1}\varphi + \int_{-n}^{n} |w|^{p-1}\varphi + \int_{-n}^{n} f_k(|w'|)\varphi + \int_{-n}^{n} \frac{\psi}{k}\varphi$$

for all $\varphi \in \mathbb{E}_0^1(-n, n) = \{u \in H_0^1(-n, n); u(t) = u(-t) \text{ a.e.}\}$.

The use of the E-weak solution will be central in our argumentation to obtain an *even* solution to the problem (P_n^k). This symmetry – being even – will also be beneficial in the use of our comparison principle, which is stated as follows:

Theorem 2.11 ([3, Theorem 3.1]). *Let $\sigma : (0, +\infty) \to \mathbb{R}$ be a continuous function such that the mapping $(0, +\infty) \ni s \mapsto \frac{\sigma(s)}{s}$ is strictly decreasing and $\rho > 0$. Suppose that there exist even functions $v, w \in C^2(-\rho, \rho) \cap C[-\rho, \rho]$ such that:*

1. *$(A(w)w')' - w + \sigma(w) \leq 0 \leq (A(v)v') - v + \sigma(v)$ in $(-\rho, \rho)$;*

2. *$v, w \geq 0$ in $(-\rho, \rho)$ and $v(\rho) \leq w(\rho)$;*

3. *$\{x \in (-\rho, \rho); v(x) = 0\}$ and $\{x \in (-\rho, \rho); w(x) = 0\}$ have null measure in $\mathbb{R}$;*

4. *$v' \cdot w' \geq 0$ in $(-\rho, \rho)$;*

5. *$v', w' \in L^\infty(-\rho, \rho)$.*

Then $v \leq w$ in $(-\rho, \rho)$.

Proof. The same as [3, p. 2419, Thm 3.1]. □

Turns out that, obtaining an E-weak solution enables us to recuperate a weak solution in the usual sense:

Definition 2.12. We will call $w \in H_0^1(-n, n)$ a *weak solution* of (P_n^k) if

$$\int_{-n}^{n} A(w)w'v' + \int_{-n}^{n} wv = \int_{-n}^{n} \lambda a_1 |w|^{q-1}v + \int_{-n}^{n} |w|^{p-1}v + \int_{-n}^{n} f_k(|w'|)v + \int_{-n}^{n} \frac{\psi}{k}v$$

for all $v \in H_0^1(-n, n)$.

This is achieved by

Lemma 2.13 ([3, Lemma 4.1]). *Let $w \in H_0^1(-n, n)$ be an E-weak solution of (P_n^k). Then w is a weak solution of (P_n^k).*

Proof. See [3, p. 2421, Lemma 4.1]. □

The subset $\mathbb{E}_0^1(-n, n) \subset H_0^1(-n, n)$ can be understood as the set of radial symmetric functions in $\mathbb{R}$. One can prove without difficulties the following properties of $\mathbb{E}_0^1(-n, n)$:

i) it is a Hilbert space;

ii) it is separable;

iii) it has an orthonormal basis.

Let $\mathbb{E}_0^1(-n,n) = \{u \in H_0^1(-n,n); u(t) = u(-t) \text{ a.e.}\}$ and $(e_l)_{l=1}^\infty$ be an orthonormal basis of $\mathbb{E}_0^1(-n,n)$.

Define $V_M = \text{span}\{e_1,\dots,e_M\}$; then for every $u \in V_M$ there exist $\xi_1,\dots,\xi_M$ in $\mathbb{R}$ such that $u = \sum_{i=1}^M \xi_i e_i$. By means of $T : V_M \to \mathbb{R}^M$, $T(u) = T(\sum_{i=1}^M \xi_i e_i) = (\xi_1,\dots,\xi_M)$, which is a linear isomorphism and preserve norm, we may define $\mathfrak{F} : \mathbb{R}^M \to \mathbb{R}^M$ such that

$$\mathfrak{F}(\xi) = (\mathfrak{F}_1(\xi),\dots,\mathfrak{F}_M(\xi)) \tag{2.2}$$

and

$$\mathfrak{F}_j(\xi) = \int_{-n}^n A(u)u'e_j' + \int_{-n}^n u e_j - \int_{-n}^n \lambda a_1 |u|^{q-1} e_j - \int_{-n}^n |u|^{p-1} e_j - \int_{-n}^n f_k(|u'|) e_j - \int_{-n}^n \frac{\psi}{k} e_j,$$

where $j \in \{1,\dots,M\}$ and $u = T^{-1}(\xi)$, for all $\xi \in \mathbb{R}^M$.

Lemma 2.14. *The function $\mathfrak{F}$ is continuous.*

Remark 2.15. Our proof will use the fact that, if (x_n) is a sequence that converges to x and, for all subsequence (x_{n_l}) of (x_n), there exist a subsequence $(x_{n_{l_k}})$ of (x_{n_l}) such that $\mathfrak{F}(x_{n_{l_k}})$ converges to $\mathfrak{F}(x)$, then $\mathfrak{F}(x_n)$ converges to $\mathfrak{F}(x)$.

Proof. Given $\xi = (\xi_1,\dots,\xi_M) \in \mathbb{R}^M$, let $(\xi_l)_{l=1}^\infty$ be a sequence in $\mathbb{R}^M$ such that $\|\xi_l - \xi\|_{\mathbb{R}^M} \to 0$. By means of T we can identify $T^{-1}(\xi) = u = \sum_{i=1}^M e_i \xi_i$ and $T^{-1}(\xi_l) = u_l = \sum_{i=1}^M e_i \xi_i^l$. Since T is isometry we have that $\|u_l - u\|_{W^{1,2}} \to 0$. That is, $\|u_l - u\|_{L^2} \to 0$ and $\|u_l' - u'\|_{L^2} \to 0$. Taking a subsequence, if necessary, we may assume that

$$u_l(x) \to u(x) \quad \text{a.e. on } (-n,n),$$
$$u_l'(x) \to u'(x) \quad \text{a.e. on } (-n,n),$$

and $|u_l(x)| \le h_1(x)$, $|u_l'(x)| \le h_2(x)$ a.e. on $(-n,n)$, with $h_1, h_2 \in L^2(-n,n)$. Let $j \in \{1,2,\dots,M\}$, we will prove that $\mathfrak{F}_j(\xi_l) \to \mathfrak{F}_j(\xi)$.

$$\left| \int_{-n}^n A(u_l)u_l'e_j' - \int_{-n}^n A(u)u'e_j' \right| \le \int_{-n}^n \left(|u_l'||A(u_l) - A(u)| + |A(u)||u_l' - u'| \right) |e_j'|, \tag{2.3}$$

since $|u_l'(x)||A(u_l(x)) - A(u(x))||e_j'(x)| \to 0$ a.e. and $|A(u(x))||u_l'(x) - u'(x)||e_j'(x)| \to 0$ a.e., by the Dominated Convergence Theorem (D.C.T) the left side of (2.3) tends to zero as $l \to +\infty$.

$$\left| \int_{-n}^n u_l e_j - \int_{-n}^n u e_j \right| \le \int_{-n}^n |u_l - u||e_j| \to 0 \quad \text{by (D.C.T)}. \tag{2.4}$$

$$\left| \int_{-n}^n [\lambda a_1(|u_l|^{q-1} - |u|^{q-1}) + (|u_l|^{p-1} - |u|^{p-1}) + (f_k(u_l') - f_k(u'))]e_j \right|$$
$$\le \int_{-n}^n \lambda |a_1| \left| |u_l|^{q-1} - |u|^{q-1} \right| |e_j| + \int_{-n}^n \left| |u_l|^{p-1} - |u|^{p-1} \right| |e_j|$$
$$+ \int_{-n}^n \left| f_k(|u_l'|) - f_k(|u'|) \right| |e_j|, \tag{2.5}$$

since that $|u_l|^{q-1} \to |u|^{q-1}$ a.e. and $|u_l|^{p-1} \to |u|^{p-1}$ a.e., (D.C.T) implies that the first two integrals above converge to zero. Using the second item of Theorem 2.2, we have

$$\int_{-n}^n \left| f_k(|u_l'|) - f_k(|u'|) \right| |e_j| \le \int_{-n}^n c(k)|u_l' - u'||e_j|. \tag{2.6}$$

Then, by (D.C.T), (2.6) converges to 0 as $l \to +\infty$.

These estimations show us that for every subsequence (ξ_{l_k}) of (ξ_l), there exists a subsequence $(\xi_{l_{k_n}})$ of (ξ_{l_k}) that $\mathfrak{F}_j(\xi_{l_{k_n}}) \to \mathfrak{F}_j(\xi)$. Therefore $\mathfrak{F}_j(\xi_l) \to \mathfrak{F}_j(\xi)$. $\qquad\square$

Proposition 2.16. *There exist* $\lambda^* > 0$ *and* $k^* \in \mathbb{N}$ *for which the problem* (P_n^k) *admits a nontrivial weak solution for every* $\lambda \in (0, \lambda^*)$ *and* $k \geq k^*$.

Remark 2.17. We will, in fact, search for an E-weak solution; but as seen in Lemma 2.13 this will also be a weak solution.

Proof. Our aim is to use Lemma 2.9, with the function $\mathfrak{F}$ defined in (2.2). Given $\xi \in \mathbb{R}^M$, we have that

$$\langle \mathfrak{F}(\xi), \xi \rangle = \int_{-n}^{n} A(u)|u'|^2 + \int_{-n}^{n} |u|^2 - \int_{-n}^{n} \lambda a_1 |u|^{q-1} u - \int_{-n}^{n} |u|^{p-1} u$$
$$- \int_{-n}^{n} f_k(|u'|)u - \int_{-n}^{n} \frac{\psi}{k} u. \quad (2.7)$$

In the following, we will estimate the above integrals. We have that

$$\int_{-n}^{n} \lambda a_1 |u|^{q-1} u \leq \lambda \|a_1\|_{L^s(\mathbb{R})} \|u\|_{L^2}^q \leq \lambda C_2 \|u\|_{W^{1,2}}^q, \quad (2.8)$$

$$\int_{-n}^{n} \frac{\psi}{k} u \leq \frac{\|\psi\|_{L^2(-n,n)} \|u\|_{W^{1,2}}}{k}. \quad (2.9)$$

Now let $\tilde{u} : \mathbb{R} \to \mathbb{R}$ be the extension by zero of u, then

$$\int_{-n}^{n} |u|^{p-1} u \leq \int_{-n}^{n} |u|^p = \int_{-n}^{n} |u|^2 |u|^{p-2} \quad (2.10)$$

$$\leq \|\tilde{u}\|_{L^\infty(\mathbb{R})}^{p-2} \int_{-n}^{n} |u|^2 \quad (2.11)$$

$$= \|\tilde{u}\|_{L^\infty(\mathbb{R})}^{p-2} \|u\|_{L^2}^2 \quad (2.12)$$

$$\leq C^{p-2} \|u\|_{W^{1,2}}^{p-2} \|u\|_{W^{1,2}}^2 = C^{p-2} \|u\|_{W^{1,2}}^p. \quad (2.13)$$

Where C is the constant for the embedding $W^{1,2}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$.

Define

$$\Omega_\geq = \left\{ s \in (-n, n) : |u'(s)| \geq \frac{1}{k} \right\} \quad \text{and} \quad \Omega_\leq = \left\{ s \in (-n, n) : 0 < |u'(s)| \leq \frac{1}{k} \right\}.$$

Then

$$\int_{-n}^{n} f_k(|u'|)u = \int_{\Omega_\geq} f_k(|u'|)u + \int_{\Omega_\leq} f_k(|u'|)u.$$

Notice that by Lemma 2.4,

$$\int_{\Omega_\leq} f_k(|u'|)u \leq \int_{\Omega_\leq} C_1 |u'||u| \leq \int_{\Omega_\leq} \frac{C_1}{k} |u|$$

$$\leq \frac{C_1}{k} \int_{-n}^{n} |u| \leq \frac{C_1 (2n)^{1/2}}{k} \|u\|_{L^2}$$

$$\leq \frac{C_1 (2n)^{1/2}}{k} \|u\|_{W^{1,2}}.$$

To estimate the integral over $\Omega_\geq$, consider the following cases :

Case 1. $2 < \theta < 3$.

Using Lemma 2.4, we have

$$\int_{\Omega_{\geq}} f_k(|u'|)u \leq \int_{\Omega_{\geq}} C_1|u'|^{\theta-1}|u| \leq \int_{-n}^{n} C_1|u'|^{\theta-1}|u|$$

$$\leq C_1\left(\int_{-n}^{n}|u|^w\right)^{\frac{1}{w}}\left(\int_{-n}^{n}|u'|^2\right)^{\frac{\theta-1}{2}}$$

$$\leq C_1\left(\int_{\mathbb{R}}|\tilde{u}|^2|\tilde{u}|^{w-2}\right)^{\frac{1}{w}}\|u'\|_{L^2}^{\theta-1}$$

$$\leq C_1\|\tilde{u}\|_{L^{\infty}(\mathbb{R})}^{\frac{w-2}{w}}\|u\|_{L^2}^{\frac{2}{w}}\|u'\|_{L^2}^{\theta-1}$$

$$\leq C_1 C^{\frac{w-2}{w}}\|u\|_{W^{1,2}}^{\frac{w-2}{w}}\|u\|_{W^{1,2}}^{\frac{2}{w}}\|u\|_{W^{1,2}}^{\theta-1} = C_1 C^{\frac{w-2}{w}}\|u\|_{W^{1,2}}^{\theta}.$$

Where $w = \left(\frac{2}{\theta-1}\right)' = \frac{2}{3-\theta} > 2$.

Case 2. $\theta = 3$.

$$\int_{\Omega_{\geq}} f_k(|u'|)u \leq \int_{\Omega_{\geq}} C_1|u'|^2|u| \leq \int_{-n}^{n} C_1|u'|^2|u|$$

$$\leq C_1\|\tilde{u}\|_{L^{\infty}(\mathbb{R})}\|u'\|_{L^2}^2 \leq C_1 C\|u\|_{W^{1,2}}\|u\|_{W^{1,2}}^2$$

$$= C_1 C\|u\|_{W^{1,2}}^3.$$

Now we are able to estimate (2.7). Notice that $\frac{w-2}{w} = \theta - 2$.

$$\langle \mathfrak{F}(\xi), \xi \rangle \geq \gamma\|u\|_{W^{1,2}}^2 - \lambda C_2\|u\|_{W^{1,2}}^q - C^{p-2}\|u\|_{W^{1,2}}^p$$
$$- C_1\max\{C^{\theta-2}, C\}\|u\|_{W^{1,2}}^{\theta} - \left(\frac{C_1(2n)^{1/2}}{k} + \frac{\|\psi\|_{L^2(-n,n)}}{k}\right)\|u\|_{W^{1,2}}.$$

Define $Z_k : \mathbb{R}^+ \to \mathbb{R}$ by

$$Z_k(x) = \gamma x^2 - \lambda C_2 x^q - C^{p-2}x^p - C_1\max\{C^{\theta-2}, C\}x^{\theta} - \left(\frac{C_1(2n)^{1/2}}{k} + \frac{\|\psi\|_{L^2(-n,n)}}{k}\right)x.$$

We would like to find $r > 0$ such that

$$\gamma r^2 - C^{p-2}r^p - C_1\max\{C^{\theta-2}, C\}r^{\theta} > \frac{r^2}{2}\gamma \tag{2.14}$$

or equivalently,

$$\frac{\gamma}{2} > C^{p-2}r^{p-2} + C_1\max\{C^{\theta-2}, C\}r^{\theta-2}.$$

For this, if we take

$$\delta_1 = \min\left\{\left(\frac{\gamma}{4C^{p-2}}\right)^{1/(p-2)}, \left(\frac{\gamma}{4C_1\max\{C^{\theta-2}, C\}}\right)^{1/(\theta-2)}\right\},$$

then for $0 < r < \delta_1$ (2.14) is true. Consequently,

$$Z_k(r) \geq \frac{r^2}{2}\gamma - \lambda C_2\delta_1^q - \left(\frac{C_1(2n)^{1/2}}{k} + \frac{\|\psi\|_{L^2(-n,n)}}{k}\right)\delta_1.$$

Define $\rho_1 = \frac{r^2}{2}\gamma - \lambda C_2 \delta_1^q$. We will adjust $\lambda > 0$ so that $\rho_1 > 0$; for this if $\rho_1 > 0$ it would imply that

$$\frac{r^2}{2}\gamma - \lambda C_2 \delta_1^q > 0 \Leftrightarrow \frac{r^2\gamma}{2C_2\delta_1^q} > \lambda.$$

Take $\lambda^* = \frac{r^2\gamma}{2C_2\delta_1^q}$ and $0 < \lambda < \lambda^*$. Thus, $\rho_1 > 0$ and we can find $k^* \in \mathbb{N}$ such that for $k > k^*$, $\rho_1 > \left(\frac{C_1(2n)^{1/2}}{k} + \frac{\|\psi\|_{L^2(-n,n)}}{k}\right)\delta_1 > 0$. Therefore, for $0 < r < \delta_1$, $0 < \lambda < \lambda^*$ and $k > k^*$

$$Z_k(r) > 0,$$

and so, with $\|u\|_{W^{1,2}} = r$,

$$\langle \mathfrak{F}(\xi), \xi \rangle > 0. \tag{2.15}$$

By Lemma 2.9, there exists $y_M \in B[0,r]$ such that $\mathfrak{F}(y_M) = 0$ that is, identifying $v_M = T^{-1}(y_M)$, for all $j \in \{1, \dots, M\}$

$$\int_{-n}^{n} A(v_M)v_M' e_j' + \int_{-n}^{n} v_M e_j \tag{2.16}$$
$$= \int_{-n}^{n} \lambda a_1 |v_M|^{q-1} e_j + \int_{-n}^{n} |v_M|^{p-1} e_j + \int_{-n}^{n} f_k(|v_M'|)e_j + \int_{-n}^{n} \frac{\psi}{k} e_j.$$

Therefore (2.16) holds for all $\varphi \in V_M$, because $\{e_1, \dots, e_M\}$ is a basis of V_M. Notice that

$$\|v_M\|_{W^{1,2}} \le r \quad \text{for all } M \in \mathbb{N}. \tag{2.17}$$

Remark 2.18. Our choice of r does not depend on M,n, λ or k. This free determination of r will be useful further down in the argumentation, because using the embedding $W^{1,2}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ we will be able to obtain a uniform upper bound, in the norm of $L^\infty(\mathbb{R})$, for the sequence of solutions of the problem (P_n^k). Then this upper bound will naturally be transferred to also bound the sequence of solution of (P_n).

Since $\|v_M\|_{W^{1,2}} \le r$ there is $v_0 \in \mathbb{E}_0^1(-n,n)$ such that $v_M \rightharpoonup v_0$ in $H_0^1(-n,n)$. By the compact embedding $W^{1,2}(-n,n) \hookrightarrow L^2(-n,n)$ we conclude $v_M \to v_0$ in $L^2(-n,n)$. Our goal is to show that v_0 is a weak solution of (P_n^k). Let $\Gamma_M : V_M \to V_M^*$ and $B_M : V_M \to V_M^*$ be defined by

$$\langle \Gamma_M(v), \varphi \rangle = \int_{-n}^{n} A(v)v'\varphi' \tag{2.18}$$

and

$$\langle B_M(v), \varphi \rangle = \int_{-n}^{n} \left(-v + \lambda a_1 |v|^{q-1} + |v|^{p-1} + f_k(|v'|) + \frac{\psi}{k} \right) \varphi. \tag{2.19}$$

Hence, $\langle \Gamma_M(v_M) - B_M(v_M), \varphi \rangle = 0$ for all $\varphi \in V_M$.

Denoting $P_M : \mathbb{E}_0^1(-n,n) \to V_M$ the projection of $\mathbb{E}_0^1(-n,n)$ onto V_M, (that is, if $u = \sum_{i=1}^{\infty} \alpha_i e_i$ then $P_M(u) = \sum_{i=1}^{M} \alpha_i e_i$) we have

$$\langle \Gamma_M(v_M) - B_M(v_M), v_M - P_M v_0 \rangle = 0,$$

so

$$\langle \Gamma_M(v_M), v_M - P_M v_0 \rangle = \langle B_M(v_M), v_M - P_M v_0 \rangle \tag{2.20}$$
$$= \int_{-n}^{n} \left(-v_M + \lambda a_1 |v_M|^{q-1} + |v_M|^{p-1} + f_k(|v_M'|) + \frac{\psi}{k} \right) (v_M - P_M v_0).$$

Letting $M \to \infty$ one can see without difficulties that $\langle \Gamma_M(v_M), v_M - P_M v_0 \rangle \to 0$. This convergence allows us to prove the following

Lemma 2.19. $v_M \to v_0$ *strongly, i.e. in the norm of* $H_0^1(-n, n)$.

Remark 2.20. The idea to consider the operators Γ_M and B_M was an inspiration from the arguments presented in [7].

Proof. The limit $\|v_M - v_0\|_{L^2(-n,n)} \to 0$ has been established before, thus we will focus our efforts demonstrating the same for $\|v'_M - v'_0\|_{L^2(-n,n)}$. Let $\Phi_M, \Phi, \Psi_M, \zeta_M \in (\mathbb{E}_0^1(-n, n))^*$ be given by

$$\Phi_M(w) = \int_{-n}^{n} A(v_M) v'_0 w' \tag{2.21}$$

$$\Phi(w) = \int_{-n}^{n} A(v_0) v'_0 w' \tag{2.22}$$

$$\Psi_M(w) = \int_{-n}^{n} A(v_M) P_M v'_0 w' \tag{2.23}$$

$$\zeta_M(w) = \int_{-n}^{n} A(v_0) P_M v'_0 w'. \tag{2.24}$$

Then, by a straightforward calculation, $|\Phi_M - \Phi| \to 0,$ $|\Psi_M - \Phi_M| \to 0$ and $|\zeta_M - \Phi| \to 0$ in $(\mathbb{E}_0^1(-n, n))^*$. Thus, $|\Psi_M - \zeta_M| \to 0$ in $(\mathbb{E}_0^1(-n, n))^*$, since $|\Psi_M - \zeta_M| \le |\Psi_M - \Phi_M| + |\Phi_M - \Phi| + |\Phi - \zeta_M|$. Writing $\Psi_M = (\Psi_M - \zeta_M) + \zeta_M$ yields that $\Psi_M \to \Phi$ in $(\mathbb{E}_0^1(-n, n))^*$. Remembering the weak convergence $v_M \rightharpoonup v_0$ one can conclude $(v_M - P_M v_0) \rightharpoonup 0$ in $\mathbb{E}_0^1(-n, n)$ because for all $f \in (\mathbb{E}_0^1(-n, n))^*$

$$|f(v_M) - f(P_M v_0)| \le |f(v_M) - f(v_0)| + \|f\| \|v_0 - P_M v_0\|_{W^{1,2}}.$$

Consequently, letting $M \to \infty$, $\Psi_M(v_M - P_M v_0) \to \Phi(0) = 0$. This means that

$$\int_{-n}^{n} A(v_M) P_M v'_0 (v'_M - P_M v'_0) \to 0. \tag{2.25}$$

Also, rewriting (2.20)

$$\int_{-n}^{n} A(v_M) v'_M (v'_M - P_M v'_0) \to 0 \quad \text{as } M \to \infty. \tag{2.26}$$

Therefore, from (2.26)–(2.25)

$$\int_{-n}^{n} A(v_M)(v'_M - P_M v'_0)^2 \to 0 \quad \text{as } M \to \infty. \tag{2.27}$$

Since $A(x) \ge \gamma > 0$ for all $x \in \mathbb{R}$ we conclude $\|v'_M - P_M v'_0\|_{L^2(-n,n)} \to 0$ as $M \to \infty$. Then $\|v'_M - v'_0\|_{L^2(-n,n)} \to 0$ as result of $\|v'_M - v'_0\|_{L^2(-n,n)} \le \|v'_M - P_M v'_0\|_{L^2(-n,n)} + \|v'_0 - P_M v'_0\|_{L^2(-n,n)}$, proving the lemma. □

We know that for every $\varphi \in V_M$

$$\int_{-n}^{n} A(v_M) v'_M \varphi' + \int_{-n}^{n} v_M \varphi \tag{2.28}$$

$$= \int_{-n}^{n} \lambda a_1 |v_M|^{q-1} \varphi + \int_{-n}^{n} |v_M|^{p-1} \varphi + \int_{-n}^{n} f_k(|v'_M|) \varphi + \int_{-n}^{n} \frac{\psi}{k} \varphi.$$

By the previous lemma, taking a subsequence if necessary, we may assume that $v'_M(x)$ converges a.e. to $v'_0(x)$ and there exists $h \in L^2(-n, n)$ such that $|v'_M(x)| \le h(x)$ a.e. Then notice that

$$\left| \int_{-n}^{n} \left(A(v_M)v'_M - A(v_0)v'_0 \right) \varphi' \right| \le \left(\int_{-n}^{n} |A(v_M)v'_M - A(v_0)v'_0|^2 \right)^{1/2} \|\varphi'\|_{L^2} \qquad (2.29)$$

and exists $Q > 0$ such that $\|v_M\|_\infty < Q$ for all $M \in \mathbb{N}$, because v_M converges to v_0 in $C^0[-n, n]$ due to the embedding $W^{1,2}(-n, n) \hookrightarrow C^0[-n, n]$. We can suppose Q big enough so that $Q > 2(r + A(0))$ and we take $\tilde{A} = \sup_{x \in [-Q,Q]} A(x)$. Since

$$|A(v_M(x))v'_M(x) - A(v_0(x))v'_0(x)| \to 0 \quad \text{a.e.} \qquad (2.30)$$

and

$$
\begin{aligned}
|A(v_M(x))v'_M(x) - A(v_0(x))v'_0(x)|^2 &\le \left(|A(v_M(x))v'_M(x)| + |A(v_0(x))v'_0(x)| \right)^2 \\
&= |A(v_M(x))|^2 |v'_M(x)|^2 \\
&\quad + 2|A(v_M(x))||A(v_0(x))||v'_M(x)||v'_0(x)| \\
&\quad + |A(v_0(x))|^2 |v'_0(x)|^2 \\
&\le \tilde{A}^2 Q^2 h^2(x) + 2\tilde{A}^2 Q^2 |v'_0(x)| h(x) \\
&\quad + \tilde{A}^2 Q^2 |v'_0(x)|^2
\end{aligned}
$$

almost everywhere, we conclude by (D.C.T) that

$$\int_{-n}^{n} A(v_M)v'_M \varphi' \to \int_{-n}^{n} A(v_0)v'_0 \varphi' \quad \text{as } M \to \infty. \qquad (2.31)$$

Also, by direct calculation, the following convergences are true

$$\int_{-n}^{n} v_M \varphi \to \int_{-n}^{n} v_0 \varphi \qquad (2.32)$$

$$\int_{-n}^{n} \lambda a_1 |v_M|^{q-1} \varphi \to \int_{-n}^{n} \lambda a_1 |v_0|^{q-1} \varphi \qquad (2.33)$$

$$\int_{-n}^{n} |v_M|^{p-1} \varphi \to \int_{-n}^{n} |v_0|^{p-1} \varphi \qquad (2.34)$$

$$\int_{-n}^{n} f_k(|v'_M|) \varphi \to \int_{-n}^{n} f_k(|v'_0|) \varphi \qquad (2.35)$$

as $M \to \infty$. Thus, for every $\varphi \in V_M$

$$\int_{-n}^{n} A(v_0)v'_0 \varphi' + \int_{-n}^{n} v_0 \varphi = \int_{-n}^{n} \lambda a_1 |v_0|^{q-1} \varphi + \int_{-n}^{n} |v_0|^{p-1} \varphi + \int_{-n}^{n} f_k(|v'_0|) \varphi + \int_{-n}^{n} \frac{\psi}{k} \varphi. \qquad (2.36)$$

Furthermore, for every $u \in \mathbb{E}_0^1(-n, n)$, it follows that

$$\int_{-n}^{n} A(v_0)v'_0 P_M u' \to \int_{-n}^{n} A(v_0)v'_0 u' \qquad (2.37)$$

$$\int_{-n}^{n} v_0 P_M u \to \int_{-n}^{n} v_0 u \qquad (2.38)$$

$$\int_{-n}^{n} \lambda a_1 |v_0|^{q-1} P_M u \to \int_{-n}^{n} \lambda a_1 |v_0|^{q-1} u \qquad (2.39)$$

$$\int_{-n}^{n} |v_0|^{p-1} P_M u \to \int_{-n}^{n} |v_0|^{p-1} u \qquad (2.40)$$

$$\int_{-n}^{n} f_k(|v'_0|) P_M u \to \int_{-n}^{n} f_k(|v'_0|) u \qquad (2.41)$$

as $M \to \infty$. Thus, for every $u \in \mathbb{E}_0^1(-n, n)$

$$\int_{-n}^{n} A(v_0)v_0'u' + \int_{-n}^{n} v_0 u = \int_{-n}^{n} \lambda a_1 |v_0|^{q-1} u + \int_{-n}^{n} |v_0|^{p-1} u + \int_{-n}^{n} f_k(|v_0'|)u + \int_{-n}^{n} \frac{\psi}{k} u. \quad (2.42)$$

So v_0 is an *E-weak solution* of (P_n^k); by Lemma 2.13 v_0 is also a weak solution. Notice that

$$\|v_0\|_{W^{1,2}} \leq r,$$

and our choice of r does not depend on n, λ or k. This finishes the proof of Proposition 2.16.   □

In what follows we will make $k \to \infty$ thus we can consider $\psi \equiv 1$, because the term $\frac{\psi}{k}$ will converge to 0 as $k \to \infty$.

Proposition 2.21. *The above weak solution v_0 satisfies:*

1. *there exist $\beta(L/\gamma)$ and $\hat{C}(L/\gamma, n)$, such that $v_0 \in C^{1,\beta}[-n, n] \cap C^2(-n, n)$ and*

$$|v_0|_{1+\beta} \leq \hat{C},$$

    where

$$L > 2\max\{Cr + \lambda a_1(n)|Cr|^{q-1} + |Cr|^{p-1} + 1, 2C_1, A(Cr), \tilde{A}\};$$

2. $v_0(t) \geq 0$.

Proof. To prove *1*, we will use [12, Theorem 1]. Let $F : [-n, n] \times [-Cr, Cr] \times \mathbb{R} \to \mathbb{R}$ be defined by $F(x, z, p) = A(z)p$, where C is the embedding constant for $W^{1,2}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, and $B(x, z, p) = z - (\lambda a_1(x)|z|^{q-1} + |z|^{p-1} + f_k(|p|) + \frac{1}{k})$ be defined in the same domain. Then, problem (P_n^k) may be rewritten as

$$\text{div}_x F(x, u(x), u'(x)) + B(x, u(x), u'(x)) = 0.$$

In order to use [12, Theorem 1] we must verify the existence of nonnegative constants l, L, M_0, m, κ with $l \leq L$ such that

$$\frac{\partial F}{\partial p}(x, z, p)\xi^2 \geq l(\kappa + |p|)^m \xi^2, \quad (2.43)$$

$$\left|\frac{\partial F}{\partial p}(x, z, p)\right| \leq L(\kappa + |p|)^m, \quad (2.44)$$

$$|F(x, z, p) - F(y, w, p)| \leq L(1 + |p|)^{m+1} \cdot |z - w|, \quad (2.45)$$

$$|B(x, z, p)| \leq L(1 + |p|)^{m+2}, \quad (2.46)$$

for all $(x, z, p) \in \{-n, n\} \times [-M_0, M_0] \times \mathbb{R}$, $w \in [-M_0, M_0]$ and $\xi \in \mathbb{R}$. Since $\frac{\partial F}{\partial p}(x, z, p) = A(z)$, inequality (2.43) follows from $A(z)\xi^2 \geq \gamma \xi^2$, that is, $l = \gamma$.

To prove the remaining inequalities take $M_0 = Cr$,

$$T > \max\{Cr + \lambda a_1(n)|Cr|^{q-1} + |Cr|^{p-1} + 1, 2C_1, A(Cr), \tilde{A}\},$$

$L = 2T, \kappa = 0$ and $m = 0$, where $\tilde{A}$ is the Lipchitz constant of A. Then:

(2.44)
$$\left|\frac{\partial F}{\partial p}(x,z,p)\right| = A(z) \le A(Cr) < L;$$

(2.45)
$$|F(x,z,p) - F(y,w,p)| = |A(z)p - A(w)p| \le \tilde{A}|p||z-w| \le L(1+|p|)|z-w|;$$

(2.46)
$$
\begin{aligned}
|B(x,z,p)| &= \left| z - \left(\lambda a_1(x)|z|^{q-1} + |z|^{p-1} + f_k(|p|) + \frac{1}{k} \right) \right| \qquad (2.47)\\
&\le Cr + \lambda a_1(n)|Cr|^{q-1}\\
&\quad + |Cr|^{p-1} + 1/k + C_1(1+|p|^{\theta-1})\\
&\le T + C_1(1 + (1+|p|)^{\theta-1})\\
&\le T + 2C_1(1+|p|)^2\\
&\le T(1 + (1+|p|)^2)\\
&\le 2T(1+|p|)^2 = L(1+|p|)^2.
\end{aligned}
$$

Therefore, by [12, Theorem 1] there exists $\beta \in (0,1)$ and a constant $\hat{C}$, independent of k, such that $v_0 \in C^{1,\beta}([-n,n])$ and

$$|v_0|_{1+\beta} \le \hat{C}. \qquad (2.48)$$

It follows from [6, p. 317, Chap. 6, Theorem 4] that $v_0 \in W^{2,2}(-n,n)$ and since v_0 is a weak solution of (P_n^k) we have

$$v_0'' = \frac{v_0 - \lambda a_1|v_0|^{q-1} - |v_0|^{p-1} - f_k(|v_0'|) - 1/k - A'(v_0)|v_0'|^2}{A(v_0)} \qquad (2.49)$$

showing that v_0'' is continuous, thus $v_0 \in C^2(-n,n)$.

To prove that $v_0(t) \ge 0$ for all $t \in (-n,n)$ we first notice that $v_0^-(t) = \max\{0, -v_0(t)\} \in H_0^1(-n,n)$. Using $v_0^-(t)$ as a test function in the definition of weak solution provides

$$
-\int_{-n}^{n} A(v_0)|v_0^-|^2 - \int_{-n}^{n} |v_0^-|^2 = \int_{-n}^{n} \lambda a_1|v_0|^{q-1}v_0^- + \int_{-n}^{n} |v_0|^{p-1}v_0^-
$$
$$
+ \int_{-n}^{n} f_k(|v_0'|)v_0^- + \int_{-n}^{n} \frac{1}{k}v_0^-. \qquad (2.50)
$$

Then $-\gamma\|v_0^-\|_{W^{1,2}}^2 \ge 0$, thus $\|v_0^-\|_{W^{1,2}} = 0$ implying $v_0^- \equiv 0$ a.e. Since v_0 is continuous, $v_0(t) \ge 0$ for all $t \in (-n,n)$. This finishes the proof of Proposition 2.21. $\qquad \square$

Thus, by Proposition 2.16 and Proposition 2.21 we obtain the proof of Theorem 2.8.

2.2 Constructing a solution to problem (P_n)

Let v_k be the (strong) solution of problem (P_n^k), obtained just above, with k varying. By the previous constructions, we have that $\|v_k\|_{W^{1,2}(-n,n)} \le r$ independent of k, as noticed in Remark 2.18. Then there exists $u_n \in H_0^1(-n,n)$, $\|u_n\|_{W^{1,2}(-n,n)} \le r$, so that v_k has a subsequence

converging weakly in $H_0^1(-n, n)$ to u_n. From now on v_k will denote this subsequence. Since the function

$$H_0^1(-n, n) \ni w \mapsto \int_{-n}^{n} A(u_n)u_n'w'$$

belongs to $(H_0^1(-n, n))^*$, we have, by the weak convergence, that

$$\int_{-n}^{n} A(u_n)u_n'(v_k - u_n)' \to 0 \quad \text{as } k \to \infty.$$

This convergence will be useful in our next task: to prove that $v_k \to u_n$ strongly in $H_0^1(-n, n)$.

Lemma 2.22. *The following convergence is true*

$$\int_{-n}^{n} A(u_n)v_k'(v_k - u_n)' \to 0 \quad \text{as } k \to \infty.$$

Proof. We might write

$$\int_{-n}^{n} A(u_n)v_k'(v_k - u_n)' = \int_{-n}^{n} [A(u_n) - A(v_k) + A(v_k)]v_k'(v_k' - u_n')$$

$$= \underbrace{\int_{-n}^{n} [A(u_n) - A(v_k)]v_k'(v_k' - u_n')}_{I_1} + \underbrace{\int_{-n}^{n} A(v_k)v_k'(v_k' - u_n')}_{I_2}$$

and analyze I_1 and I_2 separately.

Analysis of I_2   By the weak formulation of (P_n^k)

$$\int_{-n}^{n} A(v_k)v_k'(v_k' - u_n')$$

$$= \underbrace{\int_{-n}^{n} -v_k(v_k - u_n) + \lambda a_1|v_k|^{q-1}(v_k - u_n) + |v_k|^{p-1}(v_k - u_n) + \frac{(v_k - u_n)}{k}}_{E_1}$$

$$+ \underbrace{\int_{-n}^{n} f_k(|v_k'|)(v_k - u_n)}_{E_2}.$$

Since we have compact injection of $H_0^1(-n, n)$ onto $L^2(-n, n)$, the weak convergence of v_k to u_n in $H_0^1(-n, n)$ implies $\|v_k - u_n\|_{L^2} \to 0$. Thus, it is straightforward to see that (E_1) converges to 0 as $k \to \infty$. Remains to verify what happens with (E_2) in the limit. We have that

$$\int_{-n}^{n} f_k(|v_k'|)(v_k - u_n) \le \int_{-n}^{n} C_1(|v_k'|^{\theta-1} + |v_k'|)|v_k - u_n|$$

by Lemma 2.4. Using Proposition 2.21, item *1*, that is, the estimation $|v_k|_{1,\beta} \le \hat{C}$ which is independent of k, we have that

$$|v_k'|^{\theta-1} + |v_k'| \le (\hat{C})^{\theta-1} + \hat{C}.$$

Then,

$$\int_{-n}^{n} f_k(|v_k'|)(v_k - u_n) \le C_1[(\hat{C})^{\theta-1} + \hat{C}] \int_{-n}^{n} |v_k - u_n|$$

$$\le \underbrace{(2n)^{1/2}C_1[(\hat{C})^{\theta-1} + \hat{C}]\|v_k - u_n\|_{L^2}}_{\to 0 \text{ as } k \to \infty}.$$

Thus, $\lim_{k \to \infty} I_2(k) = 0$.

Analysis of I_1   We also have that $\lim_{k\to\infty} I_1(k) = 0$, as one can see through

$$
\left| \int_{-n}^{n} [A(u_n) - A(v_k)] v_k'(v_k' - u_n') \right| \leq \int_{-n}^{n} |A(u_n) - A(v_k)||v_k'||v_k' - u_n'|
$$
$$
\leq \hat{C} \int_{-n}^{n} |A(u_n) - A(v_k)||v_k' - u_n'|
$$
$$
\leq \hat{C}\tilde{A} \int_{-n}^{n} |u_n - v_k||v_k' - u_n'|
$$
$$
\leq \hat{C}\tilde{A} \|u_n - v_k\|_{L^2} \|v_k' - u_n'\|_{L^2}
$$
$$
\leq \hat{C}\tilde{A} 2r \|u_n - v_k\|_{L^2}.
$$

Thus,

$$
\int_{-n}^{n} A(u_n) u_n'(v_k - u_n)' \to 0 \quad \text{as } k \to \infty \tag{2.51}
$$

$$
\int_{-n}^{n} A(u_n) v_k'(v_k - u_n)' \to 0 \quad \text{as } k \to \infty. \tag{2.52}
$$

Subtracting (2.52) from (2.51) we have

$$
\int_{-n}^{n} A(u_n)(v_k' - u_n')^2 \to 0 \quad \text{as } k \to \infty \tag{2.53}
$$

implying that $v_k' \to u_n'$ in $L^2(-n,n)$, since γ is a uniform lower-bound for A. Hence $v_k \to u_n$ in $H_0^1(-n,n)$.   □

Remark 2.23. Since $v_k \to u_n$ in $H_0^1(-n,n)$ we conclude that u_n is also an even function; due to the embedding $W^{1,2}(-n,n) \hookrightarrow C[-n,n]$.

Proposition 2.24. *The function u_n satisfies:*

    1. u_n is strictly positive in $(-n,n)$;

    2. u_n is a solution to (P_n).

Proof.

Item 1.   Let $\tilde{a} := \inf_{x \in [-n,n]} a_1(x)$. We will divide our argument into two cases:

Remark 2.25. This division of cases is a geometric argument that we borrowed from [3].

Case 1. There exists a subsequence $(v_{k_i})_{i \in \mathbb{N}}$ of (v_k) such that $v_{k_i}' \geq 0$ in $(-n,0)$ for all i. Consider the problem

$$
\begin{cases}
-(A(u)u')' + u = \lambda \tilde{a} |u|^{q-1} & \text{in } (-n,n) \\
u > 0 & \text{in } (-n,n) \\
u(-n) = u(n) = 0.
\end{cases} \tag{2.54}
$$

Since $v_{k_i}' \geq 0$ in $(-n,0)$ we get that $v_{k_i} > 0$ in $(-n,0)$, because, due to Proposition 2.21, v_{k_i} is an even solution of (P_n^k), thus *it can not be identically zero in an interval* and $v_{k_i} \geq 0$; i.e., supposing the existence of $x_i \in (-n,0)$ such that $v_{k_i}(x_i) = 0$ implies the existence of $y_i \in (-n,0)$ such

that $v'_{k_i}(y_i) < 0$, which would be a contradiction. Thus, we see that v_{k_i} is a sup-solution for equation (2.54). Let ϕ_1 be an even and positive eigenfunction for the eigenvalue problem

$$\begin{cases} -u'' = \lambda_1 u & \text{in } (-n, n) \\ u(-n) = u(n) = 0 \end{cases} \tag{2.55}$$

where $\lambda_1 = \frac{\pi^2}{(2n)^2}$. Thus, choosing τ such that

$$\frac{\tau^{2-q}(1 + \gamma\lambda_1)}{\lambda\tilde{a}} \leq \phi_1^{q-2}$$

we have that $\tau\phi_1$ is as sub-solution of (2.54). By Theorem 2.11

$$v_{k_i}(t) \geq \tau\phi_1(t) \qquad \forall t \in (-n, n),$$

therefore, in the limit,

$$u_n(t) \geq \tau\phi_1(t) > 0 \qquad \forall t \in (-n, n).$$

Case 2. There exists a subsequence $(v_{k_i})_{i \in \mathbb{N}}$ of (v_k) and there exists a sequence $(z_i)_{i \in \mathbb{N}} \subset (-n, 0)$ such that $v'_{k_i}(z_i) < 0$.

Remark 2.26. Although the geometric argument is an inspiration from [3], we still need to adjust it to our necessity. Lemma 2.27 is one such adjustment.

Lemma 2.27. *Let* $x \in (-n, n)$ *such that* $v''_{k_i}(x) \geq 0$, *then* $v_{k_i}(x) > (\lambda\tilde{a})^{\frac{1}{2-q}}$.

Proof. Since v_{k_i} is a solution for the problem (P_n^k), with $k = k_i$, for all t

$$- A'(v_{k_i}(t))|v'_{k_i}(t)|^2 - A(v_{k_i}(t))v''_{k_i}(t) + v_{k_i}(t)$$
$$= \lambda a_1(t)|v_{k_i}(t)|^{q-1} + |v_{k_i}(t)|^{p-1} + f_{k_i}(|v'_{k_i}(t)|) + \frac{1}{k_i}$$
$$\geq \lambda a_1(t)|v_{k_i}(t)|^{q-1} + |v_{k_i}(t)|^{p-1} + \frac{1}{k_i}.$$

Here we used that $\text{sign}(f_{k_i}(s)) = \text{sign}(s)$, thus $f_{k_i}(|v'_{k_i}|) \geq 0$. Using that $|v_{k_i}|^{p-1} \geq 0$ and $a_1(t) \geq \tilde{a} > 0$ we obtain:

$$- A'(v_{k_i}(t))|v'_{k_i}(t)|^2 - A(v_{k_i}(t))v''_{k_i}(t) + v_{k_i}(t) \geq \lambda\tilde{a}|v_{k_i}(t)|^{q-1} + \frac{1}{k_i}. \tag{2.56}$$

Then, with $t = x$,

$$- A(v_{k_i}(x))v''_{k_i}(x) \geq \lambda\tilde{a}|v_{k_i}(x)|^{q-1} - v_{k_i}(x) + A'(v_{k_i}(x))|v'_{k_i}(x)|^2 + \frac{1}{k_i}$$
$$> \lambda\tilde{a}|v_{k_i}(x)|^{q-1} - v_{k_i}(x).$$

Where, in the last inequality, we used that A is non-decreasing, $|v'_{k_i}| \geq 0$ and $1/k_i > 0$. Notice that the resulting estimation is strict because $1/k_i > 0$. By hypotheses $v''_{k_i}(x) \geq 0$, then $-A(v_{k_i}(x))v''_{k_i}(x) \leq 0$. Using the previous inequality,

$$v_{k_i}(x) > \lambda\tilde{a}|v_{k_i}(x)|^{q-1},$$

thus $v_{k_i}(x) \neq 0$ and $v_{k_i}(x) > (\lambda\tilde{a})^{\frac{1}{2-q}}$. $\square$

Now, in order to use this lemma, we ought to find a $x_i \in (-n, n)$ such that $v_{k_i}''(x_i) \geq 0$. Using the fact that v_{k_i} is even and $v_{k_i}'(z_i) < 0$ we have that $v_{k_i}'(-z_i) > 0$. Let $x_i = \min_{x \in [z_i, -z_i]} v_{k_i}(x)$ and notice that $x_i \neq z_i$ and $x_i \neq -z_i$; indeed, there exist $\delta > 0$ such that, if $x \in (z_i, z_i + \delta) \cup (-z_i - \delta, -z_i)$, then $v_{k_i}(x) < v_{k_i}(z_i) = v_{k_i}(-z_i)$. Hence $x_i \in (z_i, -z_i)$ and $v_{k_i}'(x_i) = 0$; therefore $v_{k_i}''(x_i)$ must be *greater or equal than 0*, because if $v_{k_i}''(x_i) < 0$ there would be $\xi > 0$ such that, for $x \in (x_i, x_i + \xi) \subset (z_i, -z_i)$, $v_{k_i}'(x) < 0$; and for this neighborhood $v_{k_i}(x) < v_{k_i}(x_i)$ – a contradiction with the minimality of x_i. Thus, $v_{k_i}''(x_i) \geq 0$. By Lemma 2.27, we obtain for all i

$$v_{k_i}(x_i) > (\lambda \tilde{a})^{\frac{1}{2-q}}.$$

From the compactness of $[-n, n]$, there exist $x_0 \in [-n, n]$ such that $x_i \to x_0$ when $i \to \infty$; taking a subsequence if necessary. Then

$$u_n(x_0) = \lim_{i \to \infty} v_{k_i}(x_i) \geq (\lambda \tilde{a})^{\frac{1}{2-q}} > 0.$$

Finally, we will conclude *item 1* showing that, also in this case, u_n is strictly positive in $(-n, n)$.

Suppose by contradiction that there exists $y \in (-n, n)$ such that $u_n(y) = 0$. Let $(d, s) \subset (-n, n)$ be the biggest interval containing y satisfying the property: if $x \in (d, s)$ then $u_n(x) < \frac{(\lambda \tilde{a})^{\frac{1}{2-q}}}{2}$. Since $u_n(x_0) = u_n(-x_0) > \frac{(\lambda \tilde{a})^{\frac{1}{2-q}}}{2}$ we have that $d \neq -n$ or $s \neq n$. Thus we can suppose without loss of generality that $d > -n$, because on the contrary, we would apply our following arguments using the interval (d', s'), where $d' = -s$ and $s' = -d$, and the point $y' = -y$.

By the maximality of (d, s) and the continuity of u_n we have that $u_n(d) = \frac{(\lambda \tilde{a})^{\frac{1}{2-q}}}{2}$. Since $u_n(x) < \frac{(\lambda \tilde{a})^{\frac{1}{2-q}}}{2}$ for all $x \in (d, s)$ and v_{k_i} converges uniformly to u_n, there exists $i_1 \in \mathbb{N}$ such that, for $i > i_1$ and $x \in (d, s)$,

$$v_{k_i}(x) < (\lambda \tilde{a})^{\frac{1}{2-q}}.$$

Then, by Lemma 2.27 $v_{k_i}''(x) < 0$. Using that $u_n(d) = \frac{(\lambda \tilde{a})^{\frac{1}{2-q}}}{2}$, there exist $i_2 \in \mathbb{N}$ such that $i > i_2$ implies

$$v_{k_i}(d) > \frac{(\lambda \tilde{a})^{\frac{1}{2-q}}}{4}.$$

Let $i_0 > \max\{i_1, i_2\}$ and define $f : (d, s) \to \mathbb{R}$ by

$$f(x) = \frac{(\lambda \tilde{a})^{\frac{1}{2-q}}}{4} \cdot \frac{x - s}{d - s}.$$

We have that $f(d) = \frac{(\lambda \tilde{a})^{\frac{1}{2-q}}}{4}$ and $f(s) = 0$. Let $U_i(x) = v_{k_i}(x) - f(x)$ for $i \geq i_0$, then

$$\begin{cases} U_i''(x) < 0, & \text{for } x \in (d, s) \\ U_i(d) > 0, U_i(s) = v_{k_i}(s) \geq 0. \end{cases} \tag{2.57}$$

By the maximum principle, the minimum of U_i is reached on the border of the interval (d, s), implying that $U_i(x) > 0$ for all $x \in (d, s)$, that is, $v_{k_i}(x) > f(x)$ for all $x \in (d, s)$ and $i \geq i_0$. Thus, taking $x = y$ and making $i \to \infty$, we obtain

$$u_n(y) \geq f(y) > 0,$$

which is a contradiction.

Item 2. Since the estimation from Proposition 2.21 item *1* holds, that is,

$$|v_k|_{1+\beta} \leq \hat{C}$$

for all $k \in \mathbb{N}$; and, for all $1 < \alpha < \beta$, we have compact embedding $C^{1,\beta}[-n, n] \hookrightarrow C^{1,\alpha}[-n, n]$, we may assume, taking a subsequence, if necessary, that there exist $\widetilde{u_n} \in C^{1,\alpha}[-n, n]$ such that $v_k \to \widetilde{u_n}$ in $C^{1,\alpha}[-n, n]$ as $k \to \infty$. Thus,

$$v_k \to u_n \quad \text{in } C^0[-n, n] \text{ as } k \to \infty$$
$$v_k \to \widetilde{u_n} \quad \text{in } C^{1,\alpha}[-n, n] \text{ as } k \to \infty.$$

Then for all $x \in [-n, n]$ we have

$$u_n(x) = \lim_{k \to \infty} v_k(x) = \widetilde{u_n}(x),$$

i.e., $u_n = \widetilde{u_n} \in C^{1,\alpha}[-n, n]$.

Considering the definition of weak solution, for all $\varphi \in H_0^1(-n, n)$

$$\int_{-n}^n A(v_k)v_k'\varphi' + \int_{-n}^n v_k\varphi = \int_{-n}^n (\lambda a_1|v_k|^{q-1} + |v_k|^{p-1})\varphi + \int_{-n}^n f_k(|v_k'|)\varphi + \int_{-n}^n \frac{\varphi}{k}.$$

By (D.C.T), it is straightforward to see that the following convergences are true:

$$\int_{-n}^n A(v_k)v_k'\varphi' \to \int_{-n}^n A(u_n)u_n'\varphi',$$
$$\int_{-n}^n v_k\varphi \to \int_{-n}^n u_n\varphi,$$
$$\int_{-n}^n (\lambda a_1|v_k|^{q-1} + |v_k|^{p-1})\varphi \to \int_{-n}^n (\lambda a_1|u_n|^{q-1} + |u_n|^{p-1})\varphi,$$
$$\int_{-n}^n \frac{\varphi}{k} \to 0,$$

as $k \to \infty$. Let us examine the remaining integral. First notice that $f_k(|v_k'|)$ converges uniformly to $g(|u_n'|)$; indeed, g is uniformly continuous in compacts, then for the compact $[-\hat{C}, \hat{C}]$ given $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - y| < \delta$, then

$$|g(x) - g(y)| < \frac{\epsilon}{2}. \tag{2.58}$$

Also there exists $k_0 \in \mathbb{N}$ such that $k > k_0$ implies

$$||v_k'(x)| - |u_n'(x)|| < \delta \qquad \forall x \in [-n, n]. \tag{2.59}$$

thus for $k > k_0$

$$|g(|v_k'(x)|) - g(|u_n'(x)|)| < \frac{\epsilon}{2} \qquad \forall x \in [-n, n]. \tag{2.60}$$

In the perspective of Theorem 2.2, f_k converges to g uniformly in bounded sets; since $\|u_n'\|_\infty \leq \hat{C}$, for $x \in [-\hat{C}, \hat{C}]$ there exist $k_1 \in \mathbb{N}$ such that $k > k_1$ implies

$$|f_k(x) - g(x)| < \frac{\epsilon}{2} \qquad \forall x \in [-\hat{C}, \hat{C}] \tag{2.61}$$

and with all these ingredients we obtain the uniform convergence, because for $k > \max\{k_0, k_1\}$

$$|f_k(|v_k'(x)|) - g(|u_n'(x)|)| \leq |f_k(|v_k'(x)|) - g(|v_k'(x)|)| + |g(|v_k'(x)|) - g(|u_n'(x)|)|$$
$$< \epsilon \qquad \forall x \in [-n, n].$$

Thus, by (D.C.T)

$$\int_{-n}^{n} f_k(|v_k'|)\varphi \to \int_{-n}^{n} g(|u_n'|)\varphi$$

as $k \to \infty$. All these convergences together show that u_n is a weak solution for the problem (P_n).

From [6, Pag. 317, Chap. 6, Theorem 4] we conclude that $u_n \in W^{2,2}(-n,n)$; and similarly, to the argument showed in equation (2.49) we obtain that $u_n'' \in C^0(-n,n)$. Thus, u_n is a strong solution to the problem (P_n). $\qquad\square$

3 Asymptotic solution to problem (P_n)

In this section we will briefly study the solution's behavior of problem (P_n) when $\lambda \to 0$ or $\lambda \to \lambda^*$. Our main result is the following:

Theorem 3.1. *Denoting by u_λ the solution of (P_n):*

1. *as $\lambda \to 0$, we get that $\|u_\lambda\|_{W^{1,2}} \to 0$;*

2. *one can take $\lambda = \lambda^*$ and still obtain a solution to problem (P_n).*

Proof. Let v_k be the strong solution of problem (P_n^k), with $\psi \equiv 1$. In the previous section, we proved, among other things, that $v_k \to u_\lambda$ as $k \to \infty$, (taking a subsequence, if necessary). Using v_k as test function in the Definition 2.12, we get:

$$\int_{-n}^{n} A(v_k)|v_k'|^2 + \int_{-n}^{n} |v_k|^2 = \int_{-n}^{n} \lambda a_1 |v_k|^{q-1}v_k + \int_{-n}^{n} |v_k|^{p-1}v_k + \int_{-n}^{n} f_k(|v_k'|)v_k + \int_{-n}^{n} \frac{v_k}{k}.$$

Thus, following the estimations done in Proposition 2.16, we can estimate these integrals to obtain

$$\gamma\|v_k\|_{W^{1,2}}^2 \le \lambda C_2 \|v_k\|_{W^{1,2}}^q + C^{p-2}\|v_k\|_{W^{1,2}}^p + C_1 \max\{C^{\theta-2},C\}\|v_k\|_{W^{1,2}}^\theta$$
$$+ \left(\frac{C_1(2n)^{1/2}}{k} + \frac{(2n)^{1/2}}{k}\right)\|v_k\|_{W^{1,2}}.$$

Rearranging we obtain

$$\|v_k\|_{W^{1,2}}^2 \left(\gamma - C^{p-2}\|v_k\|_{W^{1,2}}^{p-2} - C_1 \max\{C^{\theta-2},C\}\|v_k\|_{W^{1,2}}^{\theta-2}\right)$$
$$\le \lambda C_2 \|v_k\|_{W^{1,2}}^q + \left(\frac{C_1(2n)^{1/2}}{k} + \frac{(2n)^{1/2}}{k}\right)\|v_k\|_{W^{1,2}}. \quad (3.1)$$

Notice that $\|v_k\|_{W^{1,2}} \le r$ independent of k and

$$r < \min\left\{\left(\frac{\gamma}{4C^{p-2}}\right)^{1/(p-2)}, \left(\frac{\gamma}{4C_1 \max\{C^{\theta-2},C\}}\right)^{1/(\theta-2)}\right\}.$$

Thus,

$$\|v_k\|_{W^{1,2}}^2 \le \frac{2}{\gamma}\left(\lambda C_2 r^q + \left(\frac{C_1(2n)^{1/2}}{k} + \frac{(2n)^{1/2}}{k}\right)r\right). \quad (3.2)$$

Making $k \to \infty$ we end up with

$$\|u_\lambda\|_{W^{1,2}}^2 \leq \lambda \cdot \left(\frac{2C_2 r^q}{\gamma} \right). \tag{3.3}$$

Then, as $\lambda \to 0$ we see that $\|u_\lambda\|_{W^{1,2}}^2 \to 0$. This proves item *1*.

　　To prove item *2*, one can take a sequence (λ_n) in $(0, \lambda^*)$ such that $\lambda_n \to \lambda^*$. Noticing that $\|u_{\lambda_n}\|_{W^{1,2}} \leq r$ independent of λ_n, one can obtain a candidate $u_{\lambda^*} \in H_0^1(-n,n)$ such that u_{λ_n} converges weakly to u_{λ^*} in $H_0^1(-n,n)$. Then, following similar argumentation as exposed in Section 2, one can prove that u_{λ^*} is an even, positive solution to (P_n) with $\lambda = \lambda^*$. □

Remark 3.2. Let

$$\tilde{r} = \min \left\{ r, \lambda^{1/2} \left(\frac{2C_2 r^q}{\gamma} \right)^{1/2} \right\}.$$

Then, by inequality (3.3) and the estimations from Proposition 2.16, we get that $\|u_\lambda\|_{W^{1,2}} \leq \tilde{r}$. Notice that $\tilde{r} \to 0$ as $\lambda \to 0$.

Remark 3.3. For now, on we will resume the previous notation, that is, we will call the strong solution of problem (P_n) by u_n. This will be useful in the next section.

4　Solution in $\mathbb{R}$

To obtain the homoclinic solution, we can proceed similarly as in [3]. However, we present a slightly different approach.

　　To obtain a solution defined in $\mathbb{R}$ we will utilize a subsequence construction wrapping it up with the arguments presented in Section 2. The reader should notice that the notation "u_n" used for the solution of (P_n), previously obtained, in $(-n,n)$ is not accidental: extending u_n by zero out of $[-n,n]$ we obtain a sequence (u_n) in $H^1(\mathbb{R})$. Throughout this section, we will use u_n to denote the solution "u_n" and its extension. Also, one can see that $\|u_n\|_{H^1(\mathbb{R})} = \|u_n\|_{W^{1,2}(-n,n)} \leq \tilde{r}$ for all n.

　　Let $K_1 = [-1,1]$; then for all $n \geq 1$ we have that $u_n^1 := u_n|_{K_1}$ is well defined and $u_n^1 \in H^1(-1,1)$. By the limitation $\|u_n^1\|_{W^{1,2}(-1,1)} \leq \tilde{r}$ there exists a subsequence $u_{n,1}$ and $s_1 \in H^1(-1,1)$ such that $u_{n,1} \rightharpoonup s_1$ in $H^1(-1,1)$. Notice that the compact injection $H^1(-1,1) \hookrightarrow C^0[-1,1]$ implies that, passing to a subsequence, $u_{n,1} \to s_1$ in $C^0[-1,1]$.

　　Let $K_2 = [-2,2]$. Taking n in the set of indices of the subsequence $u_{n,1}$, for $n \geq 2$ we have that $u_n^2 := u_n|_{K_2}$ is well defined and $\|u_n^2\|_{W^{1,2}(-2,2)} \leq \tilde{r}$. Thus, there exists a subsequence $u_{n,2}$ of u_n^2 and $s_2 \in H^1(-2,2)$ such that $u_{n,2} \rightharpoonup s_2$ in $H^1(-2,2)$.

　　Repeating the same argument, by induction we get that for all $j \in \mathbb{N}$ there exists a subsequence $u_{n,j}$ of $u_{n,j-1}$ and $s_j \in H^1(-j,j)$ such that $u_{n,j} \rightharpoonup s_j$ in $H^1(-j,j)$. Notice that $\|s_j\|_{H^1(-j,j)} \leq \tilde{r}$ for all $j \in \mathbb{N}$.

Remark 4.1. $u_{n,j}$ is the subsequence of u_n that converges weakly in $H^1(-j,j)$ to s_j. As mentioned, this weak convergence implies convergence in $C^0[-j,j]$ which gives us, in particular, that s_j is an even function, since $u_{n,j}$ is even for all $n \in \mathbb{N}$.

Lemma 4.2. $s_j|_{[1-j,j-1]} = s_{j-1}$

Proof. Given $x \in [1-j, j-1]$ we have that

$$s_{j-1}(x) = \lim_{n \to \infty} u_{n,j}(x) = s_j(x),$$

because $u_{n,j}$ is a subsequence of $u_{n,j-1}$.   □

Fix $j \in \mathbb{N}$; from here and forward we will focus our attention on proving that, in fact, s_j *is smooth and positive.*

Define $W : [-n, n] \times [-Cr, Cr] \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$W(x, z, p) = z - (\lambda a_1(x)|z|^{q-1} + |z|^{p-1} + g(|p|))$$

one can see that the estimation (2.47) also holds, with W taking part as B (remember that C is the constant for the embedding $W^{1,2}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$). Then, for any $n \geq j$, by Theorem 1 from [12] there exist $\hat{C}(j) > 0$ and $0 < \beta \leq 1$ such that

$$|u_n|_{1+\beta} \leq \hat{C}(j) \quad \text{in } C^{1,\beta}[-j, j].$$

Taking $0 < \alpha < \beta \leq 1$ we get (see the argumentation on *item 2* Proposition 2.24)

$$u_{n,j} \to s_j \quad \text{in } C^{1,\alpha}[-j, j].$$

Let $\tilde{a}_j := \inf_{x \in [-j,j]} a_1(x)$. We will use the arguments presented on *Item 1* from Proposition 2.24 to prove that s_j *is strictly positive on the interval* $[-j, j]$.

Case 1. There exists a subsequence $(u_{n_i,j})_{i \in \mathbb{N}}$ of $(u_{n,j})$ such that $u'_{n_i,j} \geq 0$ in $(-j, 0)$ for all i.

The analysis of this case follows exactly the same parameters of **Case 1** from *Item 1*, Proposition 2.24. The main difference is the change of $\tilde{a}$ to $\tilde{a}_j$.

Case 2. For all subsequence of $(u_{n,j})$ there exists a sub-subsequence $(u_{n_i,j})_{i \in \mathbb{N}}$ and exists a sequence $(z_i)_{i \in \mathbb{N}} \subset (-j, 0)$ such that $u'_{n_i,j}(z_i) < 0$.

For this case we can reformulate Lemma 2.27 as follows: If $x \in (-j, j)$ and $u''_{n_i,j}(x) \geq 0$, then $u_{n_i,j}(x) > (\lambda \tilde{a}_j)^{\frac{1}{2-q}}$. This is true because we already know that $u_{n_i,j}$ is strictly positive in $(-j, j)$, then the estimation

$$-A'(u_{n_i,j}(t))|u'_{n_i,j}(t)|^2 - A(u_{n_i,j}(t))u''_{n_i,j}(t) + u_{n_i,j}(t) > \lambda \tilde{a}_j |u_{n_i,j}(t)|^{q-1} \tag{4.1}$$

is immediately established. The remaining argumentation is similar.

Thus, we conclude that $s_j > 0$, as in Proposition 2.24. At last, let $\varphi \in C_0^\infty(-j, j)$. Then

$$\int_{-j}^{j} A(u_{n,j})u'_{n,j}\varphi' + \int_{-j}^{j} u_{n,j}\varphi = \int_{-j}^{j} (\lambda a_1 |u_{n,j}|^{q-1} + |u_{n,j}|^{p-1})\varphi + \int_{-j}^{j} g(|u'_{n,j}|)\varphi.$$

When $n \to \infty$ we get

$$\int_{-j}^{j} A(s_j)s'_j\varphi' + \int_{-j}^{j} s_j\varphi = \int_{-j}^{j} (\lambda a_1 |s_j|^{q-1} + |v|^{p-1})\varphi + \int_{-j}^{j} g(|s'_j|)\varphi.$$

Since $\varphi \in C_0^\infty(-j, j)$ is arbitrary, we conclude that s_j is a weak solution for the problem

$$-(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + g(|u'(t)|) \tag{4.2}$$

in $(-j, j)$; by [6, sect. 6.3, Theorem 1] we have that $s_j \in H^2_{\text{loc}}(-j, j)$, thus, using the same arguments as in (2.49), $s_j \in C^2(-j, j)$.

Now we will construct our candidate solution of problem (1.1). Define $w_n = u_{n,n}$, that is, w_n is the diagonal sequence. Notice that, for $n \geq j$, w_n is a subsequence of $u_{n,j}$; thus $w_n|_{[-j,j]} \to s_j$ in $C^{1,\alpha}[-j, j]$ for all $j \in \mathbb{N}$. Let $v(x)$ be defined by

$$v(x) = \lim_{n \to \infty} w_n(x).$$

Then, $v(x) = s_j(x)$ for $x \in [-j, j]$. Since $\mathbb{R} = \bigcup_{j \in \mathbb{N}}[-j, j]$, by Lemma 4.2, v is well defined in $\mathbb{R}$. Using the properties of s_j obtained just above, and the fact that $v|_{[-j,j]} = s_j$ for all $j \in \mathbb{N}$, we conclude that $v \in C^2(\mathbb{R})$, $\|v\|_{H^1(\mathbb{R})} \leq \tilde{r}$ and v is a positive, even solution of problem (1.1). From [4, p. 214, Corol. 8.9] we get the homoclinic condition.

4.1 Asymptotic solution

Throughout our previous argumentation, we fixed $\lambda \in (0, \lambda^*]$. Denote by v_λ the strong solution – obtained above – to the problem (1.1). We will analyze the behavior of v_λ as $\lambda \to 0$.

Proposition 4.3. *As $\lambda \to 0$, $v_\lambda \to 0$ in $C^0(\mathbb{R})$.*

Proof. By [4, Theorem 8.8.], we obtain

$$\|v_\lambda\|_{L^\infty(\mathbb{R})} \leq C\|v_\lambda\|_{H^1(\mathbb{R})} \leq C\tilde{r}. \tag{4.3}$$

But remember that

$$\tilde{r} = \min\left\{ r, \lambda^{1/2}\left(\frac{2C_2 r^q}{\gamma} \right)^{1/2} \right\}.$$

Then, as $\lambda \to 0$, we obtain that $\|v_\lambda\|_{L^\infty(\mathbb{R})} \to 0$, completing the proof of Theorem 1.1 . $\square$

4.2 Proof of Proposition 1.2

We proceed to prove Proposition 1.2 that says that there is no solution of (1.1) for λ large.

Proof. Suppose on the contrary that $\lambda^* = \infty$. In this way there is a sequence $\lambda_n \to \infty$ and corresponding solutions $v_{\lambda_n} > 0$ in $\mathbb{R}$ given by Theorem 1.1.

Fix $R > 0$ and define $\mathcal{P}(t, s) = \lambda a_1(t)s^{q-1} + s^{p-1}$ and $\tilde{a}_R = \inf_{(-R,R)} a_1(t)$. Define also

$$\Lambda = \lambda\, \tilde{a}_R.$$

We claim that there is a constant $C_\Lambda > 0$ such that

$$\mathcal{P}(t, s) \geq \Lambda s^{q-1} + s^{p-1} \geq C_\Lambda s \quad \text{for } s > 0, \ t \in (-R, R).$$

Consider the function $\mathcal{Q}(s) = (\Lambda s^{q-1} + s^{p-1})s^{-1}$. Then $\mathcal{Q}(s) \to \infty$ as $s \to 0^+$ and as $s \to \infty$. The minimum value of $\mathcal{Q}$ is achieved at the unique point

$$m = \left(\Lambda\, \frac{2-q}{p-2} \right)^{\frac{1}{p-q}}.$$

Thus $C_\Lambda = \mathcal{Q}(m)$.

Let $\sigma_1 > 0$ and $\varphi_1 > 0$, respectively, the first eigenvalue and the first eigenfunction satisfying

$$\begin{cases} -\varphi_1'' = \sigma_1 \varphi_1 & \text{in } (-R, R) \\ \varphi_1(-R) = \varphi_1(R) = 0. \end{cases}$$

Since C_Λ increases as λ_n increases, there is λ_0 such that the corresponding constant satisfies $C_{\Lambda_0} \geq A(C\tilde{r})\sigma_1 + A(C\tilde{r})\delta + 1$, for all $\delta \in (0, 1)$. Hence, by (H_2)–(H_3) and (4.3), the solution $v_{\lambda_0} > 0$ in $\mathbb{R}$ of (1.1) associated to λ_0 satisfies

$$\begin{cases} -v_{\lambda_0}'' \geq \frac{(C_{\Lambda_0}-1)}{A(v_{\lambda_0})} v_{\lambda_0} \geq \frac{(C_{\Lambda_0}-1)}{A(C\tilde{r})} v_{\lambda_0} \geq (\sigma_1 + \delta) v_{\lambda_0} & \text{in } (-R, R) \\ v_{\lambda_0}(-R), v_{\lambda_0}(R) \geq 0. \end{cases}$$

Otherwise, taking $\varepsilon > 0$ small enough we obtain $\varepsilon\varphi_1 < v_0$ in $(-R, R)$ and

$$\begin{cases} -(\varepsilon\varphi_1)'' = (\varepsilon\sigma_1)\varphi_1 \leq (\sigma_1 + \delta)\varphi_1 & \text{in } (-R, R) \\ \varphi_1(-R) = \varphi_1(R) = 0. \end{cases}$$

By the method of subsolution and supersolution, there is a solution $\varepsilon\varphi_1 < \omega < v_0$ in $B_R(0)$ of

$$\begin{cases} -\omega'' = (\sigma_1 + \delta)\omega & \text{in } (-R, R) \\ \omega(-R) = \omega(R) = 0. \end{cases}$$

Hence there is a contradiction to the fact that σ_1 is isolated. Therefore, $\lambda^* < \infty$, indeed. $\qquad \square$

Acknowledgements

References

[1] B. Abdellaoui, A. Dall'Aglio, I. Peral, Some remarks on elliptic problems with critical growth in the gradient, *J. Differential Equations* **222**(2006), No. 1, 21–62. https://doi.org/10.1016/j.jde.2005.02.009; MR2200746

[2] B. Abdellaoui, I. Peral, A. Primo, Elliptic problems with a Hardy potential and critical growth in the gradient: non-resonance and blow-up results, *J. Differential Equations* **239**(2007), No. 2, 386–416. https://doi.org/10.1016/j.jde.2007.05.010; MR2344278

[3] C. O. Alves, P. C. Carrião, L. F. Faria, Existence of homoclinic solutions for a class of second order ordinary differential equations, *Nonlinear Anal. Real World Appl.* **12**(2011), No. 4, 2416–2428. https://doi.org/10.1016/j.nonrwa.2011.02.018; MR2801030

[4] H. Brézis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer, New York London, 2011. https://doi.org/10.1007/978-0-387-70914-7; MR2759829

[5] A. L. A. de Araujo, L. F. O. Faria, J. L. F. M. Gurjão, Positive solutions of nonlinear elliptic equations involving supercritical Sobolev exponents without Ambrosetti and Rabinowitz condition, *Calc. Var. Partial Differential Equations* **59**(2020), No. 5, Paper No. 147, 18 pp. https://doi.org/10.1007/s00526-020-01800-x; MR4135644

[6] L. Evans, *Partial differential equations*, American Mathematical Society, Providence, R.I, 2010. MR2597943

[7] L. F. O. Faria, O. H. Miyagaki, M. Tanaka, Existence of positive solution for problems with (p,q)-laplacian and convection term in $\mathbb{R}^n$, *Bound. Value Probl.* **2016**, Paper No. 158, 20 pp. https://doi.org/10.1186/s13661-016-0665-9; MR3542817

[8] L. F. O. Faria, M. Montenegro, Existence of solution for a supercritical nonlinear Schrödinger equation, *Complex Var. Elliptic Equ.* **68**(2023), No. 1, 1–28. https://doi.org/10.1080/17476933.2021.1968383>; MR4519260

[9] L. Jeanjean, B. Sirakov, Existence and multiplicity for elliptic problems with quadratic growth in the gradient, *Comm. Partial Differential Equations* **38**(2013), No. 2, 244–264. https://doi.org/10.1080/03605302.2012.738754; MR3009079

[10] N. D. Kazarinoff, Nonlinear oscillations, dynamical systems, and bifurcations of vector fields (John Guckenheimer and Philip Holmes), *SIAM Rev.* **26**(1984), No. 4, 600–601. https://doi.org/10.1137/1026128

[11] S. Kesavan, *Topics in functional analysis and application*s, 1st Edition, Wiley, New York, 1989. MR990018

[12] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.* **12**(1988), No. 11, 1203–1219. https://doi.org/10.1016/0362-546x(88)90053-3; MR969499

[13] L. Malaguti, C. Marcelli, N. Partsvania, On transitional solutions of second order nonlinear differential equations, *J. Math. Anal. Appl.* **303**(2005), No. 1, 258–273. https://doi.org/10.1016/j.jmaa.2004.08.032; MR2113880

[14] C. Marcelli, F. Papalini, Heteroclinic connections for fully non-linear non-autonomous second-order differential equations, *J. Differential Equations* **241**(2007), No. 1, 160–183. https://doi.org/10.1016/j.jde.2007.05.038; MR2356214

[15] P. J. Martínez-Aparicio, Ecuaciones elípticas y parabólicas singulares con crecimiento natural en el gradiente, Ph.D. thesis, Universidad de Granada, 2009.

[16] A. H. Salas, J. E. Castillo H., Exact solution to Duffing equation and the pendulum equation, *Appl. Math. Sci.* **8**(2014), No. 176, 8781–8789. https://doi.org/10.12988/ams.2014.44243

[17] W. A. Strauss, On weak solutions of semi-linear hyperbolic equations, *An. Acad. Brasil. Ci.* **42**(1970), 645–651. MR306715

[18] Y. Zhang, Homoclinic solutions for a forced generalized Liénard system, *Adv. Difference Equ.* **2012**, 2012:94, 11 pp. https://doi.org/10.1186/1687-1847-2012-94; MR2971765

[19] C. Zhu, W. Zhang, Computation of bifurcation manifolds of linearly independent homoclinic orbits, *J. Differential Equations* **245**(2008), No. 7, 1975–1994. https://doi.org/10.1016/j.jde.2008.06.029; MR2433494