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Normalized solutions for Kirchhoff-type equations with combined nonlinearities: the L^2-critical case

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Abstract. In this paper, we consider the existence of normalized solutions for the following Kirchhoff-type problem:

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u=\lambda u+|u|^{p-2}u+\mu|u|^{q-2}u \quad \text{in } \mathbb{R}^N,$$

with prescribed L^2-norm:

$$\int_{\mathbb{R}^N}|u|^2dx=c^2,$$

where $N=2,3$, $a\geq 0$, $b>0$ and $c>0$ are constants, $\lambda\in\mathbb{R}$, $2<q<p=2+\frac{8}{N}$ and $\mu>0$. The number $2+\frac{8}{N}$ behaves as the L^2 -critical exponent for the above problem. We prove the multiplicity of normalized solutions for the above Kirchhoff-type problem with L^2-critical nonlinearity (that is, $p=2+\frac{8}{N}$) in the two cases: $2<q<2+\frac{4}{N}$ and $2+\frac{4}{N}<q<2+\frac{8}{N}$.

Keywords: Kirchhoff equation, constrained minimization, variational method, Pohozaev manifold.

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1 Introduction and main results

In this paper, we investigate the multiplicity of normalized solutions for the following Kirchhoff-type problem

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u=\lambda u+|u|^{p-2}u+\mu|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

with prescribed mass

$$\int_{\mathbb{R}^N}|u|^2dx=c^2,$$

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where $N = 2, 3$, $a \geq 0$, $b, c > 0$, $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier, $2 < q < p = 2 + \frac{8}{N}$ and $\mu > 0$. Let $L^s(\mathbb{R}^N)(1 \leq s < +\infty)$ be the Lebesgue space with norm $|u|_s = (\int_{\mathbb{R}^N} |u|^s dx)^{1/s}$, $H^1(\mathbb{R}^N)$ be the Hilbert space with the norm $\|u\| = \left(\int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2)dx\right)^{\frac{1}{2}}$.

Problem (1.1) is a special form of the following Kirchhoff problem

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u = f(x, u) \quad \text{in } \mathbb{R}^N,$$

which is also a variant of Dirichlet problem

$$\begin{cases} -\left(a + b\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. It is well-known that problem (1.2) appears naturally in the context of physics. Problem (1.2) is the stationary case of a nonlinear wave equation

$$u_{tt} - \left(a + b\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = f(x, u), \tag{1.3}$$

first proposed by Kirchhoff [9] in 1883. Problem (1.3) is a generalization of the classical D'Alembert's wave equation which describes free vibrations of elastic strings. The parameters in problem (1.3) have specific physical meaning: f is the external force, a is related to the intrinsic properties of the string, and u means the displacement while b denotes the initial tension. Since then, problem (1.3) has received much attention, see [1, 11, 12, 14, 15] and the references therein. Since Lions in [11] proposed an abstract functional analysis framework, Kirchhoff type problem has been intensively studied during the last decades. From a mathematical perspective, problem (1.2) is not a pointwise identity as the appearance of the nonlocal term $\int_{\Omega} |\nabla u|^2 dx$. The nonlocal term causes some mathematical difficulties and the investigation of problem (1.2) is more interesting and challenging. Such a nonlocal model also appears in other fields as biological systems describing a process depending on the average of itself, for example one species' population density.

A way to study problem (1.1) is to search for solutions with L^2-norm constraint, and such solutions are known as normalized solutions and $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier. In addition, the study of L^2-norm constraint problem can give a better insight of the dynamical properties, like orbital stability or instability, and can describe attractive Bose-Einstein condensate. Normalized solutions of problem (1.1) can be obtained by looking for critical points of the energy functional $E_{a,\mu}(u)$ constrained on S_c, where

$$E_{a,\mu}(u) := \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx - \frac{\mu}{q}\int_{\mathbb{R}^N} |u|^q dx,$$

and

$$S_c := \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = c^2\right\}.$$

Many interesting results on the normalized solutions of Kirchhoff problem are also obtained not long ago, see [2,3,5,6,8,13,17,23]. Especially, many experts considered the existence of normalized solutions for problem (1.1) with combined nonlinearities. For the case $\mu > 0$, under different ranges of p and q, Li and Lou in [10] proved a multiplicity result for problem (1.1). In detail, if $2 < q < \frac{10}{3}$, $\frac{14}{3} < p < 6$ and $\mu < \min\{\mu', \mu''\}$, two solutions for problem (1.1) were obtained. If $\frac{14}{3} < q < p < 6$, problem (1.1) has a mountain pass type solution. Hu and Mao in [6] considered the following minimization problem

$$m_{a,c} = \inf_{u \in S_c} E_{a,\mu}(u), \tag{1.4}$$

and they proved that if $2 < q < \frac{10}{3}$ and $2 < q < p \leq \frac{14}{3}$, problem (1.4) has a minimizer for every $c \in (0, c_p^*)$. At the same time, when c satisfies the suitable conditions, the nonexistence of minimizers for problem (1.4) was considered in the following four cases: (i) $q = \frac{10}{3}$ and $p = \frac{14}{3}$; (ii) $\frac{10}{3} = q < p < \frac{14}{3}$; (iii) $2 < q < p = \frac{14}{3}$; (iv) $2 < q < p, \frac{14}{3} < p < 6$. Moreover, if $\frac{14}{3} < q < p < 6$, they also obtained the existence of normalized solutions for problem (1.1) by using constraint minimization on a suitable submanifold of S_c. For the Sobolev critical case (that is, $p = 6$), Feng, Liu and Zhang in [3] proved the existence and multiplicity of normalized solutions for problem (1.1) under suitable assumptions on μ and c for the following four cases: $2 < q < \frac{10}{3}$, $q = \frac{10}{3}$, $\frac{10}{3} < q < \frac{14}{3}$, $\frac{14}{3} \leq q < p = 6$. Some similar results were also obtained in [10,23]. For the case $\mu = 0$, the existence, multiplicity and uniqueness of normalized solutions for problem (1.1) have been considered in [13,19–22]. For the case $\mu < 0$, we refer to [2,6], and for the nonlinear Kirchhoff-type equations in high dimensions see [8].

As far as we known, there are few papers to consider the existence and multiplicity of normalized solutions for problem (1.1) with L^2-critical nonlinearity (that is $p = 2 + \frac{8}{N}$) in the two cases: $2 < q < 2 + \frac{4}{N}$ and $2 + \frac{4}{N} < q < 2 + \frac{8}{N}$. The object of this paper is to prove the existence and multiplicity of normalized solutions for problem (1.1) in those cases under suitable assumptions on μ and a.

Before stating the main results of this paper, let us recall the Gagliardo–Nirenberg inequality (see [18]): for any $s \in [2, \frac{2N}{N-2})$ if $N \geq 3$ and $s \geq 2$ if $N = 1, 2$, we have

$$\frac{1}{s}|u|_s^s \leq \frac{1}{2|Q_s|_2^{s-2}}|\nabla u|_2^{s\gamma_s}|u|_2^{s-s\gamma_s}, \tag{1.5}$$

where $\gamma_s := \frac{N(s-2)}{2s}$ and with equality only for $u = Q_s$, and up to translations, Q_s is the unique positive solution of

$$-\frac{N(s-2)}{4}\Delta u + \left(1 + \frac{s-2}{4}(2-N)\right) u = |u|^{s-2}u \quad \text{in } \mathbb{R}^N,$$

and satisfies

$$|\nabla Q_s|_2^2 = |Q_s|_2^2 = \frac{2}{s}|Q_s|_s^s.$$

Especially, let $p = 2 + \frac{8}{N}$, define

$$c^* := \left(\frac{b|Q_p|_2^{\frac{8}{N}}}{2}\right)^{\frac{N}{8-2N}}.$$

For $s = p = 2 + \frac{8}{N}$ and for any $u \in S_c$, we have

$$\frac{1}{p}|u|_p^p \leq \frac{1}{4}\frac{2}{c^2}\left(\frac{c}{|Q_p|_2}\right)^{\frac{8}{N}}|\nabla u|_2^4 = \frac{b}{4}\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}|\nabla u|_2^4. \tag{1.6}$$

Set

$$\mu_* := \frac{2a|Q_q|_2^{q-2}}{(4-q\gamma_q)c^{q-q\gamma_q}} \left(\frac{2a(2-q\gamma_q)}{b(4-q\gamma_q))\left((\frac{c}{c^*})^{\frac{8-2N}{N}}-1\right)} \right)^{\frac{2-q\gamma_q}{2}}. \tag{1.7}$$

Now, our main results are following.

Theorem 1.1. *Let $2 < q < 2+\frac{4}{N}$, $p = 2+\frac{8}{N}$, $c > c^*$ and $0 < \mu < \mu_*$. Then problem (1.1) has two radial solutions, denoted by $\tilde{u}_{c,\mu}$ and $\hat{u}_{c,\mu}$. Moreover, $\tilde{u}_{c,\mu}$ is a local minimizer of the functional $E_{a,\mu}$ on the set*

$$\mathcal{A}_{R_0} := \{u \in S_{c,r} : |\nabla u|_2^2 < R_0\}$$

for a suitable $R_0 = R_0(c,\mu) > 0$ with $E_{a,\mu}(\tilde{u}_{c,\mu}) < 0$ and $\tilde{u}_{c,\mu}$ solves problem (1.1) for some $\tilde{\lambda}_{c,\mu} < 0$, and $\hat{u}_{c,\mu}$ is a critical point of mountain pass type for $E_{a,\mu}$ with $E_{a,\mu}(\hat{u}_{c,\mu}) > 0$ and $\hat{u}_{c,\mu}$ solves problem (1.1) for some $\hat{\lambda}_{c,\mu} < 0$.

Theorem 1.2. *If $2+\frac{4}{N} < q < p = 2+\frac{8}{N}$, $\mu > 0$ and $c < c^*$, we have the following results:*

(i) if $a = 0$, $m_{0,c} := \inf_{u\in S_c} E_{0,\mu}(u)$ has a radial minimizer $\tilde{u}$, and $\tilde{u}$ solves problem (1.1) for some $\tilde{\lambda} < 0$.

(ii) let $\bar{a} = \frac{b}{2}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)\left(\frac{2}{N(q-2)}-\frac{1}{4}\right)|\nabla\tilde{u}|_2^2 > 0$, for any $a \in (0,\bar{a})$, problem (1.1) has two radial solutions, the one is a global minimizer $\tilde{u}_{c,a}$ with $\tilde{\lambda}_{c,a} < 0$, and the other is the mountain pass type solution $u_{c,a}$ with $\lambda_{c,a} < 0$.

Remark 1.3. Theorem 1.1 complements [6, Theorem 1.2], where Hu and Mao considered the case $c \in (0,c^*)$ and obtained a minimizer of the functional $E_{a,\mu}$ on S_c. However, we deal with the case $c > c^*$ and obtain two solutions for problem (1.1) under suitable assumptions on the constant $\mu > 0$. In the proof of Theorem 1.1, since the functional $E_{a,\mu}$ is not bounded from below on S_c for $c > c^*$, we will restrict the functional $E_{a,\mu}$ on the Pohozaev set $\mathcal{P}_{c,\mu}$. We can get a local minimizer for $E_{a,\mu}|_{\mathcal{P}_{c,\mu}}$ and use mountain pass theorem to get the second critical point. We emphasis that (1.7) has been used to ensure that $\mathcal{P}_{c,\mu}$ is a smooth manifold and the existence of mountain pass type solution.

To the best of our knowledge, Hu and Mao in [6] proved that if $N = 3$, $\frac{10}{3} < q < \frac{14}{3}$, $p = \frac{14}{3}$, $c < c^*$ and $\mu > 0$ satisfy appropriate condition, problem (1.1) with has no minimizer. However, we try to prove the existence of normalized solution for problem (1.1) with $2+\frac{4}{N} < q < p = 2+\frac{8}{N}$ for the suitable constant $a > 0$. Furthermore, there are few results about the existence of normalized solutions to degenerate Kirchhoff equations, that is, $a = 0$, so we first establish the existence of minimizer $m_{0,c} = \inf_{u\in S_c} E_{0,\mu}(u) < 0$, which is a normalized solution of the degenerate Kirchhoff equation. And then, we establish $m_{a,c} := \inf_{u\in S_{c,r}} E_{a,\mu}(u) < 0$ with the help of the minimizer of $m_{0,c}$. At last, we will prove the existence of the second solution with the mountain pass type for problem (1.1).

To overcome the lack of compactness, we work in $H_r^1(\mathbb{R}^N)$. Although the energy functional $E_{a,\mu}$ has a bounded Palais–Smale sequence on the mass constraint set $S_{c,r}$, unfortunately, we can not deduce whether $E_{a,\mu}$ satisfies the Palais–Smale condition. To overcome this difficulty, in the proof of Theorem 1.1, we will constrain the energy functional $E_{a,\mu}$ on a submanifold of $S_{c,r}$ corresponding to the Pohozaev identity. In the proof of (2) of Theorem 1.2, we use Jeanjean's method in [7] and construct an auxiliary map $I_{a,\mu}(u,\tau) := E_{a,\mu}(\tau \star u)$, which has the same type of geometric structure on $S_{c,r} \times \mathbb{R}$ as $E_{a,\mu}$ on $S_{c,r}$.

2 Preliminaries

In this section, we will introduce some notations, then we recall a version of linking theorem. Finally, we give the compactness analysis of Palais–Smale sequences for $E_{a,\mu}|_{S_{c,r}}$. Let

$$H_r^1(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : u(|x|) = u(x)\},$$
$$S_{c,r} := S_c \cap H_r^1(\mathbb{R}^N) = \{u \in S_c : u(x) = u(|x|)\}.$$

For $u \in S_c$, and $\tau \in \mathbb{R}$, define the fiber map preserving the L^2-norm

$$(\tau \star u)(x) := e^{\frac{N}{2}\tau} u(e^\tau x) \quad \text{for any } x \in \mathbb{R}^N.$$

We introduce the auxiliary functional $I_{a,\mu} : H^1(\mathbb{R}^N) \times \mathbb{R}^+ \to \mathbb{R}$ by

$$I_{a,\mu}(u,\tau) := E_{a,\mu}(\tau \star u) = \frac{e^{2\tau}a}{2}|\nabla u|_2^2 + \frac{e^{4\tau}b}{4}|\nabla u|_2^4 - \frac{e^{4\tau}}{p}|u|_p^p - \mu\frac{e^{\gamma_q q\tau}}{q}|u|_q^q, \tag{2.1}$$

then we easily see that the functional $I_{a,\mu}$ is of class C^1. In addition, we define the Pohozaev set by

$$\mathcal{P}_{c,\mu} = \{u \in S_{c,r} : P_\mu(u) = 0\}$$

with

$$P_\mu(u) = a|\nabla u|_2^2 + b|\nabla u|_2^4 - \frac{4}{p}|u|_p^p - \mu\gamma_q|u|_q^q.$$

Lemma 2.1 ([4, Theorem 2.7]). *Let φ be a C^1-functional on a complete connected C^1-Finsler manifold X and consider a homotopy-stable family $\mathcal{F}$ with extended boundary B. Set*

$$c = c(\varphi, \mathcal{F}) = \inf_{A\in\mathcal{F}} \max_{x\in A} \varphi(x)$$

and let F be a closed subset of X satisfying

$$A \cap F \backslash B \neq \emptyset \quad \text{for every } A \in \mathcal{F} \tag{2.2}$$

and

$$\sup_{x\in B} \varphi(x) \le c \le \inf_{x\in F} \varphi(x). \tag{2.3}$$

Then, for any sequence of sets $(A_n)_n \in \mathcal{F}$ such that $\lim_{n\to\infty} \sup_{A_n} \varphi = c$, there exists a sequence $(x_n)_n$ in $X\backslash B$ such that

$$\lim_{n\to\infty} \varphi(x_n) = c, \quad \lim_{n\to\infty} \|d\varphi(x_n)\| = 0, \quad \lim_{n\to\infty} \operatorname{dist}(x_n, F) = 0, \quad \lim_{n\to\infty} \operatorname{dist}(x_n, A_n) = 0.$$

Lemma 2.2. *Let $a > 0$, $b > 0$, $c > 0$, $\mu > 0$, $2 < q < p = 2 + \frac{8}{N}$. Let $\{u_n\} \subset S_{c,r}$ be a bounded Palais–Smale sequence for $E_{a,\mu}|_{S_{c,r}}$ at energy level $m \neq 0$ with $P_\mu(u_n) \to 0$ as $n \to \infty$. Then up to a subsequence $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$. Moreover, $u \in S_{c,r}$ and u is a radial solution for problem (1.1) for some $\lambda < 0$.*

Proof. The proof is divided into three steps.

Step 1: Lagrange multipliers $\lambda_n \to \lambda$ in $\mathbb{R}$. Since $H_r^1(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $s \in (2, \frac{2N}{N-2})$, from the boundedness of Palais–Smale sequence $\{u_n\}$, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and $u \in H_r^1(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \quad \text{in } H_r^1(\mathbb{R}^N), \qquad u_n \to u \quad \text{in } L^s(\mathbb{R}^N), \qquad u_n \to u \quad \text{a.e. on } \mathbb{R}^N.$$

Because $\{u_n\}$ is a Palais–Smale sequence of $E_{a,\mu}|_{S_{c,r}}$, by the Lagrange multipliers rule, there exists $\lambda_n \in \mathbb{R}$ such that

$$(a+b|\nabla u_n|_2^2)\int_{\mathbb{R}^N}\nabla u_n\nabla\varphi dx-\mu\int_{\mathbb{R}^N}|u_n|^{q-2}u_n\varphi dx \\ -\int_{\mathbb{R}^N}|u_n|^{p-2}u_n\varphi dx-\lambda_n\int_{\mathbb{R}^N}u_n\varphi dx=o_n(1) \quad (2.4)$$

for every $\varphi \in H^1(\mathbb{R}^N)$, where $o_n(1) \to 0$ as $n \to \infty$. In particular, taking $\varphi = u_n$ in (2.4), we have

$$\lambda_n c^2 = a|\nabla u_n|_2^2 + b|\nabla u_n|_2^4 - \mu|u_n|_q^q - |u_n|_p^p + o_n(1).$$

The boundedness of $\{u_n\}$ in $H_r^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ implies that $\{\lambda_n\}$ is bounded as well. Hence, up to a subsequence, we have $\lambda_n \to \lambda \in \mathbb{R}$.

Step 2: $\lambda < 0$ and $u \not\equiv 0$. Recalling that $P_\mu(u_n) \to 0$, we have

$$\lambda_n c^2 = \mu(\gamma_q - 1)|u_n|_q^q + (\gamma_p - 1)|u_n|_p^p + o_n(1),$$

hence, let $n \to \infty$, we have

$$\lambda c^2 = \mu(\gamma_q - 1)|u|_q^q + (\gamma_p - 1)|u|_p^p.$$

Since $\mu > 0$ and $0 < \gamma_q, \gamma_p < 1$, we deduce that $\lambda \le 0$, with "=" if and only if $u \equiv 0$. If $\lambda_n \to 0$, we have $\lim_{n\to\infty}|u_n|_p^p = 0 = \lim_{n\to\infty}|u_n|_q^q$. Using again $P_\mu(u_n) \to 0$, we have $E_{a,\mu}(u_n) \to 0$, which is a contradiction with $E_{a,\mu}(u_n) \to m \neq 0$ and thus $\lambda_n \to \lambda < 0$ and $u \not\equiv 0$.

Step 3: $u_n \to u$ in $H^1(\mathbb{R}^N)$. Since $u_n \rightharpoonup u \not\equiv 0$ in $H^1(\mathbb{R}^N)$, we get $B := \lim_{n\to\infty}|\nabla u_n|_2^2 \ge |\nabla u|_2^2 > 0$. Then, (2.4) implies that

$$(a+bB)\int_{\mathbb{R}^N}\nabla u\nabla\varphi dx-\mu\int_{\mathbb{R}^N}|u|^{q-2}u\varphi dx-\int_{\mathbb{R}^N}|u|^{p-2}u\varphi dx-\lambda\int_{\mathbb{R}^N}u\varphi dx=0 \quad (2.5)$$

for any $\varphi \in H^1(\mathbb{R}^N)$. Combining (2.4) with (2.5) and taking $\varphi = u_n - u$, we obtain

$$(a+bB)|\nabla(u_n-u)|_2^2 - \lambda|u_n-u|_2^2 \to 0 \quad \text{as } n \to \infty.$$

Since $\lambda < 0$, we conclude that $\{u_n\}$ converges strongly in $H^1(\mathbb{R}^N)$. □

3 Proof of Theorem 1.1

In this section, we deal with the case $2 < q < 2 + \frac{4}{N}$, $p = 2 + \frac{8}{N}$, $c > c^*$, $\mu > 0$ and prove Theorem 1.1. First of all, it is well known that any critical point of the functional $E_{a,\mu}$ belongs to $\mathcal{P}_{c,\mu}$. Conversely, if $u \in \mathcal{P}_{c,\mu}$, we get $\partial_\tau I_{a,\mu}(u,0) = 0$. Now, we consider the decomposition of $\mathcal{P}_{c,\mu}$ into the disjoint union $\mathcal{P}_{c,\mu} = \mathcal{P}_{c,\mu}^+ \cup \mathcal{P}_{c,\mu}^0 \cup \mathcal{P}_{c,\mu}^-$, where

$$\mathcal{P}_{c,\mu}^+ := \{u \in \mathcal{P}_{c,\mu} : 2a|\nabla u|_2^2 + 4b|\nabla u|_2^4 > \mu q\gamma_q^2|u|_q^q + p\gamma_p^2|u|_p^p\} = \{u \in \mathcal{P}_{c,\mu} : \partial_{\tau\tau}I_{a,\mu}(u,0) > 0\},$$

$$\mathcal{P}_{c,\mu}^0 := \{u \in \mathcal{P}_{c,\mu} : 2a|\nabla u|_2^2 + 4b|\nabla u|_2^4 = \mu q\gamma_q^2|u|_q^q + p\gamma_p^2|u|_p^p\} = \{u \in \mathcal{P}_{c,\mu} : \partial_{\tau\tau}I_{a,\mu}(u,0) = 0\},$$

$$\mathcal{P}_{c,\mu}^- := \{u \in \mathcal{P}_{c,\mu} : 2a|\nabla u|_2^2 + 4b|\nabla u|_2^4 < \mu q\gamma_q^2|u|_q^q + p\gamma_p^2|u|_p^p\} = \{u \in \mathcal{P}_{c,\mu} : \partial_{\tau\tau}I_{a,\mu}(u,0) < 0\}.$$

By (1.5) and (1.6), we have

$$E_{a,\mu}(u) \geq \frac{a}{2}|\nabla u|_2^2 + \frac{b}{4}\left(1 - \left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)|\nabla u|_2^4 - \mu\frac{c^{q-\gamma_q q}}{2|Q_q|_2^{q-2}}|\nabla u|_2^{\gamma_q q} \tag{3.1}$$

for every $u \in S_{c,r}$. Therefore, to understand the geometry of the functional $E_{a,\mu}|_{S_{c,r}}$, it is useful to consider the function $h : \mathbb{R}^+ \to \mathbb{R}$:

$$h(t) := \frac{a}{2}t + \frac{b}{4}\left(1 - \left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)t^2 - \mu\frac{c^{q-\gamma_q}}{2|Q_q|_2^{q-2}}t^{\frac{\gamma_q q}{2}}.$$

Now, we study the properties of $h(t)$.

Lemma 3.1. *Let $c > c^*$, $2 < q < 2 + \frac{4}{N}$, $p = 2 + \frac{8}{N}$, $0 < \mu < \mu_*$, where μ_* is defined in (1.7), the function h has a local strict minimum at negative level and a global strict maximum at positive level. Moreover, there exist $0 < R_0 < R_1$, both depending on c and μ, such that $h(R_0) = 0 = h(R_1)$ and $h(t) > 0$ for any $t \in (R_0, R_1)$.*

Proof. Since

$$h(t) = t^{\frac{\gamma_q q}{2}}\left(\frac{a}{2}t^{1-\frac{\gamma_q q}{2}} + \frac{b}{4}\left(1 - \left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)t^{2-\frac{\gamma_q q}{2}} - \mu\frac{c^{q-q\gamma_q}}{2|Q_q|_2^{q-2}}\right)$$

for $t > 0$, we have $h(t) > 0$ if and only if

$$\varphi(t) > \mu\frac{c^{q-q\gamma_q}}{2|Q_q|_2^{q-2}}, \quad \text{with } \varphi(t) := \frac{a}{2}t^{1-\frac{\gamma_q q}{2}} + \frac{b}{4}\left(1 - \left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)t^{2-\frac{\gamma_q q}{2}}.$$

It is not difficult to check that φ has a unique critical point $\bar{t}$ on $(0, \infty)$, which is a global maximum point at positive level:

$$\bar{t} := \frac{2a(2 - q\gamma_q)}{b(4 - q\gamma_q)\left((\frac{c}{c^*})^{\frac{8-2N}{N}} - 1\right)},$$

and the maximum level is

$$\varphi(\bar{t}) = \frac{a}{(4 - q\gamma_q)}\left(\frac{2a(2 - q\gamma_q)}{b(4 - q\gamma_q)\left((\frac{c}{c^*})^{\frac{8-2N}{N}} - 1\right)}\right)^{1-\frac{q\gamma_q}{2}} > 0.$$

From $0 < \frac{\gamma_q q}{2} < 1$, $\mu > 0$ and $c > c^*$, it is obvious that $\lim_{t\to 0^+} h(t) = 0^-$ and $\lim_{t\to+\infty} h(t) = -\infty$. Therefore, h is positive on an open interval (R_0, R_1) if $\varphi(\bar{t}) > \mu\frac{c^{q-q\gamma_q}}{2|Q_q|_2^{q-2}}$, which is ensured by

$$0 < \mu < \mu_* := \frac{2a|Q_q|_2^{q-2}}{(4 - q\gamma_q)c^{q-q\gamma_q}}\left(\frac{2a(2 - q\gamma_q)}{b(4 - q\gamma_q)\left((\frac{c}{c^*})^{\frac{8-2N}{N}} - 1\right)}\right)^{1-\frac{q\gamma_q}{2}}.$$

It follows immediately that h has a global maximum at positive level in (R_0, R_1). Moreover, since $\lim_{t\to 0^+} h(t) = 0^-$, there exists a local minimum point at negative level in $(0, R_0)$. The fact that h has no other critical points can be verified observing that $h'(t) = 0$ if and only if

$$\psi(t) = \mu\frac{\gamma_q q c^{q-q\gamma_q}}{2|Q_q|_2^{q-2}} \quad \text{with } \psi(t) := at^{\frac{2-q\gamma_q}{2}} + b\left(1 - \left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)t^{\frac{4-q\gamma_q}{2}}.$$

Clearly ψ has only one critical point, which is a strict maximum, and hence the above equation has at most two solutions, which necessarily are the local minimum and the global maximum of h previously found. □

We now study the structure of the Pohozaev manifold $\mathcal{P}_{c,\mu}$. Recalling the decomposition of $\mathcal{P}_{c,\mu} = \mathcal{P}^+_{c,\mu} \cup \mathcal{P}^0_{c,\mu} \cup \mathcal{P}^-_{c,\mu}$.

Lemma 3.2. *If $2 < q < 2 + \frac{4}{N}$, $p = 2 + \frac{8}{N}$ and $0 < \mu < \mu_*$, then $\mathcal{P}^0_{c,\mu} = \emptyset$ and $\mathcal{P}_{c,\mu}$ is a smooth manifold of codimension 2 in $H^1(\mathbb{R}^N)$.*

Proof. Otherwise, let $u \in \mathcal{P}^0_{c,\mu}$, from $P_{c,\mu}(u) = 0$ and $\partial_{\tau\tau} I_{a,\mu}(u, 0) = 0$, we have

$$a|\nabla u|_2^2 + b|\nabla u|_2^4 - \mu\gamma_q|u|_q^q - \frac{4}{p}|u|_p^p = 0,$$

$$2a|\nabla u|_2^2 + 4b|\nabla u|_2^4 - \mu q\gamma_q^2|u|_q^q - p\gamma_p^2|u|_p^p = 0.$$

By (1.5), we obtain

$$(2 - q\gamma_q)a|\nabla u|_2^2 + (4 - q\gamma_q)b|\nabla u|_2^4 = \gamma_p(p\gamma_p - q\gamma_q)|u|_p^p \le (4 - q\gamma_q)b\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}|\nabla u|_2^4,$$

$$2a|\nabla u|_2^2 = \mu\gamma_q(4 - q\gamma_q)|u|_q^q \le \mu q\gamma_q(4 - q\gamma_q)\frac{c^{q-q\gamma_q}}{2|Q_q|_2^{q-2}}|\nabla u|_2^{q\gamma_q}.$$

Then, the lower and upper bounds of $|\nabla u|_2$ are given by

$$\left(\frac{a(2 - q\gamma_q)}{b(4 - q\gamma_q)\left((\frac{c}{c_*})^{\frac{8-2N}{N}} - 1\right)}\right)^{\frac{1}{2}} \le |\nabla u|_2 \le \left(\frac{\mu q\gamma_q(4 - q\gamma_q)c^{q-q\gamma_q}}{4a|Q_q|_2^{q-2}}\right)^{\frac{1}{2-q\gamma_q}},$$

which leads to

$$\mu > \frac{4a|Q_q|_2^{q-2}}{q\gamma_q(4 - q\gamma_q)c^{q-q\gamma_q}}\left(\frac{a(2 - q\gamma_q)}{b(4 - q\gamma_q)\left((\frac{c}{c_*})^{\frac{8-2N}{N}} - 1\right)}\right)^{\frac{2-q\gamma_q}{2}} > \mu_*,$$

which contradicts to $0 < \mu < \mu_*$, hence, $\mathcal{P}^0_{c,\mu} = \emptyset$. $\mathcal{P}_{c,\mu}$ is a smooth manifold of codimension 2 in $H^1(\mathbb{R}^N)$, see proof of [16, Lemma 5.2]. □

Lemma 3.3. *Let $a > 0, b > 0, 2 < q < 2 + \frac{4}{N}$, $p = 2 + \frac{8}{N}$, $0 < \mu < \mu^*$, if $u \in P_{c,\mu}$ is a critical point for $E_{a,\mu}|_{P_{c,\mu}}$, then u is a critical point for $E_{a,\mu}|_{S_{c,r}}$, where μ^* is defined in (1.7).*

Proof. From Lemma 3.2, we deduce that $\mathcal{P}_{c,\mu}$ is a smooth manifold of codimension 2 in $H^1(\mathbb{R}^N)$ and $\mathcal{P}^0_{c,\mu} = \emptyset$. If $u \in \mathcal{P}_{c,\mu}$ is a critical point for $E_{a,\mu}|_{\mathcal{P}_{c,\mu}}$, then by the Lagrange multipliers rule, there exists $\lambda, \xi \in \mathbb{R}$ such that

$$\left\langle E'_{a,\mu}(u), \varphi\right\rangle - \lambda\int_{\mathbb{R}^N} u\varphi dx - \xi\left\langle P'_\mu(u), \varphi\right\rangle = 0, \quad \forall\varphi \in H^1\left(\mathbb{R}^N\right).$$

So u solves

$$-((1 - 2\xi)a + (1 - 4\xi)b|\nabla u|_2^2)\Delta u - \lambda u + \mu(\xi q\delta_q - 1)|u|^{q-2}u + (p\xi\gamma_p - 1)|u|^{p-2}u = 0.$$

Combining with the Pohozaev identity, we have

$$(1-2\xi)a|\nabla u|_2^2 + (1-4\xi)b|\nabla u|_2^4 + \mu\gamma_q(\xi q\gamma_q - 1)|u|_q^q + \gamma_p(p\xi\gamma_p - 1)|u|_p^p = 0.$$

Since $u \in \mathcal{P}_{c,\mu}$ and $u \notin \mathcal{P}^0_{c,\mu}$, we deduce from $\xi(2a|\nabla u|_2^2 + 4b|\nabla u|_2^4 - \mu q\gamma_q^2|u|_q^q - \gamma_p^2 p|u|_p^p) = 0$ that $\xi = 0$. □

The manifold $\mathcal{P}_{c,\mu}$ is then divided into two components $\mathcal{P}^+_{c,\mu}$ and $\mathcal{P}^-_{c,\mu}$, having disjoint closure.

Lemma 3.4. *For every $u \in S_{c,r}$, we have*

(i) if $\frac{b}{4}|\nabla u|_2^4 \geq \frac{1}{p}|u|_p^p$, the function $I_{a,\mu}(u,\cdot)$ has a critical point $s_u \in \mathbb{R}$ and a zero $c_u \in \mathbb{R}$, with $s_u < c_u$;

(ii) if $\frac{b}{4}|\nabla u|_2^4 < \frac{1}{p}|u|_p^p$, the function $I_{a,\mu}(u,\cdot)$ has exactly two critical points $s_u < t_u \in \mathbb{R}$ and two zeros $c_u < d_u \in \mathbb{R}$, with $s_u < c_u < t_u < d_u$;

(iii) $\int_{\mathbb{R}^N} |\nabla(\tau \star u)|^2 \leq R_0$ for every $\tau \leq c_u$, and

$$E_{a,\mu}(s_u \star u) = \min\left\{E_{a,\mu}(\tau \star u) : \tau \in \mathbb{R} \text{ and } \int_{\mathbb{R}^N} |\nabla(\tau \star u)|^2 dx < R_0\right\} < 0; \tag{3.2}$$

(iv) For any $u \in S_{c,r}$ with $\frac{b}{4}|\nabla u|_2^4 < \frac{1}{p}|u|_p^p$, we have

$$E_{a,\mu}(t_u \star u) = \max\{E_{a,\mu}(\tau \star u) : \tau \in \mathbb{R}\} > 0, \tag{3.3}$$

and $I_{a,\mu}$ is strictly decreasing and concave on $\tau \in (t_u, +\infty)$;

(v) The maps $u \in S_{c,r} \mapsto s_u \in \mathbb{R}$ and $u \in S_{c,r} \mapsto t_u \in \mathbb{R}$ are of class C^1.

Proof. We recall that by (3.1)

$$I_{a,\mu}(u,\tau) = E_{a,\mu}(\tau \star u) \geq h\left(\int_{\mathbb{R}^N} |\nabla(\tau \star u)|^2 dx\right) = h\left(e^{2\tau}\int_{\mathbb{R}^N} |\nabla u|^2 dx\right).$$

Thus, the function $I_{a,\mu}(u,\cdot)$ is positive on $(C(R_0), C(R_1))$ with

$$(C(R_0), C(R_1)) := \left(\frac{1}{2}\ln\left(R_0 / \int_{\mathbb{R}^N} |\nabla u|^2 dx\right), \frac{1}{2}\ln\left(R_1 / \int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\right).$$

If $\frac{b}{4}|\nabla u|_2^4 \geq \frac{1}{p}|u|_p^p$, from (2.1), $I_{a,\mu}(u,\tau) \to +\infty$ as $\tau \to +\infty$, and $I_{a,\mu}(u,\tau) \to 0^-$ as $\tau \to -\infty$. Hence, it follows that $I_{a,\mu}$ has at least a critical point s_u, with s_u local minimum point on $(-\infty, C(R_0))$ at negative level, and $I_{a,\mu}$ has at least a zero point c_u with $s_u < c_u < C(R_0)$. Note that $\partial_\tau I_{a,\mu}(u,\tau) = 0$ reads

$$\phi(\tau) = \mu\gamma_q|u|_q^q \quad \text{with } \phi(\tau) := ae^{\frac{4-N(q-2)}{2}\tau}|\nabla u|_2^2 + be^{\frac{8-N(q-2)}{2}\tau}|\nabla u|_2^4 - \frac{4}{p}e^{\frac{8-N(q-2)}{2}\tau}|u|_p^p. \tag{3.4}$$

But $\phi(\tau)$ is increasing on $(-\infty, +\infty)$, hence, $I_{a,\mu}$ has exactly a critical point s_u and a zero point c_u.

If $\frac{b}{4}|\nabla u|_2^4 < \frac{1}{p}|u|_p^p$, $I_{a,\mu}(u,\tau) \to -\infty$ as $\tau \to +\infty$ and ϕ has a unique maximum point, and $I_{a,\mu}(u,\tau) \to 0^-$ as $\tau \to -\infty$. Therefore, we conclude that $I_{a,\mu}$ has exactly two critical points:

s_u, local minimum on $(-\infty, C(R_0))$ at negative level, and t_u, global maximum at positive level, which also gives (3.3).

From $s_u < C(R_0)$, then it holds that

$$\int_{\mathbb{R}^N} |\nabla(s_u \star u)|^2 dx = e^{2s_u} \int_{\mathbb{R}^N} |\nabla u|^2 dx < R_0.$$

In addition, we have $s_u \star u \in \mathcal{P}_{c,\mu}$, $t_u \star u \in \mathcal{P}_{c,\mu}$, and $\tau \star u \in \mathcal{P}_{c,\mu}$ implies $\tau \in \{s_u, t_u\}$. By minimality and $\mathcal{P}^0_{c,\mu} = \emptyset$, we have $\partial_{\tau\tau} I_{a,\mu}(u, s_u) > 0$, that is, $s_u \star u \in \mathcal{P}^+_{c,\mu}$. In the same way, $t_u \star u \in \mathcal{P}^-_{c,\mu}$. In particular, $I_{a,\mu}(u, \cdot)$ is concave on $[t_u, +\infty)$.

Finally, we show that $u \mapsto s_u$ and $u \mapsto t_u$ are of class C^1. To this end, we apply the implicit function theorem on the C^1 function $\Phi(u,\tau) := \partial_\tau I_{a,\mu}(u,\tau)$. We see $\Phi(u, s_u) = 0$ and $\partial_\tau \Phi(u, s_u) = \partial_{\tau\tau} I_{a,\mu}(u, s_u) > 0$, and the fact that it is not possible to pass with continuity from $\mathcal{P}^+_{c,\mu}$ to $\mathcal{P}^-_{c,\mu}$ (since $\mathcal{P}^0_{c,\mu} = \emptyset$). By the same argument, we have that $u \mapsto t_u$ is of C^1. □

From the proof of Lemma 3.4, we see that $s_u < C(R_0) < t_u$ and

$$\int_{\mathbb{R}^N} |\nabla(s_u \star u)|^2 dx < R_0 < \int_{\mathbb{R}^N} |\nabla(t_u \star u)|^2 dx,$$

which implies

$$\mathcal{P}^+_{c,\mu} \subseteq \{u \in S_{c,r} : |\nabla u|_2^2 < R_0\}$$

and

$$\mathcal{P}^-_{c,\mu} \subseteq \{u \in S_{c,r} : |\nabla u|_2^2 > R_0\}.$$

For $k > 0$, let us set

$$\mathcal{A}_k := \{u \in S_{a,r} : |\nabla u|_2^2 < k\},$$

and

$$M_{c,\mu} := \inf_{u \in \mathcal{A}_{R_0}} E_{a,\mu}(u).$$

As an immediate lemma, we have:

Lemma 3.5. $\sup_{\mathcal{P}^+_{c,\mu}} E_{a,\mu} \leq 0 \leq \inf_{\mathcal{P}^-_{c,\mu}} E_{a,\mu}$.

Lemma 3.6. *It results that $M_{c,\mu} \in (-\infty, 0)$, that*

$$M_{c,\mu} = \inf_{\mathcal{P}_{c,\mu}} E_{a,\mu} = \inf_{\mathcal{P}^+_{c,\mu}} E_{a,\mu}, \quad \textit{and that} \quad M_{c,\mu} < \inf_{\mathcal{A}_{R_0} \setminus \mathcal{A}_{R_0-\rho}} E_{a,\mu}$$

for $\rho > 0$ small enough.

Proof. For any $u \in \mathcal{A}_{R_0}$, we have

$$E_{a,\mu}(u) \geq h(|\nabla u|_2^2) \geq \min_{t \in [0,R_0]} h(t) > -\infty,$$

and hence $M_{c,\mu} > -\infty$. Moreover, for any $u \in S_{c,r}$, we have $|\nabla(\tau \star u)|_2^2 < R_0$ and $E_{a,\mu}(\tau \star u) < 0$ for $\tau \ll -1$, and hence $M_{c,\mu} < 0$.

Now, $M_{c,\mu} \leq \inf_{\mathcal{P}^+_{c,\mu}} E_{a,\mu}$ from $\mathcal{P}^+_{c,\mu} \subset \mathcal{A}_{R_0}$. On the other hand, if $u \in \mathcal{A}_{R_0}$, then $s_u \star u \in \mathcal{P}^+_{c,\mu} \subset \mathcal{A}_{R_0}$, and

$$E_{a,\mu}(s_u \star u) = \min\left\{E_{a,\mu}(\tau \star u) : \tau \in \mathbb{R} \text{ and } \int_{\mathbb{R}^N} |\nabla(\tau \star u)|^2 dx < R_0\right\} \leq E_{a,\mu}(u),$$

which implies that $\inf_{\mathcal{P}^+_{c,\mu}} E_{a,\mu} \le M_{c,\mu}$. To prove that $\inf_{\mathcal{P}^+_{c,\mu}} E_{a,\mu} = \inf_{\mathcal{P}_{c,\mu}} E_{a,\mu}$, it is sufficient to recall that $E_{a,\mu}(u) > 0$ on $\mathcal{P}^-_{c,\mu}$.

Finally, by the continuity of h, there exists $\rho > 0$ such that $h(t) \ge \frac{M_{c,\mu}}{2}$ for any $t \in [R_0 - \rho, R_0]$. Therefore, we have

$$E_{a,\mu}(u) \ge h(|\nabla u|_2^2) \ge \frac{M_{c,\mu}}{2} > M_{c,\mu}$$

for every $u \in S_{c,r}$ with $R_0 - \rho \le |\nabla u|_2^2 \le R_0$. □

Lemma 3.7. *$M_{c,\mu}$ can be achieved by some $\tilde{u}_{c,\mu} \in S_{c,r}$. Moreover, $\tilde{u}_{c,\mu}$ is an interior local minimizer for $E_{a,\mu}|_{\mathcal{A}_{R_0}}$, and $\tilde{u}_{c,\mu}$ solves problem (1.1) for some $\tilde{\lambda}_{c,\mu} < 0$. Moreover, $\tilde{u}_{c,\mu}$ is a ground state of $E_{a,\mu}|_{S_{c,r}}$, any ground state of $E_{a,\mu}|_{S_{c,r}}$ is a local minimizer of $E_{a,\mu}$ on A_{R_0}.*

Proof. Let us consider a minimizing sequence $\{v_n\}$ for $E_{a,\mu}|_{\mathcal{A}_{R_0}}$. By Lemma 3.4, there exists a sequence $\{s_{v_n}\}$ such that $s_{v_n} \star v_n \in \mathcal{P}^+_{c,\mu}$ and

$$E_{a,\mu}(s_{v_n} \star v_n) = \min \left\{ E_{a,\mu}(\tau \star s_{v_n}) : \tau \in \mathbb{R} \text{ and } \int_{\mathbb{R}^N} |\nabla(\tau \star s_{v_n})|^2 dx < R_0 \right\} < E_{a,\mu}(v_n),$$

where the last inequality follows from $v_n \in \mathcal{A}_{R_0}$. Besides, we also see that

$$\int_{\mathbb{R}^N} |\nabla(s_{v_n} \star v_n)|^2 dx < R_0,$$

furthermore, by Lemma 3.6, we have

$$\int_{\mathbb{R}^N} |\nabla(s_{v_n} \star v_n)|^2 dx < R_0 - \rho.$$

Once again by Lemma 3.6, it holds that

$$M_{c,\mu} = \inf_{\mathcal{P}_{c,\mu}} E_{a,\mu} = \inf_{\mathcal{P}^+_{a,\mu}} E_{a,\mu}.$$

Setting $u_n = s_{v_n} \star v_n$ and using the Ekeland's variational principle, we may assume that $\{u_n\}$ is a Palais–Smale sequence for $E_{a,\mu}$ on $S_{c,r}$ and $P_\mu(u_n) = 0$. Hence, we have

$$E_{a,\mu}(u_n) = \frac{a}{4}|\nabla u_n|_2^2 - \frac{\mu}{q}\left(1 - \frac{N(q-2)}{8}\right)|u_n|_q^q = M_{c,\mu} + o_n(1).$$

It results to

$$\frac{a}{4}|\nabla u_n|_2^2 \le (M_{c,\mu} + 1) + \frac{\mu}{q}\left(1 - \frac{N(q-2)}{8}\right)\frac{c^{q-\frac{N(q-2)}{2}}}{2|Q_q|_2^{q-2}}|\nabla u_n|_2^{\frac{N(q-2)}{2}}, \tag{3.5}$$

which gives $\{|\nabla u_n|_2\}$ is bounded, hence, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. From Lemma 2.2, up to a subsequence, $u_n \to \tilde{u}_{c,\mu}$ strongly in $H^1(\mathbb{R}^N)$, and $\tilde{u}_{c,\mu}$ solves problem (1.1) for some $\tilde{\lambda}_{c,\mu} < 0$. Moreover, we have $\int_{\mathbb{R}^N} |\nabla \tilde{u}_{c,\mu}|^2 dx < R_0 - \rho$ and $\tilde{u}_{c,\mu}$ is an interior local minimizer for $M_{c,\mu}$.

Since any critical point of $E_{a,\mu}|_{S_{c,r}}$ lies in $\mathcal{P}_{c,\mu}$ and $M_{c,\mu} = \inf_{\mathcal{P}_{c,\mu}} E_{a,\mu}$, we see that $\tilde{u}_{c,\mu}$ is a ground state for $E_{a,\mu}|_{S_{c,r}}$. It only remains to prove that any ground state of $E_{a,\mu}|_{S_{c,r}}$ is a local minimizer of $E_{a,\mu}$ in A_{R_0}. Let u be a critical point of $E_{a,\mu}|_{S_{c,r}}$ with $E_{a,\mu}(u) = M_{c,\mu} = \inf_{\mathcal{P}_{c,\mu}} E_{a,\mu}$. Since $E_{a,\mu}(u) < 0 < \inf_{\mathcal{P}^-_{c,\mu}} E_{a,\mu}$, necessarily $u \in \mathcal{P}^+_{c,\mu}$. Then Lemma 3.6 implies that $\mathcal{P}^+_{c,\mu} \subset A_{R_0}$. This leads to $|\nabla u|_2 < R_0$, and as a consequence u is a local minimizer for $E_{a,\mu}|_{A_{R_0}}$. Lemma 3.4 implies that $E_{a,\mu}(u) \le 0$ for any $u \in \mathcal{P}^+_{c,\mu}$, and $|\nabla u|_2^2 < R_0$. Hence, u is a local minimizer for $E_{a,\mu}|_{A_{R_0}}$. □

In the following, we focus on the existence of a second critical point for $E_{a,\mu}|_{S_{c,r}}$. Let

$$\widetilde{Q}_p(x) := c\frac{Q_p(x)}{|Q_p|_2}, \qquad Q_p^\tau(x) := c\frac{e^{\frac{N}{2}\tau}Q_p(e^\tau x)}{|Q_p|_2} \quad \text{for any } \tau > 0,$$

we have $\widetilde{Q}_p(x), Q_p^\tau(x) \in S_{c,r}$.

Lemma 3.8. *If $2 < q < 2 + \frac{4}{N}$, $p = 2 + \frac{8}{N}$, and $c > c^*$, we have $\int_{\mathbb{R}^N} |\nabla Q_p^\tau|^2 dx \to +\infty$ and $I_{a,\mu}(\widetilde{Q}_p, \tau) \to -\infty$ as $\tau \to +\infty$.*

Proof. A straightforward calculation shows that

$$\int_{\mathbb{R}^N} |\nabla Q_p^\tau|^2 dx = e^{2\tau} \int_{\mathbb{R}^N} |\nabla \widetilde{Q}_p|^2 dx.$$

From (1.5) with $s = p$ and (2.1), we have

$$\begin{aligned}
&I_{a,\mu}(\widetilde{Q}_p, \tau) \\
&= \frac{ae^{2\tau}}{2}\int_{\mathbb{R}^N} |\nabla \widetilde{Q}_p|^2 dx + \frac{be^{4\tau}}{4}\left(\int_{\mathbb{R}^N} |\nabla \widetilde{Q}_p|^2 dx\right)^2 - \frac{e^{4\tau}}{p}\int_{\mathbb{R}^N} |\widetilde{Q}_p|^p dx - \mu\frac{e^{\gamma_q q\tau}}{q}\int_{\mathbb{R}^N} |\widetilde{Q}_p|^q dx \\
&= \frac{ae^{2\tau}}{2}\frac{c^2|\nabla Q_p|_2^2}{|Q_p|_2^2} - \mu\frac{e^{\gamma_q q\tau}}{q}\frac{c^q}{|Q_p|_2^q}|Q_p|_q^q + c^4e^{4\tau}\left(\frac{b}{4}\frac{|\nabla Q_p|_2^4}{|Q_p|_2^4} - \frac{1}{4}\frac{2}{c^2}\left(\frac{c}{|Q_p|_2}\right)^{\frac{8}{N}}\frac{2|Q_p|_p^p}{q|Q_p|_2^2}\right) \\
&= \frac{ac^2e^{2\tau}}{2} - \mu\frac{e^{\gamma_q q\tau}}{q}\frac{c^q}{|Q_p|_2^q}|Q_p|_q^q + \frac{bc^4e^{4\tau}}{4}\left(1 - \left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right),
\end{aligned}$$

from $c > c^*$, we have $I_{a,\mu}(\widetilde{Q}_p, \tau) \to -\infty$ as $\tau \to +\infty$. □

Lemma 3.9. *Suppose that $E_{a,\mu}(u) < M_{c,\mu}$. Then the value t_u defined by Lemma 3.4 is negative.*

Lemma 3.10. *It results that*

$$\tilde{\sigma}_{c,\mu} = \inf_{u \in \mathcal{P}_{c,\mu}^-} E_{a,\mu}(u) > 0.$$

We introduce the minimax class

$$\Gamma := \left\{\gamma \in C([0,1], S_{c,r}) : \gamma(0) \in \mathcal{P}_{c,\mu}^+ \text{ with } \frac{b}{4}|\nabla\gamma(0)|_2^4 < \frac{1}{p}|\gamma(0)|_p^p,\ E_{a,\mu}(\gamma(1)) \le 2M_{c,\mu}\right\},$$

then $\Gamma \neq \emptyset$. In fact, we have $s_{\widetilde{Q}_p} \star \widetilde{Q}_p \in \mathcal{P}_{c,\mu}^+$ by Lemma 3.4 and $E_{a,\mu}(\tau \star \widetilde{Q}_p) \to -\infty$ as $\tau \to +\infty$ by Lemma 3.8, and $\tau \mapsto \tau \star \widetilde{Q}_p$ is continuous. Thus, we can define the minimax value

$$\sigma_{c,\mu} := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} E_{a,\mu}(\gamma(t)).$$

Lemma 3.11. *$\sigma_{c,\mu} > 0$ can be achieved by some $\hat{u}_{c,\mu} \in S_{c,r}$, and $\hat{u}_{c,\mu}$ solves problem (1.1) for some $\hat{\lambda}_{c,\mu} < 0$.*

Proof. Since we want to use Lemma 2.1, next we verify the conditions of Lemma 2.1 one by one. Let us set

$$\mathcal{F} := \Gamma, \quad A := \gamma([0,1]), \quad F := \mathcal{P}_{c,\mu}^- \quad \text{and} \quad B := \mathcal{P}_{c,\mu}^+ \cup E_{a,\mu}^{2M_{c,\mu}},$$

where $E_{a,\mu}^c := \{u \in S_{c,r} : E_{a,\mu}(u) \le c\}$.

We first show that $\mathcal{F}$ is homotopy-stable family with extended boundary B: for any $\gamma \in \Gamma$ and any $\eta \in C([0,1] \times S_{c,r}; S_{c,r})$ satisfying $\eta(t,u) = u, (t,u) \in (0 \times S_{c,r}) \cup ([0,1] \times B)$, we want to get $\eta(1,\gamma(t)) \in \Gamma$. In fact, let $\tilde{\gamma}(t) = \eta(1,\gamma(t))$, then $\tilde{\gamma}(0) = \eta(1,\gamma(0)) = \gamma(0) \in \mathcal{P}^+_{c,\mu}$. Besides, $\tilde{\gamma}(1) = \eta(1,\gamma(1)) = \gamma(1) \in E^{2M_{c,\mu}}_{a,\mu}$. Therefore, we have $\eta(1,\gamma(t)) \in \Gamma$.

Next we verify the condition (2.2): by Lemma 3.5 and Lemma 3.9, we know $F \cap B = \emptyset$ and hence $F \backslash B = F$. We claim that

$$A \cap (F \backslash B) = A \cap F = \gamma([0,1]) \cap \mathcal{P}^-_{c,\mu} \neq \emptyset, \quad \forall \gamma \in \Gamma. \tag{3.6}$$

Indeed, since $\gamma(0) \in \mathcal{P}^+_{c,\mu}$ with $\frac{b}{4}|\nabla\gamma(0)|_2^4 < \frac{1}{p}|\gamma(0)|_p^p$, we know $s_{\gamma(0)} = 0$ (see the definition of s_u in Lemma 3.4) and hence $t_{\gamma(0)} > s_{\gamma(0)} = 0$. On the other hand, since $E_{a,\mu}(\gamma(1)) \leq 2M_{c,\mu} < M_{c,\mu}$ (see Lemma 3.6), we by Lemma 3.8 have $t_{\gamma(1)} < 0$. By Lemma 3.4, we know $t_{\gamma(\tau)}$ is continuous in τ. It follows that for every $\gamma \in \Gamma$ there exists $\tau_\gamma \in (0,1)$ such that $t_{\gamma(\tau_\gamma)} = 0$, that is, $\gamma(\tau_\gamma) \in \mathcal{P}^-_{c,\mu}$, and hence $A \cap (F \backslash B) \neq \emptyset$.

Finally, we verify the condition (2.3), that is, we need to show

$$\inf_{\mathcal{P}^-_{c,\mu}} E_{a,\mu} \geq \sigma_{c,\mu} \geq \sup_{\mathcal{P}^+_{c,\mu} \cup E^{2M_{c,\mu}}_{a,\mu}} E_{a,\mu}.$$

By (3.6), for every $\gamma \in \Gamma$, we have

$$\max_{t\in[0,1]} E_{a,\mu}(\gamma(t)) \geq \inf_{\mathcal{P}^-_{c,\mu}} E_{a,\mu},$$

so that $\sigma_{c,\mu} \geq \tilde{\sigma}_{c,\mu}$. On the other hand, if $u \in \mathcal{P}^-_{c,\mu}$ with $\frac{b}{4}|\nabla u|_2^4 < \frac{1}{p}|u|_p^p$, then for $s_1 \gg 1$ large enough

$$\gamma_u : \tau \in [0,1] \mapsto ((1-\tau)s_u + \tau s_1) \star u \in S_{c,r}$$

is a path in Γ. Since $u \in \mathcal{P}^-_{c,\mu}$, we know $t_u = 0$ is a global maximum point for $I_{a,\mu}$, and deduce that

$$E_{a,\mu}(u) \geq \max_{t\in[0,1]} E_{a,\mu}\left(\gamma_u(t)\right) \geq \sigma_{c,\mu},$$

which implies that $\tilde{\sigma}_{c,\mu} \geq \sigma_{c,\mu}$. Thus, we get $\sigma_{c,\mu} = \tilde{\sigma}_{c,\mu} > 0$. By Lemma 3.5, we know $E_{a,\mu}(u) \leq 0$ for any $u \in \mathcal{P}^+_{c,\mu} \cup E^{2M_{c,\mu}}_{a,\mu}$, hence we get (2.3). From Lemma 2.1, we obtain a Palais–Smale sequence $\{u_n\}$ for the functional $E_{a,\mu}$ on $S_{c,r}$ and $P_\mu\left(u_n\right) \to 0$. Similar to (3.5), $\{u_n\}$ is bounded. Hence, from Lemma 2.2, up to a subsequence, $u_n \to \hat{u}_{c,\mu}$ strongly in $H^1(\mathbb{R}^N)$, and $\hat{u}_{c,\mu}$ solves problem (1.1) for some $\hat{\lambda}_{c,\mu} < 0$. □

Proof of Theorem 1.1. Theorem 1.1 comes from Lemma 3.7 and Lemma 3.11. □

4 Proof of Theorem 1.2

In this section, we deal with the case $2 + \frac{4}{N} < q < p = 2 + \frac{8}{N}$, $\mu > 0$, $a \geq 0$ and prove Theorem 1.2. We first consider the existence of normalized ground state solution for the degenerate Kirchhoff-type equations, that is, $a = 0$, by the following minimization problem:

$$m_{0,c} = \inf_{u\in S_c} E_{a,\mu}(u).$$

And then, we discuss the the existence of normalized solutions for the nondegenerate Kirchhoff-type equations, that is, $a > 0$.

Lemma 4.1. *If $a \geq 0$, $2 + \frac{4}{N} < q < p = 2 + \frac{8}{N}$ and $c < c^*$, the functional $E_{a,\mu}$ is coercive on S_c. Moreover, $m_{0,c} < 0$.*

Proof. Utilizing (1.5) and (1.6), we see that for any $u \in S_c$,

$$E_{a,\mu}(u) \geq \frac{b}{4}\left(1 - \left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)|\nabla u|_2^4 - \mu \frac{c^{q-q\gamma_q}}{2|Q_q|_2^{q-2}}|\nabla u|_2^{q\gamma_q},$$

hence, from $2 < \gamma_q q < 4$ and $c < c^*$, we obtain that the functional $E_{a,\mu}$ is coercive on S_c.

For any $u \in S_c$, set $u^t(x) = t^{\frac{N}{2}}u(tx)$ for any $t > 0$, then $u^t \in S_c$ and

$$m_{0,c} \leq E_{0,\mu}(u^t) = \frac{b}{4}|\nabla u|_2^4 t^4 - \frac{1}{p}|u|_p^p t^4 - \frac{\mu}{q}|u|_q^q t^{\gamma_q q} \to 0^- \quad \text{as } t \to 0^+,$$

hence, from $\mu > 0$ and $2 < \gamma_q q < 4$, we obtain $m_{0,c} < 0$. □

In order to prove that the minimizer of $m_{a,c}$ can be obtained, we now give two lemmas.

Lemma 4.2. *If $m_{a,c} < 0$, we have $m_{a,c} < m_{a,\gamma} + m_{a,c-\gamma}$ for any $0 < \gamma < c$.*

Proof. The proof is similar to [19, Lemma 2.5], so we omit it. □

Corollary 4.3. *$m_{a,c}$ is strictly decreasing in $c \in (0, +\infty)$.*

Lemma 4.4. *Let $c < c^*$, $m_{0,c} := \inf u \in S_c E_{0,\mu}(u)$ has a radial minimizer $\tilde{u}$, and $\tilde{u}$ solves problem (1.1) for some $\tilde{\lambda} < 0$.*

Proof. Let $\{u_n\} \subset S_c$ be a minimizing sequence of $m_{0,c} < 0$, it can easily see that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ by Lemma 4.1. Since $E_{0,\mu}$ is even, we can suppose that $u_n \geq 0$. Moreover, let u_n^* be the symmetric radial decreasing rearrangement of u_n, up to subsequence, we may assume that there exists $\tilde{u} \in H_r^1(\mathbb{R}^N)$ such that

$$u_n^* \rightharpoonup \tilde{u} \text{ in } H^1(\mathbb{R}^N), \quad u_n^* \to \tilde{u} \text{ in } L^s(\mathbb{R}^N), \quad s \in (2, 2^*), \quad u_n^*(x) \to \tilde{u}(x) \text{ a.e. in } \mathbb{R}^N. \tag{4.1}$$

Hence, we have

$$E_{0,\mu}(\tilde{u}) \leq \liminf_{n\to\infty} E_{0,\mu}(u_n^*) \leq \liminf_{n\to\infty} E_{0,\mu}(u_n) = m_{0,c}, \qquad |\tilde{u}|_2^2 \leq c^2.$$

From $E_{0,\mu}(\tilde{u}) \leq m_{0,c} < 0$, it follows that $\tilde{u} \not\equiv 0$. By Corollary 4.3, it must hold that

$$E_{0,\mu}(\tilde{u}) = m_{0,c}, \qquad |\tilde{u}|_2^2 = c^2.$$

By the Lagrange multiplier rule, there is $\tilde{\lambda} \in \mathbb{R}$ such that

$$-b|\nabla \tilde{u}|_2^2 \Delta \tilde{u} = \tilde{\lambda}\tilde{u} + |\tilde{u}|^{p-2}\tilde{u} + \mu|\tilde{u}|^{q-2}\tilde{u},$$

and then, combining with the Pohozaev identity, we have

$$\tilde{\lambda}|\tilde{u}|_2^2 = \frac{4-p}{p}|\tilde{u}|_p^p + \frac{\mu}{2q}(N(q-2) - 2q)|\tilde{u}|_q^q,$$

which implies $\tilde{\lambda} < 0$ from $2 + \frac{4}{N} < q < p = 2 + \frac{8}{N}$. □

Lemma 4.5. *Let $2+\frac{4}{N}<q<p=2+\frac{8}{N}$, there is a constant $\bar{a}=\bar{a}(b,c,q)>0$ such that for any $a\in(0,\bar{a})$, we have*

$$m_{a,c}:=\inf_{u\in S_c}E_{a,\mu}(u)<0.$$

Proof. From [21, Lemma 2.1], $\tilde{u}$ satisfies the following Pohozeav identity:

$$b|\nabla\tilde{u}|_2^4-\frac{4}{p}|\tilde{u}|_p^p-\mu\frac{N(q-2)}{2q}|\tilde{u}|_q^q=0,$$

and it follows that

$$\begin{aligned}
m_{0,c}&=E_{0,\mu}(\tilde{u})\\
&=\frac{b}{4}|\nabla\tilde{u}|_2^4-\frac{1}{p}|\tilde{u}|_p^p-\mu\frac{1}{q}|\tilde{u}|_q^q\\
&=\left(\frac{1}{4}-\frac{2}{N(q-2)}\right)b|\nabla\tilde{u}|_2^4+\left(\frac{8}{N(q-2)}-1\right)\frac{1}{p}|\tilde{u}|_p^p\\
&<0.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
E_{a,\mu}(\tilde{u})&=\frac{a}{2}|\nabla\tilde{u}|_2^2+E_{0,\mu}(\tilde{u})\\
&=\frac{a}{2}|\nabla\tilde{u}|_2^2+\left(\frac{1}{4}-\frac{2}{N(q-2)}\right)b|\nabla\tilde{u}|_2^4+\left(\frac{8}{N(q-2)}-1\right)\frac{1}{p}|\tilde{u}|_p^p\\
&\le\frac{a}{2}|\nabla\tilde{u}|_2^2+\frac{b}{4}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)\left(\frac{1}{4}-\frac{2}{N(q-2)}\right)|\nabla\tilde{u}|_2^4.
\end{aligned}$$

Let

$$\bar{a}=\frac{b}{2}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)\left(\frac{2}{N(q-2)}-\frac{1}{4}\right)|\nabla\tilde{u}|_2^2,$$

for any $a\in(0,\bar{a})$, we have $E_{a,\mu}(\tilde{u})<0$, and hence $m_{a,c}\leqslant E_{a,\mu}(\tilde{u})<0$. □

Lemma 4.6. *Let $0<a<\bar{a}$ and $c<c^*$, $m_{a,c}:=\inf_{u\in S_c}E_{a,\mu}(u)$ has a radial minimizer $\tilde{u}_{c,a}$, and $\tilde{u}_{c,a}$ solves problem (1.1) for some $\tilde{\lambda}_{c,a}<0$.*

Proof. The proof is similar with that of Lemma 4.4, and we omit it. □

Lemma 4.7. *Let $0<a<\bar{a}$, $2+\frac{4}{N}<q<p=2+\frac{8}{N}$ and $c<c^*$, there exists $0<K_{c,a}<\frac{|\nabla\tilde{u}_{c,a}|_2^2}{2}$ small enough such that*

$$0<\sup_{u\in\mathcal{A}}E_{a,\mu}(u)<\inf_{u\in\mathcal{B}}E_{a,\mu}(u),$$

where $\mathcal{A}=\{u\in S_{c,r}:|\nabla u|_2^2<K_{c,a}\}$, $\mathcal{B}=\{u\in S_{c,r}:|\nabla u|_2^2=2K_{c,a}\}$.

Proof. Let $K>0$ be arbitrary but fixed and suppose that $u,v\in S_{c,r}$ satisfies

$$|\nabla u|_2^2<K\quad\text{and}\quad|\nabla v|_2^2=2K.$$

From (1.5), we have

$$\begin{aligned}
E_{a,\mu}(v)-E_{a,\mu}(u)&\ge E_{a,\mu}(v)-\frac{a}{2}|\nabla u|_2^2-\frac{b}{4}|\nabla u|_2^4\\
&\ge\frac{aK}{2}+\frac{3bK^2}{4}-b\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}K^2-\mu\frac{c^{q-q\gamma_q}}{|Q_q|_2^{q-2}}(2K)^{\frac{N(q-2)-4}{4}}\\
&=K\left(\frac{a}{2}+\left(\frac{3}{4}-\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)bK-\mu\frac{c^{q-q\gamma_q}}{|Q_q|_2^{q-2}}(2K)^{\frac{N(q-2)-4}{4}}\right),
\end{aligned}$$

and
$$E_{a,\mu}(u) \geq \frac{a}{2}|\nabla u|_2^2 + \frac{b}{4}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)|\nabla u|_2^4 - \mu\frac{c^{q-q\gamma_q}}{2|Q_q|_2^{q-2}}|\nabla u|_2^{q\gamma_q}.$$

In summary, we can choose sufficiently small constant $0 < K_{c,a} < \frac{|\nabla \tilde{u}_{c,a}|_2^2}{2}$ such that
$$0 < \sup_{u\in\mathcal{A}} E_{a,\mu}(u) < \inf_{u\in\mathcal{B}} E_{a,\mu}(u)$$

where $\mathcal{A} = \{u \in S_{c,r} : |\nabla u|_2^2 < K_{c,a}\}$, $\mathcal{B} = \{u \in S_{c,r} : |\nabla u|_2^2 = 2K_{c,a}\}$. □

Let $u \in S_{c,r}$ be arbitrary and fixed, it is easy to see that $|\nabla(\tau \star u)|_2^2 \to 0$ and $I_{a,\mu}(u,\tau) \to 0^+$ as $\tau \to 0^+$. Hence, there exists $\hat{u}_{c,a} \in S_{c,r}$ such that $|\nabla \hat{u}_{c,a}|_2^2 < K_{c,a}$ and $E_{a,\mu}(\hat{u}_{c,a}) > 0$. Combining with Lemma 4.7, we can construct the minimax value for the functionals $E_{a,\mu}$ and $I_{a,\mu}$:
$$\widetilde{\gamma}_c = \inf_{\widetilde{h}\in\widetilde{\Gamma}_c}\max_{t\in[0,1]} I_{a,\mu}(\widetilde{h}(t))$$

with $\widetilde{\Gamma}_c = \{\widetilde{h} \in C([0,1], S_{c,r} \times \mathbb{R}) : \widetilde{h}(0) = (\hat{u}_{c,a}, 0),\ \widetilde{h}(1) = (\tilde{u}_{c,a}, 0)\}$, and
$$\gamma_c = \inf_{h\in\Gamma_c}\max_{t\in[0,1]} E_{a,\mu}(h(t))$$

with $\Gamma_c = \{h \in C([0,1], S_{c,r}) : h(0) = \hat{u}_{c,a},\ h(1) = \tilde{u}_{c,a}\}$, where $\tilde{u}_{c,a}$ is obtained in Lemma 4.6. We have the following lemma.

Lemma 4.8. *If $0 < a < \bar{a}$, $2 + \frac{4}{N} < q < p = 2 + \frac{8}{N}$ and $c < c^*$, we have*
$$\widetilde{\gamma}_c = \gamma_c \geq \max\{E_{a,\mu}(\hat{u}_{c,a}),\ E_{a,\mu}(\tilde{u}_{c,a})\} := \delta_c > 0.$$

Proof. For any $\widetilde{h} \in \widetilde{\Gamma}_c$, we can write it into
$$\widetilde{h}(t) = (\widetilde{h}_1(t), \widetilde{h}_2(t)) \in S_{c,r} \times \mathbb{R}.$$

Setting $h(t) = \widetilde{h}_2(t) \star \widetilde{h}_1(t)$, we have $h(t) \in \Gamma_c$ and
$$\max_{t\in[0,1]} I_{a,\mu}(\widetilde{h}(t)) = \max_{t\in[0,1]} E_{a,\mu}(\widetilde{h}_2(t) \star \widetilde{h}_1(t)) = \max_{t\in[0,1]} E_{a,\mu}(h(t)),$$

which implies $\widetilde{\gamma}_c \geq \gamma_c$. On the other hand, for any $h \in \Gamma_c$, set $\widetilde{h}(t) = (h(t), 0)$, we get $\widetilde{h} \in \widetilde{\Gamma}_c$ and
$$\max_{t\in[0,1]} I_{a,\mu}(\widetilde{h}(t)) = \max_{t\in[0,1]} E_{a,\mu}(h(t)),$$

which provides that $\gamma_c \geq \widetilde{\gamma}_c$. Thus, we have $\widetilde{\gamma}_c = \gamma_c$. Finally, $\gamma_c \geq \max\{E_{a,\mu}(\hat{u}_{c,a}), E_{a,\mu}(\tilde{u}_{c,a})\} > 0$ follows from the definition of γ_c. □

In what follows, we give the relationship between the Palais–Smale sequence for the functional $I_{a,\mu}$ and that of the functional $E_{a,\mu}$.

Lemma 4.9. *There exists a sequence $\{(v_n, \tau_n)\} \subset S_{c,r} \times \mathbb{R}^+$ such that for $n \to \infty$, we have*

(1) $I_{a,\mu}(v_n, \tau_n) \to \widetilde{\gamma}_c$,

(2) $I'_{a,\mu}|_{S_{c,r}\times\mathbb{R}}(v_n, \tau_n) \to 0$, *i.e., it holds that*

$$\partial_\tau I_{a,\mu}(v_n, \tau_n) \to 0 \quad \textit{and} \quad \langle \partial_u I_{a,\mu}(v_n, \tau_n), \widetilde{\varphi} \rangle \to 0$$

for any

$$\widetilde{\varphi} \in T_{v_n} = \left\{ \widetilde{\varphi} \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} v_n \widetilde{\varphi} dx = 0 \right\}.$$

In addition, setting $u_n(x) = \tau_n \star v_n(x)$, *then for* $n \to \infty$ *we get*

(i) $E_{a,\mu}(u_n) \to \gamma_c$,

(ii) $P_\mu(u_n) \to 0$,

(iii) $E'_{a,\mu}|_{S_{c,r}}(u_n) \to 0$, *i.e., it holds that* $\langle E'_{a,\mu}(u_n), \varphi \rangle \to 0$ *for any*

$$\varphi \in T_{u_n} = \left\{ \varphi \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} u_n \varphi dx = 0 \right\}.$$

Proof. According to the construction of $\widetilde{\gamma}_c$, we know that the conclusions (1) and (2) follow directly from Ekeland's Variational Principle. Next we mainly prove (i)–(iii).

For (i), it is obvious from

$$E_{a,\mu}(u_n) = E_{a,\mu}(\tau_n \star v_n) = I_{a,\mu}(v_n, \tau_n)$$

and $\widetilde{\gamma}_c = \gamma_c$.

For (ii), we first have

$$\begin{aligned}
\partial_\tau I_{a,\mu}(v_n, \tau_n) &= e^{2\tau_n} a |\nabla v_n|_2^2 + e^{4\tau_n} b |\nabla v_n|_2^4 - \mu e^{\gamma_q q \tau_n} \gamma_q |v_n|_q^q - e^{4\tau_n} \frac{4}{p} |v_n|_p^p \\
&= a|\nabla(\tau_n \star v_n)|_2^2 + b|\nabla(\tau_n \star v_n)|_2^4 - \mu\gamma_q |\tau_n \star v_n|_q^q - \frac{4}{p} |\tau_n \star v_n|_p^p \\
&= a|\nabla u_n|_2^2 + b|\nabla u_n|_2^4 - \mu\gamma_q |u_n|_q^q - \frac{4}{p}|u_n|_p^p \\
&= P_\mu(u_n).
\end{aligned}$$

Thus, (ii) is a consequence of $\partial_\tau I_{a,\mu}(v_n, \tau_n) \to 0$ as $n \to \infty$.

For (iii), by the definition of the functional $I_{a,\mu}$, we have

$$\begin{aligned}
\langle \partial_u I_{a,\mu}(v_n, \tau_n), \widetilde{\varphi} \rangle = e^{2\tau_n} a \int_{\mathbb{R}^N} \nabla v_n \nabla \widetilde{\varphi} dx + e^{4\tau_n} b |\nabla v_n|_2^2 \int_{\mathbb{R}^N} \nabla v_n \nabla \widetilde{\varphi} dx \\
- \mu e^{\gamma_q q \tau_n} \int_{\mathbb{R}^N} |v_n|^{q-2} v_n \widetilde{\varphi} dx - e^{4\tau_n} \int_{\mathbb{R}^N} |v_n|^{p-2} v_n \widetilde{\varphi} dx,
\end{aligned}$$

where

$$\widetilde{\varphi} \in T_{v_n} = \left\{ \widetilde{\varphi} \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} v_n \widetilde{\varphi} dx = 0 \right\}.$$

On the other hand, for any

$$\varphi \in T_{u_n} = \left\{ \varphi \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} u_n \varphi dx = 0 \right\},$$

from $u_n(x) = \tau_n \star v_n(x)$, we have

$$\begin{aligned}
&\langle E'_{a,\mu}(u_n), \varphi\rangle \\
&= a\int_{\mathbb{R}^N} \nabla u_n \nabla\varphi dx + b|\nabla u_n|_2^2 \int_{\mathbb{R}^N} \nabla u_n \nabla\widetilde{\varphi} dx - \mu\int_{\mathbb{R}^N} |u_n|^{q-2}u_n\varphi dx - \int_{\mathbb{R}^N} |u_n|^{p-2}u_n\varphi dx \\
&= e^{2\tau_n} a\int_{\mathbb{R}^N} \nabla v_n e^{-\frac{N\tau_n}{2}} \nabla\varphi(e^{-\tau_n}x)dx + e^{4\tau_n} b|\nabla v_n|_2^2 \int_{\mathbb{R}^N} \nabla v_n e^{-\frac{N\tau_n}{2}} \nabla\varphi(e^{-\tau_n}x)dx \\
&\quad - \mu e^{\gamma_q q\tau_n}\int_{\mathbb{R}^N} |v_n(x)|^{q-2}v_n(x)e^{-\frac{N\tau_n}{2}}\varphi(e^{-\tau_n}x)dx \\
&\quad - e^{4\tau_n}\int_{\mathbb{R}^N} |v_n(x)|^{p-2}v_n(x)e^{-\frac{N\tau_n}{2}}\varphi(e^{-\tau_n}x)dx.
\end{aligned}$$

Setting

$$\widetilde{\varphi}(x) = e^{-\frac{N\tau_n}{2}}\varphi(e^{-\tau_n}x),$$

we get (iii) if we could show $\widetilde{\varphi} \in T_{v_n}$. In fact, $\widetilde{\varphi} \in T_{v_n}$ comes from the following equalities:

$$0 = \int_{\mathbb{R}^N} u_n\varphi dx = \int_{\mathbb{R}^N} e^{\frac{N\tau_n}{2}} v_n(e^{\tau_n}x)\varphi(x)dx = \int_{\mathbb{R}^N} v_n(x)e^{-\frac{N\tau_n}{2}}\varphi(e^{-\tau_n}x)dx = \int_{\mathbb{R}^N} v_n\widetilde{\varphi} dx. \qquad \square$$

Lemma 4.10. *$\gamma_c > 0$ can be achieved by some $u_{c,a} \in S_{c,r}$, and $u_{c,a}$ is a radial solution of problem (1.1) for some $\lambda_c < 0$.*

Proof. By Lemma 4.1 and Lemma 4.9, we obtain a bounded Palais–Smale sequence $\{u_n\} \subset S_{c,r}$ for $E_{a,\mu}|_{S_{c,r}}$ at level $\gamma_c > 0$ such that $P_\mu(u_n) \to 0$ as $n \to \infty$. By Lemma 2.2, we have $u_n \to u_{c,a}$ in $H_r^1(\mathbb{R}^N)$, and $u_{c,a} \in S_{c,r}$ is a radial solution of problem (1.1) for some $\lambda_c < 0$. $\square$

Proof of Theorem 1.2. Theorem 1.2 comes from Lemma 4.4, Lemma 4.6 and Lemma 4.10. $\square$

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