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Normal forms for retarded functional differential equations associated with zero-double-Hopf singularity with 1 : 1 resonance

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Abstract. This manuscript introduces a framework that focuses on the singularity of a zero-double-Hopf system with 1 : 1 resonance in general retarded differential equations (RDDs). Initially, practical algorithms are proposed to identify the zero-double-Hopf singularity and the associated generalized eigenspace that corresponds to zero and two pairs of purely imaginary eigenvalues. Subsequently, by utilizing center manifold reduction and normal form techniques, we derive a reduced form of parameterized retarded differential systems up to third-order terms.

Keywords: retarded differential equations, zero-double-Hopf with 1 : 1 resonance, center manifold, normal form.

2020 Mathematics Subject Classification: 34K17, 34K18, 34C20, 34C23.

1 Introduction

In this paper, our primary objective is to analyze the zero-double-Hopf bifurcation with 1 : 1 resonance in relation to the following equation:

$$\dot{z}(t) = A(\epsilon)z(t) + B(\epsilon)z(t-\tau) + F(z(t), z(t-\tau), \epsilon), \tag{1.1}$$

where $z \in \mathbb{R}^n$, $\epsilon \in \mathbb{R}^m$, $A(\epsilon)$, $B(\epsilon) \in C^2(\mathbb{R}^m, \mathcal{M}_{n\times n}(\mathbb{R}))$ and $F \in C^3(\mathbb{R}^{2n+m}, \mathbb{R}^n)$ satisfies

$$F(0,0,\epsilon) = \frac{\partial F}{\partial x}(0,0,\epsilon) = \frac{\partial F}{\partial y}(0,0,\epsilon) = 0.$$

The characteristic equation of (1.1) at $(z, \epsilon) = (0, 0)$ is

$$\Delta(\lambda) \equiv \det(\lambda I_n - A - e^{-\lambda\tau}B) = 0, \tag{1.2}$$

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where I_n is the $n \times n$ identity matrix, $A = A(0)$ and $B = B(0)$.

To understand the dynamic behavior of the given differential system (1.1), it is crucial to examine the root distribution in equation (1.2). Several cases can arise, including:

- All roots of equation (1.2) have negative real parts, except for a double or triple zero root, respectively. In these scenarios, if the transversality condition is satisfied, the system (1.1) undergoes a Bogdanov–Takens bifurcation or a triple-zero bifurcation, respectively. The Bogdanov–Takens bifurcation in neutral differential systems was studied in [4], while both the Bogdanov–Takens and triple-zero bifurcations for neutral functional differential equations with multiple delays were explored in [2].

- All roots of equation (1.2) possess negative real parts, except for a pair of purely imaginary roots. In this case, if the transversality condition is met, the system (1.1) undergoes a Hopf bifurcation. This case has been discussed in [10].

- All roots of equation (1.2) exhibit negative real parts, except for a pair of purely imaginary roots and a simple zero root. If the transversality condition is satisfied, the system (1.1) experiences a zero-Hopf bifurcation. This case has been examined for delayed differential equations in [11] and for neutral differential equations in [1].

- All roots of equation (1.2) have negative real parts, except for two pairs of purely imaginary roots $\pm i w_1$ and $\pm i w_2$. In this situation, a double-Hopf bifurcation may occur. The corresponding normal form for scalar DDE has been computed in [3], while the same has been derived for systems of delay differential equations in [9] for the case where $\frac{w_1}{w_2} \notin \mathbb{Q}$.

- All roots of equation (1.2) possess negative real parts, except for two pairs of purely imaginary roots $\pm i w_1$ and $\pm i w_2$, where $w_1 = w_2$. In [14], a double-Hopf bifurcation with 1:1 resonance in a van der Pol oscillator has been studied. Where, the authors established explicit conditions for the characteristic equation to have a pair of purely imaginary roots with multiplicity 2 and they derived the corresponding normal forms up to order 2. In [12], authors presented a framework for studying the double-Hopf singularity with 1:1 resonance in general delayed differential equations. They also derived the corresponding normal form up to the third-order terms. To illustrate the application of their study, they applied these findings to a van der Pol oscillator with delayed feedback.

Explicit expressions for the eigenspace, its dual, and the coefficients of the normal form related to the zero-double-Hopf singularity in retarded differential equations have not been provided thus far. In [7], the authors presented the second-order normal form associated with the zero-Hopf singularity for one-dimensional delayed differential equations. However, it has been demonstrated in [8] that the second-order normal form is insufficient for determining and analyzing the bifurcation diagrams of the zero-Hopf singularity. Consequently, to obtain comprehensive bifurcation diagrams for this singularity, it becomes necessary to compute the third-order normal form, which poses greater challenges and complexities compared to the second-order normal form.

The remaining sections of this paper are organized as follows: Section 2 establishes specific conditions for the examined system to guarantee the existence of the zero-double-Hopf singularity. Section 3 applies the theory of normal forms to compute the normal form up to third-order terms for this singularity. The concluding remarks are presented in the final section of the manuscript.

2 Existence of the zero-double-Hopf with 1 : 1 resonance singularity

In this section, our examination focuses on the existence of the zero-double-Hopf singularity with 1 : 1 resonance in the analyzed retarded differential equation, considering the case where $\epsilon \in \mathbb{R}^3$. We make use of the concepts and notations introduced in [5,6,13] for our investigation.

The system (1.1) can be expressed in the following form:

$$\dot{z}(t) = L(\epsilon)z_t + F(z_t,\epsilon) = L(0)z_t + \hat{F}(z_t,\epsilon), \qquad z_t(\theta) = z(t+\theta), \qquad -\tau \leq \theta \leq 0, \tag{2.1}$$

with $\hat{F}(z_t,\epsilon) = (L(\epsilon) - L(0))z_t + F(z_t,\epsilon)$, $L(\epsilon)\phi = \int_{-\tau}^{0} d[\eta_\epsilon(\theta)]\phi(\theta)$, for all $\phi \in C = C([-\tau,0],\mathbb{R}^n)$, with supreme norm.

In particular, $Lz_t = L(0)z_t = Az(t) + Bz(t-\tau) = \int_{-\tau}^{0} d[\eta_0(\theta)]z_t(\theta)$.

The function η_ϵ is a bounded variation matrix-valued function defined on the interval $[-\tau,0]$ as follows:

$$\eta_\epsilon(\theta) = \begin{cases} A(\epsilon) + B(\epsilon), & \theta = 0, \\ B(\epsilon), & -\tau < \theta < 0, \\ 0, & \theta = -\tau. \end{cases}$$

Let us consider the linear system given by

$$\dot{z}(t) = L(0)z_t. \tag{2.2}$$

As stated in [13], the infinitesimal generator for the solution semigroup defined by the system (2.2) can be represented as:

$$\mathcal{A}_0\phi = \dot{\phi},$$
$$\mathbf{D}(\mathcal{A}_0) = \left\{ \phi \in C : \frac{d\phi}{d\theta} \in C, \dot{\phi}(0) = \mathrm{L}(0)\phi \right\}.$$

The adjoint innner product on $C \times C^*$ is defined by:

$$\langle \psi, \phi \rangle = \psi(0)\phi(0) - \int_{-\tau}^{0} \int_{0}^{s} \psi(\theta - s)d[\eta_0(s)]\phi(\theta)d\theta.$$

where $C^* = C([0,\tau],\mathbb{R}^{n*})$, with $\mathbb{R}^{n*}$ is the space of all row n-vector.

The adjoint $\mathcal{A}_0^*$ of $\mathcal{A}_0$ is defined as follows:

$$\mathcal{A}_0^*\psi = -\dot{\psi},$$
$$\mathbf{D}(\mathcal{A}_0^*) = \left\{ \psi \in C^*, \frac{d\phi}{d\theta} \in C^*, -\dot{\psi}(0) = \int_{-\tau}^{0} \psi(-\theta)d[\eta_0(\theta)] \right\}.$$

Now, it is necessary to impose the following hypotheses:

(**A1**): The infinitesimal generator $\mathcal{A}_0$ possesses a pair of purely imaginary eigenvalues $\lambda = \pm iw$ $(w > 0)$ with an algebraic multiplicity of 2 and a geometric multiplicity of 1.

(**A2**): The infinitesimal generator $\mathcal{A}_0$ has a unique eigenvalue $\lambda = 0$.

(**A3**): All eigenvalues of the infinitesimal generator $\mathcal{A}_0$ exhibit negative real parts, except for the simple zero eigenvalue and the two pairs of purely imaginary eigenvalues.

Let P denote the eigenspace of $\mathcal{A}_0$, and let P^* represent the adjoint space of P.

The space $C = C([-\tau, 0], \mathbb{R}^n)$ can be decomposed as $C = P \oplus Q$, where $Q = \{\phi \in C : \langle \psi, \phi \rangle = 0, \forall\ \psi \in P^*\}$.

Let ϕ_1, ϕ_2, and ϕ_3 denote eigenvectors of P.

Therefore, we have $\mathcal{A}_0\phi_1 = iw\phi_1$, $(\mathcal{A}_0 - iwI_n)\phi_2 = \phi_1$, and $\mathcal{A}_0\phi_3 = 0$.
By employing the definition of $\mathcal{A}_0$, we derive the following expressions:

$$\begin{cases} \dot{\phi}_1(\theta) = iw\phi_1(\theta), & -\tau \le \theta < 0, \\ \dot{\phi}_1(0) = L_0\phi_1(\theta), & \theta = 0, \end{cases} \qquad \begin{cases} \dot{\phi}_2(\theta) = iw\phi_2(\theta) + \phi_1(\theta), & -\tau \le \theta < 0, \\ iw\phi_2(0) + \phi_1(0) = L_0\phi_2(\theta), & \theta = 0 \end{cases}$$

and

$$\begin{cases} \dot{\phi}_3(\theta) = 0, & -\tau \le \theta < 0, \\ \dot{\phi}_3(0) = L_0\phi_3(\theta), & \theta = 0. \end{cases}$$

Hence, the eigenvectors ϕ_1, ϕ_2, and ϕ_3 can be written as follows: $\phi_1(\theta) = e^{iw\theta}\phi_1^0$, $\phi_2(\theta) = e^{iw\theta}(\phi_2^0 + \theta\phi_1^0)$, and $\phi_3(\theta) = \phi_3^0$, where $\phi_1^0, \phi_2^0 \in \mathbb{C}^n \setminus \{0\}$ and $\phi_3^0 \in \mathbb{R}^n \setminus \{0\}$. These vectors satisfy the following equations:

$$iw\phi_1^0 = (A + e^{-iw\tau}B)\phi_1^0, \quad (\tau e^{-iw\tau}B + I_n)\phi_1^0 = (A + e^{-iw\tau}B - iwI_n)\phi_2^0 \quad \text{and} \quad (A+B)\phi_3^0 = 0.$$

Consequently, we have $P = \text{span}\{\phi_1, \phi_2, \bar{\phi}_1, \bar{\phi}_2, \phi_3\}$.

Now, let ψ_1, ψ_2, and ψ_3 denote the eigenvectors of $\mathcal{A}_0^*$.

Hence, we have $\mathcal{A}_0^*\psi_2 = -iw\psi_2$, $(\mathcal{A}_0^* + iwI_n)\psi_1 = \psi_2$, and $\mathcal{A}_0^*\psi_3 = 0$.
Accordingly, the eigenvectors ψ_1, ψ_2, and ψ_3 can be represented as follows:
$\psi_2(s) = e^{-iws}\psi_2^0$, $\psi_1(s) = e^{-iws}(\psi_1^0 - s\psi_2^0)$ and $\psi_3(s) = \psi_3^0$, where $\psi_1^0, \psi_2^0 \in \mathbb{C}^{n*} \setminus \{0\}$ and $\psi_3^0 \in \mathbb{R}^{n*} \setminus \{0\}$ satisfying the following equations:

$$\psi_2^0(A + e^{iw\tau}B) = -iw\psi_2^0, \quad \psi_2^0(\tau e^{iw\tau}B + I_n) = \psi_1^0(A + e^{iw\tau}B + iwI_n) \quad \text{and} \quad \psi_3^0(A+B) = 0.$$

So, $P^* = \text{span}\{\bar{\psi}_1, \bar{\psi}_2, \psi_1, \psi_2, \psi_3\}$.

It is crucial to emphasize that the eigenvectors of P and P^* must fulfill the following conditions:

$$\langle \bar{\psi}_1, \phi_1 \rangle = \langle \psi_1, \bar{\phi}_1 \rangle = \langle \bar{\psi}_2, \phi_2 \rangle = \langle \psi_2, \bar{\phi}_2 \rangle = \langle \phi_3, \psi_3 \rangle = 1 \tag{2.3}$$

and

$$\langle \psi_1, \phi_1 \rangle = \langle \psi_1, \phi_2 \rangle = \langle \psi_1, \bar{\phi}_2 \rangle = \langle \psi_1, \phi_3 \rangle = 0, \tag{2.4}$$
$$\langle \bar{\psi}_1, \bar{\phi}_1 \rangle = \langle \bar{\psi}_1, \phi_2 \rangle = \langle \bar{\psi}_1, \bar{\phi}_2 \rangle = \langle \bar{\psi}_1, \phi_3 \rangle = 0, \tag{2.5}$$
$$\langle \psi_2, \phi_1 \rangle = \langle \psi_2, \bar{\phi}_1 \rangle = \langle \psi_2, \phi_2 \rangle = \langle \psi_2, \phi_3 \rangle = 0, \tag{2.6}$$
$$\langle \bar{\psi}_2, \phi_1 \rangle = \langle \bar{\psi}_2, \bar{\phi}_1 \rangle = \langle \bar{\psi}_2, \bar{\phi}_2 \rangle = \langle \bar{\psi}_2, \phi_3 \rangle = 0, \tag{2.7}$$
$$\langle \psi_3, \phi_1 \rangle = \langle \psi_3, \bar{\phi}_1 \rangle = \langle \psi_3, \phi_2 \rangle = \langle \psi_3, \bar{\phi}_2 \rangle = 0. \tag{2.8}$$

Therefore, we can appropriately select values for ϕ_1^0, ϕ_2^0, $\bar{\phi}_1^0$, $\bar{\phi}_2^0$, ϕ_3^0, $\bar{\psi}_1^0$, $\bar{\psi}_2^0$, ψ_1^0, ψ_2^0, and ψ_3^0 to ensure the satisfaction of equations (2.3), (2.4), (2.5), (2.6), (2.7) and (2.8).

Let $\Phi = (\phi_1, \phi_2, \bar{\phi}_1, \bar{\phi}_2, \phi_3)$ and $\Psi = (\bar{\psi}_1, \bar{\psi}_2, \psi_1, \psi_2, \psi_3)^T$, then $\dot{\Phi} = \Phi J$ and $\dot{\Psi} = -J\Psi$, where

$$J = \begin{pmatrix} iw & 1 & 0 & 0 & 0 \\ 0 & iw & 0 & 0 & 0 \\ 0 & 0 & -iw & 1 & 0 \\ 0 & 0 & 0 & -iw & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

3 Calculation of the normal form

In this section, our focus is on calculating the normal form up to the third order associated with the zero-double-Hopf singularity. Our approach is based on the methodology introduced by Faria and Magalhães [5,6].

We assume that hypotheses (**A1**), (**A2**), and (**A3**) are satisfied.

Let BC the enlarged space of C, which is defined as follows:

$$BC = \{\phi : [-\tau, 0] \to \mathbb{R}^n : \ \phi \text{ uniformly continuous on } [-\tau, 0) \\ \text{and with a possible jump discontinuity at } 0\}.$$

An element $\psi \in BC$ can be expressed as: $\psi = \phi + X_0 v$, with $\phi \in C$, $v \in \mathbb{R}^n$ and

$$X_0(\theta) = \begin{cases} 0, & -\tau \le \theta < 0, \\ I_n, & \theta = 0. \end{cases}$$

Let π be the projection defined as

$$\pi : BC \to P$$
$$\phi + X_0 v \mapsto \Phi[< \Psi, \phi > + \Psi(0)v].$$

The differential system (2.1) can be reformulated as

$$\begin{cases} \dot{x} = Jx + \sum_{j\ge 2} \frac{1}{j!} f_j^1(x, y, \epsilon) \\ \dot{y} = \mathcal{A}_{Q^1} y + \sum_{j\ge 2} \frac{1}{j!} f_j^2(x, y, \epsilon) \end{cases} \tag{3.1}$$

where $f_j^1(x,y,\epsilon) = \Psi(0)\hat{F}_j(\Phi x + y, \epsilon)$ and $f_j^2(x,y,\epsilon) = (I - \pi)\hat{F}_j(\Phi x + y, \epsilon)$, $\hat{F}(\Phi x + y, \epsilon) = \sum_{j\ge 2} \frac{1}{j!}\hat{F}_j(\Phi x + y, \epsilon)$ and $z_t = \Phi x + y$, with $x \in \mathbb{C}^5$ and $y \in Q^1 = \{\phi \in Q : \dot{\phi} \in C\}$, and $\mathcal{A}_{Q^1} \subset \ker(\pi)$, $\mathcal{A}_{Q^1}\phi = \dot{\phi} + X_0(L(0)\phi - \dot{\phi}(0))$.

The normal form associated with P of the system (3.1) can be represented as follows on its center manifold:

$$\dot{x} = Jx + \frac{1}{2} g_2^1(x, 0, \epsilon) + \frac{1}{6} g_3^1(x, 0, \epsilon) + \text{h.o.t.}, \tag{3.2}$$

where g_2^1 and g_3^1 are the second and third order terms in (x, ϵ), respectively.

Let M_j be the operator defined in $V_j^8(\mathbb{C}^5 \times \ker \pi)$ with the range in the same space by $M_j(f, g) = (M_j^1 f, M_j^2 g)$, with $V_j(Y)$ is the the space of homogeneous polynomials with degree j, for a normed space Y, where

$$M_j^1 f = M_j^1 \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} = D_x f(x, \epsilon) Jx - Jf(x, \epsilon),$$

$$M_j^2 g = M_j^2 g(x, \epsilon) = D_x g(x, \epsilon) Jx - \mathcal{A}_{Q^1} g(x, \epsilon),$$

with $f(x,\epsilon) \in V_j^8(\mathbb{C}^5)$ and $(x,\epsilon) \in V_j^8(\ker \pi)$.

By using the above notation, $g_2^1(x,0,\epsilon)$ and $g_3^1(x,0,\epsilon)$ can be written as follows:

$$g_2^1(x,0,\epsilon) = \text{Proj}_{\ker(M_2^1)} f_2^1(x,0,\epsilon) = \text{Proj}_{S^1} f_2^1(x,0,\epsilon) + \mathcal{O}(|\epsilon|^2)$$
$$g_3^1(x,0,\epsilon) = \text{Proj}_{\ker(M_3^1)} \tilde{f}_3^1(x,0,\epsilon) = \text{Proj}_{S^2} \tilde{f}_3^1(x,0,\epsilon) + \mathcal{O}(|\epsilon|^2 x),$$

where

$$\tilde{f}_3^1(x,0,\epsilon) = f_3^1(x,0,\epsilon) + \frac{3}{2}[(D_x f_2^1)(x,0,\epsilon)U_2^1(x,\epsilon) + (D_y f_2^1)(x,0,\epsilon)U_2^2(x,\epsilon)]$$

U_2^1 and U_2^2 are defined by:

$$U_2^1(x,\epsilon) = (M_2^1)^{-1}\,\text{Proj}_{Im(M_2^1)} f_2^1(x,0,\epsilon) \quad \text{and} \quad (M_2^2 U_2^2(x,\epsilon)) = f_2^2(x,0,\epsilon).$$

$\ker(M_2^1)$ is spanned by

$$\epsilon_j x_2 e_1,\ x_2x_5e_1,\ \epsilon_j x_4 e_3,\ x_4x_5e_3,\ \epsilon_1\epsilon_2 e_5,\ \epsilon_1\epsilon_3 e_5,\ \epsilon_2\epsilon_3 e_5,\ \epsilon_j^2 e_5,\ \epsilon_j x_5 e_5,\ x_2x_4e_5, x_5^2 e_5,$$

for $j = 1,2,3$, with $(e_1,e_2,e_3,e_4,e_5)^T$ being the canonical basis of $\mathbb{R}^5$.

$\ker(M_3^1)$ is spanned by:

$$\begin{aligned}
&\epsilon_j^2 x_2 e_1,\ x_2^2 x_4 e_1,\ x_5^2 x_2 e_1,\ \epsilon_j x_2 x_5 e_1,\ \epsilon_1\epsilon_2 x_2 e_1,\ \epsilon_1\epsilon_3 x_2 e_1,\ \epsilon_2\epsilon_3 x_2 e_1,\\
&\epsilon_j^2 x_4 e_3,\ x_4^2 x_2 e_3,\ x_5^2 x_4 e_3,\ \epsilon_j x_4 x_5 e_3,\ \epsilon_1\epsilon_2 x_4 e_3,\ \epsilon_1\epsilon_3 x_4 e_3,\ \epsilon_2\epsilon_3 x_4 e_3,\\
&\epsilon_j^3 e_5,\ \epsilon_1^2\epsilon_2 e_5,\ \epsilon_1^2\epsilon_3 e_5,\ \epsilon_2^2\epsilon_1 e_5,\ \epsilon_2^2\epsilon_3 e_5,\ \epsilon_3^2\epsilon_1 e_5,\ \epsilon_3^2\epsilon_1 e_5,\ \epsilon_j^2 x_5 e_5,\ x_5^3 e_5,\\
&\epsilon_j x_5^2 e_5,\ \epsilon_j x_2 x_4 e_5,\ \epsilon_1\epsilon_2 x_5 e_5,\ \epsilon_1\epsilon_3 x_5 e_5,\ \epsilon_2\epsilon_3 x_5 e_5,\ x_2x_4x_5e_5,
\end{aligned}$$

for $j = 1,2,3$, with $(e_1,e_2,e_3,e_4,e_5)^T$ being the canonical basis of $\mathbb{R}^5$.

Consequently, S^1 and S^2 are spanned respectively by

$$\epsilon_j x_2 e_1, x_2x_5e_1, \epsilon_j x_4 e_3, x_4x_5e_3, \epsilon_j x_5 e_5, x_2x_4e_5, x_5^2 e_5$$

and

$$x_2^2 x_4 e_1,\ x_5^2 x_2 e_1,\ x_4^2 x_2 e_3,\ x_5^2 x_4 e_3,\ x_5^3 e_5 x_2 x_4 x_5 e_5, \quad \text{for } j = 1,2,3.$$

Write

$$\begin{aligned}
\frac{1}{2}\hat{F}_2(z_t,\epsilon) = {}& A_1\epsilon_1 z(t) + A_2\epsilon_2 z(t) + A_3\epsilon_3 z(t) + B_1\epsilon_1 z(t-\tau) + B_2\epsilon_2 z(t-\tau) + B_3\epsilon_3 z(t-\tau)\\
&+ \sum_{i=1}^{n} E_i z_i(t) z(t-\tau) + \sum_{i=1}^{n} F_i z_i(t) z(t) + \sum_{i=1}^{n} K_i z_i(t-\tau) z(t-\tau),
\end{aligned}$$

with $A(\epsilon) = A + \epsilon_1 A_1 + \epsilon_2 A_2 + \epsilon_3 A_3 + O(|\epsilon|^2)$ and $B(\epsilon) = B + \epsilon_1 B_1 + \epsilon_2 B_2 + \epsilon_3 B_3 + O(|\epsilon|^2)$. So

$$\begin{aligned}
\frac{1}{2}\hat{F}_2(\Phi x,\epsilon) = {}& H_1\epsilon_1 x_1 + H_2\epsilon_2 x_1 + H_3\epsilon_3 x_1 + H_4\epsilon_1 x_2 + H_5\epsilon_2 x_2 + H_6\epsilon_3 x_2 + H_7\epsilon_1 x_3 + H_8\epsilon_2 x_3\\
&+ H_9\epsilon_3 x_3 + H_{10}\epsilon_1 x_4 + H_{11}\epsilon_2 x_4 + H_{12}\epsilon_3 x_4 + H_{13}\epsilon_1 x_5 + H_{14}\epsilon_2 x_5\\
&+ H_{15}\epsilon_3 x_5 + H_{16}x_1^2 + H_{17}x_2^2 + H_{18}x_3^2 + H_{19}x_4^2 + H_{20}x_5^2 + H_{21}x_1x_2 + H_{22}x_1x_3\\
&+ H_{23}x_1x_4 + H_{24}x_1x_5 + H_{25}x_2x_3 + H_{26}x_2x_4 + H_{27}x_2x_5 + H_{28}x_3x_4 + H_{29}x_3x_5\\
&+ H_{30}x_4x_5 + \mathcal{O}(|\epsilon|^2|x|),
\end{aligned}$$

and

$$\frac{1}{2}\hat{F}_2(\Phi x+y,\epsilon)=\frac{1}{2}\hat{F}_2(\Phi x,\epsilon)+\sum_{j=1}^{5}\sum_{k=1}^{n}(R_{jk}x_jy_k(0)+S_{jk}x_jy_k(-\tau))$$
$$+\sum_{j=1}^{n}(T_jy_j^2(0)+Q_jy_j^2(-\tau))+\sum_{j,k=1}^{n}(P_{jk}y_j(0)y_k(-\tau))+\mathcal{O}(|\epsilon|^2|x|), \qquad (3.3)$$

with A_1, A_2, A_3, B_1, B_2, B_3, E_j, F_j, K_j, R_{jk}, S_{jk}, P_{jk}, T_j and Q_j, for $1\le j,k\le n$ are coefficient matrix. The values of H_i for $1\le i\le 30$ are provided in the appendix.

Write

$$\frac{1}{6}\hat{F}_3(z_t,0)=\sum_{i,j=1}^{n}\Omega^1_{i,j}z_i(t)z_j(t)z(t)+\sum_{i,j=1}^{n}\Omega^2_{i,j}z_i(t)z_j(t-\tau)z(t-\tau)$$
$$+\sum_{i,j=1}^{n}\Omega^3_{i,j}z_i(t-\tau)z_j(t)z(t)+\sum_{i,j=1}^{n}\Omega^4_{i,j}z_i(t-\tau)z_j(t-\tau)z(t-\tau),$$

$\Omega^1_{i,j}$, $\Omega^2_{i,j}$, $\Omega^3_{i,j}$ and $\Omega^4_{i,j}$, for $1\le i,j\le n$, are coefficient matrices. So

$$\begin{aligned}\frac{1}{6}\hat{F}_3(\Phi x,0)=&\,G_1x_1^3+G_2x_2^3+G_3x_3^3+G_4x_4^3+G_5x_5^3+G_6x_1^2x_2+G_7x_1^2x_3+G_8x_1^2x_4+G_9x_1^2x_5\\&+G_{10}x_2^2x_1+G_{11}x_2^2x_3+G_{12}x_2^2x_4+G_{13}x_2^2x_5+G_{14}x_3^2x_1+G_{15}x_3^2x_2+G_{16}x_3^2x_4\\&+G_{17}x_3^2x_5+G_{18}x_4^2x_1+G_{19}x_4^2x_2+G_{20}x_4^2x_3+G_{21}x_4^2x_5+G_{22}x_5^2x_1+G_{23}x_5^2x_2\\&+G_{24}x_5^2x_3+G_{25}x_5^2x_4+G_{26}x_1x_2x_3+G_{27}x_1x_2x_4+G_{28}x_1x_2x_5+G_{29}x_1x_3x_4\\&+G_{30}x_1x_3x_5+G_{31}x_1x_4x_5+G_{32}x_2x_3x_4+G_{33}x_2x_3x_5+G_{34}x_2x_4x_5\\&+G_{35}x_3x_4x_5+\mathcal{O}(|\epsilon|^2|x|).\end{aligned}$$

The values of G_i for $i=1,2,\ldots,35$ are provided in the appendix.

We have:

$$\frac{1}{2}g_2^1(x,0,\epsilon)=\frac{1}{2}\operatorname{Proj}_{S^1}f_2^1(x,0,\epsilon)=\frac{1}{2}\operatorname{Proj}_{S^1}\Psi(0)\hat{F}_2(\Phi x,\epsilon)=\begin{pmatrix}\alpha_1x_2+\alpha_2x_2x_5\\0\\\alpha_3x_4+\alpha_4x_4x_5\\0\\\alpha_6x_5+\alpha_7x_2x_4+\alpha_8x_5^2\end{pmatrix},$$

where $\alpha_1=\bar{\psi}_1^0(H_4\epsilon_1+H_5\epsilon_2+H_6\epsilon_3)$, $\alpha_2=\bar{\psi}_1^0H_{27}$, $\alpha_3=\psi_1^0(H_{10}\epsilon_1+H_{11}\epsilon_2+H_{12}\epsilon_3)$, $\alpha_4=\psi_1^0H_{30}$, $\alpha_5=\psi_3^0(H_{13}\epsilon_1+H_{14}\epsilon_2+H_{15}\epsilon_3)$, $\alpha_6=\psi_3^0H_{26}$ and $\alpha_7=\psi_3^0H_{20}$.

Remark 3.1. We remark that $H_4=\bar{H}_{10}$, $H_5=\bar{H}_{11}$, $H_6=\bar{H}_{12}$, $H_{27}=\bar{H}_{30}$. So, $\alpha_1=\bar{\alpha}_3$ and $\alpha_2=\bar{\alpha}_4$.

Therefore,

$$\frac{1}{2}g_2^1(x,0,\epsilon)=\begin{pmatrix}\alpha_1x_2+\alpha_2x_2x_5\\0\\\bar{\alpha}_1x_4+\bar{\alpha}_2x_4x_5\\0\\\alpha_6x_5+\alpha_7x_2x_4+\alpha_8x_5^2\end{pmatrix}.$$

Now, we will calculate the term $g_3^1(x,0,\epsilon)$.

$$\begin{aligned}\frac{1}{6}g_3^1(x,0,\epsilon) &= \frac{1}{6}\operatorname{Proj}_{\ker(M_3^1)}\tilde{f}_3^1(x,0,\epsilon)\\ &= \frac{1}{6}\operatorname{Proj}_{S^2}f_3^1(x,0,0)+\frac{1}{4}\operatorname{Proj}_{S^2}[(D_xf_2^1)(x,0,0)U_2^1(x,0)\\ &\quad +(D_yf_2^1)(x,0,0)U_2^2(x,0)]+\text{h.o.t.}\end{aligned}$$

First,

$$\frac{1}{6}\operatorname{Proj}_{S^2}f_3^1(x,0,0)=\frac{1}{6}\operatorname{Proj}_{S^2}\Psi(0)\hat{F}_3(\Phi x,0)=\begin{pmatrix}\beta_1x_2^2x_4+\beta_2x_2x_5^2\\ 0\\ \beta_3x_4^2x_2+\beta_4x_4x_5^2\\ 0\\ \beta_5x_5^3+\beta_6x_2x_4x_5\end{pmatrix},$$

where $\beta_1=\bar{\psi}_1^0G_{12}$, $\beta_2=\bar{\psi}_1^0G_{23}$, $\beta_3=\psi_1^0G_{19}$, $\beta_4=\psi_1^0G_{25}$, $\beta_5=\psi_3^0G_5$ and $\beta_6=\psi_3^0G_{34}$.

Since $f_2(x,0,0)=\Psi(0)\hat{F}_2(\Phi x,0)$, then:

$$U_2^1(x,0)=U_2^1(x,\epsilon)\mid_{\epsilon=0}=(M_2^1)^{-1}\operatorname{Proj}_{Im(M_2^1)}f_2^1(x,0,0).$$

Therefore,

$$\frac{1}{4}\operatorname{Proj}_{S^2}[(D_xf_2^1)(x,0,0)U_2^1(x,0)]=\begin{pmatrix}\gamma_1x_2^2x_4+\gamma_2x_2x_5^2\\ 0\\ \gamma_3x_4^2x_2+\gamma_4x_4x_5^2\\ 0\\ \gamma_5x_5^3+\gamma_6x_2x_4x_5\end{pmatrix},$$

where

$$\begin{aligned}\gamma_1=\frac{1}{2}\bar{\psi}_1^0H_{21}\Bigg[&-\bar{\psi}_1^0\left(\frac{2}{(iw)^3}H_{22}+\frac{1}{(iw)^2}H_{23}+\frac{1}{(iw)^2}H_{25}+\frac{1}{iw}H_{26}\right)\\ &+\bar{\psi}_2^0\left(\frac{5}{(iw)^4}H_{22}+\frac{2}{(iw)^3}H_{23}+\frac{1}{(iw)^3}H_{25}+\frac{1}{(iw)^2}H_{26}\right)\Bigg]\\ &+\frac{1}{2}\bar{\psi}_1^0H_{23}\Bigg[\bar{\psi}_1^0\left(\frac{2}{(iw)^3}H_{16}+\frac{1}{iw}H_{17}-\frac{1}{(iw)^2}H_{21}\right)\\ &\qquad+\bar{\psi}_2^0\left(\frac{4}{(iw)^4}H_{16}+\frac{1}{(iw)^2}H_{17}-\frac{1}{(iw)^3}H_{21}\right)\Bigg]\\ &-\bar{\psi}_1^0H_{17}\bar{\psi}_2^0\left(\frac{1}{(iw)^3}H_{22}+\frac{1}{(iw)^2}H_{23}+\frac{1}{iw}H_{26}\right)+\frac{1}{2iw}\bar{\psi}_1^0H_{26}\bar{\psi}_2^0H_{17}\\ &+\frac{1}{2}\bar{\psi}_1^0H_{25}\Bigg[\psi_1^0\left(\frac{2}{(iw)^3}H_{22}-\frac{1}{(iw)^2}H_{23}-\frac{1}{(iw)^2}H_{25}+\frac{1}{iw}H_{26}\right)\\ &\qquad+\psi_2^0\left(\frac{4}{(iw)^4}H_{22}-\frac{2}{(iw)^3}H_{23}-\frac{1}{(iw)^3}H_{25}+\frac{1}{(iw)^2}H_{26}\right)\Bigg]\\ &+\frac{1}{2}\bar{\psi}_1^0H_{28}\Bigg[\psi_1^0\left(\frac{2}{(iw)^3}H_{16}+\frac{1}{iw}H_{17}-\frac{1}{(iw)^2}H_{21}\right)\\ &\qquad+\psi_2^0\left(\frac{6}{(iw)^4}H_{16}+\frac{1}{(3iw)^2}H_{17}-\frac{2}{(3iw)^3}H_{21}\right)\Bigg]\\ &+\bar{\psi}_1^0H_{19}\psi_2^0\left(\frac{2}{(3iw)^3}H_{16}+\frac{1}{3iw}H_{17}-\frac{2}{(3iw)^2}H_{21}\right)\end{aligned}$$

$$+\frac{1}{2}\bar{\psi}_1^0 H_{26}\psi_2^0\Big(\frac{1}{(iw)^3}H_{22}-\frac{1}{(iw)^2}H_{23}-\frac{1}{(iw)^2}H_{25}+\frac{1}{iw}H_{26}\Big)$$
$$+\frac{1}{2}\bar{\psi}_1^0 H_{30}\psi_3^0\Big(\frac{2}{(2iw)^3}H_{16}+\frac{1}{2iw}H_{17}-\frac{1}{(2iw)^2}H_{21}\Big),$$

$$\gamma_2=\frac{1}{2}\bar{\psi}_1^0 H_{21}\Big(-\frac{1}{iw}\bar{\psi}_1^0+\frac{1}{(iw)^2}\bar{\psi}_2^0\Big)H_{20}-\bar{\psi}_1^0 H_{17}\bar{\psi}_2^0 H_{20}+\frac{1}{2}\bar{\psi}_1^0 H_{25}\Big(\frac{1}{iw}\psi_1^0+\frac{1}{(iw)^2}\psi_2^0\Big)H_{20}$$
$$+\bar{\psi}_1^0 H_{29}\Big[\psi_1^0\Big(\frac{1}{2iw}H_{27}-\frac{1}{(2iw)^2}H_{24}\Big)+\psi_2^0\Big(\frac{1}{(2iw)^2}H_{27}-\frac{2}{(2iw)^3}H_{24}\Big)\Big]+\frac{1}{2iw}\bar{\psi}_1^0 H_{26}\psi_2^0 H_{20}$$
$$+\frac{1}{2}\bar{\psi}_1^0 H_{30}\psi_2^0\Big(\frac{1}{2iw}H_{27}-\frac{1}{(2iw)^2}H_{24}\Big)+\bar{\psi}_1^0 H_{20}\psi_3^0\Big(\frac{1}{iw}H_{27}-\frac{1}{(iw)^2}H_{24}\Big)$$

$$\gamma_3=\bar{\gamma}_1,\qquad \gamma_4=\bar{\gamma}_2$$

$$\gamma_5=\frac{1}{2}\psi_3^0 H_{24}\Big(-\frac{1}{iw}\bar{\psi}_1^0+\frac{1}{(iw)^2}\bar{\psi}_2^0\Big)H_{20}-\frac{1}{2iw}\psi_3^0 H_{27}\bar{\psi}_2^0 H_{20}$$
$$+\frac{1}{2}\psi_3^0 H_{29}\Big(\frac{1}{iw}\psi_1^0+\frac{1}{(iw)^2}\psi_2^0\Big)H_{20}+\frac{1}{2iw}\psi_3^0 H_{30}\psi_2^0 H_{20}$$

$$\gamma_6=\frac{1}{2}\psi_3^0 H_{21}\Big[-\bar{\psi}_1^0\Big(\frac{1}{iw}\frac{1}{2iw}H_{29}+\frac{1}{2iw}H_{30}\Big)+\bar{\psi}_2^0\Big(\frac{1}{iw}\frac{1}{(2iw)^2}H_{29}+\frac{1}{(2iw)^2}H_{30}\Big)\Big]$$
$$+\frac{1}{2}\psi_3^0 H_{24}\Big[-\bar{\psi}_1^0\Big(\frac{2}{(iw)^3}H_{22}+\frac{1}{(iw)^2}H_{23}+\frac{1}{(iw)^2}H_{25}+\frac{1}{iw}H_{26}\Big)$$
$$+\bar{\psi}_2^0\Big(\frac{5}{(iw)^4}H_{22}+\frac{2}{(iw)^3}H_{23}+\frac{1}{(iw)^3}H_{25}+\frac{1}{(iw)^2}H_{26}\Big)\Big]$$
$$-\frac{1}{2iw}\psi_3^0 H_{17}\bar{\psi}_2^0 H_{30}-\frac{1}{2}\psi_3^0 H_{27}\bar{\psi}_2^0\Big(\frac{1}{(iw)^3}H_{22}+\frac{1}{(iw)^2}H_{23}+\frac{1}{iw}H_{26}\Big)$$
$$+\frac{1}{2}\psi_3^0 H_{28}\Big[\psi_1^0\Big(\frac{1}{2iw}H_{27}-\frac{1}{(2iw)^2}H_{24})+\psi_2^0\Big(\frac{1}{(2iw)^2}H_{27}-\frac{2}{(2iw)^3}H_{24}\Big)\Big]$$
$$+\frac{1}{2}\psi_3^0 H_{29}\Big[\psi_1^0\Big(\frac{2}{(iw)^3}H_{22}-\frac{1}{(iw)^2}H_{23}-\frac{1}{(iw)^2}H_{25}+\frac{1}{iw}H_{26}\Big)$$
$$+\psi_2^0\Big(\frac{4}{(iw)^4}H_{22}-\frac{2}{(iw)^3}H_{23}-\frac{2}{(iw)^3}H_{25}+\frac{1}{(iw)^2}H_{26}\Big)\Big]$$
$$+\psi_3^0 H_{19}\psi_2^0\Big(\frac{1}{2iw}H_{27}-\frac{1}{(2iw)^2}H_{24}\Big)$$
$$+\frac{1}{2}\psi_3^0 H_{30}\psi_2^0\Big(\frac{1}{(iw)^3}H_{22}-\frac{1}{(iw)^2}H_{23}-\frac{1}{(iw)^2}H_{25}+\frac{1}{iw}H_{26}\Big)$$

To compute $\mathrm{Proj}_{S^2} D_y f_2^1(x,0,\epsilon)U_2^2(x,0)$, we define $h=h(x)(\theta)=U_2^2$ and write

$$h(\theta)=\begin{pmatrix}h^{(1)}(\theta)\\ h^{(2)}(\theta)\\ \vdots\\ h^{(n)}(\theta)\end{pmatrix}=h_{20000}x_1^2+h_{02000}x_2^2+h_{00200}x_3^2+h_{00020}x_4^2+h_{00002}x_5^2+h_{11000}x_1x_2$$
$$+h_{10100}x_1x_3+h_{10010}x_1x_4+h_{10001}x_1x_5+h_{01100}x_2x_3+h_{01010}x_2x_4$$
$$+h_{01001}x_2x_5+h_{00110}x_3x_4+h_{00101}x_3x_5+h_{00011}x_4x_5$$

$$= \begin{pmatrix} h^{(1)}_{20000}x_1^2 + h^{(1)}_{02000}x_2^2 + h^{(1)}_{00200}x_3^2 + \cdots + h^{(1)}_{00011}x_4x_5 \\ h^{(2)}_{20000}x_1^2 + h^{(2)}_{02000}x_2^2 + h^{(2)}_{00200}x_3^2 + \cdots + h^{(2)}_{00011}x_4x_5 \\ \vdots \\ h^{(n)}_{20000}x_1^2 + h^{(n)}_{02000}x_2^2 + h^{(n)}_{00200}x_3^2 + \cdots + h^{(n)}_{00011}x_4x_5 \end{pmatrix}$$

where h_{20000},h_{02000}, h_{00200}, h_{00020}, h_{00002}, h_{11000}, h_{10100}, h_{10010}, h_{10001}, h_{01100}, h_{01010}, h_{01001}, h_{00110}, h_{00101}, $h_{00011} \in Q^1$.

The coefficients of h can be determined by solving the equation $(M_2^2 h)(x) = f_2^2(x, 0, 0)$, which can also be written as:

$$D_x hJx - \mathcal{A}_{Q^1}(h) = (I - \pi)\hat{F}_2(\Phi x, 0).$$

Next, by utilizing the definition of $\mathcal{A}_{Q^1}$ and the projection π, we derive the following system of equations:

$$\begin{cases} \dot{h} - D_x hJx = \Phi(\theta)\Psi(0)\hat{F}_2(\Phi x, 0), \\ \dot{h}(0) - Lh = \hat{F}_2(\Phi x, 0). \end{cases} \tag{3.4}$$

Here, $\dot{h}$ denotes the derivative of h with respect to θ. We have:

$$\begin{aligned} \hat{F}_2(\Phi x, 0) = 2(&H_{16}x_1^2 + H_{17}x_2^2 + H_{18}x_3^2 + H_{19}x_4^2 + H_{20}x_5^2 + H_{21}x_1x_2 \\ &+ H_{22}x_1x_3 + H_{23}x_1x_4 + H_{24}x_1x_5 + H_{25}x_2x_3 + H_{26}x_2x_4 \\ &+ H_{27}x_2x_5 + H_{28}x_3x_4 + H_{29}x_3x_5 + H_{30}x_4x_5). \end{aligned}$$

By comparing the coefficients of each monomial, we establish the following relationships: $h_{20000} = \bar{h}_{00200}$, $h_{02000} = \bar{h}_{00020}$, $h_{00110} = \bar{h}_{11000}$, $h_{00101} = \bar{h}_{10001}$, $h_{01001} = \bar{h}_{00011}$ and $h_{10010} = \bar{h}_{01100}$. By substituting (3.4) into the expressions, we find that the coefficients h_{20000}, h_{00200}, h_{10100}, h_{10010}, h_{10001}, h_{00101}, h_{00002}, h_{11000} and h_{00110} satisfy the following equations:

$$\begin{cases} \dot{h}_{20000} - 2iwh_{20000} = 2\Phi(\theta)\Psi(0)H_{16}, \\ \dot{h}_{20000}(0) - L(h_{20000}) = 2H_{16}, \end{cases} \tag{3.5}$$

$$\begin{cases} \dot{h}_{02000} - (h_{11000} + 2iwh_{02000}) = 2\Phi(\theta)\Psi(0)H_{17}, \\ \dot{h}_{02000}(0) - L(h_{02000}) = 2H_{17}, \end{cases} \tag{3.6}$$

$$\begin{cases} \dot{h}_{00002} = 2\Phi(\theta)\Psi(0)H_{20}, \\ \dot{h}_{00002}(0) - L(h_{00002}) = 2H_{20}, \end{cases} \tag{3.7}$$

$$\begin{cases} \dot{h}_{11000} - 2(h_{20000} + iwh_{11000}) = 2\Phi(\theta)\Psi(0)H_{21}, \\ \dot{h}_{11000}(0) - L(h_{11000}) = 2H_{21}, \end{cases} \tag{3.8}$$

$$\begin{cases} \dot{h}_{10100} = 2\Phi(\theta)\Psi(0)H_{22}, \\ \dot{h}_{10100}(0) - L(h_{10100}) = 2H_{22}, \end{cases} \tag{3.9}$$

$$\begin{cases} \dot{h}_{10010} - h_{10100} = 2\Phi(\theta)\Psi(0)H_{23}, \\ \dot{h}_{10010}(0) - L(h_{10010}) = 2H_{23}, \end{cases} \tag{3.10}$$

$$\begin{cases} \dot{h}_{10001} - iwh_{10001} = 2\Phi(\theta)\Psi(0)H_{24}, \\ \dot{h}_{10001}(0) - L(h_{10001}) = 2H_{24}, \end{cases} \tag{3.11}$$

$$\begin{cases} \dot{h}_{01010} - (h_{10010} + h_{01100}) = 2\Phi(\theta)\Psi(0)H_{26}, \\ \dot{h}_{00101}(0) - L(h_{00101}) = 2H_{26}, \end{cases} \tag{3.12}$$

$$\begin{cases} \dot{h}_{01001} - (h_{10001} + iwh_{01001}) = 2\Phi(\theta)\Psi(0)H_{27}, \\ \dot{h}_{00110}(0) - L(h_{00110}) = 2H_{27}. \end{cases} \tag{3.13}$$

By utilizing equations (3.3), we deduce that

$$\frac{1}{4}\,\mathrm{Proj}_{S^2}\, D_y f_2^1 \,|_{y=0,\epsilon=0}\, U_2^2 = \begin{pmatrix} \sigma_1 x_2^2 x_4 + \sigma_2 x_2 x_5^2, \\ 0 \\ \sigma_3 x_4^2 x_2 + \sigma_4 x_4 x_5^2, \\ 0 \\ \sigma_5 x_5^3 + \sigma_6 x_2 x_4 x_5 \end{pmatrix}$$

where

$$\begin{aligned}
\sigma_1 &= \frac{1}{2}\bar{\psi}_1(0)\sum_{j=1}^{n}(R_{2j}h_{01010}^{(j)}(0) + S_{2j}h_{01010}^{(j)}(-\tau) + R_{4j}h_{02000}^{(j)}(0) + S_{4j}h_{02000}^{(j)}(-\tau)),\\
\sigma_2 &= \frac{1}{2}\bar{\psi}_1(0)\sum_{j=1}^{n}(R_{2j}h_{00002}^{(j)}(0) + S_{2j}h_{00002}^{(j)}(-\tau) + R_{5j}h_{01001}^{(j)}(0) + S_{5j}h_{01001}^{(j)}(-\tau)),\\
\sigma_3 &= \frac{1}{2}\psi_1(0)\sum_{j=1}^{n}(R_{2j}h_{00020}^{(j)}(0) + S_{2j}h_{00020}^{(j)}(-\tau) + R_{4j}h_{01010}^{(j)}(0) + S_{4j}h_{01010}^{(j)}(-\tau)),\\
\sigma_4 &= \frac{1}{2}\psi_1(0)\sum_{j=1}^{n}(R_{4j}h_{00002}^{(j)}(0) + S_{4j}h_{00002}^{(j)}(-\tau) + R_{5j}h_{00011}^{(j)}(0) + S_{5j}h_{00011}^{(j)}(-\tau)),\\
\sigma_5 &= \frac{1}{2}\psi_3(0)\sum_{j=1}^{n}(R_{5j}h_{00002}^{(j)}(0) + S_{5j}h_{00002}^{(j)}(-\tau)),\\
\sigma_6 &= \frac{1}{2}\psi_3(0)\sum_{j=1}^{n}(R_{2j}h_{00011}^{(j)}(0) + S_{2j}h_{00011}^{(j)}(-\tau) + R_{5j}h_{01010}^{(j)}(0) + S_{5j}h_{01010}^{(j)}(-\tau)\\
&\quad + R_{4j}h_{01001}^{(j)}(0) + S_{4j}h_{01001}^{(j)}(-\tau)).
\end{aligned}$$

Remark 3.2. $\sigma_1 = \bar{\sigma}_3$ and $\sigma_2 = \bar{\sigma}_4$.

We can explicitly determine the expressions of h_{20000}, h_{02000}, h_{00002}, h_{11000}, h_{10100}, h_{10010}, h_{10001}, h_{01010} and h_{01001} in the same way detailed in [1,11].

By consolidating all the obtained results, we reach the following conclusion:

$$\frac{1}{6}g_3^1(x,0,\epsilon) = \begin{pmatrix} (\beta_1+\gamma_1+\sigma_1)x_2^2x_4 + (\beta_2+\gamma_2+\sigma_2)x_2x_5^2 \\ 0 \\ (\beta_3+\gamma_3+\sigma_3)x_4^2x_2 + (\beta_4+\gamma_4+\sigma_4)x_4x_5^2 \\ 0 \\ (\beta_5+\gamma_5+\sigma_5)x_5^3 + (\beta_6+\gamma_6+\sigma_6)x_2x_4x_5 \end{pmatrix}.$$

Consequently, the system described in equation (3.2) can be reformulated as:

$$\begin{cases}
\dot{x}_1 = iwx_1 + x_2 + \alpha_1 x_2 + \alpha_2 x_2 x_5 + (\beta_1+\gamma_1+\sigma_1)x_2^2x_4 + (\beta_2+\gamma_2+\sigma_2)x_2x_5^2, \\
\dot{x}_2 = iwx_2, \\
\dot{x}_3 = -iwx_3 + x_4 + \alpha_3 x_4 + \alpha_4 x_4 x_5 + (\beta_3+\gamma_3+\sigma_3)x_4^2x_2 + (\beta_4+\gamma_4+\sigma_4)x_4x_5^2, \\
\dot{x}_4 = -iwx_4, \\
\dot{x}_5 = \alpha_6 x_5 + \alpha_7 x_2 x_4 + \alpha_8 x_5^2 + (\beta_5+\gamma_5+\sigma_5)x_5^3 + (\beta_6+\gamma_6+\sigma_6)x_2x_4x_5.
\end{cases} \tag{3.14}$$

Given that $x_1 = \bar{x}_3$ and $x_2 = \bar{x}_4$, the system (3.14) can be expressed equivalently as:

$$\begin{cases} \dot{x}_1 = iwx_1 + x_2 + \alpha_1 x_2 + \alpha_2 x_2 x_5 + (\beta_1 + \gamma_1 + \sigma_1)x_2^2 x_4 + (\beta_2 + \gamma_2 + \sigma_2)x_2 x_5^2 \\ \dot{x}_2 = iwx_2, \\ \dot{x}_3 = -iwx_3 + x_4 + \bar{\alpha}_1 x_4 + \bar{\alpha}_2 x_4 x_5 + (\bar{\beta}_1 + \bar{\gamma}_1 + \bar{\sigma}_1)x_4^2 x_2 + (\bar{\beta}_2 + \bar{\gamma}_2 + \bar{\sigma}_2)x_4 x_5^2, \\ \dot{x}_4 = -iwx_4, \\ \dot{x}_5 = \alpha_6 x_5 + \alpha_7 x_2 x_4 + \alpha_8 x_5^2 + (\beta_5 + \gamma_5 + \sigma_5)x_5^3 + (\beta_6 + \gamma_6 + \sigma_6)x_2 x_4 x_5. \end{cases} \tag{3.15}$$

Theorem 3.3. *Assuming the validity of assumptions (**A1**), (**A2**), and (**A3**), the retarded differential system (1.1) can be equivalently represented by the reduced system (3.15).*

4 Conclusion

Despite the extensive literature on various types of singularities in retarded differential equations (RDDs), such as the Bogdanov–Takens singularity, Hopf singularity, zero-Hopf singularity, saddle-node singularity, and double-Hopf singularity, the zero-double-Hopf singularity in RDDs remains relatively unexplored. This paper addresses this research gap by providing explicit conditions for the occurrence of the zero-double-Hopf singularity with 1 : 1 resonance in general RDDs. By employing the normal form theory proposed by Faria and Magalhães, we transform the considered RDDs into a system of three ordinary differential equations. Detailed calculations and formulas are presented, facilitating their implementation in symbolic computation systems.

However, an important question arises: How can we analyze the bifurcation diagram associated with this singularity to examine and understand the dynamics of various systems modeled by delay differential equations? Answering this question will be the focus of our future research endeavors.

A Appendix

In this part, we will define the notations:

$$\begin{aligned}
H_1 &= A_1\phi_1(0) + B_1\phi_1(-\tau), H_2 = A_2\phi_1(0) + B_2\phi_1(-\tau), H_3 = A_3\phi_1(0) + B_3\phi_1(-\tau),\\
H_4 &= A_1\phi_2(0) + B_1\phi_2(-\tau), H_5 = A_2\phi_2(0) + B_2\phi_2(-\tau), H_6 = A_3\phi_2(0) + B_3\phi_2(-\tau),\\
H_7 &= A_1\bar{\phi}_1(0) + B_1\bar{\phi}_1(-\tau), H_8 = A_2\bar{\phi}_1(0) + B_2\bar{\phi}_1(-\tau), H_9 = A_3\bar{\phi}_1(0) + B_3\bar{\phi}_1(-\tau),\\
H_{10} &= A_1\bar{\phi}_2(0) + B_1\bar{\phi}_2(-\tau), H_{11} = A_2\bar{\phi}_2(0) + B_2\bar{\phi}_2(-\tau), H_{12} = A_3\bar{\phi}_2(0) + B_3\bar{\phi}_2(-\tau),\\
H_{13} &= A_1\phi_3(0) + B_1\phi_3(-\tau), H_{14} = A_2\phi_3(0) + B_2\phi_3(-\tau), H_{15} = A_3\phi_3(0) + B_3\phi_3(-\tau),\\
H_{16} &= \sum_{i=1}^{n}(E_i\phi_{1i}(0)\phi_1(-\tau) + F_i\phi_{1i}(0)\phi_1(0) + K_i\phi_{1i}(-\tau)\phi_1(-\tau)),\\
H_{17} &= \sum_{i=1}^{n}(E_i\phi_{2i}(0)\phi_2(-\tau) + F_i\phi_{2i}(0)\phi_2(0) + K_i\phi_{2i}(-\tau)\phi_2(-\tau)),\\
H_{18} &= \sum_{i=1}^{n}(E_i\bar{\phi}_{1i}(0)\bar{\phi}_1(-\tau) + F_i\bar{\phi}_{1i}(0)\bar{\phi}_1(0) + K_i\bar{\phi}_{1i}(-\tau)\bar{\phi}_1(-\tau)),\\
H_{19} &= \sum_{i=1}^{n}(E_i\bar{\phi}_{2i}(0)\bar{\phi}_2(-\tau) + F_i\bar{\phi}_{2i}(0)\bar{\phi}_2(0) + K_i\bar{\phi}_{2i}(-\tau)\bar{\phi}_2(-\tau)),
\end{aligned}$$

$$
\begin{aligned}
H_{20} = {} & \sum_{i=1}^{n}(E_i\phi_{3i}(0)\phi_3(-\tau) + F_i\phi_{3i}(0)\phi_3(0) + K_i\phi_{3i}(-\tau)\phi_3(-\tau)),\\
H_{21} = {} & \sum_{i=1}^{n}\big(E_i(\phi_{1i}(0)\phi_2(-\tau) + \phi_{2i}(0)\phi_1(-\tau)) + F_i(\phi_{1i}(0)\phi_2(0) + \phi_{2i}(0)\phi_1(0))\\
& + K_i(\phi_{1i}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\phi_1(-\tau))\big),\\
H_{22} = {} & \sum_{i=1}^{n}\big(E_i(\phi_{1i}(0)\bar{\phi}_1(-\tau) + \bar{\phi}_{1i}(0)\phi_1(-\tau)) + F_i(\phi_{1i}(0)\bar{\phi}_1(0)\\
& + \bar{\phi}_{1i}(0)\phi_1(0)) + K_i(\phi_{1i}(-\tau)\bar{\phi}_1(-\tau) + \bar{\phi}_{1i}(-\tau)\phi_1(-\tau))\big),\\
H_{23} = {} & \sum_{i=1}^{n}\big(E_i(\bar{\phi}_{2i}(0)\phi_1(-\tau) + \phi_{1i}(0)\bar{\phi}_2(-\tau)) + F_i(\bar{\phi}_{2i}(0)\phi_1(0) + \phi_{1i}(0)\bar{\phi}_2(0))\\
& + K_i(\bar{\phi}_{2i}(-\tau)\phi_1(-\tau) + \phi_{1i}(-\tau)\bar{\phi}_2(-\tau))\big),\\
H_{24} = {} & \sum_{i=1}^{n}\big(E_i(\phi_{1i}(0)\phi_3(-\tau) + \phi_{3i}(0)\phi_1(-\tau)) + F_i(\phi_{1i}(0)\phi_3(0) + \phi_{3i}(0)\phi_1(0))\\
& + K_i(\phi_{1i}(-\tau)\phi_3(-\tau) + \phi_{3i}(-\tau)\phi_1(-\tau))\big),\\
H_{25} = {} & \sum_{i=1}^{n}\big(E_i(\bar{\phi}_{1i}(0)\phi_2(-\tau) + \phi_{2i}(0)\bar{\phi}_1(-\tau)) + F_i(\bar{\phi}_{1i}(0)\phi_2(0) + \phi_{2i}(0)\bar{\phi}_1(0))\\
& + K_i(\bar{\phi}_{1i}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\bar{\phi}_1(-\tau))\big),\\
H_{26} = {} & \sum_{i=1}^{n}\big(E_i(\phi_{2i}(0)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(0)\phi_2(-\tau)) + F_i(\phi_{2i}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(0)\phi_2(0))\\
& + K_i(\phi_{2i}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(-\tau)\phi_2(-\tau))\big),\\
H_{27} = {} & \sum_{i=1}^{n}\big(E_i(\phi_{2i}(0)\phi_3(-\tau) + \phi_{3i}(0)\phi_2(-\tau)) + F_i(\phi_{2i}(0)\phi_3(0) + \phi_{3i}(0)\phi_2(0))\\
& + K_i(\phi_{2i}(-\tau)\phi_3(-\tau) + \phi_{3i}(-\tau)\phi_2(-\tau))\big),\\
H_{28} = {} & \sum_{i=1}^{n}\big(E_i(\bar{\phi}_{1i}(0)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(0)\bar{\phi}_1(-\tau)) + F_i(\bar{\phi}_{1i}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(0)\bar{\phi}_1(0))\\
& + K_i(\bar{\phi}_{1i}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_1(-\tau))\big),\\
H_{29} = {} & \sum_{i=1}^{n}\big(E_i(\bar{\phi}_{1i}(0)\phi_3(-\tau) + \phi_{3i}(0)\bar{\phi}_1(-\tau)) + F_i(\bar{\phi}_{1i}(0)\phi_3(0) + \phi_{3i}(0)\bar{\phi}_1(0))\\
& + K_i(\bar{\phi}_{1i}(-\tau)\phi_3(-\tau) + \phi_{3i}(-\tau)\bar{\phi}_1(-\tau))\big),\\
H_{30} = {} & \sum_{i=1}^{n}\big(E_i(\bar{\phi}_{2i}(0)\phi_3(-\tau) + \phi_{3i}(0)\bar{\phi}_2(-\tau)) + F_i(\bar{\phi}_{2i}(0)\phi_3(0) + \phi_{3i}(0)\bar{\phi}_2(0))\\
& + K_i(\bar{\phi}_{2i}(-\tau)\phi_3(-\tau) + \phi_{3i}(-\tau)\bar{\phi}_2(-\tau))\big).\\
G_1 = {} & \sum_{i,j=1}^{n}\big[\Omega^1_{i,j}\phi_{1i}(0)\phi_{1j}(0)\phi_1(0) + \Omega^2_{i,j}\phi_{1i}(0)\phi_{1j}(-\tau)\phi_1(-\tau) + \Omega^3_{i,j}\phi_{1i}(-\tau)\phi_{1j}(0)\phi_1(0)\\
& + \Omega^4_{i,j}\phi_{1i}(-\tau)\phi_{1j}(-\tau)\phi_1(-\tau)\big],\\
G_2 = {} & \sum_{i,j=1}^{n}\big[\Omega^1_{i,j}\phi_{2i}(0)\phi_{2j}(0)\phi_2(0) + \Omega^2_{i,j}\phi_{2i}(0)\phi_{2j}(-\tau)\phi_2(-\tau) + \Omega^3_{i,j}\phi_{2i}(-\tau)\phi_{2j}(0)\phi_2(0)\\
& + \Omega^4_{i,j}\phi_{2i}(-\tau)\phi_{2j}(-\tau)\phi_2(-\tau)\big],
\end{aligned}
$$

$$
\begin{aligned}
G_3 = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\bar{\phi}_{1i}(0)\bar{\phi}_{1j}(0)\bar{\phi}_1(0) + \Omega^2_{i,j}\bar{\phi}_{1i}(0)\bar{\phi}_{1j}(-\tau)\bar{\phi}_1(-\tau) + \Omega^3_{i,j}\bar{\phi}_{1i}(-\tau)\bar{\phi}_{1j}(0)\bar{\phi}_1(0)\\
&+ \Omega^4_{i,j}\bar{\phi}_{1i}(-\tau)\bar{\phi}_{1j}(-\tau)\bar{\phi}_1(-\tau)\big],\\
G_4 = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\bar{\phi}_{2i}(0)\bar{\phi}_{2j}(0)\bar{\phi}_2(0) + \Omega^2_{i,j}\bar{\phi}_{2i}(0)\bar{\phi}_{2j}(-\tau)\bar{\phi}_2(-\tau) + \Omega^3_{i,j}\bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(0)\bar{\phi}_2(0)\\
&+ \Omega^4_{i,j}\bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(-\tau)\bar{\phi}_2(-\tau)\big],\\
G_5 = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\phi_{3i}(0)\phi_{3j}(0)\phi_3(0) + \Omega^2_{i,j}\phi_{3i}(0)\phi_{3j}(-\tau)\phi_3(-\tau) + \Omega^3_{i,j}\phi_{3i}(-\tau)\phi_{3j}(0)\phi_3(0)\\
&+ \Omega^4_{i,j}\phi_{3i}(-\tau)\phi_{3j}(-\tau)\phi_3(-\tau)\big],\\
G_6 = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\phi_{1i}(0)\phi_{1j}(0)\phi_2(0) + \phi_{1i}(0)\phi_{2j}(0)\phi_1(0) + \phi_{2i}(0)\phi_{1j}(0)\phi_1(0)\big)\\
&+ \Omega^2_{i,j}\big(\phi_{1i}(0)\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{1i}(0)\phi_{2j}(-\tau)\phi_1(-\tau) + \phi_{2i}(0)\phi_{1j}(-\tau)\phi_1(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\phi_{1i}(-\tau)\phi_{1j}(0)\phi_2(0) + \phi_{1i}(-\tau)\phi_{2j}(0)\phi_1(0) + \phi_{2i}(-\tau)\phi_{1j}(0)\phi_1(0)\big)\\
&+ \Omega^4_{i,j}\big(\phi_{1i}(-\tau)\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{1i}(-\tau)\phi_{2j}(-\tau)\phi_1(-\tau) + \phi_{2i}(-\tau)\phi_{1j}(-\tau)\phi_1(-\tau)\big)\big],\\
G_7 = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\phi_{1i}(0)\phi_{1j}(0)\bar{\phi}_1(0) + \phi_{1i}(0)\bar{\phi}_{1j}(0)\phi_1(0) + \bar{\phi}_{1i}(0)\phi_{1j}(0)\phi_1(0)\big)\\
&+ \Omega^2_{i,j}\big(\phi_{1i}(0)\phi_{1j}(-\tau)\bar{\phi}_1(-\tau) + \phi_{1i}(0)\bar{\phi}_{1j}(-\tau)\phi_1(-\tau) + \bar{\phi}_{1i}(0)\phi_{1j}(-\tau)\phi_1(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\phi_{1i}(-\tau)\phi_{1j}(0)\bar{\phi}_1(0) + \phi_{1i}(-\tau)\bar{\phi}_{1j}(0)\phi_1(0) + \bar{\phi}_{1i}(-\tau)\phi_{1j}(0)\phi_1(0)\big)\\
&+ \Omega^4_{i,j}\big(\phi_{1i}(-\tau)\phi_{1j}(-\tau)\bar{\phi}_1(-\tau) + \phi_{1i}(-\tau)\bar{\phi}_{1j}(-\tau)\phi_1(-\tau) + \bar{\phi}_{1i}(-\tau)\phi_{1j}(-\tau)\phi_1(-\tau)\big)\big],\\
G_8 = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\phi_{1i}(0)\phi_{1j}(0)\bar{\phi}_2(0) + \phi_{1i}(0)\bar{\phi}_{2j}(0)\phi_1(0) + \bar{\phi}_{2i}(0)\phi_{1j}(0)\phi_1(0)\big)\\
&+ \Omega^2_{i,j}\big(\phi_{1i}(0)\phi_{1j}(-\tau)\bar{\phi}_2(-\tau) + \phi_{1i}(0)\bar{\phi}_{2j}(-\tau)\phi_1(-\tau) + \bar{\phi}_{2i}(0)\phi_{1j}(-\tau)\phi_1(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\phi_{1i}(-\tau)\phi_{1j}(0)\bar{\phi}_2(0) + \phi_{1i}(-\tau)\bar{\phi}_{2j}(0)\phi_1(0) + \bar{\phi}_{2i}(-\tau)\phi_{1j}(0)\phi_1(0)\big)\\
&+ \Omega^4_{i,j}\big(\phi_{1i}(-\tau)\phi_{1j}(-\tau)\bar{\phi}_2(-\tau) + \phi_{1i}(-\tau)\bar{\phi}_{2j}(-\tau)\phi_1(-\tau) + \bar{\phi}_{2i}(-\tau)\phi_{1j}(-\tau)\phi_1(-\tau)\big)\big],\\
G_9 = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\phi_{1i}(0)\phi_{1j}(0)\phi_3(0) + \phi_{1i}(0)\phi_{3j}(0)\phi_1(0) + \phi_{3i}(0)\phi_{1j}(0)\phi_1(0)\big)\\
&+ \Omega^2_{i,j}\big(\phi_{1i}(0)\phi_{1j}(-\tau)\phi_3(-\tau) + \phi_{1i}(0)\phi_{3j}(-\tau)\phi_1(-\tau) + \phi_{3i}(0)\phi_{1j}(-\tau)\phi_1(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\phi_{1i}(-\tau)\phi_{1j}(0)\phi_3(0) + \phi_{1i}(-\tau)\phi_{3j}(0)\phi_1(0) + \phi_{3i}(-\tau)\phi_{1j}(0)\phi_1(0)\big)\\
&+ \Omega^4_{i,j}\big(\phi_{1i}(-\tau)\phi_{1j}(-\tau)\phi_3(-\tau) + \phi_{1i}(-\tau)\phi_{3j}(-\tau)\phi_1(-\tau) + \phi_{3i}(-\tau)\phi_{1j}(-\tau)\phi_1(-\tau)\big)\big],\\
G_{10} = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\phi_{1i}(0)\phi_{2j}(0)\phi_2(0) + \phi_{2i}(0)\phi_{1j}(0)\phi_2(0) + \phi_{2i}(0)\phi_{2j}(0)\phi_1(0)\big)\\
&+ \Omega^2_{i,j}\big(\phi_{1i}(0)\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(0)\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2i}(0)\phi_{2j}(-\tau)\phi_1(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\phi_{1i}(-\tau)\phi_{2j}(0)\phi_2(0) + \phi_{2i}(-\tau)\phi_{1j}(0)\phi_2(0) + \phi_{2i}(-\tau)\phi_{2j}(0)\phi_1(0)\big)\\
&+ \Omega^4_{i,j}\big(\phi_{1i}(-\tau)\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\phi_{2j}(-\tau)\phi_1(-\tau)\big)\big],
\end{aligned}
$$

$$
\begin{aligned}
G_{11} = \sum_{i,j=1}^{n} \big[& \Omega^1_{i,j}\big(\bar\phi_{1i}(0)\phi_{2j}(0)\phi_2(0) + \phi_{2i}(0)\bar\phi_{1j}(0)\phi_2(0) + \phi_{2i}(0)\phi_{2j}(0)\bar\phi_1(0)\big) \\
& + \Omega^2_{i,j}\big(\bar\phi_{1i}(0)\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(0)\bar\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2i}(0)\phi_{2j}(-\tau)\bar\phi_1(-\tau)\big) \\
& + \Omega^3_{i,j}\big(\bar\phi_{1i}(-\tau)\phi_{2j}(0)\phi_2(0) + \phi_{2i}(-\tau)\bar\phi_{1j}(0)\phi_2(0) + \phi_{2i}(-\tau)\phi_{2j}(0)\bar\phi_1(0)\big) \\
& + \Omega^4_{i,j}\big(\bar\phi_{1i}(-\tau)\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\bar\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\phi_{2j}(-\tau)\bar\phi_1(-\tau)\big)\big], \\
G_{12} = \sum_{i,j=1}^{n} \big[& \Omega^1_{i,j}\big(\bar\phi_{2i}(0)\phi_{2j}(0)\phi_2(0) + \phi_{2i}(0)\bar\phi_{2j}(0)\phi_2(0) + \phi_{2i}(0)\phi_{2j}(0)\bar\phi_2(0)\big) \\
& + \Omega^2_{i,j}\big(\bar\phi_{2i}(0)\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(0)\bar\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(0)\phi_{2j}(-\tau)\bar\phi_2(-\tau)\big) \\
& + \Omega^3_{i,j}\big(\bar\phi_{2i}(-\tau)\phi_{2j}(0)\phi_2(0) + \phi_{2i}(-\tau)\bar\phi_{2j}(0)\phi_2(0) + \phi_{2i}(-\tau)\phi_{2j}(0)\bar\phi_2(0)\big) \\
& + \Omega^4_{i,j}\big(\bar\phi_{2i}(-\tau)\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\bar\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\phi_{2j}(-\tau)\bar\phi_2(-\tau)\big)\big], \\
G_{13} = \sum_{i,j=1}^{n} \big[& \Omega^1_{i,j}\big(\phi_{3i}(0)\phi_{2j}(0)\phi_2(0) + \phi_{2i}(0)\phi_{3j}(0)\phi_2(0) + \phi_{2i}(0)\phi_{2j}(0)\phi_3(0)\big) \\
& + \Omega^2_{i,j}\big(\phi_{3i}(0)\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(0)\phi_{3j}(-\tau)\phi_2(-\tau) + \phi_{2i}(0)\phi_{2j}(-\tau)\phi_3(-\tau)\big) \\
& + \Omega^3_{i,j}\big(\phi_{3i}(-\tau)\phi_{2j}(0)\phi_2(0) + \phi_{2i}(-\tau)\phi_{3j}(0)\phi_2(0) + \phi_{2i}(-\tau)\phi_{2j}(0)\phi_3(0)\big) \\
& + \Omega^4_{i,j}\big(\phi_{3i}(-\tau)\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\phi_{3j}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\phi_{2j}(-\tau)\phi_3(-\tau)\big)\big], \\
G_{14} = \sum_{i,j=1}^{n} \big[& \Omega^1_{i,j}\big(\phi_{1i}(0)\bar\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(0)\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(0)\bar\phi_{1j}(0)\phi_1(0)\big) \\
& + \Omega^2_{i,j}\big(\phi_{1i}(0)\bar\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(0)\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(0)\bar\phi_{1j}(-\tau)\phi_1(-\tau)\big) \\
& + \Omega^3_{i,j}\big(\phi_{1i}(-\tau)\bar\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(-\tau)\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(-\tau)\bar\phi_{1j}(0)\phi_1(0)\big) \\
& + \Omega^4_{i,j}\big(\phi_{1i}(-\tau)\bar\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(-\tau)\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(-\tau)\bar\phi_{1j}(-\tau)\phi_1(-\tau)\big)\big], \\
G_{15} = \sum_{i,j=1}^{n} \big[& \Omega^1_{i,j}\big(\phi_{2i}(0)\bar\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(0)\phi_{2j}(0)\bar\phi_1(0) + \bar\phi_{1i}(0)\bar\phi_{1j}(0)\phi_2(0)\big) \\
& + \Omega^2_{i,j}\big(\phi_{2i}(0)\bar\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(0)\phi_{2j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(0)\bar\phi_{1j}(-\tau)\phi_2(-\tau)\big) \\
& + \Omega^3_{i,j}\big(\phi_{2i}(-\tau)\bar\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(-\tau)\phi_{2j}(0)\bar\phi_1(0) + \bar\phi_{1i}(-\tau)\bar\phi_{1j}(0)\phi_2(0)\big) \\
& + \Omega^4_{i,j}\big(\phi_{2i}(-\tau)\bar\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(-\tau)\phi_{2j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(-\tau)\bar\phi_{1j}(-\tau)\phi_2(-\tau)\big)\big], \\
G_{16} = \sum_{i,j=1}^{n} \big[& \Omega^1_{i,j}\big(\bar\phi_{2i}(0)\bar\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(0)\bar\phi_{2j}(0)\bar\phi_1(0) + \bar\phi_{1i}(0)\bar\phi_{1j}(0)\bar\phi_2(0)\big) \\
& + \Omega^2_{i,j}\big(\bar\phi_{2i}(0)\bar\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(0)\bar\phi_{2j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(0)\bar\phi_{1j}(-\tau)\bar\phi_2(-\tau)\big) \\
& + \Omega^3_{i,j}\big(\bar\phi_{2i}(-\tau)\bar\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(-\tau)\bar\phi_{2j}(0)\bar\phi_1(0) + \bar\phi_{1i}(-\tau)\bar\phi_{1j}(0)\bar\phi_2(0)\big) \\
& + \Omega^4_{i,j}\big(\bar\phi_{2i}(-\tau)\bar\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(-\tau)\bar\phi_{2j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(-\tau)\bar\phi_{1j}(-\tau)\bar\phi_2(-\tau)\big)\big], \\
G_{17} = \sum_{i,j=1}^{n} \big[& \Omega^1_{i,j}\big(\phi_{3i}(0)\bar\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(0)\phi_{3j}(0)\bar\phi_1(0) + \bar\phi_{1i}(0)\bar\phi_{1j}(0)\phi_3(0)\big) \\
& + \Omega^2_{i,j}\big(\phi_{3i}(0)\bar\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(0)\phi_{3j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(0)\bar\phi_{1j}(-\tau)\phi_3(-\tau)\big) \\
& + \Omega^3_{i,j}\big(\phi_{3i}(-\tau)\bar\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(-\tau)\phi_{3j}(0)\bar\phi_1(0) + \bar\phi_{1i}(-\tau)\bar\phi_{1j}(0)\phi_3(0)\big) \\
& + \Omega^4_{i,j}\big(\phi_{3i}(-\tau)\bar\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(-\tau)\phi_{3j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(-\tau)\bar\phi_{1j}(-\tau)\phi_3(-\tau)\big)\big],
\end{aligned}
$$

$$
\begin{aligned}
G_{18} = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\phi_{1i}(0)\bar{\phi}_{2j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(0)\phi_{1j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(0)\bar{\phi}_{2j}(0)\phi_1(0)\big)\\
&+ \Omega^2_{i,j}\big(\phi_{1i}(0)\bar{\phi}_{2j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(0)\phi_{1j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(0)\bar{\phi}_{2j}(-\tau)\phi_1(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\phi_{1i}(-\tau)\bar{\phi}_{2j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(-\tau)\phi_{1j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(0)\phi_1(0)\big)\\
&+ \Omega^4_{i,j}\big(\phi_{1i}(-\tau)\bar{\phi}_{2j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(-\tau)\phi_{1j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(-\tau)\phi_1(-\tau)\big)\big],\\
G_{19} = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\phi_{2i}(0)\bar{\phi}_{2j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(0)\phi_{2j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(0)\bar{\phi}_{2j}(0)\phi_2(0)\big)\\
&+ \Omega^2_{i,j}\big(\phi_{2i}(0)\bar{\phi}_{2j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(0)\phi_{2j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(0)\bar{\phi}_{2j}(-\tau)\phi_2(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\phi_{2i}(-\tau)\bar{\phi}_{2j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(-\tau)\phi_{2j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(0)\phi_2(0)\big)\\
&+ \Omega^4_{i,j}\big(\phi_{2i}(-\tau)\bar{\phi}_{2j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(-\tau)\phi_{2j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(-\tau)\phi_2(-\tau)\big)\big],\\
G_{20} = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\bar{\phi}_{1i}(0)\bar{\phi}_{2j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(0)\bar{\phi}_{1j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(0)\bar{\phi}_{2j}(0)\bar{\phi}_1(0)\big)\\
&+ \Omega^2_{i,j}\big(\bar{\phi}_{1i}(0)\bar{\phi}_{2j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(0)\bar{\phi}_{1j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(0)\bar{\phi}_{2j}(-\tau)\bar{\phi}_1(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\bar{\phi}_{1i}(-\tau)\bar{\phi}_{2j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{1j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(0)\bar{\phi}_1(0)\big)\\
&+ \Omega^4_{i,j}\big(\bar{\phi}_{1i}(-\tau)\bar{\phi}_{2j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{1j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(-\tau)\bar{\phi}_1(-\tau)\big)\big],\\
G_{21} = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\phi_{3i}(0)\bar{\phi}_{2j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(0)\phi_{3j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(0)\bar{\phi}_{2j}(0)\phi_3(0)\big)\\
&+ \Omega^2_{i,j}\big(\phi_{3i}(0)\bar{\phi}_{2j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(0)\phi_{3j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(0)\bar{\phi}_{2j}(-\tau)\phi_3(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\phi_{3i}(-\tau)\bar{\phi}_{2j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(-\tau)\phi_{3j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(0)\phi_3(0)\big)\\
&+ \Omega^4_{i,j}\big(\phi_{3i}(-\tau)\bar{\phi}_{2j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(-\tau)\phi_{3j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(-\tau)\phi_3(-\tau)\big)\big],\\
G_{22} = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\phi_{1i}(0)\phi_{3j}(0)\phi_3(0) + \phi_{3i}(0)\phi_{1j}(0)\phi_3(0) + \phi_{3i}(0)\phi_{3j}(0)\phi_1(0)\big)\\
&+ \Omega^2_{i,j}\big(\phi_{1i}(0)\phi_{3j}(-\tau)\phi_3(-\tau) + \phi_{3i}(0)\phi_{1j}(-\tau)\phi_3(-\tau) + \phi_{3i}(0)\phi_{3j}(-\tau)\phi_1(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\phi_{1i}(-\tau)\phi_{3j}(0)\phi_3(0) + \phi_{3i}(-\tau)\phi_{1j}(0)\phi_3(0) + \phi_{3i}(-\tau)\phi_{3j}(0)\phi_1(0)\big)\\
&+ \Omega^4_{i,j}\big(\phi_{1i}(-\tau)\phi_{3j}(-\tau)\phi_3(-\tau) + \phi_{3i}(-\tau)\phi_{1j}(-\tau)\phi_3(-\tau) + \phi_{3i}(-\tau)\phi_{3j}(-\tau)\phi_1(-\tau)\big)\big],\\
G_{23} = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\phi_{2i}(0)\phi_{3j}(0)\phi_3(0) + \phi_{3i}(0)\phi_{2j}(0)\phi_3(0) + \phi_{3i}(0)\phi_{3j}(0)\phi_2(0)\big)\\
&+ \Omega^2_{i,j}\big(\phi_{2i}(0)\phi_{3j}(-\tau)\phi_3(-\tau) + \phi_{3i}(0)\phi_{2j}(-\tau)\phi_3(-\tau) + \phi_{3i}(0)\phi_{3j}(-\tau)\phi_2(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\phi_{2i}(-\tau)\phi_{3j}(0)\phi_3(0) + \phi_{3i}(-\tau)\phi_{2j}(0)\phi_3(0) + \phi_{3i}(-\tau)\phi_{3j}(0)\phi_2(0)\big)\\
&+ \Omega^4_{i,j}\big(\phi_{2i}(-\tau)\phi_{3j}(-\tau)\phi_3(-\tau) + \phi_{3i}(-\tau)\phi_{2j}(-\tau)\phi_3(-\tau) + \phi_{3i}(-\tau)\phi_{3j}(-\tau)\phi_2(-\tau)\big)\big],\\
G_{24} = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\bar{\phi}_{1i}(0)\phi_{3j}(0)\phi_3(0) + \phi_{3i}(0)\bar{\phi}_{1j}(0)\phi_3(0) + \phi_{3i}(0)\phi_{3j}(0)\bar{\phi}_1(0)\big)\\
&+ \Omega^2_{i,j}\big(\bar{\phi}_{1i}(0)\phi_{3j}(-\tau)\phi_3(-\tau) + \phi_{3i}(0)\bar{\phi}_{1j}(-\tau)\phi_3(-\tau) + \phi_{3i}(0)\phi_{3j}(-\tau)\bar{\phi}_1(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\bar{\phi}_{1i}(-\tau)\phi_{3j}(0)\phi_3(0) + \phi_{3i}(-\tau)\bar{\phi}_{1j}(0)\phi_3(0) + \phi_{3i}(-\tau)\phi_{3j}(0)\bar{\phi}_1(0)\big)\\
&+ \Omega^4_{i,j}\big(\bar{\phi}_{1i}(-\tau)\phi_{3j}(-\tau)\phi_3(-\tau) + \phi_{3i}(-\tau)\bar{\phi}_{1j}(-\tau)\phi_3(-\tau) + \phi_{3i}(-\tau)\phi_{3j}(-\tau)\bar{\phi}_1(-\tau)\big)\big],
\end{aligned}
$$

$$
\begin{aligned}
G_{25} = \sum_{i,j=1}^{n} \big[& \Omega^1_{i,j}\big(\bar{\phi}_{2i}(0)\phi_{3j}(0)\phi_3(0) + \phi_{3i}(0)\bar{\phi}_{2j}(0)\phi_3(0) + \phi_{3i}(0)\phi_{3j}(0)\bar{\phi}_2(0)\big) \\
& + \Omega^2_{i,j}\big(\bar{\phi}_{2i}(0)\phi_{3j}(-\tau)\phi_3(-\tau) + \phi_{3i}(0)\bar{\phi}_{2j}(-\tau)\phi_3(-\tau) + \phi_{3i}(0)\phi_{3j}(-\tau)\bar{\phi}_2(-\tau)\big) \\
& + \Omega^3_{i,j}\big(\bar{\phi}_{2i}(-\tau)\phi_{3j}(0)\phi_3(0) + \phi_{3i}(-\tau)\bar{\phi}_{2j}(0)\phi_3(0) + \phi_{3i}(-\tau)\phi_{3j}(0)\bar{\phi}_2(0)\big) \\
& + \Omega^4_{i,j}\big(\bar{\phi}_{2i}(-\tau)\phi_{3j}(-\tau)\phi_3(-\tau) + \phi_{3i}(-\tau)\bar{\phi}_{2j}(-\tau)\phi_3(-\tau) + \phi_{3i}(-\tau)\phi_{3j}(-\tau)\bar{\phi}_2(-\tau)\big)\big], \\
G_{26} = \sum_{i,j=1}^{n} \Bigg[& \Omega^1_{i,j}\Bigg(\phi_{1i}(0)\big(\phi_{2j}(0)\bar{\phi}_1(0) + \bar{\phi}_{1j}(0)\phi_2(0)\big) + \phi_{2i}(0)\big(\phi_{1j}(0)\bar{\phi}_1(0) + \bar{\phi}_{1j}(0)\phi_1(0)\big) \\
& + \bar{\phi}_{1i}(0)\big(\phi_{1j}(0)\phi_2(0) + \phi_{2j}(0)\phi_1(0)\big)\Bigg) + \Omega^2_{i,j}\Bigg(\phi_{1i}(0)\big(\phi_{2j}(-\tau)\bar{\phi}_1(-\tau) + \bar{\phi}_{1j}(-\tau)\phi_2(-\tau)\big) \\
& + \phi_{2i}(0)\big(\phi_{1j}(-\tau)\bar{\phi}_1(-\tau) + \bar{\phi}_{1j}(-\tau)\phi_1(-\tau)\big) \\
& + \bar{\phi}_{1i}(0)\big(\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2j}(-\tau)\phi_1(-\tau)\big)\Bigg) \\
& + \Omega^3_{i,j}\Bigg(\phi_{1i}(-\tau)\big(\phi_{2j}(0)\bar{\phi}_1(0) + \bar{\phi}_{1j}(0)\phi_2(0)\big) + \phi_{2i}(-\tau)\big(\phi_{1j}(0)\bar{\phi}_1(0) + \bar{\phi}_{1j}(0)\phi_1(0)\big) \\
& + \bar{\phi}_{1i}(-\tau)\big(\phi_{1j}(0)\phi_2(0) + \phi_{2j}(0)\phi_1(0)\big)\Bigg) \\
& + \Omega^4_{i,j}\Bigg(\phi_{1i}(-\tau)\big(\phi_{2j}(-\tau)\bar{\phi}_1(-\tau) + \bar{\phi}_{1j}(-\tau)\phi_2(-\tau)\big) \\
& + \phi_{2i}(-\tau)\big(\phi_{1j}(-\tau)\bar{\phi}_1(-\tau) + \bar{\phi}_{1j}(-\tau)\phi_1(-\tau)\big) \\
& + \bar{\phi}_{1i}(-\tau)\big(\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2j}(-\tau)\phi_1(-\tau)\big)\Bigg)\Bigg], \\
G_{27} = \sum_{i,j=1}^{n} \Bigg[& \Omega^1_{i,j}\Bigg(\phi_{1i}(0)\big(\phi_{2j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\phi_2(0)\big) + \phi_{2i}(0)\big(\phi_{1j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\phi_1(0)\big) \\
& + \bar{\phi}_{2i}(0)\big(\phi_{1j}(0)\phi_2(0) + \phi_{2j}(0)\phi_1(0)\big)\Bigg) + \Omega^2_{i,j}\Bigg(\phi_{1i}(0)\big(\phi_{2j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\phi_2(-\tau)\big) \\
& + \phi_{2i}(0)\big(\phi_{1j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\phi_1(-\tau)\big) \\
& + \bar{\phi}_{2i}(0)\big(\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2j}(-\tau)\phi_1(-\tau)\big)\Bigg) \\
& + \Omega^3_{i,j}\Bigg(\phi_{1i}(-\tau)\big(\phi_{2j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\phi_2(0)\big) + \phi_{2i}(-\tau)\big(\phi_{1j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\phi_1(0)\big) \\
& + \bar{\phi}_{2i}(-\tau)\big(\phi_{1j}(0)\phi_2(0) + \phi_{2j}(0)\phi_1(0)\big)\Bigg) \\
& + \Omega^4_{i,j}\Bigg(\phi_{1i}(-\tau)\big(\phi_{2j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\phi_2(-\tau)\big) \\
& + \phi_{2i}(-\tau)\big(\phi_{1j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\phi_1(-\tau)\big) \\
& + \bar{\phi}_{2i}(-\tau)\big(\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2j}(-\tau)\phi_1(-\tau)\big)\Bigg)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{28} = \sum_{i,j=1}^{n} \Bigg[& \Omega^1_{i,j} \Big(\phi_{1i}(0)\big(\phi_{2j}(0)\phi_3(0) + \phi_{3j}(0)\phi_2(0)\big) + \phi_{2i}(0)\big(\phi_{1j}(0)\phi_3(0) + \phi_{3j}(0)\phi_1(0)\big) \\
& + \phi_{3i}(0)\big(\phi_{1j}(0)\phi_2(0) + \phi_{2j}(0)\phi_1(0)\big) \Big) + \Omega^2_{i,j} \Big(\phi_{1i}(0)\big(\phi_{2j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\phi_2(-\tau)\big) \\
& + \phi_{2i}(0)\big(\phi_{1j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\phi_1(-\tau)\big) \\
& + \phi_{3i}(0)\big(\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2j}(-\tau)\phi_1(-\tau)\big) \Big) \\
& + \Omega^3_{i,j} \Big(\phi_{1i}(-\tau)\big(\phi_{2j}(0)\phi_3(0) + \phi_{3j}(0)\phi_2(0)\big) + \phi_{2i}(-\tau)\big(\phi_{1j}(0)\phi_3(0) + \phi_{3j}(0)\phi_1(0)\big) \\
& + \phi_{3i}(-\tau)\big(\phi_{1j}(0)\phi_2(0) + \phi_{2j}(0)\phi_1(0)\big) \Big) \\
& + \Omega^4_{i,j} \Big(\phi_{1i}(-\tau)\big(\phi_{2j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\phi_2(-\tau)\big) \\
& + \phi_{2i}(-\tau)\big(\phi_{1j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\phi_1(-\tau)\big) \\
& + \phi_{3i}(-\tau)\big(\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2j}(-\tau)\phi_1(-\tau)\big) \Big) \Bigg], \\
G_{29} = \sum_{i,j=1}^{n} \Bigg[& \Omega^1_{i,j} \Big(\phi_{1i}(0)\big(\bar\phi_{2j}(0)\bar\phi_1(0) + \bar\phi_{1j}(0)\bar\phi_2(0)\big) + \bar\phi_{2i}(0)\big(\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1j}(0)\phi_1(0)\big) \\
& + \bar\phi_{1i}(0)\big(\phi_{1j}(0)\bar\phi_2(0) + \bar\phi_{2j}(0)\phi_1(0)\big) \Big) + \Omega^2_{i,j} \Big(\phi_{1i}(0)\big(\bar\phi_{2j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1j}(-\tau)\bar\phi_2(-\tau)\big) \\
& + \bar\phi_{2i}(0)\big(\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1j}(-\tau)\phi_1(-\tau)\big) \\
& + \bar\phi_{1i}(0)\big(\phi_{1j}(-\tau)\bar\phi_2(-\tau) + \bar\phi_{2j}(-\tau)\phi_1(-\tau)\big) \Big) \\
& + \Omega^3_{i,j} \Big(\phi_{1i}(-\tau)\big(\bar\phi_{2j}(0)\bar\phi_1(0) + \bar\phi_{1j}(0)\bar\phi_2(0)\big) + \bar\phi_{2i}(-\tau)\big(\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1j}(0)\phi_1(0)\big) \\
& + \bar\phi_{1i}(-\tau)\big(\phi_{1j}(0)\bar\phi_2(0) + \bar\phi_{2j}(0)\phi_1(0)\big) \Big) \\
& + \Omega^4_{i,j} \Big(\phi_{1i}(-\tau)\big(\bar\phi_{2j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1j}(-\tau)\bar\phi_2(-\tau)\big) \\
& + \bar\phi_{2i}(-\tau)\big(\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1j}(-\tau)\phi_1(-\tau)\big) \\
& + \bar\phi_{1i}(-\tau)\big(\phi_{1j}(-\tau)\bar\phi_2(-\tau) + \bar\phi_{2j}(-\tau)\phi_1(-\tau)\big) \Big) \Bigg], \\
G_{30} = \sum_{i,j=1}^{n} \Bigg[& \Omega^1_{i,j} \Big(\phi_{1i}(0)\big(\phi_{3j}(0)\bar\phi_1(0) + \bar\phi_{1j}(0)\phi_3(0)\big) + \phi_{3i}(0)\big(\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1j}(0)\phi_1(0)\big) \\
& + \bar\phi_{1i}(0)\big(\phi_{1j}(0)\phi_3(0) + \phi_{3j}(0)\phi_1(0)\big) \Big) + \Omega^2_{i,j} \Big(\phi_{1i}(0)\big(\phi_{3j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1j}(-\tau)\phi_3(-\tau)\big) \\
& + \phi_{3i}(0)\big(\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1j}(-\tau)\phi_1(-\tau)\big) + \bar\phi_{1i}(0)\big(\phi_{1j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\phi_1(-\tau)\big) \Big) \\
& + \Omega^3_{i,j} \Big(\phi_{1i}(-\tau)\big(\phi_{3j}(0)\bar\phi_1(0) + \bar\phi_{1j}(0)\phi_3(0)\big) + \phi_{3i}(-\tau)\big(\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1j}(0)\phi_1(0)\big) \\
& + \bar\phi_{1i}(-\tau)\big(\phi_{1j}(0)\phi_3(0) + \phi_{3j}(0)\phi_1(0)\big) \Big)
\end{aligned}
$$

$$+ \Omega^4_{i,j}\Big(\phi_{1i}(-\tau)\big(\phi_{3j}(-\tau)\bar{\phi}_1(-\tau) + \bar{\phi}_{1j}(-\tau)\phi_3(-\tau)\big)$$
$$+ \phi_{3i}(-\tau)\big(\phi_{1j}(-\tau)\bar{\phi}_1(-\tau) + \bar{\phi}_{1j}(-\tau)\phi_1(-\tau)\big)$$
$$+ \bar{\phi}_{1i}(-\tau)\big(\phi_{1j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\phi_1(-\tau)\big)\Big)\Big],$$

$$
\begin{aligned}
G_{31} = \sum_{i,j=1}^{n} \Bigg[&\Omega^1_{i,j}\Big(\phi_{1i}(0)\big(\phi_{3j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\phi_3(0)\big) + \phi_{3i}(0)\big(\phi_{1j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\phi_1(0)\big) \\
&+ \bar{\phi}_{2i}(0)\big(\phi_{1j}(0)\phi_3(0) + \phi_{3j}(0)\phi_1(0)\big)\Big) + \Omega^2_{i,j}\Big(\phi_{1i}(0)\big(\phi_{3j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\phi_3(-\tau)\big) \\
&+ \phi_{3i}(0)\big(\phi_{1j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\phi_1(-\tau)\big) + \bar{\phi}_{2i}(0)\big(\phi_{1j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\phi_1(-\tau)\big)\Big) \\
&+ \Omega^3_{i,j}\Big(\phi_{1i}(-\tau)\big(\phi_{3j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\phi_3(0)\big) + \phi_{3i}(-\tau)\big(\phi_{1j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\phi_1(0)\big) \\
&+ \bar{\phi}_{2i}(-\tau)\big(\phi_{1j}(0)\phi_3(0) + \phi_{3j}(0)\phi_1(0)\big)\Big) \\
&+ \Omega^4_{i,j}\Big(\phi_{1i}(-\tau)\big(\phi_{3j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\phi_3(-\tau)\big) \\
&+ \phi_{3i}(-\tau)\big(\phi_{1j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\phi_1(-\tau)\big) \\
&+ \bar{\phi}_{2i}(-\tau)\big(\phi_{1j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\phi_1(-\tau)\big)\Big)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{32} = \sum_{i,j=1}^{n} \Bigg[&\Omega^1_{i,j}\Big(\bar{\phi}_{1i}(0)\big(\phi_{2j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\phi_2(0)\big) + \phi_{2i}(0)\big(\bar{\phi}_{1j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\bar{\phi}_1(0)\big) \\
&+ \bar{\phi}_{2i}(0)\big(\bar{\phi}_{1j}(0)\phi_2(0) + \phi_{2j}(0)\bar{\phi}_1(0)\big)\Big) + \Omega^2_{i,j}\Big(\bar{\phi}_{1i}(0)\big(\phi_{2j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\phi_2(-\tau)\big) \\
&+ \phi_{2i}(0)\big(\bar{\phi}_{1j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\bar{\phi}_1(-\tau)\big) + \bar{\phi}_{2i}(0)\big(\bar{\phi}_{1j}(-\tau)\phi_2(-\tau) + \phi_{2j}(-\tau)\bar{\phi}_1(-\tau)\big)\Big) \\
&+ \Omega^3_{i,j}\Big(\bar{\phi}_{1i}(-\tau)\big(\phi_{2j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\phi_2(0)\big) + \phi_{2i}(-\tau)\big(\bar{\phi}_{1j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\bar{\phi}_1(0)\big) \\
&+ \bar{\phi}_{2i}(-\tau)\big(\bar{\phi}_{1j}(0)\phi_2(0) + \phi_{2j}(0)\bar{\phi}_1(0)\big)\Big) \\
&+ \Omega^4_{i,j}\Big(\bar{\phi}_{1i}(-\tau)\big(\phi_{2j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\phi_2(-\tau)\big) \\
&+ \phi_{2i}(-\tau)\big(\bar{\phi}_{1j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\bar{\phi}_1(-\tau)\big) \\
&+ \bar{\phi}_{2i}(-\tau)\big(\bar{\phi}_{1j}(-\tau)\phi_2(-\tau) + \phi_{2j}(-\tau)\bar{\phi}_1(-\tau)\big)\Big)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{33} = \sum_{i,j=1}^{n} \Bigg[&\Omega^1_{i,j}\Big(\bar{\phi}_{1i}(0)\big(\phi_{2j}(0)\phi_3(0) + \phi_{3j}(0)\phi_2(0)\big) + \phi_{2i}(0)\big(\bar{\phi}_{1j}(0)\phi_3(0) + \phi_{3j}(0)\bar{\phi}_1(0)\big) \\
&+ \phi_{3i}(0)\big(\bar{\phi}_{1j}(0)\phi_2(0) + \phi_{2j}(0)\bar{\phi}_1(0)\big)\Big) + \Omega^2_{i,j}\Big(\bar{\phi}_{1i}(0)\big(\phi_{2j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\phi_2(-\tau)\big) \\
&+ \phi_{2i}(0)\big(\bar{\phi}_{1j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\bar{\phi}_1(-\tau)\big) + \phi_{3i}(0)\big(\bar{\phi}_{1j}(-\tau)\phi_2(-\tau) + \phi_{2j}(-\tau)\bar{\phi}_1(-\tau)\big)\Big) \\
&+ \Omega^3_{i,j}\Big(\bar{\phi}_{1i}(-\tau)\big(\phi_{2j}(0)\phi_3(0) + \phi_{3j}(0)\phi_2(0)\big) \\
&+ \phi_{2i}(-\tau)\big(\bar{\phi}_{1j}(0)\phi_3(0) + \phi_{3j}(0)\bar{\phi}_1(0)\big) \\
&+ \phi_{3i}(-\tau)\big(\bar{\phi}_{1j}(0)\phi_2(0) + \phi_{2j}(0)\bar{\phi}_1(0)\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \Omega^4_{i,j}\Big(\bar{\phi}_{1i}(-\tau)\big(\phi_{2j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\phi_2(-\tau)\big)\\
&+ \phi_{2i}(-\tau)\big(\bar{\phi}_{1j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\bar{\phi}_1(-\tau)\big)\\
&+ \phi_{3i}(-\tau)\big(\bar{\phi}_{1j}(-\tau)\phi_2(-\tau) + \phi_{2j}(-\tau)\bar{\phi}_1(-\tau)\big)\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
G_{34} = \sum_{i,j=1}^{n} &\Big[\Omega^1_{i,j}\Big(\phi_{2i}(0)\big(\phi_{3j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\phi_3(0)\big) + \phi_{3i}(0)\big(\phi_{2j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\phi_2(0)\big)\\
&+ \bar{\phi}_{2i}(0)\big(\phi_{2j}(0)\phi_3(0) + \phi_{3j}(0)\phi_2(0)\big)\Big)\\
&+ \Omega^2_{i,j}\Big(\phi_{2i}(0)\big(\phi_{3j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\phi_3(-\tau)\big)\\
&+ \phi_{3i}(0)\big(\phi_{2j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\phi_2(-\tau)\big) + \bar{\phi}_{2i}(0)\big(\phi_{2j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\phi_2(-\tau)\big)\Big)\\
&+ \Omega^3_{i,j}\Big(\phi_{2i}(-\tau)\big(\phi_{3j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\phi_3(0)\big) + \phi_{3i}(-\tau)\big(\phi_{2j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\phi_2(0)\big)\\
&+ \bar{\phi}_{2i}(-\tau)\big(\phi_{2j}(0)\phi_3(0) + \phi_{3j}(0)\phi_2(0)\big)\Big)\\
&+ \Omega^4_{i,j}\Big(\phi_{2i}(-\tau)\big(\phi_{3j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\phi_3(-\tau)\big)\\
&+ \phi_{3i}(-\tau)\big(\phi_{2j}(-\tau)\bar{\phi}_2(-\tau)\\
&+ \bar{\phi}_{2j}(-\tau)\phi_2(-\tau)\big) + \bar{\phi}_{2i}(-\tau)\big(\phi_{2j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\phi_2(-\tau)\big)\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
G_{35} = \sum_{i,j=1}^{n} &\Big[\Omega^1_{i,j}\Big(\bar{\phi}_{1i}(0)\big(\bar{\phi}_{2j}(0)\phi_3(0) + \phi_{3j}(0)\bar{\phi}_2(0)\big) + \bar{\phi}_{2i}(0)\big(\bar{\phi}_{1j}(0)\phi_3(0) + \phi_{3j}(0)\bar{\phi}_1(0)\big)\\
&+ \phi_{3i}(0)\big(\bar{\phi}_{1j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\bar{\phi}_1(0)\big)\Big)\\
&+ \Omega^2_{i,j}\Big(\bar{\phi}_{1i}(0)\big(\bar{\phi}_{2j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\bar{\phi}_2(-\tau)\big)\\
&+ \bar{\phi}_{2i}(0)\big(\bar{\phi}_{1j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\bar{\phi}_1(-\tau)\big) + \phi_{3i}(0)\big(\bar{\phi}_{1j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\bar{\phi}_1(-\tau)\big)\Big)\\
&+ \Omega^3_{i,j}\Big(\bar{\phi}_{1i}(-\tau)\big(\bar{\phi}_{2j}(0)\phi_3(0) + \phi_{3j}(0)\bar{\phi}_2(0)\big) + \bar{\phi}_{2i}(-\tau)\big(\bar{\phi}_{1j}(0)\phi_3(0) + \phi_{3j}(0)\bar{\phi}_1(0)\big)\\
&+ \phi_{3i}(-\tau)\big(\bar{\phi}_{1j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\bar{\phi}_1(0)\big)\Big)\\
&+ \Omega^4_{i,j}\Big(\bar{\phi}_{1i}(-\tau)\big(\bar{\phi}_{2j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\bar{\phi}_2(-\tau)\big)\\
&+ \bar{\phi}_{2i}(-\tau)\big(\bar{\phi}_{1j}(-\tau)\phi_3(-\tau)\\
&+ \phi_{3j}(-\tau)\bar{\phi}_1(-\tau)\big) + \phi_{3i}(-\tau)\big(\bar{\phi}_{1j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\bar{\phi}_1(-\tau)\big)\Big)\Big].
\end{aligned}
$$

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