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Quasilinear Schrödinger equations with general sublinear conditions

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Abstract. In this paper, we study the quasilinear Schrödinger equations

$$-\Delta u + V(x)u + \Delta(u^2)u = f(x,u), \qquad \forall x \in \mathbb{R}^N,$$

where $V \in C(\mathbb{R}^N;\mathbb{R})$ may change sign and f is only locally defined for $|u|$ small. Under some new assumptions on V and f, we show that the above equation has a sequence of solutions converging to zero. Some recent results in the literature are generalized and significantly improved and some examples are also given to illustrate our main theoretical results.

Keywords: variational methods, critical points, quasilinear Schrödinger equations.

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1 Introduction

The aim of this paper is to establish the existence of multiple small solutions for the following quasilinear Schrödinger equations

$$-\Delta u + V(x)u + \Delta(u^2)u = f(x,u), \quad \forall x \in \mathbb{R}^N, \tag{QSE}$$

where $V \in C(\mathbb{R}^N;\mathbb{R})$ may change sign and f is only locally defined near the origin with respect to u and satisfies some weak and general sublinear assumptions.

Quasilinear Schrödinger equations (QSE) are widely used in non-Newtonian fluids, reaction-diffusion problems and other physical phenomena. More information on the physical background of these equations can be found in [6].

In recent years, with the aid of variational methods, the existence, nonexistence and multiplicity results of various solutions for (QSE) have been extensively investigated in the literature see [1, 5, 8, 10, 13] and the references therein. Here we emphasize that in all these papers V is a positive constant or possesses some kind of periodicity or radially symmetric, and the

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nonlinear term $f(x,u)$ is always required to satisfied various growth conditions at infinity with respect to u.

Recently, Chong et al. in [8] studied the equation (QSE) and proved the existence of multiple small solutions under the following conditions:

(C_1) There exist $\delta > 0$ and $C > 0$ such that $f \in C(\mathbb{R}^N \times [-\delta, \delta], \mathbb{R}^N)$, f is odd in x and

$$|f(x,u)| \leq C|u|, \quad \text{uniformly in } x \in \mathbb{R}^N;$$

(C_2) There exist $x_0 \in \mathbb{R}^N$ and $r_0 > 0$ such that

$$\liminf_{u \to 0} \left(\inf_{x \in B_{r_0}(x_0)} \frac{F(x,u)}{|u|^2} \right) > -\infty$$

and

$$\limsup_{u \to 0} \left(\inf_{x \in B_{r_0}(x_0)} \frac{F(x,u)}{|t|^2} \right) = +\infty,$$

where

$$F(x,u) = \int_0^u f(x,s)ds.$$

(V) For all $x \in \mathbb{R}^N$, $0 < V(x)$.

Motivated by the work of Chong et al. [8] and the [17, Lemma 2.3], in [5] the authors replaced the Condition (C_2) by a weak condition and proved the existence of multiple small solutions. Precisely, they supposed the following assumption:

(C_2') There exist $x_0 \in \mathbb{R}^{\mathbb{N}}$, two sequences (δ_n), (M_n) and constants α, $r_0 > 0$ such that δ_n, $M_n > 0$ and

$$\lim_{n \to \infty} \delta_n = 0, \quad \lim_{n \to \infty} M_n = +\infty,$$

$$\frac{F(x,u)}{{\delta_n}^2} \geq M_n \quad \text{for } |x - x_0| \leq r_0 \text{ and } |u| = \delta_n,$$

$$F(x,u) \geq -\alpha u^2 \quad \text{for } |x - x_0| \leq r_0 \text{ and } |u| \leq \delta.$$

In the present paper, different from the references mentioned above, we are going to study the existence of infinitely many solutions for (QSE) without any growth condition assumed on $f(x,u)$ at infinity with respect to u and the potential $V \in C(\mathbb{R}^N; \mathbb{R})$ may change sign. In fact, we will only require that $f(x,u)$ is locally defined for u small and satisfies some general and weak sufficient sublinear condition in u and V is neither of constant sign nor periodic. More precisely, we make the following assumptions:

(V_0) There exists a constant $a_0 > 0$ such that

$$V(x) + a_0 \geq 1, \quad \forall x \in \mathbb{R}^N,$$

$$\int_{\mathbb{R}^N} (V(x) + a_0)^{-1} dx < \infty,$$

and $\{x \in \mathbb{R}^N / V(x) \equiv 0\} \supset B(0,1)$, where $B(0,1)$ is the unit ball in $\mathbb{R}^N$.

(F_1) $F \in C^1(\mathbb{R}^N \times (-\delta,\delta))$ is even, and there exists a constant $a_1 > 0$ such that
$$|f(x,u)| \le a_1, \quad \forall (x,u) \in \mathbb{R}^N \times (-\delta,\delta),$$
where $\delta > 0$.

For $\rho > 0$, $x \in B(0,1)$ satisfying $B(x,\rho) \subset B(0,1)$ and for $u \in (0,\delta)$, we define
$$\overline{F}(x,u,\rho) := \inf\left\{ \frac{F(y,u)}{u^2}\rho^2 : y \in B(x,\rho) \right\}, \tag{1.1}$$
$$\underline{F}(x,u,\rho) := \inf\left\{ \frac{F(y,mu)}{u^2}\rho^2 : y \in B(x,\rho), 0 \le m \le 1 \right\}. \tag{1.2}$$

Substituting $m = 0$ into $\frac{F(y,mu)}{u^2}\rho^2$, we see that $\underline{F}(x,u,\rho) \le 0$. We assume:

(F_2) There exists a positive integer k_0 satisfying the following condition:
For each $k \ge k_0$, there exist $\mu_k \in (-\frac{\delta}{2},0) \cup (0,\frac{\delta}{2})$, $x_{k,i} \in B(0,1)$, with $1 \le i \le 2k$ and $\rho_k > 0$ such that $B(x_{k,i},\rho_k) \subset B(0,1)$, $B(x_{k,i},\rho_k) \cap B(x_{k,j},\rho_k) = \emptyset$ for $i \ne j$ and
$$\min_{1\le i\le 2k} \overline{F}(x_{k,i},\mu_k,\rho_k) + (2^{N+1}-1) \min_{1\le i\le 2k} \underline{F}(x_{k,i},\mu_k,\rho_k) > 2^{N+2}. \tag{1.3}$$
In (1.3), N is the dimension of the domain $\mathbb{R}^N$.

Our main results reads as follows.

Theorem 1.1. *Suppose that (V_0) and (F_1), (F_2) are satisfied. Then, equation (QSE) possesses a sequence of solutions $\{u_k\}$ such that $u_k(x) \to 0$ in L^∞ as $k \to \infty$.*

Remark 1.2.

- We insist on the fact that in the hypotheses (F_1)–(F_2), the conditions on the nonlinearity $F(x,u)$ are supposed only near $u = 0$ and there are no conditions for large $|u|$. This is essential and important. Indeed, this assumptions allows us to study equations having singularity or supercritical terms as $|u| \to \infty$.
- Under (F_1)–(F_2), $F(x,u)$ can be subquadratic, superquadratic or asymptotically quadratic at infinity. Our Theorem 1.1 is in some sense an improvement for some related results in the existing literature.
- To the best of our knowledge, there is no result concerning the existence and multiplicity of solutions for the equation (QSE) with the conditions.

Corollary 1.3. *Suppose that (V_0) and (F_1) are satisfied and $\delta > 0$ be as in (F_1). We assume that there exist sequences $M_n \to \infty$ as $n \to \infty$, $u_n \in (-\frac{\delta}{2},0) \cup (0,\frac{\delta}{2})$ and $\rho_n > 0$, $v_n \in B(0,1)$ such that $B(v_n,\rho_n) \subset B(0,1)$ and a constant $c \ge 0$, satisfy*
$$F(x,u_n)\rho_n^2 \ge M_n u_n^2,\ F(x,lu_n)\rho_n^2 \ge -cu_n^2 \quad \text{for } x \in B(v_n,\rho_n),\ 0 \le l \le 1. \tag{1.4}$$
Then, equation (QSE) possesses a sequence of solutions $\{u_k\}$ such that $u_k(x) \to 0$ in L^∞ as $k \to \infty$.

Corollary 1.4. *Suppose that (V_0) and (F_1) are satisfied and $\delta > 0$ be as in (F_1). We assume that there exist sequences $u_n \in (0,\frac{\delta}{2})$, $\rho_n > 0$ and $v_n \in B(0,1)$ such that $B(v_n,\rho_n) \subset B(0,1)$, and they satisfy*
$$\lim_{n\to\infty} \overline{F}(v_n,u_n,\rho_n) = \infty, \tag{1.5}$$
$$\liminf_{n\to\infty} \underline{F}(v_n,u_n,\rho_n) > -\infty. \tag{1.6}$$
Then, equation (QSE) possesses a sequence of solutions $\{u_k\}$ such that $u_k(x) \to 0$ in L^∞ as $k \to \infty$.

Corollary 1.5. *Suppose that* (V_0), (F) *and* (F_1) *are satisfied. Then, equation* (*QSE*) *possesses a sequence of solutions* $\{u_k\}$ *such that* $u_k(x) \to 0$ *in* L^∞ *as* $k \to \infty$.

Corollary 1.6. *Suppose that* $(V_0), (F_1)$ *and*

$$\inf_{x \in B(x_0, r_0)} u^{-2} F(x,u) \to \infty \quad \text{as } u \to 0, \tag{1.7}$$

are satisfied. Then, equation (*QSE*) *possesses a sequence of solutions* $\{u_k\}$ *such that* $u_k(x) \to 0$ *in* L^∞ *as* $k \to \infty$.

2 Preliminary results and variational setting

We employ an argument inspired by the work of Costa, Wang [11], the quasilinear problem was can be established:

$$-\operatorname{div}(h^2(u)\nabla u) + h(u)h'(u)|\nabla u|^2 + V(x)u = f(x,u), \qquad x \in \mathbb{R}^N, \tag{2.1}$$

where $h : [0, +\infty) \to \mathbb{R}$ satisfying

$$h(t) = \begin{cases} \sqrt{1-2t^2} & \text{if } 0 \le t < \frac{1}{\sqrt{6}}, \\ \frac{1}{6t} + \frac{1}{\sqrt{6}} & \text{if } t \ge \frac{1}{\sqrt{6}}, \end{cases}$$

and $h(t) = h(-t)$ for $t < 0$. It deduces that $h \in C^1(\mathbb{R}, (\left(\frac{1}{\sqrt{6}}\right), 1))$ and is increasing in $(-\infty, 0)$ and decreasing in $[0, +\infty)$. Then, we define

$$H(t) := \int_0^t h(s)ds.$$

It is well known that $H(t)$ is an odd function and inverse function $H^{-1}(t)$ exists. We now summarize some properties of $H^{-1}(t)$ as follow.

Lemma 2.1 ([1])**.** *We have:*

1. $|t| \le |H^{-1}(t)| \le \sqrt{6}|t|$ *for all* $t \in \mathbb{R}$;
2. $|H(t)| \le |t|$ *for all* $t \in \mathbb{R}$;
3. $-\frac{1}{2} \le \frac{t}{h(t)} h'(t) \le 0$ *for all* $t \ge 0$.

As in [11], in the present paper we are concerned to provide that the problem (2.1) has a sequence of weak solution $\{u_n\}$ satisfying $\|u_n\|_{L^\infty} < \min\{\delta/2, \frac{1}{\sqrt{6}}\}$, in this situation

$$h(u_n) = \left(1 - 2|u_n|^2\right)^{1/2}.$$

In order to prove our main result via the critical point theory, we need to establish the variational setting for (*QSE*). Before this, we have the following remark:

Remark 2.2. Let $V_0(x) = V(x) + a_0$, $F_0(x, H^{-1}(v)) = F(x, H^{-1}(v)) + \frac{a_0}{2}(H^{-1}(v))^2$ and $F_0(x,u) := \int_0^u f_0(x,s)ds$. Consider the following equation

$$-\Delta v + V_0(x)\frac{H^{-1}(v)}{h(H^{-1}(v))} = \frac{f_0(x, H^{-1}(v))}{h(H^{-1}(v))}, \qquad \forall x \in \mathbb{R}^N. \tag{2.2}$$

Then, equation (2.2) is equivalent to equation (*QSE*). It is easy to check that the hypotheses (V_0) and (F_1), (F_2) still hold for V_0 and F_0 provided that those hold for V and F. Hence, in what follows, we always assume without loss of generality that $V(x) \geq 1$ for all $x \in \mathbb{R}^N$ and $\int_{\mathbb{R}^N}(V(x))^{-1}dx < \infty$.

In view of Remark 2.2, we consider the space $E := \{u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x)u^2dx < \infty\}$ equipped with the following inner product

$$(u,v) := \int_{\mathbb{R}^N}(\nabla u \cdot \nabla v + V(x)uv)dx.$$

Then E is a Hilbert space and we denote by $\|\cdot\|$ the associated norm. In what follows, E becomes our working space. Moreover, we write E^* for the topological dual of E, and $\langle\cdot,\cdot\rangle$: $E^* \times E \to \mathbb{R}$ for the dual pairing. Evidently, E is continuously embedded into $H^1(\mathbb{R}^N)$. Using the Sobolev embedding theorem, we immediately get the following lemma.

Lemma 2.3. *If V satisfies (V_0), then E is continuously embedded in L^1.*

Proof. By (V_0) and Hölder inequality, we have for all $u \in E$

$$\begin{aligned}
\int_{\mathbb{R}^N} |u|\,dx &= \int_{\mathbb{R}^N} \left|(V(x))^{\frac{-1}{2}}\,(V(x))^{\frac{1}{2}}\,u\right| dx \\
&\leq \int_{\mathbb{R}^N} (V(x))^{\frac{-1}{2}} \left|(V(x))^{\frac{1}{2}}\,u\right| dx \\
&\leq \left(\int_{\mathbb{R}^N} (V(x))^{-1}\,dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} V(x)u^2dx\right)^{\frac{1}{2}} \\
&\leq \left(\int_{\mathbb{R}^N} (V(x))^{-1}\,dx\right)^{\frac{1}{2}} \|u\|.
\end{aligned} \tag{2.3}$$

□

Lemma 2.4. *If V satisfies (V_0) then E is compactly embedded into L^1.*

Proof. Let $(u_n) \subset E$ be a bounded sequence such that $u_n \rightharpoonup u$ in E. We will show that $u_n \to u$ in L^1. By Hölder's inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u_n - u|\,dx \\
&= \int_{|x|\leq R} |u_n - u|\,dx + \int_{|x|>R} |u_n - u|\,dx \\
&\leq \omega R^N \left(\int_{|x|\leq R} |u_n - u|^2\,dx\right)^{\frac{1}{2}} + \int_{|x|>R} \left|(V(x))^{\frac{-1}{2}}\,(V(x))^{\frac{1}{2}}\,(u_n - u)\right| dx \\
&\leq \omega R^N \left(\int_{|x|\leq R} |u_n - u|^2\,dx\right)^{\frac{1}{2}} + \int_{|x|>R} (V(x))^{-\frac{1}{2}} \left|(V(x))^{\frac{1}{2}}\,(u_n - u)\right| dx \\
&\leq \omega R^N \left(\int_{|x|\leq R} |u_n - u|^2\,dx\right)^{\frac{1}{2}} + \left(\int_{|x|>R} (V(x))^{-1}\,dx\right)^{\frac{1}{2}} \left(\int_{|x|>R} V(x)(u_n - u)^2dx\right)^{\frac{1}{2}} \\
&\leq \omega R^N \left(\int_{|x|\leq R} |u_n - u|^2\,dx\right)^{\frac{1}{2}} + \left(\int_{|x|>R} (V(x))^{-1}\,dx\right)^{\frac{1}{2}} \|u_n - u\|,
\end{aligned} \tag{2.4}$$

where $R > 0$, ω the volume of the unit ball in $\mathbb{R}^N$. Then by (V_0) and the Sobolev embedding Theorem, for any $\varepsilon > 0$ there exits $R_0 > 0$ such that for $R > R_0$, we have

$$\int_{\mathbb{R}^N} |u_n - u|\, dx \leq \varepsilon. \qquad \square$$

Lemma 2.5 ([2]). *E is continuously embedded into $L^p(\mathbb{R}^N)$ for all $p \in [2,6]$, and hence there exists $\tau_p > 0$ such that*

$$\|v\|_{L^p(\mathbb{R}^N)} \leq \tau_p \|u\|, \qquad \forall u \in E \text{ and } p \in [2,6]. \tag{2.5}$$

3 Proofs of main results

In order to define the corresponding variational functional on our working space E, we need modify $f(x,u)$ for u outside a neighborhood of the origin to get a globally defined $\widetilde{f}(x,u)$ as follows: Choose a constant $b \in (0, \frac{\delta}{2})$ and define a cut-off function $\chi \in C(\mathbb{R}, \mathbb{R})$ satisfying

$$\chi(t) := \begin{cases} 1 & \text{if } -b \leq t \leq b \\ 0 & \text{if } t \geq 2b \end{cases} \qquad \text{and,} \quad -\frac{2}{b} \leq \chi'(t) < 0 \quad \text{for } b < |t| < 2b. \tag{3.1}$$

Let $\widetilde{f}(x,u) := \chi(u) f(t,u)$, for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$, and $\widetilde{F}(x,u) := \int_0^u \widetilde{f}(x,s)ds$, for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$. By (3.1) and assumption (F_1) we have, for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$,

$$\left|\widetilde{F}(x,u)\right| \leq a_1 |u| \quad \text{and} \quad \left|\widetilde{f}(x,u)\right| \leq a_2, \tag{3.2}$$

where a_1 is the constant given in assumption (F_1) and a_2 is a positive constant.

Remark 3.1. As we have mentioned above, it is easy to verify that the equation (3.2) becomes

$$\left|\widetilde{F}(x, H^{-1}(v))\right| \leq a_1 \left|H^{-1}(v)\right| \quad \text{and} \quad \left|\widetilde{f}(x, H^{-1}(v))\right| \leq a_2 \left|h(H^{-1}(v))\right|. \tag{3.3}$$

Now, we consider the following modified equation

$$-\Delta v + V(x) \frac{H^{-1}(v)}{h(H^{-1}(v))} = \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))}, \qquad \forall x \in \mathbb{R}^N. \tag{$\widetilde{QSE}$}$$

To find the weak solutions of ($\widetilde{QSE}$) with desired properties, we focus on a Lagrangian functional defined by

$$\Phi(v) := \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x) |H^{-1}(v)|^2 \right) dx - \Psi(H^{-1}(v)), \tag{3.4}$$

with the change of variable $v = H(u)$ and $\Psi(v) = \int_{\mathbb{R}^N} \widetilde{F}(x, H^{-1}(v)) dx$.

Lemma 3.2. *Suppose that conditions (V_0) and (F_1) are satisfied. If $v \in E$ is a critical point of Φ, then $u = H^{-1}(v) \in E$ and this u is a weak solution for ($\widetilde{QSE}$).*

Proof. Since $v \in E$ and by Lemma 2.1, we can conclude that $u = H^{-1}(v) \in E$. Furthermore, v is a critical point for Φ, it follows that

$$\int_{\mathbb{R}^N} \nabla v \nabla \varphi dx + \int_{\mathbb{R}^N} V(x) \frac{H^{-1}(v)}{h(H^{-1}(v))} \varphi dx = \int_{\mathbb{R}^N} \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \varphi dx, \quad \text{for all } \varphi \in E.$$

If we take the function $\varphi = h(u)\psi$, where $u = H^{-1}(v)$ and $\psi \in C_0^\infty(\mathbb{R}^N)$, then we can obtain

$$\int_{\mathbb{R}^N} \nabla v \nabla u h'(u)\psi dx + \int_{\mathbb{R}^N} \nabla v \nabla \psi h(u) dx + \int_{\mathbb{R}^N} V(x) u \psi dx - \int_{\mathbb{R}^N} \widetilde{f}(x,u)\psi dx = 0.$$

Then, we get

$$\int_{\mathbb{R}^N} \Big(-\operatorname{div}(h^2(u)\nabla u) + h(u)h'(u)|\nabla u|^2 + V(x)u - \widetilde{f}(x,u)\Big)\psi dx = 0. \qquad \square$$

According to [8], we know that in order to find solutions of ($\widetilde{QSE}$) it suffices to obtain the critical points of Φ. For this purpose we recall the following definitions and results (see [14, 15]).

Definition 3.3 ([15])**.** Let E be a real Banach space and $\phi \in C^1(E,\mathbb{R})$.

- ϕ is said to satisfy (PS) condition if any sequence $(u_k) \subset E$ for which $(\phi(u_k))$ is bounded and $\phi'(u_k) \to 0$ as $k \to +\infty$, possesses a convergent subsequence in E. Here $\phi'(u)$ denotes the Fréchet derivative of $\phi(u)$.
- Set $\Gamma := \{A \subset E\backslash\{0\} : A \text{ is closed and symmetric with respect to the origin}\}$. For $A \in \Gamma$, we say genus of A is n (denoted by $\sigma(A) = n$), if there is an odd mapping $\varphi \in C(A, \mathbb{R}^n\backslash\{0\})$, and n is the smallest integer with this property.

Theorem 3.4 ([14, Theorem 1])**.** *Let ϕ be an even C^1 functional on E with $\phi(0) = 0$. Suppose that ϕ satisfies the (PS) condition and*

(1) ϕ is bounded from below.

(2) For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma$ such that $sup_{u\in A_k}\phi(u) < 0$, where $\Gamma_k = \{A \in \Gamma : \sigma(A) \geq k\}$.

Then either (i) or (ii) below holds.

(i) There exists a critical point sequence (u_k) such that $\phi(u_k) < 0$ and $\lim_{k\to\infty} u_k = 0$.

(ii) There exist two critical point sequences (u_k) and (v_k) such that $\phi(u_k) = 0, u_k \neq 0, \lim_{k\to\infty} u_k = 0, \phi(v_k) < 0, \lim_{k\to\infty} \phi(v_k) = 0$, and (v_k) converges to a non-zero limit.

Lemma 3.5. *Let (V_0) and (F_1) be satisfied. Then $\Psi \in C^1(E,\mathbb{R})$, and hence $\Phi \in C^1(E,\mathbb{R})$. Moreover,*

$$\langle \Psi'(v), \varphi\rangle = \int_{\mathbb{R}^N} \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))}\varphi\, dx, \tag{3.5}$$

and

$$\begin{aligned} \langle \Phi'(v), \varphi\rangle &= \int_{\mathbb{R}^N} \Big(\nabla v \nabla \varphi + V(x)\frac{H^{-1}(v)}{h(H^{-1}(v))}\varphi\Big)dx - \langle \Psi'(v),\varphi\rangle, \\ &= \int_{\mathbb{R}^N} \Big(\nabla v \nabla \varphi + V(x)\frac{H^{-1}(v)}{h(H^{-1}(v))}\varphi\Big)dx - \int_{\mathbb{R}^N} \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))}\varphi\, dx, \end{aligned} \tag{3.6}$$

for all $v, \varphi \in E$, and nontrivial critical points of Φ on E are solutions of equation ($\widetilde{QSE}$).

Proof. First, we show that Φ and Ψ are both well defined. For any $v \in E$, by (2.3) and (3.2), we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\widetilde{F}(x, H^{-1}(v))| dx &\leq a_1 \int_{\mathbb{R}^N} |H^{-1}(v)| \, dx \\ &\leq a_1 \int_{\mathbb{R}^N} |v| \, dx \\ &\leq a_1 \left(\int_{\mathbb{R}^N} (V(x))^{-1} dx \right)^{\frac{1}{2}} \|v\|.\end{aligned}$$

This implies that Φ and Ψ are both well defined.

Next, we prove $\Psi \in C^1(E, \mathbb{R})$. For any given $v \in E$, define an associated linear operator $J(v) : E \to \mathbb{R}$ by

$$\langle J(v), \varphi \rangle = \int_{\mathbb{R}^N} \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \varphi \, dx, \quad \forall \varphi \in E.$$

By (2.3) and (3.2), there holds

$$\begin{aligned}|\langle J(v), \varphi \rangle| &= \int_{\mathbb{R}^N} \left| \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right| |\varphi| \, dx \\ &\leq a_2 \int_{\mathbb{R}^N} |\varphi| \, dx \\ &\leq a_2 \left(\int_{\mathbb{R}^N} (V(x))^{-1} \, dx \right)^{\frac{1}{2}} \|\varphi\|.\end{aligned}$$

This implies that $J(v)$ is well defined and bounded. Observing (2.3) and (3.2), for any $v, \varphi \in E$, by the Mean Value Theorem and Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned}\lim_{s \to 0} \frac{\Psi(H^{-1}(v) + s\varphi) - \Psi(H^{-1}(v))}{s} &= \lim_{s \to 0} \int_{\mathbb{R}^N} \frac{\widetilde{f}(x, H^{-1}(v) + \theta(x) s\varphi)}{h(H^{-1}(v) + \theta(x) s\varphi)} \varphi \, dx \\ &= \int_{\mathbb{R}^N} \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \varphi \, dx \\ &= \langle J(v), \varphi \rangle,\end{aligned} \tag{3.7}$$

where $\theta(x) \in [0, 1]$ depends on v, φ, s. This implies that Ψ is Gâteaux differentiable on E and the Gâteaux derivative of Ψ at $v \in E$ is $J(v)$. Now for any $\epsilon > 0$, by (V_0), there exists $R_\epsilon > 0$ such that

$$\left(\int_{|x| > R_\epsilon} (V(x))^{-1} dx \right)^{\frac{1}{2}} < \frac{\epsilon}{4a_2}. \tag{3.8}$$

For this end, we claim that if $H^{-1}(v_n) \rightharpoonup H^{-1}(v)$ in E, then for any $R > 0$, $\frac{\widetilde{f}(x, H^{-1}(v_n))}{h(H^{-1}(v_n))} \to \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))}$ in $L^2(B_R)$, where B_R denotes the ball in $\mathbb{R}^N$ centered at 0 with radius R. Arguing indirectly, by Lemma 2.5, we assume that there exist constants $R_\epsilon, \epsilon > 0$ and a subsequence $\{H^{-1}(v_{n_k})\}_{k \in \mathbb{N}}$ such that

$$H^{-1}(v_{n_k}) \to H^{-1}(v) \quad \text{in } L^2(B_{R_\epsilon}) \quad \text{and} \quad H^{-1}(v_{n_k}) \to H^{-1}(v) \quad \text{a.e. in } B_{R_\epsilon} \quad \text{as } k \to \infty, \tag{3.9}$$

but using (F_1), we have

$$\int_{|x| \leq R_\epsilon} \left| \frac{\widetilde{f}(x, H^{-1}(v_{n_k}))}{h(H^{-1}(v_{n_k}))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right|^2 dx \geq \epsilon, \quad \forall k \in \mathbb{N}. \tag{3.10}$$

By (3.9), passing to a subsequence if necessary, we can assume that

$$\sum_{k=1}^{\infty} \|H^{-1}(v_{n_k}) - H^{-1}(v)\|_{L^2(B_{R_\epsilon})} < +\infty.$$

By virtue of (3.3), we get

$$\int_{|x|\le R_\epsilon} \left| \frac{\widetilde{f}(x, H^{-1}(v_{n_k}))}{h(H^{-1}(v_{n_k}))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right|^2 dx < +\infty. \tag{3.11}$$

For the R_ϵ given above, combining (3.9), (3.11) and Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{k\to\infty} \int_{|x|\le R_\epsilon} \left| \frac{\widetilde{f}(x, H^{-1}(v_{n_k}))}{h(H^{-1}(v_{n_k}))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right|^2 dx = 0,$$

which contradicts (3.10). Thus the claim is true. Consequently, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\int_{|x|\le R_\epsilon} \left| \frac{\widetilde{f}(x, H^{-1}(v_{n_k}))}{h(H^{-1}(v_{n_k}))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right|^2 dx < \frac{\epsilon}{2}, \ \forall n \ge N_\epsilon. \tag{3.12}$$

Combining (3.3), (3.8), (3.12) and the Hölder inequality, for each $n \ge N_\epsilon$, we have

$$\begin{aligned}
\|J(v_n) - J(v)\|_{E^*} &= \sup_{\|H^{-1}(v)\|=1} |\langle J(v_n) - J(v), \varphi\rangle| \\
&\le \sup_{\|H^{-1}(v)\|=1} \left| \int_{\mathbb{R}^N} \left[\frac{\widetilde{f}(x, H^{-1}(v_n))}{h(H^{-1}(v_n))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right] \varphi dx \right| \\
&\le \sup_{\|H^{-1}(v)\|=1} \left| \int_{|x|\le R_\epsilon} \left[\frac{\widetilde{f}(x, H^{-1}(v_n))}{h(H^{-1}(v_n))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right] \varphi dx \right| \\
&\quad + \sup_{\|H^{-1}(v)\|=1} \left| \int_{|x|> R_\epsilon} \left[\frac{\widetilde{f}(x, H^{-1}(v_n))}{h(H^{-1}(v_n))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right] \varphi dx \right| \\
&\le \sup_{\|H^{-1}(v)\|=1} \left(\int_{|x|\le R_\epsilon} \left| \frac{\widetilde{f}(x, H^{-1}(v_n))}{h(H^{-1}(v_n))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} dx \right|^2 \right)^{\frac{1}{2}} \left(\int_{|x|\le R_\epsilon} |\varphi|^2 dx \right)^{\frac{1}{2}} \\
&\quad + 2a_2 \sup_{\|H^{-1}(v)\|=1} \left(\int_{|x|> R_\epsilon} (V(x))^{-1} dx \right)^{\frac{1}{2}} \left(\int_{|x|> R_\epsilon} V(x)\varphi^2 dx \right)^{\frac{1}{2}} \\
&\le \frac{\epsilon}{2} + \frac{2a_2\epsilon}{4a_2} = \epsilon.
\end{aligned}$$

This, means that J is continuous in u. Thus, $\Psi \in C^1(E, \mathbb{R})$ and (3.5) holds. Due to the form of ϕ, we know that $\Phi \in C^1(E, \mathbb{R})$ and (3.6) also holds.

Finally, a standard argument shows that nontrivial critical points of Φ on E are solutions of ($\widetilde{QSE}$) (see, e.g., [8]). The proof is completed. □

Lemma 3.6. *Let* (V_0) *and* (F_1) *be satisfied. Then* Φ *is bounded from below and satisfies (PS) condition.*

Proof. We first prove that Φ is bounded from below. Combining $(F1)$, (2.3), (3.2) and the Hölder inequality, we have

$$\begin{aligned}\Phi(v) &\geq \frac{1}{2}\|v\|^2 - a_1\int_{\mathbb{R}^N}|H^{-1}(v)|dx \\ &\geq \frac{1}{2}\|v\|^2 - a_1\left(\int_{\mathbb{R}^N}(V(x))^{-1}dx\right)^{\frac{1}{2}}\|v\|, \quad \forall v\in E,\end{aligned} \tag{3.13}$$

where a_2 is the constant given in (3.2). Then it follows that Φ is bounded from below.

Next, we show that Φ satisfies (PS)-condition.

Let $\{v_n\}\subset E$ be a (PS)-sequence, i.e.,

$$|\Phi(v_n)|\leq D_2 \quad \text{and} \quad \Phi'(v_n)\to 0 \quad \text{as } n\to\infty \tag{3.14}$$

for some $D_2>0$. By (3.13) and (3.14), we have

$$D_2 \geq \frac{1}{2}\|v_n\|^2 - a_2\left(\int_{\mathbb{R}^N}(V(x))^{-1}dx\right)^{\frac{1}{2}}\|v_n\|, \quad \forall n\in\mathbb{N}.$$

This implies that $\{v_n\}$ is bounded in E. Thus, there exists a subsequence $\{H^{-1}(v)_{n_k}\}$ such that

$$H^{-1}(v_{n_k}) \rightharpoonup H^{-1}(v_0) \quad \text{as } k\to\infty \tag{3.15}$$

for some $v_0\in E$. By Lemma 2.4, it holds that

$$H^{-1}(v_{n_k}) \to H^{-1}(v_0) \quad \text{in } L^1 \text{ as } k\to\infty. \tag{3.16}$$

This together with (3.3) yields

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}\left[\frac{\widetilde{f}(x,H^{-1}(v_{n_k}))}{h(H^{-1}(v_{n_k}))} - \frac{\widetilde{f}(x,H^{-1}(v_0))}{h(H^{-1}(v_0))}\right](H^{-1}(v_{n_k})-H^{-1}(v_0))dx\right| \\ &\quad\leq 2a_2\int_{\mathbb{R}^N}|H^{-1}(v_{n_k})-H^{-1}(v_0)|dx \to 0 \quad \text{as } k\to\infty.\end{aligned} \tag{3.17}$$

Noting that $\{\xi_n\}$ is bounded in E, we infer from (3.14) and (3.15) that

$$\langle\Phi'(\xi_{n_k})-\Phi'(\xi_0), H^{-1}(\xi_{n_k})-H^{-1}(\xi_0)\rangle \to 0 \quad \text{as } k\to\infty. \tag{3.18}$$

Combining (3.6), (3.17) and (3.18), we have

$$\begin{aligned}&\|H^{-1}(\xi_{n_k})-H^{-1}(\xi_0)\|^2 \\ &= \langle\Phi'(\xi_{n_k})-\Phi'(\xi_0), H^{-1}(\xi_{n_k})-H^{-1}(\xi_0)\rangle \\ &\quad + \int_{\mathbb{R}N}\left(\frac{\widetilde{f}(x,\xi_{n_k})}{h(H^{-1}(\xi_{n_k}))} - \frac{\widetilde{f}(x,\xi_0)}{h(H^{-1}(\xi_0))}\right)(H^{-1}(\xi_{n_k})-H^{-1}(\xi_0))dx \to 0 \quad \text{as } k\to\infty.\end{aligned} \tag{3.19}$$

This means that $H^{-1}(\xi_{n_k})\to H^{-1}(\xi_0)$ in E as $k\to\infty$. Thus Φ satisfies (PS)-condition. □

We introduce a closed symmetric set V_k as below:

$$V_k \equiv \{(l_1,l_2,\ldots,l_{2k})\in\mathbb{R}^{2k}; |l_i|\leq 1 \text{ for all } i, \operatorname{card}\{i:|l_i|=1\}\geq k\}. \tag{3.20}$$

Lemma 3.7 ([15, Lemma 4.5]). *V_k has the genus of $k+1$.*

Lemma 3.8. *Let (V_0), (F_1) and (F_2) be satisfied. Then for each $k \in \mathbb{N}$, there exists an $A_k \subseteq E$ with genus $\sigma(A_k) = k+1$ such that $\sup_{u \in A_k} \Phi(v) < 0$.*

Proof. Let $\mu_k, x_{k,i}$ and ρ_k with $k \geq k_0$ be given in assumption (F_2). Since $\Gamma_k \subset \Gamma_{k-1}$ by definition, it is enough to construct an $A_k \in \Gamma_k$ for $k \geq k_0$ such that $\sup_{u \in A_k} \Phi(u) < 0$. Fix $k \geq k_0$. Instead of $\mu_k, x_{k,i}$ and ρ_k we write μ, x_i and ρ for simplicity. Using $\overline{F}$ and $\underline{F}$ given by (1.1) and (1.2) respectively, we define

$$\overline{F}_i := \overline{F}(x_i, \mu, \rho), \qquad \underline{F}_i := \underline{F}(x_i, \mu, \rho),\ 1 \leq i \leq 2k.$$

It follows from (1.1) and (1.2) and for $x \in B(x_i, \rho)$, that

$$F(x, \mu) \geq \frac{1}{\rho^2} \overline{F}_i (H^{-1}(\mu))^2 \geq \frac{1}{\rho^2} \overline{F}_i \mu^2, \tag{3.21}$$

$$F(x, l(\mu)) \geq \frac{1}{\rho^2} \underline{F}_i (H^{-1}(\mu))^2 \geq \frac{1}{\rho^2} \underline{F}_i \mu^2, \qquad |l| \leq 1. \tag{3.22}$$

We define a function $\varphi(t)$ on $\mathbb{R}$ by $\varphi(t) = 1$ for $|t| \leq 1/2$, $\varphi(t) = 2(1-|t|)$ for $1/2 \leq |t| \leq 1$, $\varphi(t) = 0$ for $|t| \geq 1$. Put $\varphi_i(x) = \varphi(|x - x_i|/\rho)$ for $x \in \mathbb{R}^N$. Then $\varphi_i \in W^{1,\infty}(\mathbb{R}^N)$. Define $B_i := B(x_i, \rho)$ and $D_i := B(x_i, \rho/2)$. Then $0 \leq \varphi_i(x) \leq 1$ in $\mathbb{R}^N$, $\varphi_i(x) = 0$ for $x \in \mathbb{R}^N \backslash B_i$ and

$$\varphi_i(x) = 1 \quad \text{for } x \in D_i, \quad |\nabla \varphi_i(x)| \leq \frac{2}{\rho} \quad \text{for } x \in \mathbb{R}^N. \tag{3.23}$$

Let V_k be given by (3.20). We define

$$A_k := \left\{ \mu \sum_{i=1}^{2k} l_i \varphi_i(x) : \ (l_1, \ldots, l_{2k}) \in V_k \right\}.$$

Since all the supports of φ_i $(1 \leq i \leq 2k)$ are disjoint, they are linearly independent. Define $g(l_1, \ldots, l_{2k}) := \mu \sum_{i=1}^{2k} l_i \varphi_i(x)$. Then g is a mapping from V_k onto A_k and it is an odd homeomorphism. By Lemma 3.7, the genus of V_k is $k+1$ and so is A_k. Thus $A_k \in \Gamma_k$.

We shall show that $\sup_{A_k} \Phi(v) < 0$. Fix $(l_1, \ldots, l_{2k}) \in V_k$ arbitrary. Let $v := \mu \sum_{i=1}^{2k} l_i \varphi_i(x) \in A_k$ and $\mu \in (0, \frac{1}{2\sqrt{6}}\delta)$ be arbitrary. Since the support of φ_i is $\overline{B_i}$ and $B_i \cap B_j = \emptyset$ for $i \neq j$, we have

$$\begin{aligned}
\Phi(v) &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V_0(x)(H^{-1}(v))^2) dx - \int_{\mathbb{R}^N} \widetilde{F}_0(x, H^{-1}(v)) dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)(H^{-1}(v))^2) dx - \int_{\mathbb{R}^N} \widetilde{F}(x, H^{-1}(v)) dx \\
&= \sum_{i=1}^{2k} \int_{B_i} \frac{1}{2} \mu^2 |l_i|^2 |\nabla \varphi_i|^2 dx - \sum_{i=1}^{2k} \int_{B_i} F(x, H^{-1}(\mu l_i \varphi_i)) dx.
\end{aligned}$$

By the assumption (V_0) and (3.23), we have

$$\Phi(v) \leq 4k\omega \mu^2 \rho^{N-2} - \sum_{i=1}^{2k} \int_{B_i} F(x, H^{-1}(\mu l_i \varphi_i)) dx. \tag{3.24}$$

To estimate the second term, we define

$$\Lambda_1 := \{i \in \{1, \dots, 2k\} : |l_i| = 1\},$$
$$\Lambda_2 := \{i \in \{1, \dots, 2k\} : |l_i| < 1\}.$$

By the definition of V_k, the cardinal number of Λ_1 greater than or equal to k. We compute the integral of F on B_i for $i \in \Lambda_1$, and for $i \in \Lambda_2$, separately. Recall that $F(x, v)$ is even with respect to v and $\varphi_i(x) = 1$ on D_i. Clearly, the volume of D_i is $2^{-N}\omega\rho^N$. By (3.21) and (3.22), we obtain, for $i \in \Lambda_1$,

$$\begin{aligned}\int_{B_i} F(x, H^{-1}(\mu l_i \varphi_i))dx &= \int_{D_i} F(x, H^{-1}(\mu))dx + \int_{B_i \setminus D_i} F(x, H^{-1}(\mu l_i \varphi_i))dx \\ &\geq 2^{-N}\omega\mu^2\rho^{N-2}\overline{F}_i + (1 - 2^{-N})\omega\mu^2\rho^{N-2}\underline{F}_i.\end{aligned} \tag{3.25}$$

We define

$$\alpha := \min_{1 \leq i \leq 2k} \overline{F}_i, \quad \beta := \min_{1 \leq i \leq 2k} \underline{F}_i.$$

As stated after (1.2), it holds that $\underline{F}_i \leq 0$, and hence $\beta \leq 0$. We rewrite (1.3) as

$$\alpha + (2^{N+1} - 1)\beta > 2^{N+2}. \tag{3.26}$$

We reduce (3.25) to

$$\int_{B_i} F(x, \mu l_i \varphi_i)dx \geq \left[2^{-N}\alpha + (1 - 2^{-N})\beta\right]\omega\mu^2\rho^{N-2}.$$

The right hand side is positive because of (3.26) with $\beta \leq 0$. Recall that the cardinal number of Λ_1 is greather than ou equal to k. Summing up both sides of the inequality above over $i \in \Lambda_1$, we obtain

$$\sum_{i \in \Lambda_1} \int_{B_i} F(x, \mu l_i \varphi_i)dx \geq \left[2^{-N}\alpha + (1 - 2^{-N})\beta\right]k\omega\mu^2\rho^{N-2}. \tag{3.27}$$

Next, by (3.22), for $i \in \Lambda_2$, we have

$$\int_{B_i} F(x, \mu l_i \varphi_i)dx \geq \omega\mu^2\rho^{N-2}\underline{F}_i \geq \beta\omega\mu^2\rho^{N-2}. \tag{3.28}$$

Recall that the cardinal number of Λ_2 is less than or equal to k. Summing up both sides over $i \in \Lambda_2$ and using $\beta \leq 0$, we find

$$\sum_{i \in \Lambda_2} \int_{B_i} F(x, \mu l_i \varphi_i)dx \geq k\beta\omega\mu^2\rho^{N-2}. \tag{3.29}$$

The set Λ_2 may be empty. In this case, we consider the left hand side to be zero. Then the inequality above is still valid because $\beta \leq 0$. Substituting (3.27) and (3.29) into (3.24) and using (3.26), we obtain

$$\Phi(v) \leq -\left[\alpha(2^{N+1} - 1) + \beta - 2^{N+2}\right]k\omega\mu^2\rho^{N-2} < 0,$$

which implies that $\sup_{v \in A_k} \Phi(v) < 0$. □

In order to prove our main results, we further need the following lemma.

Lemma 3.9. *If $\{v_k\}$ is a critical point sequence of Φ satisfying $v_k \to 0$ in E as $k \to \infty$, then $v_k \to 0$ in $L^\infty(\mathbb{R}^N)$ as $k \to \infty$.*

Proof. Let $v \in E$ be a weak solution of ($\widetilde{QSE}$), i.e.,

$$\int_{\mathbb{R}^N} \nabla v \nabla \varphi dx + \int_{\mathbb{R}^N} V(x) \frac{H^{-1}(v)}{h(H^{-1}(v))} \varphi dx - \int_{\mathbb{R}^N} \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \varphi dx \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N). \tag{3.30}$$

Set $T > 0$, and denote

$$v_T := \begin{cases} -T, & \text{if } v \le T, \\ v, & \text{if } -T < v < T, \\ T, & \text{if } v \ge T. \end{cases} \tag{3.31}$$

Taking $\varphi = |v_T|^{2(\eta-1)} v_T$ as the text function, where $\eta > 1$ to be determined later, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |v_T|^{2(\eta-1)} \nabla v \nabla v_T dx + 2(\eta - 1) \int_{\mathbb{R}^N} |v_T|^{2(\eta-1)-1} \nabla v \nabla v_T dx \\ + \int_{\mathbb{R}^N} V(x) \frac{H^{-1}(v)}{h(H^{-1}(v))} |v_T|^{2(\eta-1)} v_T dx \\ = \int_{\mathbb{R}^N} \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} |v_T|^{2(\eta-1)} v_T dx. \end{aligned} \tag{3.32}$$

By using the facts

$$(\eta - 1) \int_{\mathbb{R}^N} |v_T|^{2(\eta-1)-1} \nabla v \nabla v_T dx \ge 0,$$

$$\int_{\mathbb{R}^N} V(x) \frac{H^{-1}(v)}{h(H^{-1}(v))} |v_T|^{2(\eta-1)} v_T dx \ge 0$$

and Lemma 2.1, we have

$$\frac{1}{\eta^2} \int_{\mathbb{R}^N} |\nabla |v_T|^\eta|^2 dx \le \int_{\mathbb{R}^N} \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} |v_T|^{2\eta-1} dx \le a_2 \int_{\mathbb{R}^N} |v|^{2\eta-1} dx. \tag{3.33}$$

On the other hand, it follows from the Sobolev inequality that

$$\frac{S}{\eta^2} \|v_T\|_{2^*\eta}^{2\eta} \le \frac{1}{\eta^2} \int_{\mathbb{R}^N} |\nabla |v_T|^\eta|^2 dx, \tag{3.34}$$

where $S = \inf\{\int_{\mathbb{R}^N} |\nabla v|^2 dx \setminus \int_{\mathbb{R}^N} |v|^{2^*} dx = 1\}$ and $2^* = 2N/(N-2)$. In what follows, by (3.33) and (3.34), we get

$$\frac{1}{\eta^2} \|v_T\|_{2^*\eta}^{2\eta} \le a_2 \int_{\mathbb{R}^N} |v|^{2\eta-1} dx. \tag{3.35}$$

From Fatou's lemma, sending $T \to \infty$ in (3.35), it follows that

$$\|v\|_{2^*\eta} \le (c\eta)^{1/\eta} \|v\|_{2\eta-1}^{(2\eta-1)/2\eta}. \tag{3.36}$$

Let us define $\eta_k := \frac{2^*\eta_k - 1}{2}$, where $k = 1, 2, \ldots.$ and $\eta_0 = \frac{2^*-1}{2}$. Next, we present the first step of Moser's iteration, which is shown below:

$$\|v\|_{\eta_1 2^*} \le (C\eta_1)^{1/\eta_1} \|v\|_{2\eta_1-1}^{(2\eta_1-1)/2\eta_1} \tag{3.37}$$

$$\le (C\eta_1)^{1/\eta_1} (C\eta_0)^{1/\eta_0(2\eta_1-1)/2\eta_1} \|v\|_{2\eta_0-1}^{(2\eta_0-1)/2\eta_0(2\eta_1-1)/2\eta_1}. \tag{3.38}$$

We can assume, without loss of generality, that $C > 1$. Moreover, for any $i < j$, we we have the inequality given by equation

$$(C\eta_i)^{(2\eta_j-1)/2\eta_j} \leq C\eta_j. \tag{3.39}$$

Using equations (3.37) and (3.39), we obtain the inequality

$$\|v\|_{\eta_1 2^*} \leq (C\eta_1)^{1/\eta_1}(C\eta_0)^{1/\eta_0}\|v\|_{2\eta_1-1}^{(2\eta_0-1)/p\eta_0(2\eta_1-1)/2\eta_1}.$$

Applying Moser's iteration method, we can now derive the following result.

$$\|v\|_{2\eta_{k+1}-1} \leq \exp\left(\sum_{i=0}^{k}\frac{\ln(C\eta_i)}{\eta_i}\right)\|v\|_{2^*}^{\mu_k},$$

where $\mu_k = \Pi_{i=0}^k \frac{2\eta_i-1}{2\eta_i}$. Taking the limit as $k \to \infty$, we obtain the following result.

$$\|v\|_\infty \leq \exp\left(\sum_{i=0}^{k}\frac{\ln(C\eta_i)}{\eta_i}\right)\|v\|_{2^*}^{\mu},$$

where $\mu = \Pi_{i=0}^k \frac{2\eta_i-1}{2\eta_i}$ $(0 < \mu < 1)$ and $\exp\left(\sum_{i=0}^k \frac{\ln(C\eta_i)}{\eta_i}\right)$ is a positive constant. This, together with the Sobolev embedding theorem, we can conclude that if v_k is a sequence of critical points of Φ such that $v_k \to 0$ strongly in E as $k \to \infty$, then v_k converges strongly to zero in $L^\infty(\mathbb{R}^N)$. □

Now we are in the position to give the proofs of our main results.

4 Proofs of Theorem 1.1 and Corollaries 1.3–1.6

The aim of this section is to establish the proofs of Theorem 1.1 and Corollaries 1.3–1.6.

4.1 Proof of Theorem 1.1

Lemmas 3.6, 3.7 and 3.8 shows that the functional Φ satisfies conditions (1) and (2) in Theorem 3.4. Therefore, there exist a sequence of nontrivial critical points (u_k) of Φ such that $\Phi(u_k) \leq 0$ for all $k \in \mathbb{N}$ and $u_k \to 0$ in E as $k \to \infty$. By virtue of Lemma 3.5, $\{u_k\}$ is a sequence of solutions of ($\widetilde{QSE}$) with $u_k \to 0$ in E as $k \to \infty$. Hence, there exists $k_0 \in \mathbb{N}$ such that u_k is a solution of (QSE) for each $k \geq k_0$.

4.2 Proof of Corollary 1.3 and 1.4

It is enough to show that (1.5) and (1.6) $\Rightarrow$ (1.4) $\Rightarrow$ (1.3). Impose (1.5) and (1.6). Then we shall construct $\mu_k, x_{k,i}$ and ρ_k satisfying (1.3). Fix k arbitrarily. Let C_n be the inscribed cube in $B(v_n, \rho_n)$. Then its edge has the length of $2\rho_n/\sqrt{N}$. Let q be the smallest positive integer satisfying $q^N \geq 2k$. We divide the cube C_n equally into q^N small cubes by planes parallel to each face of C_n and denote them by $C_{n,i}$ with $1 \leq i \leq q^N$. More precisely, denote C_n by

$$C_n := [0, a] \times \cdots \times [0, a] \quad \text{with } a := 2\rho_n/\sqrt{N}.$$

Put $I_j := [a(j-1)/q, aj/q]$ with $1 \leq j \leq q$ and define

$$I(j_1, \ldots, j_N) := I_{j_1} \times \cdots \times I_{j_N} \quad \text{with } 1 \leq j_1, \ldots, j_N \leq q.$$

This, is a cube in $\mathbb{R}^N$ and C_n is the union of all these cubes. We rename all $I(j_1,\ldots,j_N)$ to $C_{n,i}$ with $1 \le i \le q^N$. Then the edge of each $C_{n,i}$ has the length of $2\rho_n/q\sqrt{N}$. Denote the inscribed ball in $C_{n,i}$ by $B(x_{n,i}, r_n)$. Then $r_n = \rho/q\sqrt{N}$. Since $q^N \ge 2k$, $x_{n,i}$ is defined for all $1 \le i \le 2k$.

We shall show that assumption (F_2) is fulfilled with μ_k, $x_{k,i}$ and ρ_k replaced by $u_n, x_{n,i}$ and r_n, respectively, if n is large enough. It is clear that $B(x_{n,i}, r_n) \subset B(0,1)$ and $B(x_{n,i}, r_n) \cap B(x_{n,j}, r_n) = \emptyset$ when $i \ne j$. Define $M_n := \overline{F}(v_n, u_n, \rho_n)$, which implies that

$$\frac{F(x,u_n)}{u_n^2}\rho_n^2 \ge M_n \quad \text{for } x \in B(v_n, \rho_n).$$

By (1.6), there exists a $c \ge 0$ such that

$$\frac{F(x,lu_n)}{u_n^2}\rho_n^2 \ge -c \quad \text{for } x \in B(v_n,\rho_n),\ 0 \le l \le 1.$$

Then we obtain (1.4). On the other hand, substituting $\rho_n = q\sqrt{N}\, r_n$ in the two inequalities above, we have

$$\frac{NF(x,u_n)}{u_n^2}q^2r_n^2 \ge M_n, \qquad \frac{NF(x,lu_n)}{u_n^2}q^2r_n^2 \ge -c,$$

for $x \in B(v_n, \rho_n)$ and $0 \le l \le 1$. Since $B(x_{n,i}, r_n) \subset B(v_n, \rho_n)$, the inequalities above are valid for $x \in B(x_{n,i}, r_n)$ also. Taking the infimum on $B(x_{n,i}, r_n)$, we have

$$\overline{F}(x_{n,i}, u_n, r_n) \ge \frac{M_n}{Nq^2}, \qquad \underline{F}(x_{n,i}, u_n, r_n) \ge -\frac{c}{Nq^2}.$$

Then we get

$$\min_{1\le i\le 2k} \overline{F}(x_{n,i},u_n,r_n) + (2^{N+1}-1)\min_{1\le i\le 2k}\underline{F}(x_{n,i},u_n,r_n) \ge \frac{1}{Nq^2}\left(M_n - (2^{N+1}-1)c\right).$$

Since $\lim_{n\to\infty} M_n = \infty$ by (1.5), the right hand side is larger than 2^{N+2} for n large enough.

4.3 Proof of Corollary 1.5

To prove this corollary, it is enough to show that the assumption (F) implies (1.5) and (1.6). By (F) there exists a sequence u_n converging to zero such that

$$\inf_{x\in B(x_0,r_0)} u_n^{-2}F(x,u_n) \to \infty \quad \text{as } n \to \infty.$$

Put $B(x_n, r_n) := B(x_0, r_0)$ for all n. Then the above inequality shows (1.5). Also, by (F), there exists a constant $c \ge 0$ such that

$$\inf_{x\in B(x_0,r_0)} u^{-2}F(x,u) \ge -c \quad \text{for } 0 < |u| \le 1.$$

Putting $u := lu_n$, we find

$$\inf_{x\in B(x_0,r_0)} (lu_n)^{-2}F(x,lu_n) \ge -c \quad \text{for all large } n \text{ and } 0 < l \le 1,$$

which leads to

$$\inf_{x\in B(x_0,r_0)} u_n^{-2}F(x,lu_n) \ge -cl^2 \ge -c.$$

Therefore (1.6) holds.

4.4 Proof of Corollary 1.6

We observe that (1.7) implies (F). Therefore, Corollary 1.5 yields Corollary 1.6.

5 Example

For the reader's convenience, we present one example to illustrate our main results.

Let

$$V(x) = \begin{cases} 0 \text{ if } |x| \le p, \\ (p^2+1)^2(|x|-p), \text{ if } p \le |x| < p + \frac{1}{p^2+1}, \\ p^2+1, \text{ if } p + \frac{1}{p^2+1} \le |x| < p + \frac{p^2}{p^2+1}, \\ (p^2+1)^2(p+1-|x|), \text{ if } p + \frac{p^2}{p^2+1} \le |x| < p+1, \end{cases}$$

and

$$F(x,u) = \frac{a}{s}|u|^s - \frac{d(x)}{r}|u|^r, \tag{5.1}$$

where $p \in \mathbb{N}^*$, and s, r, a are constants satisfying $1 < r < 2,\ 1 < s < \frac{2}{3}(r+1),\ a > 0$ and

$$d(x) := \inf\{|x-y| : y \in \partial B(0,1)\}.$$

Then V is neither of constant sign nor periodic. Moreover, we have

$$\inf_{x \in B(x_0,r_0)} \frac{F(x,u)}{u^2} = \frac{a}{s}|u|^{-(2-s)} - \frac{D}{r}|u|^{-(2-r)} \to -\infty \quad \text{as } u \to 0,$$

for any $B(x_0,r_0) \subset B(0,1)$, where $D := \max_{|x-x_0| \le r_0} d(x) > 0$. Which implies that the assumption (C_2) and (C_2') are not satisfied. Now, we show that V and F match Theorem 1.1. Indeed, it is clear that $V(x)$ and $F(x,u)$ satisfy (V_0) and (F_1) respectively. It remains to check that $F(x,u)$ satisfies (F_2). For this purpose we assume that there exists a $\delta > 0$ such that for each $k \in \mathbb{N}$, there exist points $\xi_i \in \partial B(0,1)$ with $1 \le i \le 2k$ which satisfy $|\xi_i - \xi_j| \ge 4\delta/k$ for $i \ne j$, and δ is independent of k. Indeed, for example, choose a smooth curve on $\partial B(0,1)$ such that $g : [0,1] \to \partial B(0,1)$ is a C^1-diffeomorphism from $[0,1]$ onto $g([0,1])$. Since g^{-1} is Lipschitz continuous, there exists a $c_0 > 0$ such that $|g(t) - g(s)| \ge c_0|t-s|$ for $t, s \in [0,1]$. Put $\xi_i := g(i/2k)$ with $1 \le i \le 2k$. Then we have for $i \ne j$,

$$|\xi_i - \xi_j| = |g(i/2k) - g(j/2k)| \ge c_0|(i-j)/2k| \ge c_0/2k.$$

Define $\delta := c_0/8$. Then $|\xi_i - \xi_j| \ge 4\delta/k$ for $i \ne j$ and δ is independent of k.

Put $\rho_k := \delta/k$. For each $1 \le i \le 2k$, there exists a unique point $x_i \in B(0,1)$ such that $B(x_i, \rho_k) \subset B(0,1)$ and $\partial B(x_i,\rho_k) \cap \partial B(0,1) = \{\xi_i\}$, after replacing δ by a small constant if necessary. Since $|\xi_i - \xi_j| \ge 4\delta/k$ for $i \ne j$, $B(x_i,\rho_k) \cap B(x_j,\rho_k) = \emptyset$ for $i \ne j$. Since $d(x) \le 2\rho_k$ in $B(x_i,\rho_k)$, we have

$$F(x,u) \ge \frac{a}{s}|u|^s - \frac{2}{r}|u|^r \rho_k \quad \text{for } x \in B(x_i,\rho_k). \tag{5.2}$$

Define θ as follows

$$\frac{2}{2-s} < \theta < \frac{s}{2(s-r)} + 1 \quad \text{when } s > r, \tag{5.3}$$

$$\frac{2}{2-s} < \theta \quad \text{when } s \le r. \tag{5.4}$$

It follows from (5.3) and (5.4) and $1 < s < 2(r+1)/3$ that

$$-(2-s)\theta + 2 < 0, \qquad -(2-s)\theta + 2 < -(2-r)\theta + 3. \tag{5.5}$$

We define $\mu_k := \rho_k^\theta$. Let us compute $\overline{F}$ defined by (1.1). Using (5.2), we have

$$\overline{F}(x_i, \mu_k, \rho_k) \ge \frac{a}{s}\rho_k^{-(2-s)\theta+2} - \frac{2}{r}\rho_k^{-(2-r)\theta+3} \to \infty, \tag{5.6}$$

as $k \to \infty$ by (5.5). Using (5.2) and $\mu_k := \rho_k^\theta$, we compute

$$\frac{F(x, m\mu_k)}{\mu_k^2}\rho_k^2 \ge \frac{am^s}{s}\rho_k^{-(2-s)\theta+2} - \frac{2m^r}{r}\rho^{-(2-r)\theta+3}, \tag{5.7}$$

for $x \in B(x_i, \rho_k)$ and $0 \le m \le 1$. We put

$$\alpha_k := a\rho_k^{-(2-s)\theta+2}, \qquad \beta_k := 2\rho_k^{-(2-r)\theta+3}$$

and denote the right hand side of (5.7) by

$$g_k(m) := \frac{\alpha_k}{s}m^s - \frac{\beta_k}{r}m^r \quad \text{for } m \in [0,1].$$

We shall show that $g_k(m)$ is bounded from below by a constant independent of k and $m \in [0,1]$. By (5.6), $g_k(1) > 0$ for $k \ge k_0$ with a large k_0. We divide the proof into two cases.

- $s > r$. Then $g_k(m)$ achieves a negative minimum in $[0,1]$, which is computed as

  $$\min_{0 \le m \le 1} g_k(m) = -\frac{s-r}{sr}\alpha_k^{-\frac{r}{s-r}}\beta_k^{\frac{s}{s-r}} = -\frac{s-r}{sr}2^{\frac{s}{s-r}}a^{-\frac{r}{s-r}}\rho_k^{\nu},$$

  where

  $$\nu = \frac{1}{s-r}\Big(-2(s-r)\theta + 3s - 2r\Big).$$

  Then $\nu > 0$ because of (5.3). Thus, the minimum of g_k converges to zero as $k \to \infty$.

- $s \le r$. Since $m^s \ge m^r$, we have $g_k(m) \ge \Big((\alpha_k/s) - (\beta_k/r)\Big)m^s \ge 0$ for $k \ge k_0$ and $m \in [0,1]$.

By Cases 1 and 2, we have the inequality $g_k(m) \ge -c$ with some $c \ge 0$ independent of k and $m \in [0,1]$, which shows that $\underline{F}(x_i, \mu_k, \rho_k) \ge -c$ for all $1 \le i \le 2k$ and $k \in \mathbb{N}$. This estimate with (5.6) shows (1.3) for all large k.

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References

[1] C. O. Alves, Y. Wang, Y. Shen, Soliton solutions for a class of quasilinear Schrödinger equations with a parameter, *J. Differential Equations* **259**(2015), 318–343. https://doi.org/10.1016/j.jde.2015.02.030; MR3335928

[2] T. Bartsch, A. Pankov, Z. Q. Wang, Nonlinear Schrödinger equations with steep potential well, *Commun. Contemp. Math.* **3**(2001), 549–569. https://doi.org/10.1142/S0219199701000494; MR1869104

[3] M. Benchohra, A. Cabada, D. Seba, An existence result for nonlinear fractional differential equations on Banach spaces, *Bound. Value Probl.* **2009**, Art. ID 628916, 11 pp. https://doi.org/10.1155/2009/628916; MR2525581

[4] A. Benhassine, General and weak sufficient condition for Hamiltonian systems, *J. Elliptic Parabol Equ.* **7**(2019), 747–759. https://doi.org/10.1007/s41808-021-00114-z; MR4342647

[5] S. Bridaa, A. B. Hassine, Weak conditions for Schrödinger equations involving the *p*-Laplacian, *Rocky Mt. J. Math.*, to appear.

[6] H. Brandi, C. Manus, G. Mainfray, T. Lehner, G. Bonnaud, Relativistic and ponderomotive self-focusing of a laser beam in a radially inhomogeneous plasma, *Phys. Fluids B* **5**(1993), 3539–3550. https://doi.org/10.1063/1.860828

[7] A. Cabada, G. Wang, Positive solutions of nonlinear fractional differential equations with integral boundary value conditions, *J. Math. Anal. Appl.* **389**(2012), No. 1, 403–411. https://doi.org/10.1016/j.jmaa.2011.11.065; MR2876506

[8] D. Chong, X. Zhang, C. Huang, Multiple small solutions for Schrödinger equations involving the *p*-Laplacian and positive quasilinear term, *Electron. J. Qual. Theory Differ. Equ.* **2020**, No. 31, 1–16. https://doi.org/10.14232/ejqtde.2020.1.31; MR4098343

[9] G. Christopher, A general low-order partial regularity theory for asymptotically convex functionals with asymptotic dependence on the minimizer, *Calc. Var. Partial Differential Equations* **57**(2018), No. 6, Paper No. 167, 50 pp. https://doi.org/10.1007/s00526-018-1422-y; MR3866575

[10] M. Colin, L. Jeanjean, Solutions for a quasilinear Schrödinger equations: a dual approach, *Nonlinear Anal.* **56**(2004), No. 2, 213–226. https://doi.org/10.1016/j.na.2003.09.008; MR2029068

[11] D. Costa, Z.-Q. Wang, Multiplicity results for a class of superlinear elliptic problems, *Proc. Amer. Math. Soc.* **133**(2005), No. 2, 787–794. https://doi.org/10.1090/S0002-9939-04-07635-X; MR2113928

[12] G. Figueiredo, O. Miyagaki, S. Moreira, Nonlinear perturbations of a periodic Schrödinger equation with supercritical growth, *Z. Angew. Math. Phys.* **66**(2015), 2379–2394. https://doi.org/10.1007/s00033-015-0525-y; MR3412303

[13] C. Huang, G. Jia, Existence of positive solutions for supercritical quasilinear Schrödinger elliptic equations, *J. Math. Anal. Appl.* **472**(2019), No. 1, 705–727. https://doi.org/10.1016/j.jmaa.2018.11.048; MR3906395

[14] R. Kajikiya, A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations, *J. Funct. Anal.* **225**(2005), No. 2, 352–370. `https://doi.org/10.1016/j.jfa.2005.04.005`; MR2152503

[15] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, Vol. 65, American Mathematical Society, Provodence, RI, 1986. `https://doi.org/10.1090/cbms/065`; MR845785

[16] Y. Ye, C.-L. Tang, Existence and multiplicity of solutions for Schrödinger–Poisson equations with sign-changing potential, *Calc. Var. Partial Differential Equations* **53**(2015), 383–411. `https://doi.org/10.1007/s00526-014-0753-6`; MR3336325

[17] W. Zhang, G.-D. Li, C.-L. Tang, Infinitely many solutions for a class of sublinear Schrödinger equations, *J. Appl. Anal. Comput.* **8**(2018), No.5, 1475–1493. `https://doi.org/10.11948/2018.1475`; MR3863240