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On the preservation of Lyapunov exponents of integrally separated systems of differential equations under small nonlinear perturbations

Dedicated to Prof. Nguyen Huu Du on the occassion of his 70th birthday

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Abstract. This paper addresses the Lyapunov exponents of non-vanishing solutions to quasi-linear time-varying systems of differential equations. The linear part is not required to be regular but it is assumed to be integrally separated, which ensures that the associated Lyapunov exponents are distinct and stable. The nonlinear perturbations are assumed to be small in a certain sense, though less restrictive than the condition in Barreira and Valls' paper, *J. Differential Equations* **258**(2015), 339–361. The main result is a Perron-type theorem for upper and lower Lyapunov exponents, offering an alternative to Barreira and Valls' result. In addition, an analogous result holds for Bohl exponents.

Keywords: quasi-linear system, Lyapunov exponent, integral separation, nonlinear perturbation, Perron-type theorem.

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1 Introduction

Asymptotic behaviour of solutions is a classical topic in the qualitative theory of differential equations. It has been discussed in well-known monographs such as [5–7]. In this paper, we study the asymptotic behavior of solutions of linear time-varying ordinary differential equations (ODEs) under nonlinear perturbations

$$x'(t) = A(t)x(t) + f(t, x(t)), \quad t \in \mathbb{I} = [t_0, \infty), \tag{1.1}$$

where the coefficient $A : \mathbb{I} \to \mathbb{C}^{n\times n}$ and the nonlinear term $f : \mathbb{I} \times \mathbb{C}^n \to \mathbb{C}^n$ are continuous. The question is that if the nonlinear term f is supposed to be sufficiently small in some sense, how certain solutions of the quasi-linear ODE (1.1) behave asymptotically comparing to those

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of the unperturbed linear ODE as t tends to infinity. In the case of constant matrix A, the result is known as Perron theorem, which was established long time ago, see [6, Theorem 5, p. 97].

Theorem 1.1. *Consider the equation* (1.1) *with a constant matrix A such that*

$$\|f(t,x(t))\| \le \gamma(t)\|x(t)\|, \quad t \ge t_0,$$

where $\gamma(t)$ is a continuous nonnegative function satisfying

$$\int_t^{t+1} \gamma(s)ds \to 0, \quad t \to \infty. \tag{1.2}$$

If $x(t)$ is a bounded solution of (1.1) *then either $x(t) = 0$ for all large t or the limit*

$$\mu = \lim_{t\to\infty} \frac{1}{t} \ln \|x(t)\|$$

exists and is equal to the real part of one of the eigenvalues of A.

This is the classical version of Perron theorem. It is noted that actually Perron did prove a weaker form. This version is due to Lettenmeyer. Later Hartman and Wintner refined the proof. The number μ is called the (strict) Lyapunov exponent of the solution x [1]. Theorem 1.1 means that the Lyapunov exponent of the solution x exists and it is equal to one of the Lyapunov exponents of the linear system, i.e., no new Lyapunov exponent arises.

Recently, extensions of this result to functional differential equations [18], nonautonomous ODEs [4], and differential-algebraic equations [14] were obtained. Similar results were also obtained for difference equations [3, 17] and functional difference equations [16]. In the case of time-dependent coefficient matrix A, by using the regularity theory Barreira and Valls did prove a similar result but under a more restrictive assumption.

Theorem 1.2 ([4, Theorem 1])**.** *Consider the quasi-linear system* (1.1)*, where $A(t)$ is supposed to be given in the block diagonal form. It is assumed further that the linear subsystems associated with each block have the same and sharp Lyapunov exponents, but the Lyapunov exponents belonging to different blocks are distinct. If $x(t)$ is a solution of* (1.1) *such that*

$$\|f(t,x(t))\| \le \gamma(t)\|x(t)\|, \quad t \ge t_0,$$

where $\gamma(t)$ is a continuous nonnegative function satisfying

$$\int_t^{t+1} e^{\delta s}\gamma(s)ds \to 0, \quad t \to \infty, \tag{1.3}$$

for some $\delta > 0$ then either $x(t) = 0$ for all large t or the limit

$$\mu = \lim_{t\to\infty} \frac{1}{t} \ln \|x(t)\|$$

exists and it is equal to one of the Lyapunov exponents of the linear system.

One can see that the condition (1.3) on $\gamma(t)$ in Theorem 1.2 is much stronger than the condition (1.2) in Theorem 1.1. The variation of Lyapunov exponents under linear perturbations, i.e. $f(t,x(t)) = B(t)x(t)$ has been well investigated in the literature, see [1, Chapter 5]. The stability concept plays a key role in the preservation of Lyapunov exponents under small linear perturbations. Necessary and sufficient conditions for the stability of Lyapunov exponents have been discussed in details in [1, Chapter V]. We emphasize that neither the existence of sharp Lyapunov exponents nor the regularity does imply the stability. We note in addition that in the case of constant matrix A, the Lyapunov exponents of the linear systems are stable without any extra assumption. An analogue of the result in [4] was established for nonautonomous difference equations in [3]. Furthermore, an extension to the so-called μ-Lyapunov exponents, see [2], was obtained in [10] where more general growth rates of solutions are characterized. Some further discussions on the stability of Lyapunov exponents and computational consequences are given in [8, 9], where the numerical approximation of Lyapunov exponents is addressed.

In this paper, we present an alternative version of Perron theorem for the nonlinear system (1.1) with a time-varying coefficient A. Under an assumption that guarantees the stability of distinct (upper) Lyapunov exponents, we are able to relax the condition on the nonlinear term, i.e., the assumption on $\gamma(t)$ remains the same as in Theorem 1.1. Furthermore, we do not require the sharp Lyapunov exponents as in [4]. The proof, which differs from that in [4], relies on reducing the linear part to a diagonal system. Analogous statements are also obtained for lower Lyapunov exponents and Bohl exponents. This investigation is of particular interest because the quasi-linear system (1.1) may arise when linearizing a nonlinear system along a particular solution. As a consequence, we can obtain information about the rate at which nearby solutions converge or diverge to/from the particular solution.

The paper is organized as follows. In the next section, we provide a brief overview of the theory on Lyapunov and Bohl exponents, with a focus on the stability of Lyapunov exponents, the property of integral separation, and the relationship between them. In Section 3, we describe the asymptotic behavior of solutions under the assumption of integral separation. As the main result, Perron-type theorems that establish the exponential growth rates of solutions are then presented. In the last section, we discuss several open questions and conjectures.

2 Preliminaries

First, we recall the notion of Lyapunov exponents, which is used to characterize the asymptotic growth of functions. Then, we briefly summarize the results related to the stability of Lyapunov exponents. These results are given in details in [1].

Definition 2.1. For a non-vanishing function $f : [0,\infty) \longrightarrow \mathbb{R}^n$, the quantities $\chi^u(f) = \limsup_{t\to\infty} \frac{1}{t}\ln\|f(t)\|$, $\chi^\ell(f) = \liminf_{t\to\infty}\frac{1}{t}\ln\|f(t)\|$, are called *upper and lower Lyapunov exponents of* f, respectively. If the exact limit exists, i.e., the upper and the lower Lyapunov exponents coincide, then we say f has a sharp Lyapunov exponent.

Consider a linear system

$$x'(t) = A(t)x(t), \quad t \in \mathbb{I} = [0,\infty), \tag{2.1}$$

with a bounded and continuous matrix function A.

Definition 2.2. Given a fundamental solution matrix X of (2.1), we introduce

$$\lambda_i^u = \limsup_{t\to\infty}\frac{1}{t}\ln\|X(t)e_i\|, \qquad \lambda_i^\ell = \liminf_{t\to\infty}\frac{1}{t}\ln\|X(t)e_i\|,$$

where e_i denotes the i-th unit vector and $\|\cdot\|$ denotes the Euclidean norm. The columns of X form a *normal basis* if $\Sigma_{i=1}^n \lambda_i^u$ is minimal. The λ_i^u, $i = 1, 2, \dots, n$ belonging to a normal basis are called *(upper) Lyapunov exponents* of (2.1).

We assume that the upper Lyapunov exponents are ordered

$$-\infty < \lambda_1^u \leq \lambda_2^u \leq \cdots \leq \lambda_n^u < \infty.$$

The following result is known as Lyapunov's inequality.

Theorem 2.3 ([1, Theorem 2.5.1]). *Let $\{\lambda_i^u\}_{i=1}^n$ be the upper Lyapunov exponents of* (2.1). *Then*

$$\sum_{i=1}^n \lambda_i^u \geq \limsup_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{trace} A(s) ds. \tag{2.2}$$

Here trace A *denotes the trace of the matrix function* A.

We say that the system (2.1) is *regular* if the inequality (2.2) becomes an equality and the exact limit exists, i.e.,

$$\sum_i^n \lambda_i^u = \liminf_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{trace} A(s) ds.$$

If (2.1) is regular, then for any nontrivial solution x, the sharp Lyapunov exponent exists. Hence, we have $\lambda_i^l = \lambda_i^u$ for $i = 1, \dots, n$, i.e., the Lyapunov spectrum of (2.1) is a point spectrum. We apply the transformation $x = L(t)y$, where L is a nonsingular and continuously differentiable matrix function for $t \geq t_0$, to system (2.1), we obtain

$$y' = B(t)y, \quad B(t) = L^{-1}A(t)L(t) - L^{-1}(t)L'(t). \tag{2.3}$$

This transformation is called a kinematic similarity transformation.

Definition 2.4. The above transformation is called a Lyapunov transformation if $L(t)$, $L^{-1}(t)$ and $L'(t)$ are bounded for $t \geq t_0$.

If we apply a Lyapunov transformation to system (2.1) and obtain the new system (2.3), then we say (2.1) is reducible to (2.3). Lyapunov transformations form a group and they do not change the Lyapunov exponent.

One of the most important questions is the variation of Lyapunov exponents under small linear and nonlinear perturbations. Consider the perturbed system

$$y' = (A(t) + Q(t))y, \tag{2.4}$$

where Q is a continuous and bounded matrix function. Let the upper Lyapunov exponents of (2.4) be ordered and denoted as follows

$$-\infty < \gamma_1^u \leq \gamma_2^u \leq \cdots \leq \gamma_n^u < \infty.$$

Definition 2.5. The upper Lyapunov exponents of system (2.1) are said to be stable if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that the inequality $\sup_{t\geq t_0} \|Q(t)\| < \delta$ implies

$$|\lambda_i^u - \gamma_i^u| < \varepsilon, \quad i = 1, 2, \dots, n.$$

It is known that regularity does not ensure the stability of exponents. In order to answer the question of stability, we need the property of integral separation.

Definition 2.6. The real, bounded and continuous functions $a_1(t), a_2(t), \ldots, a_n(t)$ are said to be separated on $\mathbb{R}^+$ if there exists a constant $a > 0$ such that

$$a_{k+1}(t) - a_k(t) \geq a, \quad k = 1, 2, \ldots, n-1, \quad t \geq 0.$$

They are said to be integrally separated on $\mathbb{R}^+$ if there exist constants $a > 0$ and $d > 0$ such that

$$\int_s^t [a_{k+1}(\tau) - a_k(\tau)] d\tau \geq a(t-s) - d$$

for all $t \geq s \geq 0$, $k = 1, 2, \ldots, n-1$.

Obviously, the condition for separateness implies integral separateness, but not vice versa.

Definition 2.7. A linear system is said to be system with integral separateness if it has solutions $x_1(t)$, $x_2(t)$, $\ldots$, $x_n(t)$ such that the inequality

$$\frac{\|x_{i+1}(t)\|}{\|x_{i+1}(s)\|} : \frac{\|x_i(t)\|}{\|x_i(s)\|} \geq de^{a(t-s)}, \quad i = 1, 2, \ldots, n-1, \tag{2.5}$$

with some constants $a > 0$, $d \geq 1$, is valid for all $t \geq s$.

The definition of integral separateness implies some properties:

- integrally separated systems have different Lyapunov exponents;
- integral separateness is invariant under Lyapunov transformations;
- the solutions $x_1(t)$, $x_2(t)$, $\ldots$, $x_n(t)$ in Definition 2.7 form a normal basis.

The following important property was proven by Bylov.

Theorem 2.8 ([1, Theorem 5.3.1 and Corollary 5.3.2]). *An integrally separated system is reducible to a real diagonal one by means of a Lyapunov transformation and the diagonal is integrally separated.*

Furthermore, by the use the Steklov function and the H-transformation, see [1, pp. 153-155], we have the following result.

Theorem 2.9 ([1, Theorem 5.4.1]). *A diagonal real system with an integrally separated diagonal is reducible to a diagonal system with a separated diagonal.*

By Millionshchikov's method of rotation, a necessary and sufficient condition for the stability of distinct Lyapunov exponents was obtained.

Theorem 2.10 ([1, Theorem 5.4.8]). *If system (2.1) has n distinct Lyapunov exponents $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, then they are stable if and only if the system is integrally separated, i.e. there exists an integrally separated fundamental solution matrix.*

In addition to Lyapunov exponents, another characteristic of the asymptotic behavior of the solutions of system (2.1) introduced by Bohl [7], has more natural properties.

Definition 2.11. Let x be a nontrivial solution of system (2.1). The *(upper) Bohl exponent* $\kappa_B^u(x)$ of this solution is the greatest lower bound of all those numbers ρ for which there exist numbers N_ρ such that

$$\|x(t)\| \leq N_\rho e^{\rho(t-s)} \|x(s)\|, \quad t \geq s \geq 0.$$

If such numbers ρ do not exist, then one sets $\kappa_B^u(x) = +\infty$.

Similarly, the *lower Bohl exponent* $\kappa_B^\ell(x)$ is the least upper bound of all those numbers ρ' for which there exist numbers N'_ρ such that

$$\|x(t)\| \geq N'_\rho e^{\rho'(t-s)} \|x(s)\|, \quad 0 \leq s \leq t.$$

It is easy to verify the estimates

$$\kappa_B^\ell(x) \le \lambda^\ell(x) \le \lambda^u(x) \le \kappa_B^u(x)$$

as well as the formulas

$$\kappa_B^u(x) = \limsup_{s,t-s\to\infty} \frac{\ln\|x(t)\| - \ln\|x(s)\|}{t-s}, \qquad \kappa_B^\ell(x) = \liminf_{s,t-s\to\infty} \frac{\ln\|x(t)\| - \ln\|x(s)\|}{t-s}.$$

If $A(t)$ is *integrally bounded*, i.e., if

$$\sup_{t\ge 0} \int_t^{t+1} \|A(s)\|\, ds < \infty,$$

then the Bohl exponents are finite.

The relation between Lyapunov and Bohl exponents are as follows.

- Bohl exponents characterize the uniform growth rate of solutions, while Lyapunov exponents simply characterize the growth rate of solutions departing from $t = 0$.
- If the greatest bound of upper Lyapunov exponents for all solutions of (2.1) is negative, then the system is asymptotically stable. If the same holds for the greatest bound of the upper Bohl exponents then the system is uniformly exponentially stable.
- Unlike Lyapunov exponents, Bohl exponents are stable without any extra assumption, see [7, Theorem 4.6].

The following lemma will be used in the proof of the main theorem in Section 3.

Lemma 2.12 ([6, Lemma 1, p. 98])**.** *Let $\beta(v)$ be a continuous function at $v = v^*$ and let $\gamma(t)$ be a continuous nonnegative function such that (1.2) holds. If $v(t)$ is a solution of the differential inequality*

$$v' \ge \beta(v) - \gamma(t)$$

for $t \ge t_0$ and there exists a sequence $\tau_n \to \infty$ such that $v(\tau_n) \to v^*$, then $\beta(v^*) \le 0$. Moreover, the exact limit of $v(\tau)$ as $\tau \to \infty$ exists.

3 Perron-type theorems for integrally separated systems

First, we establish a generalization of the classical Perron theorem for time-varying systems (1.1).

Theorem 3.1. *Consider the quasi-linear system (1.1), where the associated linear system (2.1) is integrally separated and has finite exponents. If $x(t)$ is a solution of (1.1) such that*

$$\|f(t, x(t))\| \le \gamma(t)\|x(t)\|, \quad t \ge t_0,$$

where $\gamma(t)$ is a continuous nonnegative function satisfying condition (1.2), then either $x(t) = 0$ for all large t or

$$\limsup_{t\to\infty} \frac{1}{t} \ln\|x(t)\|$$

is equal to one of the upper Lyapunov exponents of the linear system.

The same statement holds true for the limit inferior and the lower Lyapunov exponents, respectively.

That is, here we assume the stability of distinct Lyapunov exponents instead of the regularity. The proof is quite similar to that in [6] for the case of a constant coefficient. While in the case of a constant A, the transformation to the Jordan canonical form is used, here thanks to the integral separation, the linear system can be transformed into a diagonal system with a separated diagonal.

Proof. Due to Theorem 2.8 and Theorem 2.9, without loss of generality, we assume that A is already of real diagonal form

$$A(t) = \operatorname{diag}\left(\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)\right),$$

and

$$\lambda_i(t) - \lambda_{i+1}(t) \geq a > 0, \quad \forall t \geq t_0, \; i = 1, 2, \ldots, n-1.$$

Then, the i-th equation ($i = 1, 2, \ldots, n$) reads

$$x_i' = \lambda_i(t) x_i + f_i(t, x),$$

which implies

$$\frac{d}{dt}|x_i|^2 = 2\lambda_i |x_i|^2 + 2\bar{x}_i f_i.$$

Here, $\bar{x}_i$ and $|x_i|$ denote the complex conjugate and the modulus of x_i, respectively. Here, the argument t of the functions is omitted for brevity. Writing $r_i = |x_i|$, we have

$$\left| \frac{d}{dt} r_i^2 - 2\lambda_i r_i^2 \right| \leq 2 r_i |f_i(t, x)|, \quad i = 1, 2, \ldots, n. \tag{3.1}$$

Putting

$$\limsup_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \lambda_i(s) ds = \mu_i,$$

it clearly holds that μ_i, $i = 1, 2, \ldots, n$, are the upper Lyapunov exponents of the linear system, and $\mu_i - \mu_{i+1} \geq a$. Let us denote

$$L_k = r_k^2, \qquad M_k = \sum_{i<k} r_i^2, \qquad N_k = \sum_{i \geq k} r_i^2,$$

we have so that $M_k + N_k = \sum_{i=1}^{n} r_i^2 = \|x\|^2$, where the Euclidean norm is used.

From (3.1), we obtain

$$\left| L_k' - 2\lambda_k L_k \right| \leq 2\gamma(t) L_k^{1/2} (M_k + N_k)^{1/2}, \tag{3.2}$$

$$M_k' \geq 2\lambda_{k-1} M_k - 2\gamma(t) M_k^{1/2} (M_k + N_k)^{1/2}, \tag{3.3}$$

$$N_k' \leq 2\lambda_k N_k + 2\gamma(t) N_k^{1/2} (M_k + N_k)^{1/2}. \tag{3.4}$$

Thus, $N_1 = \|x\|^2$ satisfies

$$2(\lambda_n(t) - \gamma(t)) N_1 \leq N_1' \leq 2(\lambda_1(t) + \gamma(t)) N_1.$$

By integration, we get for $t \geq t_1 \geq t_0$

$$e^{\int_{t_1}^{t} 2(\lambda_n(t) - \gamma(t)) ds} \|x(t_1)\| \leq \|x(t)\| \leq e^{\int_{t_1}^{t} 2(\lambda_1(t) + \gamma(t)) ds} \|x(t_1)\|.$$

This shows that if $x(t_1)$ for some $t_1 \geq t_0$, then $x(t) = 0$ for all $t \geq t_1$. Thus, we exclude this case from now on.

Consider the function

$$v(t) = v_k(t) = \frac{M_k}{M_k + N_k},$$

which is well defined for all $t \geq t_0$ and fulfills $0 \leq v \leq 1$. Furthermore, since

$$v' = \frac{M'N - MN'}{(M+N)^2} \quad \text{and} \quad v(1-v) = \frac{MN}{(M+N)^2},$$

from (3.3) and (3.4), we obtain

$$v' \geq bv(1-v) - \sqrt{2}\gamma(t), \tag{3.5}$$

where $b = b_k = 2(\lambda_{k-1} - \lambda_k) \geq 2a > 0$.

By Lemma 2.12, it follows that for each $k = 1, 2, \ldots, n$, the limit of $v_k(t)$ as $t \to \infty$ exists and equal either 0 or 1. Moreover, $v_1(t) \to 0$ as $t \to \infty$, since $M_1 \equiv 0$. Let m be the greatest value of k for which $\lim_{t\to\infty} v_k(t) = 0$. Since $M_k = L_1 + \cdots + L_{k-1}$, it follows that

$$\lim_{t\to\infty} \frac{L_k(t)}{M_k(t) + N_k(t)} = \begin{cases} 0 & \text{for } k \neq m, \\ 1 & \text{for } k = m. \end{cases} \tag{3.6}$$

Therefore, we have

$$\lim_{t\to\infty} \frac{L_k(t)}{L_m(t)} = 0 \quad \text{for } k \neq m \tag{3.7}$$

and by (3.2) with $k = m$

$$\left| \frac{d}{dt}(\ln L_m) - 2\lambda_m \right| \leq 2\gamma(t) \left(\frac{M_m + N_m}{L_m} \right)^{1/2}. \tag{3.8}$$

Note that

$$\lim_{t\to\infty} \frac{M_m(t) + N_m(t)}{L_m(t)} = 1$$

and assumption (1.2) implies $t^{-1} \int_{t_0}^t \gamma(s)ds \to 0$ as $t \to \infty$. Therefore, integrating both sides of (3.8) from t_0 to t and dividing by t, we get

$$\frac{\ln L_m(t)}{t} - \frac{2}{t}\int_{t_0}^t \lambda_m(s)ds = o(1), \quad t \to \infty.$$

This, together with the asymptotic relation

$$\frac{\ln \|x(t)\|^2}{t} = \frac{\ln L_m(t)}{t}(1 + o(1)), \quad t \to \infty,$$

implies that

$$\limsup_{t\to\infty} \frac{\ln \|x(t)\|}{t} = \limsup_{t\to\infty} \frac{\ln L_m(t)}{2t} = \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^t \lambda_m(s)ds = \mu_m,$$

which completes the proof.

If we take the limit inferior instead of the limit superior, the statement for the lower Lyapunov exponent is obtained, too. □

Remark 3.2. The condition on $f(t,x)$ in Theorem 3.1 is certainly satisfied by any solution $x(t)$ of (1.1) which tends to zero as $t \to \infty$ if

$$f(t,x) = o(\|x\|), \quad \text{for } t \to \infty, \ \|x\| \to 0.$$

In the special case of linear perturbation $f(t,x) = B(t)x$, the condition (1.2) holds in particular if $\|B(t)\| \to 0$ as $t \to \infty$ or $B(t) \in L_p[t_0, \infty)$, where $1 \le p < \infty$.

As a consequence, we obtain a theorem on the asymptotic stability of a non-stationary solution of a nonlinear system by using linearization.

Theorem 3.3. *Consider a nonlinear system*

$$y' = g(t,y), \quad t \ge t_0, \tag{3.9}$$

where g is continuous and continuously differentiable with respect to variable y. Suppose that $y^* = y^*(t)$ is a particular solution that exists on $[t_0, \infty)$. Consider the linearized system

$$x' = A(t)x, \quad A(t) = g_y(t, y^*(t)). \tag{3.10}$$

If the system (3.10) is integrally separated and all of its Lyapunov exponents are negative, then y^* is an asymptotically stable solution of (3.9).

We note the fact that all of its Lyapunov exponents are negative implies that the system (3.10) is exponentially stable, but it is not necessarily uniformly exponentially stable.

Analogously, we obtain a Perron theorem for Bohl exponents.

Theorem 3.4. *Consider the quasi-linear system (1.1), where the associated linear system (2.1) is integrally separated and has finite exponents. If $x(t)$ is a solution of (1.1) such that*

$$\|f(t,x(t))\| \le \gamma(t)\|x(t)\|, \quad t \ge t_0,$$

where $\gamma(t)$ is a continuous nonnegative function satisfying condition (1.2), then either $x(t) = 0$ for all large t or

$$\limsup_{s,t-s\to\infty} \frac{\ln\|x(t)\| - \ln\|x(s)\|}{t-s}$$

is equal to one of the upper Bohl exponents of the linear system.

The same statement holds true for the limit inferior and the lower Bohl exponents, respectively.

Proof. We proceed as in the the proof of Theorem 3.1 and integrate both sides of (3.8) from s to t and dividing by $t-s$. Then, take the limit superior/inferior as s, $t-s$ tend to ∞. □

4 Discussion

In this paper, we have derived two extended versions of the classical Perron-type theorem. Assuming integral separation, which guarantees the stability of Lyapunov exponents, we have confirmed that the Perron theorem remains valid under the same smallness condition on the nonlinear part as in the constant coefficient case. Therefore, our version of the Perron theorem differs from the one obtained in [4]. Additionally, we have established a Perron theorem for Bohl exponents as well. Extensions of these results to different growth rates and the μ-Lyapunov exponents defined in [2,10] appear to be straightforward.

Several open problems and conjectures remain. First, our results can be extended to difference equations, providing an alternative to the Perron-type theorem presented in [3]. Second, it is known that the Lyapunov exponents of linear systems may be nondistinct but still stable, as characterized in [1, Theorem 5.4.9] and [9]. In such cases, the linear system (2.1) can be reduced to block-diagonal form with upper-triangular blocks subject to additional conditions. We conjecture that Theorem 3.1 still holds, i.e., the stability of Lyapunov exponents implies their preservation under small nonlinear perturbations satisfying (1.3). This would fully generalize the classical Perron theorem to the time-varying system (1.1).

Furthermore, it is well known that Bohl exponents are stable without any additional assumptions. Therefore, we also conjecture that the result of Theorem 3.4 holds without the integral separation assumption (see the related result in [7, Chapter VII, Section 3] and [9]). However, the arguments used in the proof of Theorem 3.1 are insufficient for the last two problems, as reducibility to diagonal form no longer holds. Addressing these questions would require overcoming further technical challenges.

Lastly, recent results on the asymptotic behavior of solutions and Lyapunov exponents have been extended from ODEs to DAEs (see [11–15]). Therefore, extending the Perron-type theorem to linear time-varying DAEs under small nonlinear perturbations would also be of interest.

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