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UNIQUE SOLVABILITY OF SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH NON-LOCAL BOUNDARY CONDITIONS

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Abstract. Some general conditions sufficient for unique solvability of the boundary-value problem for a system of linear functional differential equations of the second order are established. The class of equations considered covers, in particular, linear equations with transformed argument, integro-differential equations and neutral equations. An example is presented to illustrate the general theory.

1. Problem formulation

The purpose of this paper, which has been motivated in part by the recent works [11–16,18], is to establish new general conditions sufficient for the unique solvability of the non-local boundary-value problem for systems of linear functional differential equations on the assumptions that the linear operator $l = (l_k)_{k=1}^n$, appearing in (1.1) can be estimated by certain other linear operators generating problems with conditions (1.2), (1.3) for which the statement on the integration of differential inequality holds. The precise formulation of the property mentioned is given by Definition 1.1.

The proof of the main result obtained here is based on the application of [10, Theorem 49.4], which ensures the unique solvability of an abstract equation with an operator satisfying Lipschitz-type conditions with respect to a suitable cone.

We consider the linear boundary-value problem for a second order functional differential equation

$$u''(t) = (lu)(t) + q(t), \quad t \in [a, b], \tag{1.1}$$

$$u'(a) = r_1(u), \tag{1.2}$$

$$u(a) = r_0(u), \tag{1.3}$$

where $l : W^2([a, b], \mathbb{R}^n) \to L_1([a, b], \mathbb{R}^n)$ is linear operator, $r_i : W^2([a, b], \mathbb{R}^n) \to \mathbb{R}^n$, $i = 0, 1$, are linear functionals.

By a solution of problem (1.1)–(1.3), as usual (see, e. g., [1]), we mean a vector function $u = (u_k)_{k=1}^n : [a, b] \to \mathbb{R}^n$ whose components are absolutely continuous, satisfy system (1.1) almost everywhere on the interval $[a, b]$, and possess properties (1.2), (1.3).

Definition 1.1. A linear operator $l = (l_k)_{k=1}^n : W^2([a, b], \mathbb{R}^n) \to L_1([a, b], \mathbb{R}^n)$ is said to belong to the set $\mathcal{S}_{r_0,r_1}$ if the boundary value problem (1.1), (1.2), (1.3) has a unique solution $u = (u_k)_{k=1}^n$ for any $q \in L_1([a, b], \mathbb{R}^n)$ and, moreover, the solution

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of (1.1), (1.2), (1.3) possesses the property

$$\min_{t\in[a,b]} u_k(t) \geq 0, \qquad k = 1, 2, \dots, n, \tag{1.4}$$

whenever the components of the function q, appearing in (1.1) are non-negative almost everywhere on $[a, b]$.

2. Notation

Throughout the paper, we fix a bounded interval $[a, b]$ and a natural number n. We use the following notation.

(1) $\mathbb{R} := (-\infty, \infty)$; $\|x\| := \max_{1\leq i\leq n} |x_i|$ for $x = (x_i)_{i=1}^n \in \mathbb{R}^n$.

(2) $L_1([a, b], \mathbb{R}^n)$ is the Banach space of all the Lebesgue integrable vector-functions $u : [a, b] \to \mathbb{R}^n$ with the standard norm

$$L_1([a, b], \mathbb{R}^n) \ni u \longmapsto \int_a^b \|u(\xi)\|\, d\xi.$$

(3) $W^k([a, b], \mathbb{R}^n)$, $k = 1, 2$, is set of vector functions $u = (u_i)_{i=1}^n : [a, b] \to \mathbb{R}^n$ with $u^{(k-1)}$ absolutely continuous on $[a, b]$ and the norm given by the formula

$$W^k([a, b], \mathbb{R}^n) \ni u \longmapsto \|u\|_k := \int_a^b \|u^{(k)}(\xi)\|\, d\xi + \sum_{m=0}^{k-1} \|u^{(m)}(a)\|. \tag{2.1}$$

(4) For $k = 1, 2$, $m = \overline{0, 2}$, by $W^k_{(m)}([a, b], \mathbb{R}^n)$ we denote the set of functions $u = (u_i)_{i=1}^n : [a, b] \to \mathbb{R}^n$ from $W^k([a, b], \mathbb{R}^n)$ such that the components of $u^{(m)}$ are non-negative a.e. on $[a, b]$ and $u_i^{(j)}(a) \geq 0$ for $0 \leq j \leq m - 1$, $i = 1, 2, \dots, n$.

(5) If $r_j : W^2([a, b], \mathbb{R}^n) \to \mathbb{R}^n$, $j = 0, 1$, are functionals, then the symbol $W^2_{(r_0,r_1)}([a, b], \mathbb{R}^n)$ denotes the set of all u from $W^2([a, b], \mathbb{R}^n)$ for which $u(a) = r_0(u)$ and $u'(a) = r_1(u)$.

(6) $W^2_{(m;r_0,r_1)}([a, b], \mathbb{R}^n) := W^2_{(r_0,r_1)}([a, b], \mathbb{R}^n) \cap W^2_{(m)}([a, b], \mathbb{R}^n)$ for $m = \overline{0, 2}$.

The symbols defined above will usually appear in the text in a shortened form, e. g., the sets $W^2([a, b], \mathbb{R}^n)$ and $W^2_{(m;r_0,r_1)}([a, b], \mathbb{R}^n)$ will be referred to simply as W^2 and $W^2_{(m;r_0,r_1)}$, etc. Since a, b, and n are fixed, no confusion will arise.

3. Auxiliary statements

To prove our main results, we use the following statement on the unique solvability of an equation with a Lipschitz type non-linearity established in [9] (see also [10]).

Let us consider the abstract operator equation

$$Fx = z, \tag{3.1}$$

where $F : E_1 \to E_2$ is a mapping, $\langle E_1, \|\cdot\|_{E_1}\rangle$ is a normed space, $\langle E_2, \|\cdot\|_{E_2}\rangle$ is a Banach space over the field $\mathbb{R}$, $K_i \subset E_i$, $i = 1, 2$, are closed cones, and z is an arbitrary element from E_2.

The cones K_i, $i = 1, 2$, induce natural partial orderings of the respective spaces. Thus, for each $i = 1, 2$, we write $x \leqq_{K_i} y$ and $y \geqq_{K_i} x$ if and only if $\{x, y\} \subset E_i$ and $y - x \in K_i$.

Theorem 3.1 ([10, Theorem 49.4]). *Let the cone K_2 be normal and generating. Furthermore, let $B_k : E_1 \to E_2$, $k = 1, 2$, be additive and homogeneous operators such that B_1^{-1} and $(B_1 + B_2)^{-1}$ exist and possess the properties*

$$B_1^{-1}(K_2) \subset K_1, \quad (B_1 + B_2)^{-1}(K_2) \subset K_1, \tag{3.2}$$

and, furthermore, let the order relation

$$B_1(x - y) \leqq_{K_2} Fx - Fy \leqq_{K_2} B_2(x - y) \tag{3.3}$$

be satisfied for any pair $(x, y) \in E_1^2$ such that $x \geqq_{K_1} y$.

Then equation (3.1) has a unique solution $x \in E_1$ for an arbitrary element $z \in E_2$.

Let us recall two definitions that has been used above (see, e.g., [8, 10]).

Definition 3.1. A cone $K_2 \subset E_2$ is called normal if every subset of E_2 bounded with respect to the partial ordering $\leqq_{E_2}$ generated by K_2 is also bounded with respect to the norm.

A cone K_1 is said to be generating in E_1 if an arbitrary element $x \in E_1$ can be represented in the form $x = u - v$, where $\{u, v\} \subset K_1$.

3.1. **Lemmas.** We need some technical lemmas.

Lemma 3.1. *The following assertions are true:*

(1) *$W^2_{(r_0,r_1)}$ is a closed subspace of W^2 with respect to the norm (2.1).*
(2) *The set $W^2_{(0;r_0,r_1)}$ is a cone in the space $W^2_{(r_0,r_1)}$.*
(3) *The set $W^2_{(2;0,0)}$ is a normal and generating cone in the space $W^2_{(0,0)}$.*

Proof. Assertions 1 and 2 follow immediately from the assumption that the linear functionals $r_j : W^2([a, b], \mathbb{R}^n) \to \mathbb{R}^n$, $j = 0, 1$, are bounded.

Let us check assertion 3. If $\{u_1, u_2\} \subset W^2_{(2;0,0)}$ and $\{\lambda_1, \lambda_2\} \subset [0, +\infty)$, then, obviously, $\lambda_1 u_1 + \lambda_2 u_2$ lies in $W^2_{(2;0,0)}$ as well. Suppose that $u \in W^2_{(2;0,0)}$ and $-u \in W^2_{(2;0,0)}$ simultaneously. Then, according to the definition of $W^2_{(2;0,0)}$, we have $u'' \equiv 0$ and, moreover, $u(a) = 0$, $u'(a) = 0$, whence it is obvious that $u \equiv 0$. Thus, $W^2_{(2;0,0)}$ is a cone in $W^2_{(0,0)}$.

In order to prove that the cone $W^2_{(2;0,0)}$ is normal, it is sufficient to show that every set of the form

$$\{x \in W^2_{(0,0)} : \{x - u, v - x\} \subset W^2_{(2;0,0)}\}, \tag{3.4}$$

where $\{u, v\} \subset W^2_{(0,0)}$, is bounded with respect to the norm $\|\cdot\|_2$ (see (2.1) with $k = 2$). Indeed, if an x belongs to set (3.4), then

$$u''(t) \le x''(t) \le v''(t), \qquad t \in [a, b],$$

componentwise. Therefore,

$$\|x\|_2 = \int_a^b \|x''(s)\| \, ds \le \max\{\|u\|_2, \|v\|_2\},$$

which, in view of the arbitrariness of x, implies that set (3.4) is bounded.

To prove that the cone $W^2_{(2;0,0)}$ is generating in the space $W^2_{(0,0)}$, it is sufficient to show that every element x of $W^2_{(0,0)}$ admits a majorant in $W^2_{(2;0,0)}$. Indeed, let $x \in W^2_{(0,0)}$ be arbitrary. Then x has the form

$$x(t) = \int_a^t \left(\int_a^s X(\xi) d\xi \right) ds, \qquad t \in [a, b], \tag{3.5}$$

where $X \in L_1$. Equality (3.5) implies that, componentwise,

$$x(t) \le u(t), \qquad t \in [a, b],$$

where

$$u(t) := \int_a^t \left(\int_a^s |X(\xi)|\, d\xi \right) ds, \qquad t \in [a, b]. \tag{3.6}$$

It is obvious from (3.6) that $u(a) = 0$, $u'(a) = 0$, and u'' is non-negative and, therefore, u is an element of $W^2_{(2;0,0)}$. This, due to the arbitrariness of x, proves that $W^2_{(2;0,0)}$ is generating. □

Let us define a linear operator $V_{l,r_0,r_1} : W^2_{(r_0,r_1)} \to W^2_{(0,0)}$ by putting

$$(V_{l,r_0,r_1} u)(t) := u(t) - \int_a^t \left(\int_a^s (lu)(\xi)\, d\xi \right) ds - (t-a)\, r_1(u) - r_0(u) \tag{3.7}$$

for all $u \in W^2_{(r_0,r_1)}$. Then the following assertion is straightforward.

Lemma 3.2. *A function u from W^2 is a solution of the equation*

$$(V_{l,r_0,r_1} u)(t) = \int_a^t \left(\int_a^s q(\xi) d\xi \right) ds, \qquad t \in [a, b], \tag{3.8}$$

where $q \in L_1$, if and only if it is a solution of the non-local boundary value problem (1.1)–(1.3).

The lemma below sets the relation between the property described by Definition 1.1 and the positive invertibility of operator (3.7).

Lemma 3.3. *If $l = (l_k)_{k=1}^n : W^2 \to L_1$ is a linear operator such that*

$$l \in \mathcal{S}_{r_0,r_1}, \tag{3.9}$$

then the operator V_{l,r_0,r_1} is invertible and, moreover, its inverse V^{-1}_{l,r_0,r_1} satisfies the inclusion

$$V^{-1}_{l,r_0,r_1}(W^2_{(2;0,0)}) \subset W^2_{(0;r_0,r_1)}. \tag{3.10}$$

Proof. Let the mapping l belong to the set $\mathcal{S}_{r_0,r_1}$. Given an arbitrary function $y = (y_k)_{k=1}^n \in W^2_{(0,0)}$, consider the equation

$$V_{l,r_0,r_1} u = y. \tag{3.11}$$

Since $y \in W^2_{(0,0)}$, we have that, in particular,

$$y(a) = 0, \qquad y'(a) = 0. \tag{3.12}$$

In view of assumption (3.9), there exists a unique function u such that u' is absolutely continuous, the equation

$$u''(t) = (lu)(t) + y''(t), \qquad t \in [a, b], \tag{3.13}$$

holds, and

$$u'(a) = r_1(u), \tag{3.14}$$

$$u(a) = r_0(u). \tag{3.15}$$

By Lemma 3.2, it follows that u is, in fact, the unique solution of equation (3.11). In other words, $u = V^{-1}_{l,r_0,r_1} y$ due to the arbitrariness of $y \in W^2_{(0,0)}$.

Moreover, inclusion (3.9) also guarantees that if the functions y_k, $k = 1, 2, \ldots, n$, are such that

$$y_k''(t) \ge 0, \qquad t \in [a, b],\ k = 1, 2, \ldots, n, \tag{3.16}$$

then the components of u are non-negative. Therefore, $V_{l,r_0,r_1}^{-1} y \in W^2_{(0;r_0,r_1)}$. However, relation (3.16), together with (3.12), means that $y \in W^2_{(2;0,0)}$. Since y is arbitrary, we arrive at the required inclusion (3.10). □

Lemma 3.4. *For arbitrary linear operators $p_i : W^2 \to L_1$, $i = 1, 2$, the identity*

$$V_{p_1,r_0,r_1} + V_{p_2,r_0,r_1} = 2V_{\frac{1}{2}(p_1+p_2),r_0,r_1} \tag{3.17}$$

is true.

Proof. Equality (3.17) is obtained immediately from relation (3.7). □

Remark 3.1. A linear operator $l = (l_k)_{k=1}^n : W^2 \to L_1$ belongs to the set $\mathcal{S}_{r_0,r_1}$ if problem (1.3) for the system

$$u_k'(t) = \int_a^t (p_k u)(s)ds + r_{1k}(u) + \int_a^t q_k(s)ds, \quad t \in [a, b], \ k = 1, 2, \dots, n, \tag{3.18}$$

has a unique solution $u = (u_k)_{k=1}^n$ for any $\{q_k \mid k = 1, 2, \dots, n\} \subset L_1$ and, moreover, the solution of (3.18), (1.3) possesses property (1.4) if q_k, $k = 1, 2, \dots, n$, are non-negative almost everywhere on $[a, b]$.

A number of results related to the solvability of the linear boundary-value problem (3.18), (1.3) (and therefore, by virtue of Remark 3.1, to properties of the set $\mathcal{S}_{r_0,r_1}$) can be found, for example, in [2, 4, 5, 7, 11–14, 17–19].

4. A general theorem on the solvability

The theorems presented below allow one to deduce conditions under which problem (1.3), (3.18) always has a unique solution.

Theorem 4.1. *Suppose that there exist certain linear operators $p_i = (p_{ik})_{k=1}^n : W^2 \to L_1$, $i = 1, 2$, satisfying the inclusions*

$$p_1 \in \mathcal{S}_{r_0,r_1}, \qquad \frac{1}{2}(p_1 + p_2) \in \mathcal{S}_{r_0,r_1} \tag{4.1}$$

and such that the inequalities

$$(p_{2k}u)(t) \le (l_k u)(t) \le (p_{1k}u)(t), \quad t \in [a, b], k = 1, 2, \dots, n, \tag{4.2}$$

are fulfilled for arbitrary non-negative absolutely continuous vector function $u : [a, b] \to \mathbb{R}^n$ from $W^2_{(0;r_0,r_1)}$.

Then the non-local boundary value problem (1.1)–(1.3) *has a unique solution for any $q \in L_1$.*

Proof. Let us put

$$E_1 = W^2_{(r_0,r_1)}, \qquad E_2 = W^2_{(0,0)}$$

and define a mapping $F : E_1 \to E_2$ by setting

$$(Fu)(t) := (V_{l,r_0,r_1}u)(t), \quad t \in [a, b], \tag{4.3}$$

for any u from $W^2_{(0;r_0,r_1)}$, where V_{l,r_0,r_1} is given by (3.7). Then equation (4.3) takes form (3.1) with

$$z(t) := \int_a^t \left(\int_a^s q(\xi)d\xi \right) ds, \qquad t \in [a, b].$$

Consider problem (1.3), (3.18). It is clear (see Remark 3.1) that an absolutely continuous vector function $u = (u_k)_{k=1}^n : [a, b] \to \mathbb{R}^n$ is a solution of (1.3), (3.18) if, and only if it satisfies the equation

$$V_{l,r_0,r_1} u = z. \tag{4.4}$$

Assumption (4.2) means that the estimate

$$-(p_{1k}u)(t) \leq -(l_k u)(t) \leq -(p_{2k}u)(t), \quad t \in [a, b], \tag{4.5}$$

is true for any u from $W^2_{(0;r_0,r_1)}$ and all $k = 1, 2, \dots, n$. For any such functions u the relation

$$u_k''(t) - (p_{1k}u)(t) \leq u_k''(t) - (l_k u)(t) \leq u_k''(t) - (p_{2k}u)(t), \quad t \in [a, b], \tag{4.6}$$

is true for almost all $t \in [a, b]$. Integrating (4.6), and taking property (1.2) into account, we obtain that the inequality

$$\begin{aligned} u_k'(t) - \int_a^t (p_{1k}u)(\xi)d\xi - r_{1k}(u) \leq u_k'(t) - \int_a^t (l_k u)(\xi)d\xi - r_{1k}(u) \leq \\ \leq u_k'(t) - \int_a^t (p_{2k}u)(\xi)d\xi - r_{1k}(u), \quad k = 1, 2, \dots, n, \end{aligned} \tag{4.7}$$

holds for any u from $W^2_{(0;r_0,r_1)}$ and all $k = 1, 2, \dots, n$.

Let us define the linear mappings $B_{ik} : W^2_{(r_0,r_1)} \to W^2_{(0,0)}$, $i = 1, 2$, $k = 1, 2, \dots, n$, by putting

$$(B_{ik}u)(t) := u(t) - \int_a^t \left(\int_a^s (p_{ik}u)(\xi)d\xi \right) ds - (t-a)r_1(u) - r_0(u), \qquad t \in [a, b], \tag{4.8}$$

for an arbitrary u from $W^2_{(0;r_0,r_1)}$. Then, integrating (4.7) and taking property (1.3) into account, we obtain

$$\begin{aligned} (B_{1k}u)(t) \leq u_k(t) - (t-a)r_{1k}(u) - r_0(u) - \int_a^t \left(\int_a^s (l_k u)(\xi)d\xi \right) ds \\ \leq (B_{2k}u)(t), \qquad t \in [a, b], \end{aligned} \tag{4.9}$$

for any $u = (u_k)_{k=1}^n$ from $W^2_{(0;r_0,r_1)}$ and all $k = 1, 2, \dots, n$.

Construct the mappings $B_i : W^2_{(r_0,r_1)} \to W^2_{(0,0)}$, $i = 1, 2$, according to the formula

$$W^2_{(r_0,r_1)} \ni u \longmapsto B_i u := \begin{pmatrix} B_{i1}u \\ B_{i2}u \\ \vdots \\ B_{in}u \end{pmatrix}, \qquad i = 1, 2. \tag{4.10}$$

Then, considering the definition of the mapping V_{l,r_0,r_1} (see formula (3.7)) and the sets $W^2_{(0;r_0,r_1)}$ and $W^2_{(2;0,0)}$, we see that estimates (4.6), (4.7) and (4.9) ensure the validity of the inclusion

$$\{B_2 u - V_{l,r_0,r_1}u, V_{l,r_0,r_1}u - B_1 u\} \subset W^2_{(2;0,0)} \tag{4.11}$$

for an arbitrary u from $W^2_{(0;r_0,r_1)}$.

Finally, let us define K_1 and K_2 by the formulae

$$K_1 := W^2_{(0;r_0,r_1)}, \qquad K_2 := W^2_{(2;0,0)}. \tag{4.12}$$

By Lemma 3.1, the set K_1 forms a cone in the normed space $W^2_{(r_0,r_1)}$, whereas K_2 is a normal and generating cone in the Banach space $W^2_{(0,0)}$.

According to equalities (3.7) and (4.8), we have $B_i = V_{p_i,r_0,r_1}$, $i = 1, 2$. Furthermore, it follows from Lemma 3.4 that identity (3.17) is true and, therefore,

$$B_1 + B_2 = 2V_{\frac{1}{2}(p_1+p_2),r_0,r_1}. \tag{4.13}$$

In view of assumption (4.1), Lemma 3.3 guarantees the invertibility of the operators V_{p_1,r_0,r_1} and $V_{\frac{1}{2}(p_1+p_2),r_0,r_1}$. Consequently, we have $B_1^{-1} = V^{-1}_{p_1,r_0,r_1}$ and, by (4.13), the equality $(B_1 + B_2)^{-1} = \frac{1}{2}V^{-1}_{\frac{1}{2}(p_1+p_2),r_0,r_1}$ holds. The same Lemma 3.3 ensures the positivity of the inverse operators in the sense that

$$V^{-1}_{p_1,r_0,r_1}(W^2_{(2;0,0)}) \subset W^2_{(0;r_0,r_1)},$$
$$V^{-1}_{\frac{1}{2}(p_1+p_2),r_0,r_1}(W^2_{(2;0,0)}) \subset W^2_{(0;r_0,r_1)}$$

and, hence, inclusions (3.2) are true.

Finally, in view of assumption (4.2), we see that relation (3.3) holds with F, B_1, and B_2 given by (4.3), (4.10) with respect to the cones K_1 and K_2 defined by (4.12).

Applying Theorem 3.1, we establish the unique solvability of the boundary value problem (3.18), (1.3) for arbitrary $q \in L_1$. Taking Remark 3.1 into account, we complete the proof of Theorem 4.1. □

5. Corollaries

The following statements are true.

Corollary 5.1. *Assume that there exist certain linear operators $f_i : W^2 \to L_1$, $i = 1, 2$, such that, for an arbitrary function $u = (u_k)_{k=1}^n : [a, b] \to \mathbb{R}^n$ from $W^2_{(0;r_0,r_1)}$, the inequalities*

$$|(l_k u)(t) - (f_{1k}u)(t)| \le (f_{2k}u)(t), \quad k = 1, 2, \ldots, n, \tag{5.1}$$

hold. Moreover, let the inclusions

$$f_1 + f_2 \in \mathcal{S}_{r_0,r_1}, \qquad\qquad f_1 \in \mathcal{S}_{r_0,r_1} \tag{5.2}$$

be satisfied.

Then the non-local boundary value problem (1.1)-(1.3) *has a unique solution for an arbitrary $q \in L_1$.*

Proof. This statement is proved similarly to [6, Theorem 2]. Indeed, it is obvious that, for any u from $W^2_{(0;r_0,r_1)}$, condition (5.1) is equivalent to the relation

$$-(f_{2k}u)(t) + (f_{1k}u)(t) \le (l_k u)(t) \le (f_{2k}u)(t) + (f_{1k}u)(t), \quad t \in [a, b].$$

Let us put

$$p_{ik} := f_{1k} - (-1)^i f_{2k}, \quad i = 1, 2, \tag{5.3}$$

for any $k = 1, 2, \ldots, n$. We see that, under conditions (5.1) and (5.2), the operators $p_{ik} : W^2 \to L_1$, $i = 1, 2$, defined by formulae (5.3) satisfy conditions (4.1) and (4.2) of Theorem 4.1. Application of Theorem 4.1 thus leads us to the assertion of Corollary 5.1. □

Definition 5.1. We say that an operator $p = (p_k)_{k=1}^n : W^2 \to L_1$ is *positive* if, for any $u \in W^2_{(0;r_0,r_1)}$, the inequalities

$$(p_k u)(t) \geq 0, \qquad k = 1, 2, \ldots, n,$$

are true for a. e. $t \in [a, b]$,

Corollary 5.2. *Let there exist certain positive linear operators $g_i = (g_{ik})_{k=1}^n : W^2 \to L_1$, $i = 0, 1$, which satisfy the inclusions*

$$g_0 \in \mathcal{S}_{r_0,r_1}, \qquad -\frac{1}{2} g_1 \in \mathcal{S}_{r_0,r_1}, \tag{5.4}$$

and are such that the inequalities

$$|(l_k u)(t) + (g_{1k} u)(t)| \leq (g_{0k} u)(t), \qquad t \in [a, b],\ k = 1, 2, \ldots, n, \tag{5.5}$$

hold for any function $u : [a, b] \to \mathbb{R}^n$ from $W^2_{(0;r_0,r_1)}$.

Then the boundary value problem (1.1)-(1.3) *has a unique solution for an arbitrary $q \in L_1$.*

Proof. It follows from assumption (5.5) and the positivity of the operator g_1 that the relations

$$\begin{aligned} |(l_k u)(t) + \frac{1}{2}(g_{1k} u)(t)| &= |(l_k u)(t) + (g_{1k} u)(t) - \frac{1}{2}(g_{1k} u)(t)| \\ &\leq (g_{0k} u)(t) + \frac{1}{2}|(g_{1k} u)(t)| \\ &= (g_{0k} u)(t) + \frac{1}{2}(g_{1k} u)(t) \end{aligned}$$

are true for any u from $W^2_{(0;r_0,r_1)}$. This means that $l = (l_k)_{k=1}^n$ admits estimate (5.1) with the operators f_1 and f_2 defined by the equalities

$$f_1 := -\frac{1}{2} g_1, \qquad f_2 := g_0 + \frac{1}{2} g_1. \tag{5.6}$$

Moreover, assumption (5.4) guarantees that inclusions (5.2) hold for f_1 and f_2 of form (5.6). Thus, we can apply Corollary 5.1, which leads us to the required assertion. □

Corollary 5.3. *Assume that there exist positive linear operators $p_i = (p_{ik})_{k=1}^n : W^2 \to L_1$, $i = 1, 2$, satisfying the inclusions*

$$p_1 + p_2 \in S_{r_0,r_1}, \qquad -\frac{1}{2} p_1 \in S_{r_0,r_1}$$

and such that the inequalities

$$|(l_i u)(t) + (p_{1i} u)(t)| \leq (p_{1i} u)(t) + (p_{2i} u)(t), \quad t \in [a, b],\ i = 1, 2, \ldots, n,$$

are true for an arbitrary function $u : [a, b] \to \mathbb{R}^n$ from $W^2_{(0;r_0,r_1)}$.

Then problem (1.1)-(1.3) *has a unique solution for any $q \in L_1$.*

Proof. It is sufficient to put $g_0 := p_1 + p_2$, $g_1 := p_1$, notice that g_0 and g_1 are positive, and apply Corollary 5.2. □

It should be noted that conditions of the statements presented above are optimal in a certain sense and cannot be improved. For example, assumption (5.4) cannot be replaced by any of the weaker conditions

$$(1 - \varepsilon)\, g_0 \in \mathcal{S}_{r_0,r_1}, \qquad -\frac{1}{2} g_1 \in \mathcal{S}_{r_0,r_1},$$

and

$$g_0 \in \mathcal{S}_{r_0,r_1}, \qquad\qquad -\frac{1}{2+\varepsilon} g_1 \in \mathcal{S}_{r_0,r_1},$$

where $\varepsilon > 0$, because after such a replacement the assertion of Corollary 5.2 is not true any more. The optimality of the conditions is proved by analogy to [3, 16].

6. The case of l defined on W^1

In the general case, l from equation (1.1) is given on W^2 only and, thus, the right-hand side term of equation (1.1) may contain u'', which corresponds to an equation of neutral type.

If the operator l in equation (1.1) is defined not only on W^2 but also on the entire space W^1, then a statement equivalent to Theorem 4.1 can be obtained with the help of results established in [6, 16].

Given an operator $p : W^1 \to L_1$, we put

$$(I_p u)(t) := \int_a^t (pu)(s)ds, \qquad t \in [a, b], \tag{6.1}$$

for any u from W^1, so that I_p is a map from W^1 to itself. We need the following definition [6].

Definition 6.1. Let $r : W^1 \to \mathbb{R}^n$ be a continuous linear vector functional. A linear operator $p : W^1 \to L_1$ is said to belong to the set $\mathcal{S}_r$ if the boundary value problem

$$u'(t) = (pu)(t) + v(t), \qquad t \in [a, b], \tag{6.2}$$

$$u(a) = r(u) \tag{6.3}$$

has a unique solution $u = (u_k)_{k=1}^n$ for any $v = (v_k)_{k=1}^n \in L_1$ and, moreover, the solution of (6.2), (6.3) has non-negative components provided that the functions v_k, $k = 1, 2, \ldots, n$, are non-negative almost everywhere on $[a, b]$.

In the case where the operator l, which determines the right-hand side of equation (1.1), is well defined on the entire space W^1, results of the preceding sections admit an alternative formulation. In particular, the following statements hold.

Theorem 6.1. *Suppose that there exist certain linear operators $p_i = (p_{ik})_{k=1}^n : W^1 \to L_1$, $i = 1, 2$, satisfying the inclusions*

$$I_{p_1} + r_1 \in \mathcal{S}_{r_0}, \qquad\qquad \frac{1}{2} I_{p_1+p_2} + r_1 \in \mathcal{S}_{r_0}, \tag{6.4}$$

and such that inequalities (4.2) *hold for an arbitrary u from $W^1_{(0;r_1,r_1)}$.*

Then the non-local boundary value problem (1.1)–(1.3) *has a unique solution for any $q \in L_1$.*

Theorem 6.2. *Let there exist certain positive linear operators $g_i = (g_{ik})_{k=1}^n : W^1 \to L_1$, $i = 0, 1$, which satisfy inequalities* (5.5) *for arbitrary u from $W^1_{(0;r_1,r_1)}$, and, moreover, are such that the inclusions*

$$I_{g_0} + r_1 \in \mathcal{S}_{r_0}, \qquad\qquad -\frac{1}{2} I_{g_1} + r_1 \in \mathcal{S}_{r_0} \tag{6.5}$$

hold.

Then the non-local boundary value problem (1.1)–(1.3) *has a unique solution for an arbitrary $q \in L_1$.*

The proof of Theorems 6.1 and 6.2 is based on the following

Lemma 6.1. *If $l : W^1 \to L_1$ is a bounded linear operator, then the inclusion*

$$I_l + r_1 \in \mathcal{S}_{r_0} \tag{6.6}$$

implies that $l \in \mathcal{S}_{r_0,r_1}$.

Proof. According to Definition 1.1, l belongs to $\mathcal{S}_{r_0,r_1}$ if and only if problem (1.1), (1.2), (1.3) has a unique solution for any $q \in L_1$ and, moreover, the solution is non-negative for non-negative q. By integrating (1.1), we can represent problem (1.1), (1.2), (1.3) in the equivalent form

$$u'(t) = (I_l u)(t) + r_1(u) + \int_a^t q(s)ds, \qquad t \in [a, b], \tag{6.7}$$

$$u(a) = r_0(u), \tag{6.8}$$

which, obviously, is a particular case of (6.2), (6.3) with $r := r_0$, $p := I_l + r_1$, and $v := \int_a^{\cdot} q(s)ds$. However, by virtue of Definition 6.1, the unique solvability of problem (6.7), (6.8) and the monotone dependence of its solution on q follow from inclusion (6.6). Therefore, $l \in \mathcal{S}_{r_0,r_1}$. □

7. An example of a second order equation with argument deviations

Let us consider the two-point boundary value problem for the nonlinear scalar differential equation with argument deviations

$$u''(t) = \sum_{k=1}^{N} \alpha_k(t)u(\omega_k(t)) + q(t), \quad t \in [a, b], \tag{7.1}$$

$$u'(a) = 0, \tag{7.2}$$

$$u(a) = \mu u(b), \tag{7.3}$$

where $N \geq 1$, $\mu \in \mathbb{R}$, $\{q, \alpha_1, \alpha_2, \ldots, \alpha_N\} \subset L_1$ and $\omega_1, \omega_2, \ldots, \omega_N$ are Lebesgue measurable functions mapping the interval $[a, b]$ into itself and such that

$$\omega_k(t) \leq t, \quad k = 1, 2, \ldots, N. \tag{7.4}$$

The following statement is true.

Corollary 7.1. *Let $|\mu| < 1$ and*

$$\sum_{k=1}^{N} \int_a^b \left(\int_a^s [\alpha_k(\xi)]_- d\xi \right) ds \leq 2. \tag{7.5}$$

Moreover, if $\mu \neq 0$, assume also that the inequality

$$\sum_{k=1}^{N} \int_a^b \left(\int_a^s [\alpha_k(\xi)]_+ d\xi \right) ds < -\ln|\mu| \tag{7.6}$$

is fulfilled.

Then the boundary value problem (7.1), (7.2), (7.3) *has a unique solution for any $q \in L_1$.*

In (7.5) and (7.6), we use the notation $[x]_+ := \max\{x, 0\}$ and $[x]_- := \max\{-x, 0\}$ for any $x \in \mathbb{R}$.

To prove Corollary 7.1, we use the following propositions concerning the scalar linear functional differential equation

$$u'(t) = (pu)(t) + q(t), \quad t \in [a, b], \tag{7.7}$$

where p is a map from $C := C([a, b], \mathbb{R})$ to L_1.

We shall say that p is positive if it maps the non-negative functions from C to almost everywhere non-negative elements of L_1.

Proposition 7.1 ([7, Corollary 2.1 (a)]). *Suppose that $|\mu| < 1$ and p in (7.7) is a positive Volterra operator. Let, moreover,*

$$|\mu| \exp\Big(\int_a^b (p1)(s)ds\Big) < 1. \tag{7.8}$$

Then the boundary value problem (7.7), (7.3) is uniquely solvable for an arbitrary integrable q and, moreover, the non-negativity of q implies the non-negativity of the solution.

Proposition 7.2 ([7, Theorem 2.3]). *Suppose that $|\mu| < 1$ and p in (7.7) is a Volterra operator such that $-p$ is positive and*

$$\int_a^b |(p1)(s)|ds \leq 1. \tag{7.9}$$

Then the boundary value problem (7.7), (7.3) is uniquely solvable for every integrable q. Moreover, if q is non-negative, then so does the solution of problem (7.7), (7.3).

Proof of Corollary 7.1. We shall use Theorem 6.1. Indeed, it is easy to see that problem (7.1), (7.2), (7.3) is a particular case of (1.1), (1.2), (1.3) with $n = 1$ and the operator $l : W^1 \to L_1$ given by the formula

$$(lu)(t) := \sum_{j=1}^{N} \alpha_j(t)u(\omega_j(t)), \qquad t \in [a, b], \tag{7.10}$$

and

$$r_1(u) := 0, \qquad\qquad r_0(u) := \mu u(b)$$

for any u from W^1. Let us put

$$(g_0u)(t) := \sum_{k=1}^{N} [\alpha_k(t)]_+ u(\omega_k(t)), \tag{7.11}$$

$$(g_1u)(t) := \sum_{k=1}^{N} [\alpha_k(t)]_- u(\omega_k(t)), \quad t \in [a, b], \tag{7.12}$$

for all $u \in W^1$. Then it is easy to see that inequalities (5.5) are true. Therefore, we need to make sure that $G_0 \in \mathcal{S}_{r_0}$ and $G_1 \in \mathcal{S}_{r_0}$, where

$$G_0 := I_{g_0}, \qquad\qquad G_1 := -\frac{1}{2} I_{g_1} \tag{7.13}$$

for all $u \in C$.

Indeed, it is clear from (7.10), (7.11), and (7.12) that l, g_0, and g_1 can be considered as mappings from C to L_1, so we can use Propositions 7.1 and 7.2.

Clearly, G_0 is a positive operator, which, due to assumption (7.4), is of Volterra type. It follows from (6.1), (7.11), and (7.13) that

$$\int_a^b (G_0 1)(s)ds = \sum_{k=1}^N \int_a^b \left(\int_a^t [\alpha_k(s)]_+ ds \right) dt$$

and, hence, for $\mu \neq 0$, assumption (7.6) implies the relation

$$\int_a^b (G_0 1)(s)ds < -\ln|\mu|.$$

This means that inequality (7.8) is satisfied. Applying Proposition 7.1, we show that $G_0 \in \mathcal{S}_{r_0}$. Note that if $\mu = 0$, then problem (7.7), (7.3) reduces to a Cauchy problem at the point a and, as is known in this case (see, e. g., [7]), the inclusion $G_0 \in \mathcal{S}_0$ is guaranteed by the Volterra property of G_0.

Similarly, it follows from (6.1), (7.12), and (7.13) that

$$\int_a^b (G_1 1)(s)ds = -\frac{1}{2} \sum_{k=1}^N \int_a^b \left(\int_a^t [\alpha_k(s)]_- ds \right) dt. \tag{7.14}$$

By assumption (7.4), G_1 is a Volterra operator, and it is obvious from (7.12) that $-G_1$ is positive. In view of (7.14), assumption (7.5) guarantees that (7.9) is satisfied. Consequently, by Proposition 7.2, we have $G_1 \in \mathcal{S}_{r_0}$.

Thus, we have shown that all the conditions of Theorem 6.2 are satisfied. Applying that theorem, we complete the proof. □

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