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On the asymptotic behavior of solutions of nonlinear differential equations of integer and also of non-integer order

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Abstract

We present conditions under which all solutions of the fractional differential equation with the Caputo derivative

$$^{c}D_{a}^{\alpha}x(t) = f(t, x(t)), \;\; a > 1, \;\; \alpha \in (1, 2), \tag{1}$$

are asymptotic to $at + b$ as $t \to \infty$ for some real numbers a, b.

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1 Introduction

In the asymptotic theory of $n-$th order nonlinear ordinary differential equations

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}) \tag{2}$$

the classical problem is to to establish some conditions for the existence of a solution which approach to a polynomial of degree $1 \le m \le n-1$ as $t \to \infty$. The first paper concerning this problem was published by D. Caligo [3] in 1941. He proved that if

$$|A(t)| < \frac{k}{t^{2+\rho}} \tag{3}$$

for all large t, where k, ρ are given, then any solution $y(t)$ of the linear differential equation

$$y''(t) + A(t)y(t) = 0, \;\; t > 0, \tag{4}$$

can be represented asymptotically as $y(t) = c_1 t + c_2 + o(1)$ when $t \to +\infty$, $c_1, c_2 \in \mathbb{R}$ (see [1]). The first paper on the nonlinear second order differential equations

$$y''(t) = f(t, y(t)) \tag{5}$$

was published b W. F. Trench [27] (1963). He proved a sufficient condition on the existence of a solution of the equation (5) which is asymptotic to $a + bt$ as $t \to +\infty$, for some real numbers a, b. Different conditions under which all solutions of the equation

$$y''(t) = f(t, y(t)) \tag{6}$$

is approaching to $a + bt$ as $t \to \infty$ for some real numbers a, b. The asymptotic behavior of solutions of this type of equation has been discussed by D. S. Cohen [5] (1967), J. Tong [26] (1982), T. Kusano and W. F. Trench [10] (1985) and [11] (1985) and others. This problem has been solved for the equation

$$y''(t) = f(t, y(t), y'(t)) \tag{7}$$

by F. M. Dannan [8] (1985), A. Constantin [6] (1993) and [7] (2005), Y. V. Rogovchenko [23] (1998), S. P. Rogovchenko [24] (2000), O. G. Mustafa, Y. V. Rogovchenko [19] (2002), O. Lipovan [12] (2003) and others. In the proofs of their results the key role plays the Bihari inequality (see [2]) which is a generalization of the Gronwall inequality. Some results on the existence of solutions of the n-th order differential equation

$$y^{(n)}(t) = f(t, y(t)) \tag{8}$$

approaching to a polynomial function of the degree m with $1 \leq m \leq n-1$ are proved by Ch. G. Philos, I. K. Purnaras and P. Ch. Tsamatos [21] (2004). Their proofs are based on an application of the Schauder Fixed Point Theorem. The paper by R. P. Agarwal, S. D. Djebali, T. Moussaoui and O. G. Mustafa [1] (2007) surveys the literature concerning the topic in asymptotic integration theory of ordinary differential equations. Several conditions under which all solutions of the one dimensional p-Laplacian equation

$$(|y'|^{p-1}y')' = f(t,y,y'),\ p > 1 \tag{9}$$

are asymptotic to $a + bt$ as $t \to \infty$ for some real numbers a, b are proved in [17](2008) and some sufficient conditions for the existence of solutions of the equation

$$(\Phi(y^{(n)})' = f(t,y),\ \ n \geq 1, \tag{10}$$

where $\Phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with a locally Lipschitz inverse satisfying $\Phi(0) = 0$ are given in the paper [16] (2010).

The aim of this paper is to give some conditions under which all solutions of the fractional differential equation (1) are asymptotic to $a+bt$ as $t \to \infty$ for some real numbers a, b. The proof of this result is based on a desingularization method proposed by the author in the paper [14] (see also [15]) in the study of nonlinear integral inequalities with weakly singular kernels.

2 Fractional Differential equations with the Caputo's derivative

Consider the initial value problem

$${}^{c}D_a^{\alpha}x(t) = f(t,x(t)),\ \ t \geq a > 1,\ \ 1 < \alpha < 2, \tag{11}$$

$$x(a) = c_0,\ \ x'(a) = c_2, \tag{12}$$

where

$${}^{c}D_a^{\alpha}x(t) := \frac{1}{\Gamma(2-\alpha)} \int_a^t (t-s)^{\alpha-1} x''(s)ds \tag{13}$$

is the Caputo derivative of the order $\alpha \in (1,2)$ of a C^2-scalar valued function $x(t)$ defined on the interval $[a\infty)$, $x''(t) = \frac{d^2x(t)}{dt^2}$. This definition has been given by M. Caputo in the paper [4]. For the definition of the Caputo derivative of order $\alpha \in (n-1,n), n \geq 1$ see [20] and also the monographs [18], [25]. We assume that any solution $x(t)$ of this problem exists on the interval

$[a, \infty)$. One can show that the initial value problem (11), (12) is equivalent to the integral equation

$$x(t) = c_0 + c_1(t-a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, x(s)) ds. \tag{14}$$

Since $\alpha > 1$ the function $x(t)$ is differentiable and therefore (14) yields

$$x'(t) = c_1 + \frac{1}{\Gamma(\alpha-1)} \int_a^t (t-s)^{\alpha-2} f(s, x(s)) ds. \tag{15}$$

Lemma 1 (see [22], [13]) *Let β, γ and p be positive constants such that $p(\beta-1)+1] > 0$, $p(\beta-1)+1 > 0$. Then*

$$\int_0^t (t-s)^{p(\beta-1)} s^{p(\gamma-1)} ds = t^{\Theta} B,\ t \geq 0 \tag{16}$$

where $B := B[p(\gamma-1)+1,\ p(\beta-1)+1]$, $B[\xi, \eta] = \int_a^1 s^{\xi-1}(1-s)^{\eta-1} ds$, $(\xi > 0,\ \eta > 0)$ and $\Theta = p(\beta+\gamma-2)+1$.

Theorem 1 *Suppose that $1 < \alpha < 2$, $p > 1$, $p(\alpha-2)+1 > 0$, $a > 1$, $q = \frac{p}{p-1}$ and the function $f(t, u)$ satisfies the following conditions:*

(i) $f(t, u)$ is continuous in $D = \{(t, v) \colon t \in [0, \infty),\ v \in \mathbb{R}\}$;

(ii) There are continuous nonnegative functions $h : R_+ := [0, \infty) \to R_+$, $g : R_+ \to R_+$, and $\gamma > 0$ with $p(\gamma-1)+1 > 0$ such that

$$|f(t, x)| \leq t^{\gamma-1} h(t) g\left(\frac{|x|}{t}\right), \quad t > 0, \quad (t, x) \in D, \tag{17}$$

where $\gamma = 3 - \alpha + \frac{1}{p}$, i.e. $\Theta := p(\alpha+\gamma-3)+1 = 0$ and

$$\int_a^\infty h(s)^q ds < \infty. \tag{18}$$

(iii)

$$\int_a^\infty \frac{\tau^{q-1} d\tau}{g(\tau)^q} = \infty. \tag{19}$$

Then every solution $x(t)$ of the equation (11) is asymptotic to $c + dt$ for $t \to \infty$, where $c, d \in \mathbb{R}$.

Proof. By applying the condition (17) we obtain from the (14) and (15)

$$|x(t)| \le Ct + B_1 \int_a^t (t-s)^{\alpha-1} h(s) s^{\gamma-1} g\left(\frac{|x|}{s}\right),\ )ds, \tag{20}$$

where $C = |c_1| + |c_2|$, $B_1 = \frac{1}{\Gamma(\alpha)}$, i. e.

$$\begin{aligned} |x(t)| &\le Ct + B_1 \int_a^t (t-s)^{\alpha-1} h(s) s^{\gamma-1} g\left(\frac{|x|}{s}\right) ds \\ &\le Ct + B(t-a) \int_a^t (t-s)^{\alpha-2} h(s) s^{\gamma-1} g\left(\frac{|x|}{s}\right) ds \end{aligned} \tag{21}$$

This yields the inequality

$$\frac{|x(t)|}{t} \le C + B \int_a^t (t-s)^{\alpha-2} h(s) s^{\gamma-1} g\left(\frac{|x|}{s}\right) ds. \tag{22}$$

If we denote by $z(t)$ the right-hand side of the inequality (22) we obtain the inequalities:

$$\frac{|x(t)|}{t} \le z(t), \tag{23}$$

$$|x'(t)| \le z(t). \tag{24}$$

Since the function g is nondecreasing, the inequality (23) yields

$$g\left(\frac{|x(t)|}{t}\right) \le g(z(t)) \tag{25}$$

and from (22) we obtain

$$z(t) \le 1 + C + B_1 \int_a^t (t-s)^{\beta-1} h(s) s^{\gamma-1} k(s) g\left(z(s)\right) ds, \tag{26}$$

$0 < \beta = \alpha - 1 < 1$. By applying the Hölder inequality and Lemma 1 we obtain

$$\begin{aligned} &\int_a^t (t-s)^{\beta-1} s^{\gamma-1} h(s) g(z(s)) ds \\ &\le \left(\int_a^t (t-s)^{p(\beta-1)} s^{p(\gamma-1)} ds\right)^{\frac{1}{p}} \left(\int_a^t h(s)^q g(z(s))^q\right)^{\frac{1}{q}} \\ &\le \left(\int_0^t (t-s)^{p(\beta-1)} s^{p(\gamma-1)} ds\right)^{\frac{1}{p}} \left(\int_a^t h(s)^q g(z(s))^q\right)^{\frac{1}{p}} \\ &\le BB_1 t^{\Theta} \left(\int_a^t h(s)^q g(z(s))^q\right)^{\frac{1}{q}} \le BB_1 t^{\Theta} \left(\int_a^t h(s)^q g(z(s))^q\right)^{\frac{1}{q}}, \end{aligned}$$

where $B = B[p(\gamma-1)+1, p(\beta-1)+1]$, $\Theta = p(\beta+\gamma-2)+1 = p(\alpha+\gamma-3)+1 = 0$, i.e.

$$\int_a^t (t-s)^{\beta-1} s^{\gamma-1} h(s) g(z(s)) ds \le BB_1 \left(\int_a^t h(s)^q g(z(s))^q \right)^{\frac{1}{q}}. \tag{27}$$

Using this inequality and the elementary inequality $(a+b)^q \le 2^{q-1}(a^q + b^q)$, $a, b \ge 0$ we obtain from (26)

$$z(t)^q \le 2^{q-1}[(1+C)^q + (BB_1) \int_a^t k(s)^q g(z(s))^q ds]. \tag{28}$$

If we denote $u(t) = z(t)^q$, i.e. $z(t) = u(t)^{\frac{1}{q}}$, $P_1 = 2^{q-1}[(1+c)^q]$, $Q_1 = 2^{q-1}(B_1)^q$ then

$$u(t) \le P_1 + Q_1 \int_a^t h(s)^q g(u(t)^{\frac{1}{q}})^q ds, \quad t \ge a. \tag{29}$$

Denote

$$\omega(v) = g(v^{\frac{1}{q}})^q,\ \Omega(u) = \int_{u_0}^u \frac{d\sigma}{\omega(\sigma)}, \quad u_0 = u(a). \tag{30}$$

Since $\Omega(u) = q \int_{v_0}^v \frac{\tau^{q-1} d\tau}{g(\tau)^q}$, where $v_0 = (u_0)^{\frac{1}{q}}$, $v = u^{\frac{1}{q}}$ the condition (iii) of Theorem 1 implies that $\lim_{u\to\infty} \Omega(u) = \infty$, i.e. $\Omega([u_0, \infty)) = [0, \infty)$ then by the Bihari lemma

$$u(t) \le K_0 := \Omega^{-1}[\Omega(P_1) + Q_1 \int_a^\infty h(s)^q ds] < \infty. \tag{31}$$

Since $u(t) = z(t)^{\frac{1}{q}}$ we obtain that $z(t) \le K := K_0^q$ and from (24), (25) we have

$$\frac{|x(t)|}{t} \le K_1, \quad |x'(t)| \le K_1, \quad t \ge a. \tag{32}$$

From the condition (ii) of Theorem 1 we have

$$\begin{aligned} \int_a^t (t-s)^{\beta-1} |f(s, x(s))| ds &\le \int_a^t (t-s)^{\beta-1} s^{\gamma-1} h(s) g(z(s)) ds \\ &\le z(t) \le K_1, \quad t \ge a, \end{aligned} \tag{33}$$

therefore $\int_a^\infty (t-s)^{\beta-1} |f(s, x(s))| ds$ exists. Therefore form the equality (15) it follows that $\lim_{t\to\infty} x'(t) = d$ exists and by the l'Hospital rule we conclude that

$$\lim_{t\to\infty} \frac{u(t)}{t} = \lim_{t\to 0} x'(t) = d, \tag{34}$$

so the proof is now complete.

3 Example

Let $p > 1$, $\alpha = 2 - \frac{1}{2p}$, $\gamma = 3 - \alpha - \frac{1}{p}$, i.e. $p(\alpha + \gamma - 3) + 1 = 0$, $q = \frac{p}{p-1}$, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, $g(u) = u^{\frac{q-1}{q}}[\ln(2+u)]^{\frac{1}{q}}$, $u \geq 0$, $f(t,u) = t^{\gamma-1}h(t)g(\frac{u}{t})$, where $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function with $\int_a^\infty h(s)^q ds < \infty$. Obviously $1 < \alpha < 2$, $p(\alpha - 2) + 1 = \frac{1}{2}$, $p(\gamma - 1) + 1 = 1 - \frac{1}{2p} > 0$ and

$$\int_a^\infty \frac{\tau^{q-1}}{g(\tau)^q} d\tau = \int_a^\infty \frac{1}{\ln(2+\tau)} d\tau = \infty. \tag{35}$$

Therefore the function $f(t,u)$ satisfy the conditions (i)–(iii) of Theorem 1.

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