Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **86**, 1-26; **http://www.math.u-szeged.hu/ejqtde/**

Nonlinear Dynamic Inequalities of Gronwall-Bellman Type on Time Scales

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Abstract

The main aim of this paper is to establish some new explicit bounds of solutions of a certain class of nonlinear dynamic inequalities (with and without delays) of Gronwall-Bellman type on a time scale $\mathbb{T}$ which is unbounded above. These on the one hand generalize and on the other hand furnish a handy tool for the study of qualitative as well as quantitative properties of solutions of delay dynamic equations on time scales. Some examples are considered to demonstrate the applications of the main results.

Key words: Dynamic inequalities of Gronwall-Bellman Type, dynamic equations, time scales.
MSC (**2000**): 26D15, 26D20, 39A12, 34N05.

1 Introduction

In 1919 Thomas Gronwall [8] proved that if β and u are real-valued continuous functions defined on J, where J is an interval in $\mathbb{R}$, $t_0 \in J$, and u is differentiable in the interior J^0 of J, then

$$u^{'}(t) \leq \beta(t)u(t), \text{ for } t \in J^0, \tag{1}$$

implies

$$u(t) \leq u(t_0) \exp\left(\int_{t_0}^{t} \beta(s)\right), \text{ for all } t \in J. \tag{2}$$

In 1943 Richard Bellman [4] considered the integral form of (1) and proved that if

$$u(t) \leq \alpha(t) + \int_{t_0}^{t} \beta(s)u(s)ds, \text{ for } t \in J, \tag{3}$$

then

$$u(t) \le \alpha(t) + \int_{t_0}^{t} \alpha(s)\beta(s) \exp\left(\int_s^t \beta(\theta)d\theta\right) ds, \text{ for all } t \in J, \tag{4}$$

where J is an interval in $\mathbb{R}$, $t_0 \in J$, and α, β, $u \in C(J, \mathbb{R}^+)$. If in addition $\alpha(t)$ is nondecreasing, then (3) implies

$$u(t) \le \alpha(t) \exp\left(\int_{t_0}^{t} \beta(s)ds\right), \text{ for all } t \in J. \tag{5}$$

Since the discovery of these inequalities much work has been done, and many papers which deal with new proofs, various generalizations and extensions have appeared in the literature, we refer to the results by Ou-Iang [15], Dafermos [7] and Pachpatte [16]. The inequalities of the form (4), which are called the Gronwall-Bellman type inequalities, are important tools to obtain various estimates in the theory of differential equations. For example, Ou-Iang [15] in his study of the boundedness of certain second order differential equations established the following result which is generally known as Ou-Iang's inequality: If u and f are non-negative functions defined on $[0, \infty)$ such that

$$u^2(t) \le k^2 + 2\int_0^t f(s)u(s)ds, \quad \text{for all} \quad t \in [0, \infty), \tag{6}$$

where $k \ge 0$ is a constant, then

$$u(t) \le k + \int_0^t f(s)ds, \text{ for all } t \in [0, \infty). \tag{7}$$

Dafermos [7] established a generalization of Ou-Iang's inequality in the process of investigating the connection between stability and the second law of thermodynamics. He proved that if $u \in L^\infty[0, r]$ and $f \in L^1[0, r]$ are non-negative functions satisfying

$$u^2(t) \le M^2u^2(0) + 2\int_0^t [Nf(s)u(s) + Ku^2(s)]ds, \text{ for all } t \in [0, r], \tag{8}$$

where M, N, K are non-negative constants, then

$$u(t) \le \left[Mu(0) + N\int_0^t f(s)ds\right] e^{Kt}.$$

Pachpatte [16] established the following further generalizations of the result of Dafermos [7] and proved that: If u, f, g are continuous non-negative functions on $[0,\infty)$ satisfying

$$u^2(t) \le k^2 + 2\int_0^t [f(s)u(s) + g(s)u^2(s)]ds, \text{ for all } t \in [0,\infty), \tag{9}$$

where $k \ge 0$ is a constant, then

$$u(t) \le \left(k + \int_0^t f(s)ds\right) \exp\left(\int_0^t g(s)ds\right), \text{ for all } t \in [0,\infty). \tag{10}$$

It is well known that the dynamic inequalities play important roles in the development of the qualitative theory of dynamic equations on time scales. The study of dynamic equations on time scales which goes back to its founder Stefan Hilger [9] becomes an area of mathematics and recently has received a lot of attention. The general idea is to prove a result for a dynamic equation or a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$. We assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty) \cap \mathbb{T}$. The book on the subject of time scales by Bohner and Peterson [5] summarizes and organizes much of time scale calculus. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [10]), i.e, when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. In this paper, we will refer to the (delta) integral which we can define as follows: If $G^{\Delta}(t) = g(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t g(s)\Delta s := G(t) - G(a)$. It can be shown (see [5]) that if $g \in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t)$, $t \in \mathbb{T}$. There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics. A recent cover story article in New Scientist [23] discusses several possible applications.

During the past decade a number of dynamic inequalities has been established by some authors which are motivated by some applications, for example, when studying the behavior of solutions of certain class of dynamic equations on a time scale $\mathbb{T}$, the bounds provided by earlier inequalities are inadequate in applications and we need some new and specific type of dynamic inequalities on time scales. For contributions, we refer the reader to [1], [2], [3], [5], [6], [11], [12, 13], [17], [18], [19], [20] and [21] and the references cited therein. So it is expected to see the time scale versions of the above inequalities and their extensions. The general form of (1) on the time scale $\mathbb{T}$ has been studied in [5, Theorem 6.1]. In particular, it is proved that if u, a and $p \in C_{rd}$ and $p \in \mathcal{R}^+$, then

$$u^{\Delta}(t) \leq f(t) + p(t)u(t), \quad \text{for all } t \in [t_0, \infty)_{\mathbb{T}}, \tag{11}$$

implies

$$u(t) \leq u(t_0)e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(s))f(s)\Delta s, \text{ for all } t \in [t_0, \infty)_{\mathbb{T}}, \tag{12}$$

where $\mathcal{R}^+ := \{a \in \mathcal{R} : 1 + \mu(t)a(t) > 0,\ t \in \mathbb{T}\}$ and $\mathcal{R}$ is the class of rd-continuous and regressive functions. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided f is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$, where $\sigma(t)$ is the forward jump operator defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t)f(t) \neq 0$, $t \in \mathbb{T}$. The set of all regressive functions on a time scale $\mathbb{T}$ forms an Abelian group under the addition $\oplus$ defined by $p \oplus q := p + q + \mu pq$. The exponential function $e_p(t, s)$ on time scales is defined by

$$e_p(t, s) = \exp\left(\int_{s}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \text{ for } t \in \mathbb{T}, s \in \mathbb{T}^k,$$

where $\xi_h(z)$ is the cylinder transformation, which is given by

$$\xi_h(z) = \begin{cases} \frac{\log(1+hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases}$$

Alternatively, for $p \in \mathcal{R}$ one can define the exponential function $e_p(\cdot, t_0)$, to be the unique solution of the IVP $x^{\Delta} = p(t)x$, with $x(t_0) = 1$. If $p \in \mathcal{R}$, then $e_p(t, s)$ is real-valued and nonzero on $\mathbb{T}$. If $p \in \mathcal{R}^+$, then $e_p(t, t_0)$ is always positive, $e_p(t, t) = 1$ and $e_0(t, s) = 1$. Note that

$$\begin{cases} e_p(t, t_0) = \exp(\int_{t_0}^{t} p(s)ds), & \text{if } \mathbb{T} = \mathbb{R}, \\ e_p(t, t_0) = \prod_{s=t_0}^{t-1} (1 + p(s)), & \text{if } \mathbb{T} = \mathbb{N}, \\ e_p(t, t_0) = \prod_{s=t_0}^{t-1} (1 + (q-1)sp(s)), & \text{if } \mathbb{T} = q^{\mathbb{N}_0}. \end{cases}$$

The generalizations of (11) on time scales has been studied in [17, 19] and some explicit upper bounds of the unknown function are obtained. Note that if we put $f(t) = 0$ in (11), then (11) and (12) can be considered as the time scale versions of (1) and (2). We mentioned here that the study of the general form of (11) on time scales is important in applications, especially in oscillation theory of dynamic equations on time scales. In particular, the application of the Riccati techniques on second and third order dynamic equations reduces these equations to a Riccati dynamic inequality of the form

$$w^{\Delta}(t) \leq f(t) + p(t)w(t) - q(t)w^{\lambda+1},$$

which is a generalization of (11). For contributions in this direction, we refer the reader to the book [22].

The Gronwall-Bellman dynamic inequality, which is the time scale version of (3) has been proved in [5, Theorem 6.4]. In particular it is proved that: If u, a and $p \in C_{rd}$ and $p \in \mathcal{R}^{+}$, then

$$u(t) \leq a(t) + \int_{t_0}^{t} p(s)u(s)\Delta s, \text{ for all } t \in [t_0, \infty)_{\mathbb{T}}, \tag{13}$$

implies that

$$u(t) \leq a(t) + \int_{t_0}^{t} e_p(t, \sigma(s))a(s)p(s)\Delta s, \text{ for all } t \in [t_0, \infty)_{\mathbb{T}}. \tag{14}$$

Since (14) provides an explicit bound to the unknown function $u(t)$ and a tool to the study of many qualitative as well as quantitative properties of solutions of dynamic equations, it has become one of the very few classic and most influential results in the theory and applications of dynamic inequalities. Because of its fundamental importance, over the years, many generalizations and analogous results of (14) have been established.

In [19] the author considered a dynamic inequality of the form

$$u^{p}(t) \leq a(t) + b(t)\int_{t_0}^{t} \left[f(s)u^{q}(s) - g(s)u^{p}(\sigma(s))\right]\Delta s, \; t \in [t_0, \infty)_{\mathbb{T}}, \tag{15}$$

and proved that if a, f and g are positive rd-continuous functions defined on $[t_0,\infty)_{\mathbb{T}}$, $u(t) \geq 0$, for all $t \geq t_0$, where $t_0 \geq 0$ is a fixed number, p, q are positive constants such that $p > q \geq 1$, then (15) implies for $t \in [t_0, \infty)_{\mathbb{T}}$ that

$$u(t) \leq a^{\frac{1}{p}}(t) + \frac{q}{p}a^{\frac{1}{p}-1}(t)b(t)\left[\int_{t_0}^{t} e_{\left(a^{\frac{q}{p}}f\right)}(t, \sigma(s))f(s)a^{\frac{q}{p}-1}(s)\Delta s\right]. \tag{16}$$

We note that the inequality (16) has been proved in the case when $p > q \geq 1$. So it would be interesting to find the explicit bound for u of (15) when $q > p \geq 1$. Also in [19] the author considered the dynamic inequality

$$u^{\gamma}(t) \leq a(t) + b(t)\int_{t_0}^{t}\left[f(s)u^{\delta}(s) + g(s)u^{\alpha}(s)\right]\Delta s,\ \text{for } t \in [t_0,\infty)_{\mathbb{T}},$$

when $\delta \leq \gamma$ and $\alpha \leq \gamma$, and established some explicit bounds for the function $u(t)$. The main results in [19] has been proved by employing the Bernoulli inequality [14, Bernoulli's inequality]

$$(1+x)^{\gamma} \leq 1 + \gamma x, \quad \text{for} \quad 0 < \gamma \leq 1 \text{ and } x > -1. \tag{17}$$

Following this trend and to develop the study of dynamic inequalities on time scales, we consider the general nonlinear dynamic inequality

$$u^{\gamma}(t) \leq a(t) + b(t)\int_{t_0}^{t}\left[f(s)u^{\delta}(s) + g(s)u^{\alpha}(s)\right]^{\lambda}\Delta s,\ \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \tag{18}$$

and the delay dynamic inequality

$$u^{\gamma}(t) \leq a(t) + b(t)\int_{t_0}^{t}\left[f(s)u^{\delta}(\tau(s)) + g(s)u^{\alpha}(\eta(s))\right]^{\lambda}\Delta s,\ \text{for } t \in [t_0,\infty)_{\mathbb{T}}. \tag{19}$$

For (18) and (19), we will assume the following hypotheses:

(H_1) $\begin{cases} u,\ a,\ b,\ f \text{ and } g \text{ are rd-continuous positive functions defined on } [t_0,\infty)_{\mathbb{T}}, \\ \alpha,\ \delta,\ \lambda \text{ and } \gamma \text{ are positive constants such that } \gamma \geq 1. \end{cases}$

(H_2) $a(t)$, $b(t)$ are nondecreasing functions, $\tau, \eta : \mathbb{T} \to \mathbb{T}$ such that $\tau(t) \leq t$, $\eta(t) \leq t$ and $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\eta(t) = \infty$.

Our aim in this paper is to establish some explicit bounds of the unknown function $u(t)$ of the inequality (18) and extend these results to the delay dynamic inequality (19). When $\mathbb{T} = \mathbb{R}$, the results will be different from the results established by Ou-Iang [15], Dafermos [7] and Pachpatte [16] and in a time scale $\mathbb{T}$ the results complement the results established in [19] in the sense that the results do not require the conditions $\delta \leq \gamma$ and $\alpha \leq \gamma$ and can be applied in the cases when $\delta \geq \gamma$ and $\alpha \geq \gamma$. The main results will be proved by employing the Bernoulli inequality (17), the Young inequality [14]

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},\ \text{ where } a,\ b \geq 0,\ p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \tag{20}$$

and the algebraic inequalities [14]

$$(a+b)^\lambda \le 2^{\lambda-1}(a^\lambda + b^\lambda), \text{ for } a,\ b \ge 0, \text{ and } \lambda \ge 1, \tag{21}$$

$$(a+b)^\lambda \le a^\lambda + b^\lambda, \text{ for } a,\ b \ge 0, \text{ and } 0 \le \lambda \le 1. \tag{22}$$

Some examples are considered to illustrate the main results.

2 Main Results

Before we state and prove the main results we present some basic Lemmas which play important roles in the proof of our main results in this paper.

Lemma 2.1 [6]. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Suppose that y, a, b, $p \in C_{rd}$ and b, $p \ge 0$. If*

$$y(t) \le a(t) + b(t)\int_{t_0}^{t} p(s)y(s)\Delta s, \quad \text{for all } t \in [t_0,\infty)_{\mathbb{T}}, \tag{23}$$

then

$$y(t) < a(t) + b(t)\int_{t_0}^{t} a(s)p(s)e_{bp}(t,\sigma(s))\Delta s, \quad \text{for all } t \in [t_0,\infty)_{\mathbb{T}}. \tag{24}$$

Lemma 2.2. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Let $g_i : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ for $i = 1, 2, ..., n$ be functions with $g_i(t, x_1) \le g_i(t, x_2)$ for all $t \in \mathbb{T}$ and $i = 1, 2, ..., n$, whenever $x_1 \le x_2$. Let v, $w : \mathbb{T} \to \mathbb{R}$ be differentiable with*

$$v^\Delta(t) \le \sum_{i=1}^{n} g_i(t, v(t)), \quad w^\Delta(t) \ge \sum_{i=1}^{n} g_i(t, w(t)), \text{ for all } t \in [t_0,\infty)_{\mathbb{T}}. \tag{25}$$

Then $v(t_0) < w(t_0)$ implies $v(t) \le w(t)$ for all $t \in [t_0,\infty)_{\mathbb{T}}$.

Proof. The proof is similar to the proof of Theorem 6.9 in [5] and hence is omitted.

Lemma 2.3. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Suppose that $g_i : \mathbb{R} \to \mathbb{R}$ is nondecreasing for $i = 1, 2, ..., n$ and $y : \mathbb{T} \to \mathbb{R}$ is such that $g_i(y)$ is rd-continuous. Let p_i be rd-continuous for $i = 1, 2, ..., n$ and $f : \mathbb{T} \to \mathbb{R}$ differentiable. Then*

$$y(t) \le f(t) + \sum_{i=1}^{n}\int_{t_0}^{t} p_i(s)g_i(y(s))\Delta s, \quad \text{for all } t \ge t_0, \tag{26}$$

implies $y(t) \leq x(t)$ for all $t \geq t_0$, where x solves the initial value problem

$$x^{\Delta}(t) = f^{\Delta}(t) + \sum_{i=1}^{n} p_i(t) g_i(x(t)), \quad x(t_0) = x_0 > f(t_0) > 0. \tag{27}$$

Proof. Let

$$v(t) := f(t) + \sum_{i=1}^{n} \int_{t_0}^{t} p_i(s) g_i(y(s)) \Delta s, \quad \text{for all } t \geq t_0. \tag{28}$$

Then

$$v^{\Delta}(t) := f^{\Delta}(t) + \sum_{i=1}^{n} p_i(t) g_i(y(t)), \quad \text{for all } t \geq t_0, \tag{29}$$

and $y(t) \leq v(t)$ so that

$$v^{\Delta}(t) \leq f^{\Delta}(t) + \sum_{i=1}^{n} p_i(t) g_i(v(t)), \quad \text{for all } t \geq t_0. \tag{30}$$

Since $v(t_0) = f(t_0) < x_0 = x(t_0)$, the comparison Lemma 2.2 yields $v(t) \leq x(t)$ for all $t \geq t_0$. Hence, since $y(t) \leq v(t)$, we obtain $y(t) \leq x(t)$ where x solves the initial value problem (27). The proof is complete.

Now, we are ready to state and prove the main results. First, we consider the inequality (18) and establish some explicit bounds of the unknown function $u(t)$ when $\lambda \geq 1$ and α, $\delta \leq \gamma$. For simplicity, we introduce the following notations:

$$\begin{aligned}
F(t) \; &: = 2^{2(\lambda-1)} \int_{t_0}^{t} \left[f^{\lambda}(s) \left[a^{\frac{\delta}{\gamma}}(s) \right]^{\lambda} + g^{\lambda}(s) \left[a^{\frac{\alpha}{\gamma}}(s) \right]^{\lambda} \right] \Delta s, \\
F^{\Delta}(t) \; &: = 2^{2(\lambda-1)} \left[f^{\lambda}(t) \left[a^{\frac{\delta}{\gamma}}(t) \right]^{\lambda} + g^{\lambda}(t) \left[a^{\frac{\alpha}{\gamma}}(t) \right]^{\lambda} \right], \\
G(t) \; &: = 2^{2(\lambda-1)} \left(f^{\lambda}(s) \left[\frac{\delta}{\gamma} a^{\frac{\delta}{\gamma}-1}(s) \right]^{\lambda} + g^{\lambda}(s) \left[\frac{\alpha}{\gamma} a^{\frac{\alpha}{\gamma}-1}(s) \right]^{\lambda} \right).
\end{aligned} \tag{31}$$

Theorem 2.1. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Assume that (H_1) holds, $\lambda \geq 1$ and α, $\delta \leq \gamma$. Then*

$$u^{\gamma}(t) \leq a(t) + b(t) \int_{t_0}^{t} \left[f(s) u^{\delta}(s) + g(s) u^{\alpha}(s) \right]^{\lambda} \Delta s, \text{ for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{32}$$

implies that

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma} a^{\frac{1}{\gamma}-1}(t) b(t) w(t), \text{ for all } t \in [t_0, \infty)_{\mathbb{T}}, \tag{33}$$

where $w(t)$ solves the initial value problem

$$w^{\Delta}(t) = F^{\Delta}(t) + b^{\lambda}(t) G(t) w^{\lambda}(t), \quad w(t_0) = w_0 > 0. \tag{34}$$

Proof. Define a function $y(t)$ by

$$y(t) := \int_{t_0}^{t} \Big[f(s) u^{\delta}(\tau(s)) + g(s) u^{\alpha}(\eta(s)) \Big]^{\lambda} \Delta s. \tag{35}$$

This reduces (32) to

$$u^{\gamma}(t) \leq a(t) + b(t) y(t), \quad \text{for } \ t \in [t_0, \infty)_{\mathbb{T}}. \tag{36}$$

This implies (noting that $\gamma \geq 1$) that

$$u(t) \leq (a(t) + b(t) y(t))^{\frac{1}{\gamma}}, \quad \text{for } \ t \in [t_0, \infty)_{\mathbb{T}}. \tag{37}$$

Applying the inequality (17), we see that

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma} a^{\frac{1}{\gamma}-1}(t) b(t) y(t), \text{ for } \ t \in [t_0, \infty)_{\mathbb{T}}. \tag{38}$$

From (37), we obtain

$$u^{\alpha}(t) \leq a^{\frac{\alpha}{\gamma}}(t) \left[1 + \frac{b(t) y(t)}{a(t)} \right]^{\frac{\alpha}{\gamma}}, \quad \text{for } \ t \in [t_0, \infty)_{\mathbb{T}}. \tag{39}$$

Applying inequality (17) on (39) (where $\alpha \leq \gamma$), we obtain for $t \in [t_0, \infty)_{\mathbb{T}}$ that

$$u^{\alpha}(t) \leq a^{\frac{\alpha}{\gamma}}(t) \left[1 + \frac{\alpha}{\gamma} \frac{b(t)}{a(t)} y(t) \right] = a^{\frac{\alpha}{\gamma}}(t) + \frac{\alpha}{\gamma} a^{\frac{\alpha}{\gamma}-1}(t) b(t) y(t). \tag{40}$$

Also from (37), we obtain

$$u^{\delta}(t) \leq a^{\frac{\delta}{\gamma}}(t) \left[1 + \frac{b(t) y(t)}{a(t)} \right]^{\frac{\delta}{\gamma}}, \quad \text{for } \ t \in [t_0, \infty)_{\mathbb{T}}. \tag{41}$$

Applying inequality (17) on (41) (where $\delta \leq \gamma$), we have for $t \in [t_0, \infty)_{\mathbb{T}}$ that

$$u^{\delta}(t) \leq a^{\frac{\delta}{\gamma}}(t) \left[1 + \frac{\delta}{\gamma} \frac{b(t)}{a(t)} y(t) \right] = a^{\frac{\delta}{\gamma}}(t) + \frac{\delta}{\gamma} a^{\frac{\delta}{\gamma}-1}(t) b(t) y(t). \tag{42}$$

Combining (35), (40) and (42), and applying the inequality (21) (noting that $\lambda \geq 1$), we have

$$\begin{aligned}
y(t) &= \int_{t_0}^{t} \left[f(s)u^{\delta}(s) + g(s)u^{\alpha}(s)\right]^{\lambda} \Delta s \\
&\leq 2^{\lambda-1} \int_{t_0}^{t} \left[f(s)u^{\delta}(s)\right]^{\lambda} \Delta s + 2^{\lambda-1} \int_{t_0}^{t} \left[g(s)u^{\alpha}(s)\right]^{\lambda} \Delta s \\
&\leq 2^{\lambda-1} \int_{t_0}^{t} f^{\lambda}(s) \left[a^{\frac{\delta}{\gamma}}(s) + \frac{\delta}{\gamma} a^{\frac{\delta}{\gamma}-1}(s)b(s)y(s)\right]^{\lambda} \Delta s \\
&\quad +2^{\lambda-1} \int_{t_0}^{t} g^{\lambda}(s) \left[a^{\frac{\alpha}{\gamma}}(s) + \frac{\alpha}{\gamma} a^{\frac{\alpha}{\gamma}-1}(s)b(s)y(s)\right]^{\lambda} \Delta s.
\end{aligned}$$

This implies that

$$\begin{aligned}
y(t) &\leq 2^{2(\lambda-1)} \int_{t_0}^{t} f^{\lambda}(s) \left[a^{\frac{\delta}{\gamma}}(s)\right]^{\lambda} \Delta s \\
&\quad +2^{2(\lambda-1)} \int_{t_0}^{t} f^{\lambda}(s) \left[\frac{\delta}{\gamma} a^{\frac{\delta}{\gamma}-1}(s)b(s)\right]^{\lambda} y^{\lambda}(s)\Delta s \\
&\quad +2^{2(\lambda-1)} \int_{t_0}^{t} g^{\lambda}(s) \left[a^{\frac{\alpha}{\gamma}}(s)\right]^{\lambda} \Delta s \\
&\quad + \int_{t_0}^{t} g^{\lambda}(s) \left[\frac{\alpha}{\gamma} a^{\frac{\alpha}{\gamma}-1}(s)b(s)\right]^{\lambda} y^{\lambda}(s)\Delta s \\
&= F(t) + \int_{t_0}^{t} G(s)y^{\lambda}(s)\Delta s, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.
\end{aligned}$$

Now an application of Lemma 2.3 (with $n = 1$ and $g(y) = y^{\lambda}$), gives that

$$y(t) < w(t), \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{43}$$

where $w(t)$ solves the initial value problem (34). Substituting (43) into (38), we obtain the desired inequality (33). The proof is complete.

Theorem 2.2. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Assume that (H_1) holds,* $\lambda \geq 1$ and α, $\delta \leq \gamma$. *Then (32) implies*

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + b^{\frac{1}{\gamma}}(t)w^{\frac{1}{\gamma}}(t), \text{ for all } t \in [t_0, \infty)_{\mathbb{T}}, \tag{44}$$

where $w(t)$ solves the initial value problem

$$\begin{cases} w^{\Delta}(t) = F^{\Delta}(t) + G_1(t)w^{\lambda(\frac{\delta}{\gamma})}(t) + G_2(t)w^{\lambda\left(\frac{\alpha}{\gamma}\right)}(t), \\ w(t_0) = w_0 > 0, \end{cases} \tag{45}$$

where $F(t)$ is defined as in (31) and

$$G_1(t) := 2^{2(\lambda-1)} \int_{t_0}^{t} f^{\lambda}(t) \left[b^{\frac{\delta}{\gamma}}(t) \right]^{\lambda}, \quad G_2 := 2^{2(\lambda-1)} g^{\lambda}(t) \left[b^{\frac{\alpha}{\gamma}}(t) \right]^{\lambda}. \quad (46)$$

Proof. Define a function $y(t)$ by (35) and proceed as in the proof of Theorem 2.1 to obtain

$$u(t) \leq (a(t) + b(t)y(t))^{\frac{1}{\gamma}}, \quad \text{for} \quad t \in [t_0, \infty)_{\mathbb{T}}. \quad (47)$$

Applying the inequality (22), we see that

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + b^{\frac{1}{\gamma}}(t) y^{\frac{1}{\gamma}}(t), \quad \text{for} \quad t \in [t_0, \infty)_{\mathbb{T}}. \quad (48)$$

From (47), we obtain

$$u^{\alpha}(t) \leq (a(t) + b(t)y(t))^{\frac{\alpha}{\gamma}}, \quad \text{for} \quad t \in [t_0, \infty)_{\mathbb{T}}. \quad (49)$$

Applying inequality (22) on (49) (where $\alpha \leq \gamma$), we obtain for $t \in [t_0, \infty)_{\mathbb{T}}$ that

$$u^{\alpha}(t) \leq a^{\frac{\alpha}{\gamma}}(t) + b^{\frac{\alpha}{\gamma}}(t) y^{\frac{\alpha}{\gamma}}(t). \quad (50)$$

Also from (47), we have by (22) that

$$u^{\delta}(t) \leq a^{\frac{\delta}{\gamma}}(t) + b^{\frac{\delta}{\gamma}}(t) y^{\frac{\delta}{\gamma}}(t), \quad \text{for} \quad t \in [t_0, \infty)_{\mathbb{T}}. \quad (51)$$

Combining (35), (50) and (51), and applying the inequality (21) (noting that $\lambda \geq 1$), we have

$$\begin{aligned}
y(t) &= \int_{t_0}^{t} \left[f(s) u^{\delta}(s) + g(s) u^{\alpha}(s) \right]^{\lambda} \Delta s \\
&\leq 2^{\lambda-1} \int_{t_0}^{t} \left[f(s) u^{\delta}(s) \right]^{\lambda} \Delta s + 2^{\lambda-1} \int_{t_0}^{t} \left[g(s) u^{\alpha}(s) \right]^{\lambda} \Delta s \\
&\leq 2^{\lambda-1} \int_{t_0}^{t} f^{\lambda}(s) \left[a^{\frac{\delta}{\gamma}}(s) + b^{\frac{\delta}{\gamma}}(s) y^{\frac{\delta}{\gamma}}(s) \right]^{\lambda} \Delta s \\
&\quad + 2^{\lambda-1} \int_{t_0}^{t} g^{\lambda}(s) \left[a^{\frac{\alpha}{\gamma}}(s) + b^{\frac{\alpha}{\gamma}}(s) y^{\frac{\alpha}{\gamma}}(s) \right]^{\lambda} \Delta s.
\end{aligned}$$

This implies that

$$\begin{aligned} y(t) &\leq 2^{2(\lambda-1)}\int_{t_0}^{t} f^{\lambda}(s)\left[a^{\frac{\delta}{\gamma}}(s)\right]^{\lambda}\Delta s + 2^{2(\lambda-1)}\int_{t_0}^{t} g^{\lambda}(s)\left[a^{\frac{\alpha}{\gamma}}(s)\right]^{\lambda}\Delta s \\ &\quad + 2^{2(\lambda-1)}\int_{t_0}^{t} f^{\lambda}(s)\left[b^{\frac{\delta}{\gamma}}(s)\right]^{\lambda} y^{\lambda\left(\frac{\delta}{\gamma}\right)}(s)\Delta s \\ &\quad + 2^{2(\lambda-1)}\int_{t_0}^{t} g^{\lambda}(s)\left[b^{\frac{\alpha}{\gamma}}(s)\right]^{\lambda} y^{\lambda\left(\frac{\alpha}{\gamma}\right)}(s)\Delta s \\ &= F(t) + \int_{t_0}^{t}\left[G_1(s)y^{\lambda(\frac{\delta}{\gamma})}(s) + G_2(s)y^{\lambda\left(\frac{\alpha}{\gamma}\right)}(s)\right]\Delta s, \quad t\in[t_0,\infty)_{\mathbb{T}}. \end{aligned}$$

Now an application of Lemma 2.3 (with $n = 2$, $g_1(y) = y^{\lambda(\frac{\delta}{\gamma})}$ and $g_2(y) = y^{\lambda\left(\frac{\alpha}{\gamma}\right)}$), gives that

$$y(t) < w(t), \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}}, \tag{52}$$

where $w(t)$ solves the initial value problem (45). Substituting (52) into (48), we obtain the desired inequality (44). The proof is complete.

As in the proof of Theorem 2.1 by employing the inequality (22) instead of the inequality (21), we can obtain an explicit bound for $u(t)$ when $0 \leq \lambda \leq 1$. This will be presented below in Theorem 2.3 without proof since the proof is similar to the proof of Theorem 2.1. For simplicity, we introduce the following notations:

$$\begin{aligned} F_1(t) &:= \int_{t_0}^{t}\left[f^{\lambda}(s)\left[a^{\frac{\delta}{\gamma}}(s)\right]^{\lambda} + g^{\lambda}(s)\left[a^{\frac{\alpha}{\gamma}}(s)\right]^{\lambda}\right]\Delta s, \\ F_1^{\Delta}(t) &:= f^{\lambda}(t)\left[a^{\frac{\delta}{\gamma}}(t)\right]^{\lambda} + g^{\lambda}(t)\left[a^{\frac{\alpha}{\gamma}}(t)\right]^{\lambda}, \\ G_3(t) &:= \left(f^{\lambda}(t)\left[\frac{\delta}{\gamma}a^{\frac{\delta}{\gamma}-1}(t)\right]^{\lambda} + g^{\lambda}(t)\left[\frac{\alpha}{\gamma}a^{\frac{\alpha}{\gamma}-1}(t)\right]^{\lambda}\right). \end{aligned} \tag{53}$$

Theorem 2.3. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Assume that (H_1) holds, $0 < \lambda \leq 1$, $\delta \leq \gamma$ and $\alpha \leq \gamma$. Then (32) implies that*

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma}a^{\frac{1}{\gamma}-1}(t)b(t)s(t), \quad t\in[t_0,\infty)_{\mathbb{T}}. \tag{54}$$

where $s(t)$ solves the initial value problem

$$s^{\Delta}(t) = F_1^{\Delta}(t) + G_3(t)b^{\lambda}(t)s^{\lambda}(t), \quad s(t_0) = s_0 > 0. \tag{55}$$

In the following, we apply the Young inequality (20) to find a new explicit upper bound for $u(t)$ of (32) when $\lambda \geq 1$ and $0 \leq \lambda \leq 1$. First, we consider the case when $\lambda \geq 1$ and assume that $\lambda(\alpha/\gamma) < 1$ and $\lambda(\delta/\gamma) < 1$.

Theorem 2.4. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Assume that (H_1) holds, $\lambda \geq 1$ and α, $\delta \leq \gamma$ such that $(\lambda\alpha/\gamma) < 1$ and $(\lambda\delta/\gamma) < 1$. Then (32) implies that*

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + b^{\frac{1}{\gamma}}(t) F_3^{\frac{1}{\gamma}}(t), \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{56}$$

where

$$F_3(t) := F_0(t) + \beta \int_{t_0}^{t} F_0(s) e_\beta(t, \sigma(s)) \Delta s, \ \beta = \lambda[\frac{\alpha}{\gamma} + \frac{\delta}{\gamma}],$$

$$\begin{aligned} F_0(t) \ &: = F(t) + \frac{(\gamma - \lambda\delta)}{\gamma} \int_{t_0}^{t} (G_1(s))^{\gamma/(\gamma-\lambda\delta)} \Delta s \\ &+ \frac{(\gamma - \lambda\alpha)}{\gamma} \int_{t_0}^{t} (G_2(s))^{\gamma/(\gamma-\lambda\alpha)} \Delta s, \end{aligned}$$

where F, G_1 and G_2 are defined as in (31) and (46).

Proof. Define a function $y(t)$ by (35) and proceed as in the proof of Theorem 2.2 to obtain

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + b^{\frac{1}{\gamma}}(t) y^{\frac{1}{\gamma}}(t), \text{ for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{57}$$

and

$$y(t) \leq F(t) + \int_{t_0}^{t} \left[G_1(s) y^{\lambda(\frac{\delta}{\gamma})}(s) + G_2(s) y^{\lambda\left(\frac{\alpha}{\gamma}\right)}(s) \right] \Delta s, \ t \in [t_0, \infty)_{\mathbb{T}}, \tag{58}$$

where F, G_1 and G_2 are defined as in (31) and (46). Applying the Young inequality (20) on the term $G_1(s) y^{\lambda(\frac{\delta}{\gamma})}(s)$ with $q = \gamma/\lambda\delta > 1$ and $p = \gamma/(\gamma - \lambda\delta) > 1$, we see that

$$G_1(s) y^{\lambda(\frac{\delta}{\gamma})}(s) \leq \frac{(\gamma - \lambda\delta)}{\gamma} (G_1(s))^{\gamma/(\gamma-\lambda\delta)} + (\frac{\lambda\delta}{\gamma}) y(s). \tag{59}$$

Again applying the Young inequality (20) on the term $G_2(s) y^{\lambda(\frac{\alpha}{\gamma})}(s)$ with $q = \gamma/\lambda\alpha > 1$ and $p = \gamma/(\gamma - \lambda\alpha) > 1$, we see that

$$G_2(s) y^{\lambda(\frac{\alpha}{\gamma})}(s) \leq \frac{(\gamma - \lambda\alpha)}{\gamma} (G_2(s))^{\gamma/(\gamma-\lambda\alpha)} + (\frac{\lambda\alpha}{\gamma}) y(s). \tag{60}$$

Substituting (59) and (60) into (58), we have

$$\begin{aligned} y(t) &\leq F(t)+\frac{(\gamma-\lambda\delta)}{\gamma}\int_{t_0}^{t}(G_1(s))^{\gamma/(\gamma-\lambda\delta)}\,\Delta s \\ &+\frac{(\gamma-\lambda\alpha)}{\gamma}\int_{t_0}^{t}(G_2(s))^{\gamma/(\gamma-\lambda\alpha)}\,\Delta s \\ &+[\frac{\lambda\alpha}{\gamma}+\frac{\lambda\delta}{\gamma}]\int_{t_0}^{t}y(s)\Delta s,\ \text{for all } t\in[t_0,\infty)_{\mathbb{T}}. \end{aligned}$$

From the definitions of $F_0(t)$ and β, we get that

$$y(t)\leq F_0(t)+\beta\int_{t_0}^{t}y(s)\Delta s,\quad \text{for } t\in[t_0,\infty)_{\mathbb{T}}.$$

Applying Lemma 2.1, we have

$$y(t)<F_0(t)+\beta\int_{t_0}^{t}F_0(s)e_\beta(t,\sigma(s))\Delta s,\ \text{for all } t\in[t_0,\infty)_{\mathbb{T}}. \tag{61}$$

Substituting (61) into (57), we get the desired inequality (56). The proof is complete.

Theorem 2.5. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t\in\mathbb{T}$. Assume that (H_1) holds, $0<\lambda\leq 1$ and α, $\delta\leq\gamma$. Then (32) implies that*

$$u(t)\leq a^{\frac{1}{\gamma}}(t)+\frac{1}{\gamma}a^{\frac{1}{\gamma}-1}(t)b(t)F_4(t),\quad t\in[t_0,\infty)_{\mathbb{T}}, \tag{62}$$

where

$$\begin{aligned} F_4(t) &:= F_2(t)+\lambda\int_{t_0}^{t}F_2(s)e_\lambda(t,\sigma(s))\Delta s, \\ F_2(t) &:= F_1(t)+(1-\lambda)\int_{t_0}^{t}(G_3(s))^{\frac{1}{1-\lambda}}\,\Delta s, \end{aligned}$$

where F_1 and G_3 are defined as in (53).

Proof. Define a function $y(t)$ by (35) and proceed as in the proof of Theorem 2.1 to obtain

$$u(t)\leq a^{\frac{1}{\gamma}}(t)+\frac{1}{\gamma}a^{\frac{1}{\gamma}-1}(t)b(t)y(t),\ \text{for}\ \ t\in[t_0,\infty)_{\mathbb{T}}, \tag{63}$$

and

$$y(t)\leq F_1(t)+\int_{t_0}^{t}G_3(s)y^\lambda(s)\Delta s,\quad \text{for } t\in[t_0,\infty)_{\mathbb{T}}, \tag{64}$$

where F_1 and G_3 are defined in (53). Applying the Young inequality (20) on the term $G_3(s)y^{\lambda}(s)$ with $q = \frac{1}{\lambda} > 1$ and $p = \frac{1}{1-\lambda} > 1$, we see that

$$G_3(s)y^{\lambda}(s) \leq (1-\lambda)\left(G_3(s)\right)^{\frac{1}{1-\lambda}} + \lambda\left(y^{\lambda}(s)\right)^{\frac{1}{\lambda}}.$$

This and (64) imply that

$$y(t) \leq F_1(t) + (1-\lambda)\int_{t_0}^{t}\left(G_3(s)\right)^{\frac{1}{1-\lambda}}\Delta s + \lambda\int_{t_0}^{t} y(s)\Delta s,\ t \in [t_0,\infty)_{\mathbb{T}}.$$

Using the definition of $F_2(t)$, we get that

$$y(t) \leq F_2(t) + \lambda\int_{t_0}^{t} y(s)\Delta s, \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}.$$

Applying Lemma 2.1, we have

$$y(t) < F_2(t) + \lambda\int_{t_0}^{t} F_2(s)e_{\lambda}(t,\sigma(s))\Delta s,\ \text{for all } t \in [t_0,\infty)_{\mathbb{T}}. \tag{65}$$

Substituting (65) into (63), we get the desired inequality (62). The proof is complete.

Next, in the following, we consider the delay dynamic inequality (19) and establish some explicit bounds of the unknown function $u(t)$. First, we consider the case when $\lambda = 1$ and α, $\delta \leq \gamma$. For this case, we introduce the following notations:

$$\begin{aligned}
A(t) \quad &: \ = F^*(t) + \int_{t_0}^{t} F^*(s)G^*(s)e_G(t,\sigma(s))\Delta s,\\
F^*(t) \quad &: \ = \int_{t_0}^{t}[f(s)a^{\frac{\delta}{\gamma}}(s) + g(s)a^{\frac{\alpha}{\gamma}}(s)]\Delta s,\\
G^*(t) \quad &: \ = b(t)\left[\frac{\delta}{\gamma}a^{\frac{\delta}{\gamma}-1}(t)f(t) + \frac{\alpha}{\gamma}a^{\frac{\alpha}{\gamma}-1}(t)g(t)\right].
\end{aligned}$$

Theorem 2.6. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Assume that $(H_1) - (H_2)$ hold, $\lambda = 1$ and α, $\delta \leq \gamma$. Then (19) implies that*

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma}a^{\frac{1}{\gamma}-1}(t)b(t)A(t), \quad t \in [t_0,\infty)_{\mathbb{T}}. \tag{66}$$

Proof. Define a function $y(t)$ by

$$y(t) := \int_{t_0}^{t}\left[f(s)u^{\delta}(\tau(s)) + g(s)u^{\alpha}(\delta(s))\right]\Delta s. \tag{67}$$

This reduces (19) to

$$u^{\gamma}(t) \leq a(t) + b(t)y(t), \quad \text{for} \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{68}$$

This implies that

$$u(t) \leq (a(t) + b(t)y(t))^{\frac{1}{\gamma}}, \quad \text{for} \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{69}$$

Applying the inequality (17) on (69), we see that

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma} a^{\frac{1}{\gamma}-1}(t)b(t)y(t), \text{ for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{70}$$

From (69), since $a(t)$, $b(t)$ and $y(t)$ are nondecreasing, we see that

$$u(\eta(t)) \leq (a(t) + b(t)y(t))^{\frac{1}{\gamma}}, \quad \text{for} \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{71}$$

Applying the inequality (17), we have

$$u(\eta(t)) \leq a^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma} a^{\frac{1}{\gamma}-1}(t)b(t)y(t), \quad \text{for} \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{72}$$

From (72), we obtain

$$u^{\alpha}(\eta(t)) \leq a^{\frac{\alpha}{\gamma}}(t) \left[1 + \frac{b(t)y(t)}{a(t)}\right]^{\frac{\alpha}{\gamma}}, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.$$

Applying the inequality (17) (where $\alpha \leq \gamma$), we obtain

$$u^{\alpha}(\eta(t)) \leq a^{\frac{\alpha}{\gamma}}(t) \left[1 + \frac{\alpha}{\gamma} \frac{b(t)}{a(t)} y(t)\right] = a^{\frac{\alpha}{\gamma}}(t) + \frac{\alpha}{\gamma} a^{\frac{\alpha}{\gamma}-1}(t)b(t)y(t), \tag{73}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$. Also as in (71), we may have

$$u^{\delta}(\tau(t)) \leq a^{\frac{\delta}{\gamma}}(t) \left[1 + \frac{b(t)y(t)}{a(t)}\right]^{\frac{\delta}{\gamma}}, \quad \text{for} \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{74}$$

Applying the inequality (17) (where $\delta \leq \gamma$), we have

$$u^{\delta}(\tau(t)) \leq a^{\frac{\delta}{\gamma}}(t) \left[1 + \frac{\delta}{\gamma} \frac{b(t)}{a(t)} y(t)\right] = a^{\frac{\delta}{\gamma}}(t) + \frac{\delta}{\gamma} a^{\frac{\delta}{\gamma}-1}(t)b(t)y(t), \tag{75}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$. Combining (67), (73) and (75), we see that

$$\begin{aligned} y(t) &= \int_{t_0}^{t} \Big[f(s) u^{\delta}(\tau(s)) + g(s) u^{\alpha}(\eta(s)) \Big] \Delta s \\ &\leq \int_{t_0}^{t} f(s) a^{\frac{\delta}{\gamma}}(s) \Delta s + \frac{\delta}{\gamma} \int_{t_0}^{t} f(s) a^{\frac{\delta}{\gamma}-1}(s) b(s) y(s) \Delta s \\ &\quad + \int_{t_0}^{t} g(s) a^{\frac{\alpha}{\gamma}}(s) \Delta s + \frac{\alpha}{\gamma} \int_{t_0}^{t} a^{\frac{\alpha}{\gamma}-1}(s) g(s) b(s) y(s) \Delta s \\ &= F^*(t) + \int_{t_0}^{t} G^*(s) y(s) \Delta s, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \end{aligned}$$

Now an application of Lemma 2.1 gives that

$$y(t) < F^*(t) + \int_{t_0}^{t} F^*(s) G^*(s) e_{G^*}(t, \sigma(s)) \Delta s, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{76}$$

Substituting (76) into (70), we obtain the desired inequality (66). The proof is complete.

In the following, we consider (19) and establish an upper bound for the function $u(t)$ in the case when $\lambda = 1$ and $\alpha = \delta \geq \gamma$. For simplicity, we introduce the following notations:

$$\begin{aligned} v(t) &:= 2^{\frac{\alpha}{\gamma}-1} \int_{t_0}^{t} a^{\frac{\alpha}{\gamma}}(s) \left[f(s) + g(s) \right] \Delta s, \\ R(t) &:= 2^{\frac{\alpha}{\gamma}-1} \int_{t_0}^{t} b^{\frac{\alpha}{\gamma}}(s) \left[g(s) + f(s) \right] \Delta s. \end{aligned}$$

Theorem 2.7. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Assume that $(H_1) - (H_2)$ hold, $\lambda = 1$ and $\alpha = \delta \geq \gamma$. Then (19) implies that*

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma} a^{\frac{1}{\gamma}-1}(t) b(t) V(t), \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{77}$$

where $V(t)$ solves the initial value problem

$$V^{\Delta}(t) = v^{\Delta}(t) + R(t) V^{\frac{\alpha}{\gamma}}(t), \;\; V(t_0) = V_0 > 0. \tag{78}$$

Proof. Define $y(t)$ as in (67) and proceed as in the proof of Theorem 2.3 to get

$$u^{\gamma}(t) \leq a(t) + b(t) y(t), \quad \text{for} \;\; t \in [t_0, \infty)_{\mathbb{T}}. \tag{79}$$

Applying the inequality (17), we see that

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma} a^{\frac{1}{\gamma}-1}(t) b(t) y(t), \text{ for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{80}$$

From (79), since $a(t)$, $b(t)$ and $y(t)$ are nondecreasing, we see that

$$u^\alpha(\eta(t)) \leq [a(t) + b(t)y(t)]^{\frac{\alpha}{\gamma}}, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.$$

Applying the inequality (21) (where $\alpha \geq \gamma$), we obtain

$$u^\alpha(\eta(t)) \leq 2^{\frac{\alpha}{\gamma}-1} \left[a^{\frac{\alpha}{\gamma}}(t) + b^{\frac{\alpha}{\gamma}}(t) y^{\frac{\alpha}{\gamma}}(t) \right], \text{ for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{81}$$

Also as in (81), we may have

$$u^\alpha(\tau(t)) \leq [a(t) + b(t)y(t)]^{\frac{\alpha}{\gamma}}, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.$$

Applying the inequality (21) (where $\alpha \geq \gamma$), we have

$$u^\alpha(\tau(t)) \leq 2^{\frac{\alpha}{\gamma}-1} \left[a^{\frac{\alpha}{\gamma}}(t) + b^{\frac{\alpha}{\gamma}}(t) y^{\frac{\alpha}{\gamma}}(t) \right], \text{ for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{82}$$

Combining (67), (81) and (82), we have

$$\begin{aligned} y(t) &= \int_{t_0}^t [f(s)u^\alpha(\tau(s)) + g(s)u^\alpha(\eta(s))] \Delta s \\ &\leq 2^{\frac{\alpha}{\gamma}-1} \int_{t_0}^t f(s) a^{\frac{\alpha}{\gamma}}(s) \Delta s + 2^{\frac{\alpha}{\gamma}-1} \int_{t_0}^t f(s) b^{\frac{\alpha}{\gamma}}(s) y^{\frac{\alpha}{\gamma}}(s) \Delta s \\ &\quad + 2^{\frac{\alpha}{\gamma}-1} \int_{t_0}^t g(s) a^{\frac{\alpha}{\gamma}}(s) \Delta s + 2^{\frac{\alpha}{\gamma}-1} \int_{t_0}^t g(s) b^{\frac{\alpha}{\gamma}}(s) y^{\frac{\alpha}{\gamma}}(s) \Delta s \\ &= v(t) + \int_{t_0}^t R(s) y^{\frac{\alpha}{\gamma}}(s) \Delta s, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \end{aligned}$$

Now an application of Lemma 2.3 (with $n = 1$ and $g(y) = y^{\frac{\alpha}{\gamma}}$) gives that

$$y(t) < V(t), \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{83}$$

where $V(t)$ solves the inequality (78). Substituting (83) into (38), we obtain the desired inequality (33). The proof is complete.

Remark 1 *Note that the results in Theorems 2.6, 2.7 can be extended to the cases when $\lambda \geq 1$ and $0 \leq \lambda \leq 1$. Also Theorem 2.7 can be proved as in the proof of Theorem 2.3 when $\alpha \neq \delta$. The details are left to the interested reader.*

In the following, we apply the Young inequality (20) to find a new explicit upper bound for $u(t)$ of (19) when α, $\delta \leq \gamma$. For simplicity, we introduce the following notations:

$$V_3(t) \quad : \quad = \int_{t_0}^{t} \left[f(s)(a^{\frac{\alpha}{\gamma}}(s) + \frac{b^{\frac{\alpha}{\gamma-\alpha}}(s)}{\frac{\gamma}{\gamma-\alpha}}) + g(s)(a^{\frac{\delta}{\gamma}}(s) + \frac{b^{\frac{\delta}{\gamma-\delta}}(s)}{\frac{\gamma}{\gamma-\delta}}) \right] \Delta s,$$

$$B_1(t) \quad : \quad = \int_{t_0}^{t} \left[\frac{\alpha}{\gamma} f(s) + \frac{\delta}{\gamma} g(s) \right] \Delta s.$$

Theorem 2.8. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Assume that $(H_1) - (H_2)$ hold, $\lambda = 1$ and α, $\delta \leq \gamma$. Then (19) implies that*

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma} a^{\frac{1}{\gamma}-1}(t) b(t) V_1(t), \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{84}$$

where $V_1(t)$

$$V_1(t) = V_3(t) + \int_{t_0}^{t} V_3(s) B_1(s) e_{B_1}(t, \sigma(s)) \Delta s. \tag{85}$$

Proof. Define $y(t)$ as in (67) and proceed as in the proof of Theorem 2.3 to get

$$u^{\gamma}(t) \leq a(t) + b(t) y(t), \quad \text{for} \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{86}$$

Applying the inequality (17), we see that

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma} a^{\frac{1}{\gamma}-1}(t) b(t) y(t), \text{ for } \ t \in [t_0, \infty)_{\mathbb{T}}. \tag{87}$$

From (86), since $a(t)$, $b(t)$ and $y(t)$ are nondecreasing, we see that

$$u^{\delta}(\eta(t)) \leq [a(t) + b(t) y(t)]^{\frac{\delta}{\gamma}}, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.$$

Applying the inequality (22) (where $\delta \leq \gamma$), we obtain

$$u^{\delta}(\eta(t)) \leq a^{\frac{\delta}{\gamma}}(t) + b^{\frac{\delta}{\gamma}}(t) y^{\frac{\delta}{\gamma}}(t), \text{ for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{88}$$

Applying the Young inequality (20) on the term $b^{\frac{\delta}{\gamma}}(t) y^{\frac{\delta}{\gamma}}(t)$ with $q = \frac{\gamma}{\delta} > 1$, and $p = \frac{\gamma}{\gamma-\delta} > 1$, we see that

$$b^{\frac{\delta}{\gamma}}(s) y^{\frac{\delta}{\gamma}}(s) \leq \frac{b^{\frac{\delta}{\gamma-\delta}}(s)}{\frac{\gamma}{\gamma-\delta}} + \frac{\delta}{\gamma} y(s). \tag{89}$$

This implies that

$$u^{\delta}(\eta(t)) \leq \left(a^{\frac{\delta}{\gamma}}(t) + \frac{b^{\frac{\delta}{\gamma-\delta}}(t)}{\frac{\gamma}{\gamma-\delta}} \right) + \frac{\delta}{\gamma} y(t), \text{ for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{90}$$

Also as in (90), we may prove that

$$u^{\alpha}(\tau(t)) \leq \left(a^{\frac{\alpha}{\gamma}}(t) + \frac{b^{\frac{\alpha}{\gamma-\alpha}}(t)}{\frac{\gamma}{\gamma-\alpha}} \right) + \frac{\alpha}{\gamma} y(t), \text{ for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{91}$$

Combining (67), (90) and (91), we see that

$$\begin{aligned} y(t) &= \int_{t_0}^{t} [f(s)u^{\alpha}(\tau(s)) + g(s)u^{\alpha}(\eta(s))]\, \Delta s \\ &\leq \int_{t_0}^{t} f(s) \left(a^{\frac{\alpha}{\gamma}}(s) + \frac{b^{\frac{\alpha}{\gamma-\alpha}}(s)}{\frac{\gamma}{\gamma-\alpha}} \right) \Delta s + \frac{\alpha}{\gamma} \int_{t_0}^{t} f(s)y(s)\Delta s \\ &\quad + \int_{t_0}^{t} g(s) \left(a^{\frac{\delta}{\gamma}}(s) + \frac{b^{\frac{\delta}{\gamma-\delta}}(s)}{\frac{\gamma}{\gamma-\delta}} \right) \Delta s + \frac{\delta}{\gamma} \int_{t_0}^{t} g(s)y(s)\Delta s \\ &= V_3(t) + \int_{t_0}^{t} B_1(s)y(s)\Delta s, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \end{aligned}$$

Now an application of Lemma 2.1 gives that

$$y(t) < V_3(t) + \int_{t_0}^{t} V_3(s)B_1(s)e_{B_1}(t, \sigma(s))\Delta s, \text{ for all } t \in [t_0, \infty)_{\mathbb{T}}. \tag{92}$$

Substituting (92) into (87), we obtain the desired inequality (84). The proof is complete.

Remark 2 *Note that the above results can be applied on different types of time scales. For example, if $\mathbb{T} = \mathbb{R}$, then the results in Theorems 2.8 reduce to integral inequalities and when $\mathbb{T} = \mathbb{N}$, then the results in Theorem 2.8 reduce to discrete inequalities. This means that the above results involve the integral inequalities and discrete inequalities as special cases. For more details, we refer the reader to [22].*

3 Applications

In this section, we give some examples to illustrate the main results. First, we consider the second-order half-linear delay dynamic equation

$$(r(t) \left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} + p(t)x^{\gamma}(\tau(t)) = 0, \text{ for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{93}$$

on an arbitrary time scale $\mathbb{T}$, and establish an explicit upper bound of the nonoscillatory solutions, where $\gamma \geq 1$ is a quotient of odd positive integers, p is a positive $rd-$continuous function on $\mathbb{T}$, $r(t)$ is a positive and (delta) differentiable function and the so-called delay function $\tau : \mathbb{T} \to \mathbb{T}$ satisfies $\tau(t) \leq t$ for $t \in \mathbb{T}$ and $\lim_{t\to\infty} \tau(t) = \infty$. By a solution of (93) we mean a nontrivial real–valued function $x \in C_r^1[T_x, \infty)$, $T_x \geq t_0$ which has the property that $r(t)\left(x^{\Delta}(t)\right)^{\gamma} \in C_r^1[T_x, \infty)$ and satisfies equation (93) on $[T_x, \infty)$, where C_r is the space of $rd-$continuous functions. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$, here $g^{\sigma} = g \circ \sigma$) of two differentiable function f and g

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}, \text{ and } \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}. \quad (94)$$

Lemma 3.1 [22]. *Assume that*

$$r^{\Delta}(t) \geq 0, \quad \text{and} \quad \int_{t_0}^{\infty} \tau^{\gamma}(t)p(t)\Delta t = \infty, \quad (95)$$

and

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} = \infty. \quad (96)$$

Assume that (93) has a positive solution x on $[t_0, \infty)_{\mathbb{T}}$. Then there exists a $T \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large, so that

(i) $x^{\Delta}(t) > 0$, $x^{\Delta\Delta}(t) < 0$, $x(t) > tx^{\Delta}(t)$, for $t \in [T, \infty)_{\mathbb{T}}$;

(ii) $\frac{x(t)}{t}$ is strictly decreasing on $[T, \infty)_{\mathbb{T}}$.

The following theorem gives an upper bound of nonoscillatory solutions of (93).

Theorem 3.1. *Assume that* (95) *and* (96) *hold and $x(t)$ is a nonoscillatory solution of (93). Then $x(t)$ satisfies $x(t) \leq x(t_1)e_K(t, t_1)$, where*

$$K(t) = \left[\frac{A}{\delta(t)r(t)} + \int_{t_1}^{t} \left[\frac{r(s)((\delta^{\Delta}(s))^{\gamma+1}}{\delta^{\gamma}(s)(\gamma+1)^{\gamma+1}} - \delta(s)p(s)\left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma}\right]\Delta s\right]^{\frac{1}{\gamma}}, \quad (97)$$

and $\delta(t)$ is any positive $\Delta-$differentiable function and A is a positive constant and $t_1 \in [t_0, \infty)_{\mathbb{T}}$.

Proof. Assume that there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t)$ satisfies the conclusions of Lemma 3.1 on $[t_1, \infty)_{\mathbb{T}}$ with $x(\tau(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Let $\delta(t)$

be a positive Δ differentiable function and consider the Riccati substitution

$$w(t) = \delta(t)r(t)\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma}.$$

Then by Lemma 3.1, we see that the function $w(t)$ is positive on $[t_1, \infty)_{\mathbb{T}}$. By the product rule and then the quotient rule (suppressing arguments)

$$\begin{aligned} w^{\Delta} &= \delta^{\Delta}\left(\frac{r(x^{\Delta})^{\gamma}}{x^{\gamma}}\right)^{\sigma} + \delta\left(\frac{r(x^{\Delta})^{\gamma}}{x^{\gamma}}\right)^{\Delta} \\ &= \frac{\delta^{\Delta}}{\delta^{\sigma}}w^{\sigma} + \delta\frac{x^{\gamma}(r(x^{\Delta})^{\gamma})^{\Delta} - r(x^{\Delta})^{\gamma}(x^{\gamma})^{\Delta}}{x^{\gamma}x^{\gamma\sigma}} \\ &= \frac{\delta^{\Delta}}{\delta^{\sigma}}w^{\sigma} - p\delta\left(\frac{x^{\tau}}{x^{\sigma}}\right)^{\gamma} - \delta\frac{r(x^{\Delta})^{\gamma}(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}. \end{aligned}$$

Using the fact that $\frac{x(t)}{t}$ and $r(t)(x^{\Delta}(t))^{\gamma}$ are decreasing (from Lemma 3.1) we get

$$\frac{x^{\tau}(t)}{x^{\sigma}(t)} \geq \frac{\tau(t)}{\sigma(t)}, \quad \text{and} \quad r(t)(x^{\Delta}(t))^{\gamma} \geq r^{\sigma}(t)(x^{\Delta}(t))^{\gamma\sigma}.$$

From these last two inequalities we obtain

$$w^{\Delta} \leq \frac{\delta^{\Delta}}{\delta^{\sigma}}w^{\sigma} - \delta p\left(\frac{\tau}{\sigma}\right)^{\gamma} - \delta\frac{r^{\sigma}(x^{\Delta\sigma})^{\gamma}(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}. \tag{98}$$

By the chain rule and the fact that $x^{\Delta}(t) > 0$, we obtain

$$\begin{aligned} (x^{\gamma}(t))^{\Delta} &= \gamma\int_0^1 \left[x(t) + h\mu(t)x^{\Delta}(t)\right]^{\gamma-1} dh\, x^{\Delta}(t) \\ &\geq \gamma\int_0^1 (x^{\sigma}(t))^{\gamma-1} dh\, x^{\Delta}(t) \\ &= \gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t). \end{aligned} \tag{99}$$

Using (98) and (99), we have that

$$w^{\Delta} \leq \frac{\delta^{\Delta}}{\delta^{\sigma}}w^{\sigma} - \delta p\left(\frac{\tau}{\sigma}\right)^{\gamma} - \gamma\delta\frac{r^{\sigma}(x^{\Delta\sigma})^{\gamma}x^{\Delta}}{x^{\gamma}x^{\sigma}}.$$

Since

$$x^{\Delta}(t) \geq \frac{(r^{\sigma}(t))^{\frac{1}{\gamma}}(x^{\Delta}(t))^{\sigma}}{r^{\frac{1}{\gamma}}(t)}, \quad \text{and} \quad x^{\sigma}(t) \geq x(t),$$

we get that

$$w^{\Delta} \le \frac{\delta^{\Delta}}{\delta^{\sigma}} w^{\sigma} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma} - \gamma \frac{\delta r^{\sigma(1+\frac{1}{\gamma})}}{r^{\frac{1}{\gamma}}} \left(\frac{x^{\Delta\sigma}}{x^{\sigma}}\right)^{\gamma+1}.$$

Using the definition of w we finally obtain

$$w^{\Delta} \le \frac{(\delta^{\Delta})_{+}}{\delta^{\sigma}} w^{\sigma} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma} - \gamma \frac{\delta}{(\delta^{\sigma})^{\lambda} r^{\frac{1}{\gamma}}} (w^{\sigma})^{\lambda}, \tag{100}$$

where $\lambda := \frac{\gamma+1}{\gamma}$. Define positive A and B by

$$A^{\lambda} := \frac{\gamma\delta}{(\delta^{\sigma})^{\lambda} r^{\frac{1}{\gamma}}} (w^{\sigma})^{\lambda}, \quad B^{\lambda-1} := \frac{r^{\frac{1}{\gamma+1}}}{\lambda(\gamma\delta)^{\frac{1}{\lambda}}} (\delta^{\Delta})_{+}.$$

Then, using the inequality $\lambda A B^{\lambda-1} - A^{\lambda} \le (\lambda-1) B^{\lambda}$, we get that

$$\frac{(\delta^{\Delta})_{+}}{\delta^{\sigma}} w^{\sigma} - \gamma \frac{\delta}{(\delta^{\sigma})^{\lambda} r^{\frac{1}{\gamma}}} (w^{\sigma})^{\lambda} \le \frac{r((\delta^{\Delta})_{+})^{\gamma+1}}{\delta^{\gamma}(\gamma+1)^{\gamma+1}}.$$

From this last inequality and (100), we get

$$w^{\Delta} \le \frac{r(\delta^{\Delta})^{\gamma+1}}{\delta^{\gamma}(\gamma+1)^{\gamma+1}} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma}.$$

Integrating both sides from t_1 to t we get

$$w(t) \le w(t_1) + \int_{t_1}^{t} \left[\frac{r(\delta^{\Delta})^{\gamma+1}}{\delta^{\gamma}(\gamma+1)^{\gamma+1}} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma}\right] \Delta s,$$

which leads to

$$x^{\Delta}(t) \le \left[\frac{w(t_1)}{\delta(t) r(t)} + \int_{t_1}^{t} \left[\frac{r(\delta^{\Delta})^{\gamma+1}}{\delta^{\gamma}(\gamma+1)^{\gamma+1}} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma}\right] \Delta s\right]^{\frac{1}{\gamma}} x(t). \tag{101}$$

Applying the inequality (12), we get the desired inequality (97). The proof is complete.

Remark 3 *When $\delta(t) = 1$, then $K(t)$ reduces to*

$$K_1(t) = \left[\frac{A}{r(t)} - \int_{t_1}^{t} p(s) \left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma} \Delta s\right]^{\frac{1}{\gamma}}, \tag{102}$$

and then the upper bound of $x(t)$ of (93) is given by $x(t) \le x(t_1) e_{K_1}(t, t_1)$.

Next, we consider the dynamic equation

$$(c(t)x^{\gamma}(t))^{\Delta} = a(t) + b(t)[f(t)x^{\frac{\delta}{\beta}}(t) + g(t)x^{\frac{\alpha}{\beta}}(t)]^{\beta}, \text{ for } t \in [t_0, \infty)_{\mathbb{T}}, \quad (103)$$

with $x(t_0) > 0$ and establish an upper bound for a positive solution $x(t)$. To prove the main results for equation (103), we introduce the following notations:

$$\begin{aligned} F_*(t) \quad &: \quad = \int_{t_0}^{t} \left[b(s)f^{\beta}(s)C^{\frac{\delta}{\gamma}}(s) + b(s)g^{\beta}(s)C^{\frac{\alpha}{\gamma}}(s) \right] \Delta s, \\ G_*(t) \quad &: \quad = \left(b(s)f^{\beta}(s)\frac{\delta}{\gamma}C^{\frac{\delta}{\gamma}-1}(s) + b(s)g^{\beta}(s)\frac{\alpha}{\gamma}C^{\frac{\alpha}{\gamma}-1}(s) \right), \quad (104) \\ C(t) \quad &= \quad \frac{x(t_0)}{c(t)} + \frac{1}{c(t)} \int_{t_0}^{t} a(s)\Delta s, \quad B(t) = \frac{2^{\lambda-1}}{c(t)}. \end{aligned}$$

Theorem 3.2. *Assume that a, b, c, f and g are rd-continuous positive functions defined on $[t_0,\infty)_{\mathbb{T}}$, and γ, $\beta \geq 1$ and α, $\delta \leq \gamma$. Then*

$$x(t) \leq C^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma}C^{\frac{1}{\gamma}-1}(t)B(t)W(t), \quad t \in [t_0, \infty)_{\mathbb{T}}, \quad (105)$$

where $W_1(t)$ solves

$$W^{\Delta}(t) \leq F_*^{\Delta}(t) + B(t)G_*(t)W^{\alpha}(t), \quad W(t_0) = W_0 > 0. \quad (106)$$

Proof. Since $\beta \geq 1$, we from (103) after application of (21), that

$$(c(t)x^{\gamma}(t))^{\Delta} \leq a(t) + 2^{\beta-1}b(t)[f^{\beta}(t)x^{\delta}(t) + g^{\beta}(t)x^{\alpha}(t)], \text{ for } t \in [t_0, \infty)_{\mathbb{T}}.$$

Integrating this inequality from t_0 to t, we have

$$x^{\gamma}(t) \leq C(t) + B(t) \int_{t_0}^{t} [b(s)f^{\beta}(s)x^{\delta}(s) + b(s)g^{\beta}(s)x^{\alpha}(s)]\Delta s,$$

Applying Theorem 2.1 with $\lambda = 1$, we get that

$$x(t) \leq C^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma}C^{\frac{1}{\gamma}-1}(t)B(t)W(t), \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

where $W(t)$ solves the initial value problem (106). The proof is complete.

Remark 4 *One can apply Theorem 2.4 to find an upper bound of $x(t)$ of (103) when $\beta \leq 1$. Also, one can apply the Young inequality on the term $B(t)G_*(t)W^{\alpha}(t)$ when $\alpha < 1$ and find a new explicit upper bound for the solution $x(t)$. The details are left to the interested reader.*

Acknowledgements

The author is very grateful to the anonymous referees for valuable remarks and comments which significantly contributed to the quality of the paper. This project was supported by King Saud University, Dean-ship of Scientific Research, College of Science Research Center.

References

[1] D. A. Anderson, Young's integral inequality on time scales revisited, J. Ineq. Pure Appl. Math. 8 (2007), Issue 3, Art. 64, 5pp.

[2] D. A. Anderson, Nonlinear dynamic integral inequalities in two independent variables on time scale pairs, Advances Dyn. Syst. Appl. 3 (2008), 1-13.

[3] D. A. Anderson, Dynamic double integral inequalities in two independent variables on time scales, J. Math. Inq. 2 (2008), 163-184.

[4] R. Bellman, The stability of solutions of linear differential equations, Duke Math. J. 10 (1943), 643-647.

[5] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*: An Introduction with Applications, Birkhäuser, Boston, 2001.

[6] E. Akin-Bohner, M. Bohner, F. Akin, Pachpatte inequalities on time scales, JIPAM. J. Ineq. Pure Appl. Math. 6 (2005) 1–23.

[7] C. M. Dafermos, The second law of thermodynamics and stability, Arch. Rational Mech. Anal. 70 (1979), 167–179.

[8] T. H. Gronwall, Note on the derivative with respect to a parameter of the solutions of a system of differential equations, Ann. of Math. 20 (1919), 292-296.

[9] S. Hilger, Analysis on measure chains–a unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18–56.

[10] V. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, 2001.

[11] W. N. Li, Some Pachpatte type inequalities on time scales, Comp. Math. Appl. 57 (2009), 275-282.

[12] W. N. Li, Some new dynamic inequalities on time scales, J. Math. Anal. Appl. 319 (2007), 802-814.

[13] W. N. Li and M Han, Bounds for certain nonlinear dynamic inequalities on time scales, Discrete Dyn. Nat. Soc. 2009 (2009), ID 897087, 14 pages.

[14] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publ. 1993.

[15] L. Ou-Iang, The boundedness of solutions of linear differential equations $y'' + A(t)y = 0$, Shuxue Jinzhan 3 (1957) 409–415.

[16] B. G. Pachpatte, On some new inequalities related to certain inequalities in the theory of differential equations, J. Math. Anal. Appl. 189 (1995) 128–144.

[17] S. H. Saker, Some nonlinear dynamic inequalities on time scales and applications, Journal of Math. Ineq. 4 (2010), 561-579.

[18] S. H. Saker, Opial's type inequalities on time scales and some applications, Annal. Polon. Math. (accepted).

[19] S. H. Saker, Some nonlinear dynamic inequalities on time scales, Math. Ineq. Appl. 14 (2011), 633-645.

[20] S. H. Saker, Lyapunov inequalities for half-linear dynamic equations on time scales and disconjugacy, Dyn. Contin. Discr.Impuls. Syst. Series B: Applications & Algorithms 18 (2011), 149-161.

[21] S. H. Saker, Some Opial-type inequalities on time scales, Abstr. Appl. Anal. 2011 (2011), Id 265316, 19 pages.

[22] S. H. Saker, *Oscillation Theory of Dynamic Equations on Time Scales: Second and Third Orders*, Lambert Academic Publishing, Germany (2010).

[23] V. Spedding, Taming Nature's Numbers, New Scientist, July 19 (2003), 28–31.

(Received September 17, 2011)