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Existence Results for Impulsive Semilinear Damped Differential Inclusions

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Abstract

In this paper we investigate the existence of mild solutions for first and second order impulsive semilinear evolution inclusions in real separable Banach spaces. By using suitable fixed point theorems, we study the case when the multivalued map has convex and nonconvex values.

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AMS (MOS) Subject Classifications: 34A37, 34A60, 34G20, 35R10, 47H20

1 Introduction

In this paper, we shall be concerned with the existence of mild solutions for first and second order impulsive semilinear damped differential inclusions in a real Banach space. Firstly, we consider the following first order impulsive semilinear differential inclusions of the form:

$$y' - Ay \in By + F(t,y), \quad \text{a.e. } t \in J := [0,b],\ t \neq t_k,\ k = 1,\dots,m, \tag{1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1,\dots,m, \tag{2}$$

$$y(0) = y_0, \tag{3}$$

where $F : J \times E \to P(E)$ is a multivalued map ($P(E)$ is the family of all nonempty subsets of E), A is the infinitesimal generator of a family of semigroup $\{T(t) : t \geq 0\}$, B is a bounded linear operator from E into $E, y_0 \in E, 0 < t_1 < \dots < t_m < t_{m+1} =$

$b,\ I_k \in C(E,E)\ (k=1,\dots,m),\ \Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-),\ y(t_k^+) = \lim_{h\to 0^+} y(t_k+h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k-h)$ represent the right and left limits of $y(t)$ at $t=t_k$, respectively and E a real separable Banach space with norm $|\cdot|$.

Later, we study the second order impulsive semilinear evolution inclusions of the form:

$$y'' - Ay \in By' + F(t,y), \quad \text{a.e. } t \in J := [0,b],\ t \neq t_k,\ k=1,\dots,m, \tag{4}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k=1,\dots,m, \tag{5}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)),\ k=1,\dots,m, \tag{6}$$

$$y(0)=y_0,\ y'(0)=y_1, \tag{7}$$

where F, I_k, B and y_0 are as in problem (1)–(3), A is the infinitesimal generator of a family of cosine operators $\{C(t): t\geq 0\}$, $\overline{I}_k \in C(E,E)$ and $y_1 \in E$.

The study of the dynamical buckling of the hinged extensible beam which is either stretched or compressed by axial force in a Hilbert space, can be modeled by the following hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} - \left(\alpha + \beta\int_0^L \left|\frac{\partial u}{\partial t}(\xi,t)\right|^2 d\xi\right)\frac{\partial^2 u}{\partial x^2} + g\left(\frac{\partial u}{\partial t}\right) = 0, \tag{E}$$

where $\alpha,\beta,L>0$, $u(t,x)$ is the deflection of the point x of the beam at the time t, g is a nondecreasing numerical function, and L is the length of the beam.

Equation (E) has its analogue in $\mathbb{R}^n$ and can be included in a general mathematical model

$$u'' + A^2u + M(\|A^{1/2}u\|_H^2)Au + g(u') = 0, \tag{E_1}$$

where A is a linear operator in a Hilbert space H and M and g are real functions. Equation (E) was studied by Patcheu [26] and the equation (E_1) by Matos and Pereira [25]. These equations are special cases of the equations (4), (7).

Impulsive differential and partial differential equations have become more important in recent years in some mathematical models of real phenomena, especially in control, biological or medical domains, see the mongraphs of Lakshmikantham *et al* [20], Samoilenko and Perestyuk [28], and the papers of Ahmed [2], Agur *et al* [1], Erbe *et al* [13], Goldbeter *et al* [16], Kirane and Rogovchenko [18], Liu *et al* [22], Liu and Zhang [23].

This paper will be organized as follows. In Section 2 we will recall briefly some basic definitions and preliminary facts from multivalued analysis which will be used later. In Section 3 we shall establish two existence theorems for the problem (1)–(3) when the right hand side is convex as well as nonconvex valued. In the first case a fixed point theorem due to Bohnenblust and Karlin [8] (see also [32]) is used. A fixed point theorem for contraction multivalued maps due to Covitz and Nadler [10] is applied in the second one. In Section 4 existence theorems for the both cases are presented for the problem (4)–(7) in the spirit of the analysis used in the previous section.

The special case (for B=0) of the problem (1)–(3) was studied by Benchohra *et al* in [5] by using the concept of upper and lower mild solutions combined with the semigroup theory and by Benchohra and Ntouyas in [7] with the aid to a fixed point theorem due to Martelli for condensing multivalued maps [24]. Notice that when the impulses are absent (i.e. $I_k, \overline{I}_k = 0, k = 1, \dots, m$) the problem (4)–(7) was studied by Benchohra *et al* in [6]. Hence the results of the present paper can be seen as an extension of the problems considered in [6], [5] and [7].

2 Preliminaries

We will briefly recall some basic definitions and facts from multivalued analysis that we will use in the sequel.

$C([0,b],E)$ is the Banach space of all continuous functions from $[0,b]$ into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 0 \le t \le b\}.$$

$B(E)$ is the Banach space of all linear bounded operator from E into E with norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

A measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral, see for instance, Yosida [31]).

$L^1(J,E)$ denotes the Banach space of functions $y : J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|dt.$$

Let $(X, |\cdot|)$ be a normed space and $P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}$. $P_{cp}(X) = \{Y \in P(X) : Y \text{ compact}\}$. $P_{cp,c}(X) = \{Y \in P(X) : Y \text{ compact, convex}\}$. A multivalued map $G : X \to P(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighbourhood N_0 of x_0 such that $G(N_0) \subseteq N$.

G is said to be completely continuous if $G(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.

A multivalued map $N : J \to P_{cl}(E)$ is said to be measurable, if for every $y \in E$, the function $t \longmapsto d(y, N(t)) = \inf\{|y - z| : z \in N(t)\}$ is measurable, where d is the metric induced by the norm of the Banach space E. For more details on multivalued maps see the books of Aubin and Cellina [3], Aubin and Frankowska [4], Deimling [11] and Hu and Papageorgiou [17] .

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is a strongly continuous cosine family if:

(1) $C(0) = I$ (I is the identity operator in E),

(2) $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $s, t \in \mathbb{R}$,

(3) the map $t \longmapsto C(t)y$ is strongly continuous for each $y \in E$.

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y = \int_0^t C(s)y ds, \quad y \in E, \ t \in \mathbb{R}.$$

The infinitesimal generator $A : D(A) \subseteq E \longrightarrow E$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ay = \frac{d^2}{dt^2} C(t)y\Big|_{t=0}.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books of Goldstein [15], Fattorini [14], and to the papers of Travis and Webb [29], [30]. For properties of semigroup theory, we refer the interested reader to the books of Goldstein [15] and Pazy [27].

Definition 2.1 *The multivalued map $F : J \times E \to P(E)$ is said to be an L^1-Carathéodory if*

(i) $t \longmapsto F(t, y)$ *is measurable for each* $y \in E$;

(ii) $y \longmapsto F(t, y)$ *is upper semicontinuous for almost all* $t \in J$;

(iii) For each $r > 0$, there exists $\varphi_r \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, y)\| = \sup\{|v| : v \in F(t, y)\} \leq \varphi_r(t) \quad \textit{for all } \ |y| \leq r \quad \textit{and for a.e. } \ t \in J.$$

For each $y \in C(J, E)$, define the set of selections of F by

$$S_{F(y)} = \{v \in L^1(J, E) : v(t) \in F(t, y(t) \text{ a.e. } t \in J\}.$$

The following Lemmas are crucial in the proof of our main results.

Lemma 2.2 *[21]. Let X be a Banach space. Let $F : J \times X \longrightarrow P_{cp,c}(X)$ be an L^1-Carathéodory multivalued map with $S_{F(y)} \neq \emptyset$ and let Γ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$, then the operator*

$$\Gamma \circ S_F : C(J,X) \longrightarrow P_{cp,c}(C(J,X)),\ y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F(y)})$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

Lemma 2.3 *(Bohnenblust-Karlin [8], see also [32] p. 452). Let X be a Banach space and $K \in P_{cl,c}(X)$ and suppose that the operator $G : K \longrightarrow P_{cl,c}(K)$ is upper semi-continuous and the set $G(K)$ is relatively compact in X. Then G has a fixed point in K.*

3 First Order Impulsive Differential Inclusions

In this section we are concerned with the existence of solutions for problem (1)–(3) when the right hand side has convex as well as nonconvex values. Initially we assume that $F : J \times E \to P(E)$ is compact and convex valued multivalued map. In order to define the solution of (1)–(3) we shall consider the following space

$$\begin{aligned}\Omega &= \{y : [0,b] \to E : y_k \in C(J_k, E), k = 0, \ldots, m \text{ and there exist } y(t_k^-) \\ &\quad \text{and } y(t_k^+), k = 1, \ldots, m \text{ with } y(t_k^-) = y(t_k)\}\end{aligned}$$

which is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y_k\|_{J_k}, k = 0, \ldots, m\},$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}],\ k = 0, \ldots, m$. So let us start by defining what we mean by a mild solution of problem (1)–(3).

Definition 3.1 *A function $y \in \Omega$ is said to be a mild solution of (1)–(3) if there exists a function $v \in L^1(J,E)$ such that $v(t) \in F(t, y(t))$ a.e. on J and*

$$y(t) = T(t)y_0 + \int_0^t T(t-s)B(y(s))ds + \int_0^t T(t-s)v(s) + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)).$$

Let us introduce the following hypotheses:

(H1) $F : J \times E \longrightarrow P_{cp,c}(E)$ is an L^1-Carathéodory map and for each fixed $y \in C(J,E)$ the set

$$S_{F(y)} = \{v \in L^1(J,E) : v(t) \in F(t, y(t)) \text{ a.e. } t \in J\}$$

is nonempty.

(H2) There exist constants c_k, such that $|I_k(y)| \leq c_k, k = 1, \ldots, m$ for each $y \in E$.

(H3) $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \geq 0$, which is compact for $t > 0$, and there exists a constant M such that $\|T(t)\|_{B(E)} \leq M$ for each $t \geq 0$.

(H4) There exists a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, y)\| \leq p(t)\psi(|y|) \text{ for a.e. } t \in J \text{ and each } y \in E,$$

with

$$\int_0^b m(s)ds < \int_c^\infty \frac{du}{u + \psi(u)},$$

where

$$m(t) = \max(M\|B\|_{B(E)}, Mp(t)) \text{ and } c = M|y_0| + \sum_{k=1}^{m} c_k.$$

Remark 3.2 *(i) If dim$E < \infty$ then for each $y \in C(J, E)$, $S_{F(y)} \neq \emptyset$ (see Lasota and Opial [21]).*

(ii) If dim$E = \infty$ then $S_{F(y)}$ is nonempty if and if the function $Y : J \to \mathbb{R}$ defined by

$$Y(t) = \inf\{|v| : v \in F(t, y(t))\},$$

belongs to $L^1(J, \mathbb{R})$ (see Hu and Papageorgiou [17]).

(ii) Assumption (H4) is satisfied if for example F satisfies the standard domination

$$\|F(t, y)\| \leq p(t)(1 + |y|), \ p \in L^1(J, \mathbb{R}^+), \ t \in J, \ y \in E.$$

Theorem 3.3 *Assume that hypotheses (H1)-(H4) hold. Then the IVP (1)–(3) has least one mild solution.*

Proof. Transform the problem (1)–(3) into a fixed point problem. Consider the multivalued operator $N : \Omega \longrightarrow P(\Omega)$ defined by:

$$\begin{aligned} N(y) = \Big\{ h \in \Omega : h(t) &= T(t)y_0 + \int_0^t T(t-s)B(y(s))ds \\ &+ \int_0^t T(t-s)g(s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)), \ g \in S_{F(y)} \Big\}. \end{aligned}$$

Remark 3.4 *Clearly the fixed points of N are mild solutions to (1)–(3).*

We shall show that N satisfies the assumptions of Lemma 2.3. The proof will be given in several steps. Let

$$K := \{y \in \Omega : \|y\|_\Omega \le a(t),\ t \in J\},$$

where

$$a(t) = I^{-1}\left(\int_0^t m(s)ds\right),$$

and

$$I(z) = \int_c^z \frac{du}{u + \psi(u)}.$$

It is clear that K is a closed bounded convex set. Let $k^* = \sup\{\|y\|_\Omega : y \in K\}$.

Step 1: $N(K) \subset K$.

Indeed, let $y \in K$ and fix $t \in J$. We must show that $N(y) \in K$. There exists $g \in S_{F(y)}$ such that for each $t \in J$

$$h(t) = T(t)y_0 + \int_0^t T(t-s)B(y(s))ds + \int_0^t T(t-s)g(s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)).$$

Thus

$$\begin{aligned}
|h(t)| &\le M|y_0| + M\sum_{k=1}^m c_k + \int_0^t m(s)(|y(s)| + \psi(|y(s)|))ds \\
&\le M|y_0| + M\sum_{k=1}^m c_k + \int_0^t m(s)(a(s) + \psi(a(s)))ds \\
&= M|y_0| + M\sum_{k=1}^m c_k + \int_0^t a'(s)ds \\
&= a(t),
\end{aligned}$$

since

$$\int_c^{a(s)} \frac{du}{u + \psi(u)} = \int_0^s m(\tau)d\tau.$$

Thus $N(y) \in K$; So, $N : K \to K$.

Step 2: $N(K)$ is relatively compact.

Since K is bounded and $N(K) \subset K$, it is clear that $N(K)$ is bounded. $N(K)$ is equicontinuous. Indeed, let $\tau_1, \tau_2 \in J,\ \tau_1 < \tau_2$ and $\epsilon > 0$ with $0 < \epsilon \le \tau_1 < \tau_2$. Let $y \in K$ and $h \in N(y)$. Then there exists $g \in S_{F(y)}$ such that for each $t \in J$ we have

$$h(t) = T(t)y_0 + \int_0^t T(t-s)(By(s))ds + \int_0^t T(t-s)g(s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)).$$

Then

$$
\begin{aligned}
|h(\tau_2)-h(\tau_1)| &\leq |T(\tau_2)y_0 - T(\tau_1)y_0| \\
&+ \int_0^{\tau_1-\epsilon} \|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}|By(s)|ds \\
&+ \int_{\tau_1}^{\tau_1-\epsilon} \|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}|By(s)|ds \\
&+ \int_{\tau_1}^{\tau_2} \|T(\tau_2-s)\|_{B(E)}|By(s)|ds \\
&+ \int_0^{\tau_1-\epsilon} \|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}|g(s)|ds \\
&+ \int_{\tau_1}^{\tau_1-\epsilon} \|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)}|g(s)|ds \\
&+ \int_{\tau_1}^{\tau_2} \|T(\tau_2-s)\|_{B(E)}|g(s)|ds + Mc_k(\tau_2-\tau_1) \\
&+ \sum_{0<t_k<\tau_1} c_k\|T(\tau_1-t_k)-T(\tau_2-t_k)\|_{B(E)}.
\end{aligned}
$$

The right-hand side tends to zero as $\tau_2-\tau_1 \to 0$, and ϵ sufficiently small, since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t>0$ implies the continuity in the uniform operator topology. As a consequence of the Arzelá-Ascoli theorem it suffices to show that the multivalued N maps K into a precompact set in E. Let $0<t\leq b$ be fixed and let ϵ be a real number satisfying $0<\epsilon<t$. For $y\in K$ we define

$$
\begin{aligned}
h_\epsilon(t) &= T(t)y_0 + T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)(By(s))ds \\
&+ T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)g(s)ds \\
&+ T(\epsilon)\sum_{0<t_k<t-\epsilon} T(t-t_k-\epsilon)I_k(y(t_k^-))
\end{aligned}
$$

where $g\in S_{F(y)}$. Since $T(t)$ is a compact operator, the set $H_\epsilon(t)=\{h_\epsilon(t) : h_\epsilon \in N(y)\}$ is precompact in E for every ϵ, $0<\epsilon<t$. Moreover, for every $h\in N(y)$ we have

$$
\begin{aligned}
|h_\epsilon(t)-h(t)| &\leq \|B\|_{B(E)}k^*\int_{t-\epsilon}^{t} \|T(t-s)\|_{B(E)}ds \\
&+ \int_{t-\epsilon}^{t} \|T(t-s)\|_{B(E)}|a(s)|ds \\
&+ \sum_{t-\epsilon\leq t_k<t} c_k\|T(t-t_k)\|_{B(E)}.
\end{aligned}
$$

Therefore there are precompact sets arbitrarily close to the set $\{h(t) : h \in N(y)\}$. Hence the set $\{h(t) : h \in N(y)\}$ is precompact in E.

Step 3: N has a closed graph.

Let $y_n \longrightarrow y_*$, $h_n \in N(y_n)$ and $h_n \longrightarrow h_*$. We shall prove that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $g_n \in S_{F(y_n)}$ such that for each $t \in J$

$$h_n(t) = T(t)y_0 + \int^t T(t-s)By_n(s)ds + \int_0^t T(t-s)g_n(s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-)).$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that for each $t \in J$

$$h_*(t) = T(t)y_0 + \int_0^t T(t-s)By_*(s)ds + \int_0^t T(t-s)g_*(s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k^-)).$$

Clearly since $I_k,\ k = 1, \ldots, m$ and B are continuous we have that

$$\begin{aligned}&\Big\|\Big(h_n - T(t)y_0 - \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-)) - \int_0^t T(t-s)By_n(s)ds\Big)\\ &-\Big(h_* - T(t)y_0 - \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k^-)) - \int_0^t T(t-s)By_*(s)ds\Big)\Big\|_\infty \longrightarrow 0,\end{aligned}$$

as $n \to \infty$. Consider the linear continuous operator

$$\Gamma : L^1(J,E) \longrightarrow C(J,E)$$

$$g \longmapsto \Gamma(g)(t) = \int_0^t T(t-s)g(s)ds.$$

From Lemma 2.2, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have that

$$h_n(t) - T(t)y_0 - \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-)) - \int_0^t T(t-s)By_n(s)ds \in \Gamma(S_{F(y_n)}).$$

Since $y_n \longrightarrow y_*$, it follows from Lemma 2.2 that

$$h_*(t) - T(t)y_0 - \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k^-)) - \int_0^t T(t-s)By_*(s)ds = \int_0^t T(t-s)g_*(s)ds$$

for some $g_* \in S_{F(y_*)}$.

As a consequence of Lemma 2.3 we deduce that N has a fixed point which is a mild solution of (1)–(3). ■

We present now a result for the problem (1)–(3) with a nonconvex valued right hand side.

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. Consider $H_d : P(X) \times P(X) \longrightarrow \mathrm{I\!R}_+ \cup \{\infty\}$, given by

$$H_d(\mathcal{A}, \mathcal{B}) = \max\left\{\sup_{a\in\mathcal{A}} d(a, \mathcal{B}), \sup_{b\in\mathcal{B}} d(\mathcal{A}, b)\right\},$$

where $d(\mathcal{A}, b) = \inf_{a\in\mathcal{A}} d(a, b)$, $d(a, \mathcal{B}) = \inf_{b\in\mathcal{B}} d(a, b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized (complete) metric space (see [19]).

Definition 3.5 *A multivalued operator $G : X \to P_{cl}(X)$ is called*

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(G(x), G(y)) \leq \gamma d(x, y), \quad \textit{for each } x,\ y \in X.$$

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Our considerations are based on the following fixed point theorem for contraction multivalued operators given by Covitz and Nadler in 1970 [10] (see also Deimling, [11] Theorem 11.1).

Lemma 3.6 *Let (X, d) be a complete metric space. If $G : X \to P_{cl}(X)$ is a contraction, then $FixG \neq \emptyset$.*

Let us introduce the following hypotheses:

(H5) $F : J \times E \longrightarrow P_{cp}(E)$; $(t, .) \longmapsto F(t, y)$ is measurable for each $y \in E$.

(H6) There exists constants c'_k, such that

$$|I_k(y) - I_k(\overline{y})| \leq c'_k|y - \overline{y}|, \text{for each } k = 1, \ldots, m, \text{ and for all } y, \overline{y} \in E.$$

(H7) There exists a function $l \in L^1(J, \mathrm{I\!R}^+)$ such that

$$H_d(F(t, y), F(t, \overline{y})) \leq l(t)|y - \overline{y}|, \text{ for a.e. } t \in J \text{ and all } y, \overline{y} \in E,$$

and

$$d(0, F(t, 0)) \leq l(t) \text{ for a.e. } t \in J.$$

Theorem 3.7 *Suppose that hypotheses (H3), (H5)–(H7) are satisfied. If*

$$\frac{2}{\tau} + M\sum_{k=1}^{m} c_k < 1,$$

where $\tau \in \mathrm{I\!R}^+$, then the IVP (1)–(3) has at least one mild solution.

Remark 3.8 *For each $y \in \Omega$ the set $S_{F(y)}$ is nonempty since by (H5) F has a measurable selection (see [9], Theorem III.6).*

Proof of the theorem. Transform the problem (1)–(3) into a fixed point problem. Let the multivalued operator $N : \Omega \to P(\Omega)$ defined as in Theorem 3.3. We shall show that N satisfies the assumptions of Lemma 3.6. The proof will be given in two steps.

Step 1: $N(y) \in P_{cl}(\Omega)$ for each $y \in \Omega$.

Indeed, let $(y_n)_{n\geq 0} \in N(y)$ such that $y_n \longrightarrow \tilde{y}$ in Ω. Then $\tilde{y} \in \Omega$ and there exists $g_n \in S_{F(y)}$ such that for each $t \in J$

$$y_n(t) = T(t)y_0 + \int_0^t T(t-s)By(s)ds + \int_0^t T(t-s)g_n(s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)).$$

Using the fact that F has compact values and from (H7), we may pass to a subsequence if necessary to get that g_n converges to g in $L^1(J, E)$ and hence $g \in S_{F(y)}$. Then for each $t \in J$

$$y_n(t) \longrightarrow \tilde{y}(t) = T(t)y_0 + \int_0^t T(t-s)By(s)ds + \int_0^t T(t-s)g(s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)).$$

So $\tilde{y} \in N(y)$.

Step 2: There exists $\gamma < 1$, such that

$$H_d(N(y), N(\overline{y})) \leq \gamma \|y - \overline{y}\|_\Omega \quad \text{for each} \quad y, \overline{y} \in \Omega.$$

Let $y, \overline{y} \in \Omega$ and $h \in N(y)$. Then there exists $g(t) \in F(t, y(t))$ such that for each $t \in J$

$$h(t) = T(t)y_0 + \int_0^t T(t-s)By(s)ds + \int_0^t T(t-s)g(s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)).$$

From (H7) it follows that

$$H_d(F(t, y(t)), F(t, \overline{y}(t))) \leq l(t)|y(t) - \overline{y}(t)|.$$

Hence there is $w \in F(t, \overline{y}(t))$ such that

$$|g(t) - w| \leq l(t)|y(t) - \overline{y}(t)|, \quad t \in J.$$

Consider $U : J \to P(E)$, given by

$$U(t) = \{w \in E : |g(t) - w| \leq l(t)|y(t) - \overline{y}(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}(t))$ is measurable (see Proposition III.4 in [9]), there exists a function $\overline{g}(t)$, which is a measurable selection for V. So, $\overline{g}(t) \in F(t, \overline{y}(t))$ and

$$|g(t) - \overline{g}(t)| \leq l(t)|y(t) - \overline{y}(t)|, \quad \text{for each} \quad t \in J.$$

Let us define for each $t \in J$

$$\overline{h}(t) = T(t)y_0 + \int_0^t T(t-s)By(s)ds + \int_0^t T(t-s)\overline{g}(s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(\overline{y}(t_k^-)).$$

We define on Ω an equivalent norm to $\|\cdot\|_\Omega$ by

$$\|y\|_1 = \sup_{t \in J}\{e^{-\tau L(t)}|y(t)|\} \quad \text{for all } y \in \Omega,$$

where $L(t) = \displaystyle\int_0^t \widehat{M}(s)ds$, $\tau \in \mathbb{R}^+$ and $\widehat{M}(t) = \max(M\|B\|_{B(E)}, Ml(t))$.

Then

$$\begin{aligned}
|h(t) - \overline{h}(t)| &\leq \int_0^t \widehat{M}(s)|y(s) - \overline{y}(s)|ds + \int_0^t \widehat{M}(s)|y(s) - \overline{y}(s)|\,ds \\
&+ M\sum_{k=1}^m c_k'|y(s) - \overline{y}(s)| \\
&\leq 2\int_0^t \widehat{M}(s)e^{-\tau L(s)}e^{\tau L(s)}|y(s) - \overline{y}(s)|ds \\
&+ M\sum_{k=1}^m c_k' e^{-\tau L(s)}e^{\tau L(s)}|y(s) - \overline{y}(s)| \\
&\leq 2\int_0^t (e^{\tau L(s)})'ds\|y - \overline{y}\|_1 + M\sum_{k=1}^m c_k' e^{\tau L(s)}\|y - \overline{y}\|_1 \\
&\leq \frac{2}{\tau}\|y - \overline{y}\|_1 e^{\tau L(t)} + M\sum_{k=1}^m c_k'\|y - \overline{y}\|_1 e^{\tau L(t)}.
\end{aligned}$$

Then

$$\|h - \overline{h}\|_1 \leq \left(\frac{2}{\tau} + M\sum_{k=1}^m c_k'\right)\|y - \overline{y}\|_1.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y), N(\overline{y})) \leq \left(\frac{2}{\tau} + M\sum_{k=1}^m c_k'\right)\|y - \overline{y}\|_1.$$

So, N is a contraction and thus, by Lemma 3.6, N has a fixed point y, which is a mild solution to (1)–(3). ■

4 Second Order Impulsive Differential Inclusions

In this section we study the problem (4)–(7) when the right hand side has convex and nonconvex values. We give first the definition of mild solution of the problem (4)–(7)

Definition 4.1 *A function $y \in \Omega$ is said to be a mild solution of (4)–(7) if there exists $v \in L^1(J, \mathbb{R}^n)$ such that $v(t) \in F(t, y(t))$ a.e. on J, $y(0) = y_0$, $y'(0) = y_1$ and*

$$\begin{aligned} y(t) &= (C(t) - S(t)B)y_0 + S(t)y_1 + \int_0^t C(t-s)By(s)ds \\ &+ \int_0^t S(t-s)v(s)ds + \sum_{0<t_k<t} [C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\overline{I}_k(y(t_k^-))]. \end{aligned}$$

Theorem 4.2 *Assume (H1)–(H2) and the conditions:*

(A1) There exists constants $\overline{d}_k$, such that $|\overline{I}_k(y)| \leq d_k$ for each $y \in E$, $k = 1, \dots, m$;

(A2) $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in J\}$ which is compact for $t > 0$, and there exists a constant $M_1 > 0$ such that $\|C(t)\|_{B(E)} < M_1$ for all $t \in \mathbb{R}$;

(A3) there exists a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, y)\| \leq p(t)\psi(|y|) \text{ for a.e. } t \in J \text{ and each } y \in E$$

with

$$\int_0^b \widehat{m}(s)ds < \int_{\tilde{c}}^{\infty} \frac{d\tau}{\tau + \psi(\tau)},$$

where

$$\tilde{c} = M_1(1+b)|y_0| + bM_1|y_1| + M_1 \sum_{k=1}^{m} [c_k + bd_k] \text{ and } \widehat{m}(t) = \max(M_1\|B\|, bM_1 p(t))$$

are satisfied. Then the IVP (4)-(7) has at least one mild solution.

Proof. Transform the problem (4)–(7) into a fixed point problem. Consider the multivalued operator $\overline{N} : \Omega \to P(\Omega)$ defined by:

$$\begin{aligned} \overline{N}(y) = \Big\{ h \in \Omega : h(t) &= (C(t) - S(t))y_0 + S(t)y_1 + \int_0^t C(t-s)By(s)ds \\ &+ \int_0^t S(t-s)v(s)ds + \sum_{0<t_k<t} [C(t-t_k)I_k(y(t_k^-)) \\ &+ S(t-t_k)\overline{I}_k(y(t_k^-))], \ v \in S_{F(y)} \Big\}. \end{aligned}$$

As in Theorem 3.3 we shall show that $\overline{N}$ satisfies the assumptions of Lemma 2.3. Let

$$K_1 := \{y \in \Omega : \|y\|_\Omega \leq b(t),\ t \in J\},$$

where

$$b(t) = I^{-1}\left(\int_0^t \widehat{m}(s)ds\right),$$

and

$$I(z) = \int_{\tilde{c}}^z \frac{du}{u + \psi(u)}.$$

It is clear that K is a closed bounded convex set.

Step 1: $\overline{N}(K_1) \subset K_1$.

Indeed, let $y \in K_1$ and fix $t \in J$. We must show that $\overline{N}(y) \subset K_1$. Let $h \in \overline{N}(y)$. Thus there exists $v \in S_{F(y)}$ such that for each $t \in J$

$$\begin{aligned} h(t) &= (C(t) - S(t))y_0 + S(t)y_1 + \int_0^t C(t-s)By(s)ds \\ &+ \int_0^t S(t-s)v(s)ds + \sum_{0<t_k<t} [C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\overline{I}_k(y(t_k^-))]. \end{aligned}$$

This implies by (H2) and (A1)-(A2) that for each $t \in J$ we have

$$\begin{aligned} |h(t)| &\leq (M_1 + bM_1)|y_0| + bM_1|y_1| + M_1\int_0^t |By(s)|ds + \int_0^t M_1 bp(s)\psi(|y(s)|)ds \\ &+ M_1\sum_{k=1}^m [c_k + bd_k] \\ &\leq (M_1 + bM_1)|y_0| + bM_1|y_1| + M_1\|B\|_{B(E)}\int_0^t |y(s)|ds \\ &+ M_1 b\int_0^t p(s)\psi(|y(s)|)ds + M_1\sum_{k=1}^m [c_k + bd_k] \\ &\leq (M_1 + bM_1)|y_0| + bM_1|y_1| + \int_0^t \widehat{m}(s)(b(s) + \psi(b(s))ds \\ &+ M_1\sum_{k=1}^m [c_k + bd_k] \\ &= (M_1 + bM_1)|y_0| + bM_1|y_1| + M_1\sum_{k=1}^m [c_k + bd_k] + \int_0^t b'(s)ds \\ &= b(t), \end{aligned}$$

since

$$\int_{\tilde{c}}^{b(s)} \frac{du}{u+\psi(u)} = \int_0^s \widehat{m}(\tau)d\tau.$$

Thus $\overline{N}(y) \subset K_1$; So, $\overline{N} : K_1 \to K_1$.

As in Theorem 3.3 we can show that $\overline{N}(K_1)$ is relatively compact and hence by Lemma 2.3 the operator $\overline{N}$ has a least one fixed point which is a mild solution to problem (4)–(7). ■

In this last part we consider problem (4)–(7) with a nonconvex valued right-hand side.

Theorem 4.3 *Suppose that hypotheses (H5)–(H7), (A2) and*

(A4) There exists constants $\overline{c}_k$, *such that*

$$|\overline{I}_k(y) - \overline{I}_k(\overline{y})| \leq d_k'|y-\overline{y}|, \textit{for each } k=1,\dots,m, \textit{ and for all } y,\overline{y} \in E$$

are satisfied. If

$$\frac{2}{\tau} + M_1 \sum_{k=1}^{m} [c_k' + bd_k'] < 1,$$

then the IVP (4)–(7) has at least one mild solution.

Proof. Transform the problem (4)–(7) into a fixed point problem. Consider the multivaled map $\overline{N} : \Omega \to P(\Omega)$ where $\overline{N}$ is defined as in the theorem 4.2. As in the proof of theorem 3.7 we can show that $\overline{N}$ is a closed values. Here we repeat the proof that $\overline{N}$ is a contraction i.e. there exists $\gamma < 1$, such that

$$H_d(\overline{N}(y), \overline{N}(\overline{y})) \leq \gamma \|y-\overline{y}\|_\Omega \text{ for each } y,\overline{y} \in \Omega.$$

Let $y,\overline{y} \in \Omega$ and $h \in \overline{N}(y)$. Then there exists $g(t) \in F(t,y(t))$ such that for each $t \in J$

$$\begin{aligned} h(t) &= (C(t)-S(t))y_0 + S(t)y_1 + \int_0^t C(t-s)By(s)ds + \int_0^t S(t-s)g(s)ds \\ &+ \sum_{0<t_k<t} [C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\overline{I}_k(y(t_k^-))]. \end{aligned}$$

From (H7) it follows that

$$H_d(F(t,y(t)), F(t,\overline{y}(t))) \leq l(t)|y(t)-\overline{y}(t)|.$$

Hence there is $w \in F(t,\overline{y}(t))$ such that

$$|g(t) - w| \leq l(t)|y(t)-\overline{y}(t)|, \quad t \in J.$$

Consider $U : J \to P(E)$, given by

$$U(t) = \{w \in E : |g(t) - w| \le l(t)|y(t) - \overline{y}(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}(t))$ is measurable (see Proposition III.4 in [9]), there exists a function $\overline{g}(t)$, which is a measurable selection for V. So, $\overline{g}(t) \in F(t, \overline{y}(t))$ and

$$|g(t) - \overline{g}(t)| \le l(t)|y(t) - \overline{y}(t)|, \quad \text{for each} \quad t \in J.$$

Let us define for each $t \in J$

$$\begin{aligned} \overline{h}(t) &= (C(t) - S(t))y_0 + S(t)y_1 + \int_0^t C(t-s)B\overline{y}(s)ds + \int_0^t S(t-s)\overline{g}(s)ds \\ &+ \sum_{0<t_k<t} [C(t-t_k)I_k(\overline{y}(t_k^-)) + S(t-t_k)\overline{I}_k(y(t_k^-))]. \end{aligned}$$

We define on Ω an equivalent norm by

$$\|y\|_2 = \sup_{t \in J} e^{-\tau \widetilde{L}(t)}|y(t)| \quad \text{for all } y \in \Omega,$$

where $\widetilde{L}(t) = \displaystyle\int_0^t \widetilde{M}(s)ds, \tau \in \mathbb{R}^+$ and $\widetilde{M}(t) = \max(M_1\|B\|_{B(E)}, M_1 b l(t))$. Then we have

$$\begin{aligned} |h(t) - \overline{h}(t)| &\le \int_0^t M_1|By(s) - B\overline{y}(s)|ds + \int_0^t M_1 b|g(s) - \overline{g}(s)|\,ds \\ &+ M_1 \sum_{k=1}^m |I_k(y(t_k^-)) - I_k(\overline{y}(t_k^-))| \\ &+ M_1 b \sum_{k=1}^m |\overline{I}_k(y(t_k^-)) - \overline{I}_k(\overline{y}(t_k^-))| \\ &\le \int_0^t M_1\|B\|_{B(E)}|y(s) - \overline{y}(s)|ds + \int_0^t M_1 b l(s)|y(s) - \overline{y}(s)|ds \\ &+ M_1 \sum_{k=1}^m c_k'|y(t_k) - \overline{y}(t_k))| + M_1 b \sum_{k=1}^m d_k'|y(t_k) - \overline{y}(t_k)| \\ &\le 2\int_0^t \widetilde{M}(s)e^{\tau \widetilde{L}(t)}e^{-\tau \widetilde{L}(t)}|y(s) - \overline{y}(s)|ds \\ &+ M_1 e^{\tau \widetilde{L}(t)} \sum_{k=1}^m [c_k' + b d_k']\|y - \overline{y}\|_2 \\ &\le \frac{2}{\tau}e^{\tau \widetilde{L}(t)}\|y - \overline{y}\|_2 + M_1 e^{\tau \widetilde{L}(t)} \sum_{k=1}^m [c_k' + b d_k']\|y - \overline{y}\|_2. \end{aligned}$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(\overline{N}(y), \overline{N}(\overline{y})) \leq \left(\frac{2}{\tau} + M_1 \sum_{k=1}^{m}[c'_k + bd'_k]\right) \|y - \overline{y}\|_2.$$

So, $\overline{N}$ is a contraction and thus, by Lemma 3.6, $\overline{N}$ has a fixed point y, which is a mild solution to (4)–(7). ■

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