Coexistence for a Resource-Based Growth Model with Two Resources [1]

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Abstract

We investigate the coexistence of positive steady-state solutions to a parabolic system, which models a single species on two growth-limiting, non-reproducing resources in an un-stirred chemostat with diffusion. We establish the existence of a positive steady-state solution for a range of the parameter (m, n), the bifurcation solutions and the stability of bifurcation solutions. The proof depends on the maximum principle, bifurcation theorem and perturbation theorem.

Keywords: chemostat; coexistence; local bifurcation; maximum principle.

1 Introduction

Consider the following parabolic system

$$\begin{aligned} &S_t = d_1 S_{xx} - muf(S,R), && 0 < x < 1, t > 0,\\ &R_t = d_2 R_{xx} - nug(S,R), && \\ &u_t = d_3 u_{xx} + u(mf(S,R) + cng(S,R)), && \end{aligned} \tag{1.1}$$

with the boundary conditions

$$\begin{aligned} &S_x(0,t) = -1, \quad R_x(0,t) = -1, \quad u_x(0,t) = 0,\\ &S_x(1,t) + \gamma S(1,t) = 0, \quad R_x(1,t) + \gamma R(1,t) = 0, \quad u_x(1,t) + \gamma u(1,t) = 0, \end{aligned} \tag{1.2}$$

and initial conditions

$$S(x,0) = S_0(x) \geq 0, \quad R(x,0) = R_0(x) \geq 0, \quad u(x,0) = u_0(x) \geq 0, \not\equiv 0, \tag{1.3}$$

where $f(S,R) = S/(1 + aS + bR)$, $g(S,R) = R/(1 + aS + bR)$, $m > 0$ is the maximal growth rate of species u on resource S in the absence of resource R, the constant n is defined similarly, constant c denotes the ratio of the growth yield constant of S and R. The constant $a > 0$ and $b > 0$ are the Michaelis-Menten constants, $\gamma > 0$.

Since we are only concerned with the nonnegative solutions (S, R, u) of (1.1), we can redefine the response functions f, g for $S \leq 0, R \leq 0$ without affecting our results. The un-stirred chemostat with one resource has been considered by many authors in the past decade(see [1][2][3]). Just as pointed out in [4], the un-stirred chemostat with two resources is more realistic and thus of interest, and the system (1.1) with equal diffusion rates is investigated in paper [5]. Without the assumption of equal diffusion rates, we

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obtain some estimates on the size of the coexistence region near a bifurcation point in the parameter space. The existence of positive steady-state solution of the system (1.1) is established by the maximum principle and the theorem of bifurcation, which appears in [6] to study the local solutions. The stability of bifurcation solutions is also studied via the perturbation theorem.

2 Extinction

In this section we use the maximum principle to establish conditions under which the species become extinct.

Lemma 2.1. The region $(S \geq 0, R \geq 0, u \geq 0)$ is invariant.

Proof. Consider the nutrient equation

$$\begin{aligned} &S_t = d_1 S_{xx} - muf(S, R), && 0 < x < 1, t > 0, \\ &S_x(0,t) = -1, \quad S_x(1,t) + \gamma S(1,t) = 0, \quad S(x,0) = S_0(x) \geq 0. \end{aligned}$$

For the fixed u, $\bar{S}(x,t) \equiv 0$ is a solution of the differential equation above, $S(x,0) \geq \bar{S}(x,0)$, and $-S_x(0,t) = 1 \geq 0 = \bar{S}_x(0)$. By the comparison theorem for the parabolic equation (for example see [7]), we can show that $S(x,t) \geq 0$ for all (x,t). Moreover, the boundary condition $S \not\equiv 0$ implies that $S(x,t) > 0$ for $t > 0$. Similarly, we can prove that $R(x,t) > 0$ and $u(x,t) > 0$ for all $t > 0$, and thus the proof is completed.

Let $\lambda_0^{(i)} > 0 (i = 1,2,3)$ be the principle eigenvalue of the following problem

$$\begin{aligned} &d_i \phi_{xx} + \lambda \phi = 0, && 0 < x < 1, \\ &\phi_x(0) = 0, \quad \phi_x(1) + \gamma \phi(1) = 0. \end{aligned}$$

with the eigenfunction $\phi_0^{(i)} > 0 (i = 1,2,3)$ on $[0,1]$.

Let $\bar{S}(x)$ be the unique positive solution of the following problem

$$\begin{aligned} &S_{xx} = 0, && 0 < x < 1, \\ &S_x(0) = -1, \quad S_x(1) + \gamma S(1) = 0. \end{aligned}$$

The existence and uniqueness of $\bar{S}(x)$ is standard, and by the maximum principle it is easy to show that $\bar{S} > 0$ on $[0,1]$.

Lemma 2.2. There are positive constants α_i and $K_i (i = 1,2)$ such that $S(x,t) \leq \bar{S}(x) + K_1 e^{-\alpha_1 t}$, $R(x,t) \leq \bar{S}(x) + K_2 e^{-\alpha_2 t}$, for all $x \in [0,1], t > 0$.

Proof. Let $\omega(x,t) = S(x,t) - \bar{S}(x)$, then ω satisfies

$$\begin{aligned} &\omega_t \leq d_1 \omega_{xx}, && 0 < x < 1, t > 0, \\ &\omega_x(0,t) = 0, \quad \omega_x(1,t) + \gamma \omega(1,t) = 0, && t > 0. \end{aligned}$$

Then, by the comparison theorem, we have $\omega(x,t) \leq W(x,t)$, where $W(x,t)$ is the unique solution of the linear problem

$$\begin{aligned} &W_t = d_1 W_{xx}, && 0 < x < 1, t > 0, \\ &W_x(0,t) = 0, \quad W_x(1,t) + \gamma W(1,t) = 0, \\ &W(x,0) = S(x,0) - \bar{S}(x). \end{aligned}$$

In order to estimate W, let $0 < \alpha_1 < \lambda_0^{(1)}$, $W(x,t) = \phi_0^{(1)}(x)h(x,t)e^{-\alpha_1 t}$. Then we have

$$\begin{aligned}
&h_t = d_1 h_{xx} + \frac{2d_1}{\phi_0^{(1)}}\phi_{0x}^{(1)} h_x + (\alpha_1 - \lambda_0^{(1)})h, \quad 0 < x < 1, t > 0,\\
&h_x(0,t) = 0, \quad h_x(1,t) = 0,\\
&h(x,0) = \frac{S(x,0) - \bar{S}(x)}{\phi_0^{(1)}(x)}.
\end{aligned}$$

The maximum principle ([7]) implies that

$$|h(x,t)| \le \max_{[0,1]} \frac{|S(x,0) - \bar{S}(x)|}{\phi_0^{(1)}(x)}$$

and this leads to

$$S(x,t) \le \bar{S}(x) + K_1 e^{-\alpha_1 t}, \quad x \in [0,1], \quad t > 0,$$

for some constants $K_1 > 0$. Similarly result holds for R.

Lemma 2.3. Let (S,R,u) be a solution of system (1.1)-(1.3), and suppose that $\frac{m+cn}{min(a,b)} < \lambda_0^{(3)}$. Then there are positive constants K, α such that $u(x,t) \le Ke^{-\alpha t}$.

Proof. Straightforward computation leads to

$$u_t = d_3 u_{xx} + u(mf(S,R) + cng(S,R)) \le d_3 u_{xx} + \frac{m+cn}{min(a,b)}u, \quad 0 < x < 1, t > 0.$$

Let $V(x,t)$ be the unique solution of the following problem

$$\begin{aligned}
&V_t = d_3 V_{xx} + \frac{m+cn}{min(a,b)}V, && 0 < x < 1, t > 0,\\
&V_x(0,t) = 0, \quad V_x(1,t) + \gamma V(1,t) = 0,\\
&V(x,0) = u(x,0).
\end{aligned}$$

By the comparison principle, we have $u(x,t) \le V(x,t)$. Let $V(x,t) = \phi_0^{(3)}(x)h(x,t)e^{-\alpha t}$, where $\alpha > 0$ is small enough so that $\alpha + \frac{m+cn}{min(a,b)} - \lambda_0^{(3)} < 0$. then

$$\begin{aligned}
&h_t = d_3 h_{xx} + \frac{2d_3}{\phi_0^{(3)}}\phi_{0x}^{(3)} h_x + (\alpha + \frac{m+cn}{min(a,b)} - \lambda_0^{(3)})h, \quad 0 < x < 1, t > 0,\\
&h_x(0,t) = 0, \quad h_x(1,t) = 0,\\
&h(x,0) = \frac{u(x,0)}{\phi_0^{(3)}(x)}.
\end{aligned}$$

As in the previous lemma, it follows that $|h(x,t)| \le \max_{[0,1]} \frac{|u(x,0)|}{\phi_0^{(3)}(x)}$ and the lemma follows.

3 Coexistence.

In this section we consider the coexistence of the positive steady-state solutions of the system (1.1). So we consider the elliptic system

$$\begin{aligned}
&d_1 S_{xx} - muf(S,R) = 0, && 0 < x < 1,\\
&d_2 R_{xx} - nug(S,R) = 0, && \\
&d_3 u_{xx} + u(mf(S,R) + cng(S,R)) = 0, &&
\end{aligned} \tag{3.1}$$

with the boundary conditions

$$\begin{aligned}
&S_x(0) = -1, \quad R_x(0) = -1, \quad u_x(0) = 0,\\
&S_x(1) + \gamma S(1) = 0, \quad R_x(1) + \gamma R(1) = 0, \quad u_x(1) + \gamma u(1) = 0.
\end{aligned} \tag{3.2}$$

Let $z = (d_1 S + cd_2 R + d_3 u)/(d_1 + cd_2)$, then z satisfies

$$\begin{aligned}
&z_{xx} = 0, && 0 < x < 1,\\
&z_x(0) = -1, z_x(1) + \gamma z(1) = 0, &&
\end{aligned} \tag{3.3}$$

and we have $z = (1+\gamma)/\gamma - x$.

First we give some estimates about the nonnegative solution of (3.1)-(3.2). The similar proof can be found in [4,8]. We omit the detail here.

Lemma 3.1. Suppose that (S, R, u) is a nonnegative solution of (3.1)-(3.2), then $S > 0$, $R > 0$, and either $0 < S < z$, $0 < R < z$ or $S = R = z$. Furthermore, $d_1 S + cd_2 R + d_3 u = (d_1 + cd_2)z$.

Let $s = z - S$, $r = z - R$, then by lemma 3.1, either $0 < s$, $r < z$ or $s = r = 0$, and

$$\begin{aligned}
&d_1 d_3 s_{xx} + m(d_1 s + cd_2 r) f(z-s, z-r) = 0, \quad 0 < x < 1,\\
&d_2 d_3 r_{xx} + n(d_1 s + cd_2 r) g(z-s, z-r) = 0, \quad 0 < x < 1,
\end{aligned} \tag{3.4}$$

with the boundary conditions

$$\begin{aligned}
&s_x(0) = 0, \quad r_x(0) = 0,\\
&s_x(1) + \gamma s(1) = 0, \quad r_x(1) + \gamma r(1) = 0.
\end{aligned} \tag{3.5}$$

3.1. The special case of $d_2 m = d_1 n$

In this subsection, we consider the case of $d_2 m = d_1 n$ and discuss the existence of a positive solution of (3.4).

Let $\omega = s - r$, then ω satisfies

$$\omega_{xx} - C(x)\omega = 0, 0 < x < 1, \quad \omega_x(0) = 0, \quad \omega_x(1) + \gamma\omega(1) = 0,$$

where $C(x) = m(d_1 s + cd_2 r)/(d_1 d_3(1 + a(z-s) + b(z-r)))$. It follows from the maximum principle that $\omega = 0$, which leads to $s = r$ on $[0, 1]$. Substituting $s = r$ into (3.4), we have

$$\begin{aligned}
&d_1 d_3 s_{xx} + m(d_1 + cd_2) s f(z-s, z-s) = 0, \quad 0 < x < 1,\\
&s_x(0) = 0, s_x(1) + \gamma s(1) = 0.
\end{aligned} \tag{3.6}$$

Let $\lambda_1 > 0$ and $\phi_1 > 0$ be the principle eigenvalue and eigenfunction of the following problem, with ϕ normalized so that $\int_0^1 f(z,z)\phi^2 dx = 1$

$$\phi_{xx} + \lambda_1 f(z,z)\phi = 0,\ \phi_x(0) = 0,\quad \phi_x(1) + \gamma\phi(1) = 0. \tag{3.7}$$

By the result in [8,9], we have

Theorem 3.1. There exists a unique positive solution $\bar{s}$ of (3.6), if and only if $m > d_1 d_3 \lambda_1/(d_1 + cd_2)$, moreover $0 < \bar{s} < z$, $\bar{s}$ is continuous with respect to $m \in [d_1 d_3 \lambda_1/(d_1 + cd_2), \infty]$, and $\lim\limits_{m \to (\frac{d_1 d_3 \lambda_1}{d_1 + cd_2})^+} \bar{s} = 0$ uniformly in $(0,1)$, $\lim\limits_{m\to\infty} \bar{s} = z$ *a.e.* $x \in (0,1)$.

Clearly, if $m > d_1 d_3 \lambda_1/(d_1 + cd_2)$, then $(\bar{S}, \bar{R}, \bar{u}) = (z - \bar{s}, z - \bar{s}, (d_1 + cd_2)\bar{s}/d_3)$ is the unique positive steady-state solution of (3.1)-(3.2) in the case $d_2 m = d_1 n$.

3.2. The case of $d_2 m \neq d_1 n$

In this subsection, we discuss the existence and nonexistence of a positive solution of (3.4)(3.5). First we give a basic estimate for (s, r).

Lemma 3.2. If $d_2 m \geq d_1 n$, then the solution (s, r) of (3.4)(3.5) satisfies $r \leq s \leq \dfrac{d_2 m}{d_1 n} r$.

Proof. Let $\omega = s - r$, then

$$\omega_{xx} - C(x)\omega \leq 0, 0 < x < 1, \omega_x(0) = 0, \omega_x(1) + \gamma\omega(1) = 0,$$

where $C(x) = n(d_1 s + cd_2 r)/(d_2 d_3(1 + a(z - s) + b(z - r))) \geq 0$. It follows from the maximum principle that $\omega \geq 0$, and thus $r \leq s$.

Again, let $\omega = d_1 ns - d_2 mr$, then

$$\begin{aligned}
\omega_{xx} &= \frac{mn(d_1 s + cd_2 r)}{d_3}(g(z - s, z - r) - f(z - s, z - r)) \\
&= \frac{mn(d_1 s + cd_2 r)}{d_3(1 + a(z - s) + b(z - r))}(s - r) \\
&\geq 0 \\
\omega_x(0) &= 0, \qquad \omega_x(1) + \gamma\omega(1) = 0,
\end{aligned}$$

it follows that $\omega \leq 0$, i.e. $s \leq \dfrac{d_2 m}{d_1 n} r$. Similarly, if $d_2 m \leq d_1 n$, then we have $s \leq r \leq \dfrac{d_1 n}{d_2 m} s$.

The following theorem shows that a positive solution of (3.4)(3.5) cannot exist if both m and n are too small.

Theorem 3.2. Suppose $m \leq d_1 d_3 \lambda_1/(d_1 + cd_2)$ and $n \leq d_2 d_3 \lambda_1/(d_1 + cd_2)$, then $(s, r) = (0, 0)$ is the unique nonnegative solution of (3.4)(3.5).

Proof. If $m \leq d_1 d_3 \lambda_1/(d_1 + cd_2)$ and $n \leq d_2 d_3 \lambda_1/(d_1 + cd_2)$, and (s, r) is a nontrivial nonnegative solution of (3.4)(3.5). Then it follows from the maximum principle that

$s > 0, r > 0$. If $\dfrac{d_1 n}{d_2} \le m \le \dfrac{d_1 d_3 \lambda_1}{d_1 + c d_2}$, multiplying the first equation in (3.4) by s, integrating over $(0,1)$ and using Green formula, we find

$$\begin{aligned} d_1 d_3 (\textstyle\int_0^1 s_x^2 dx + \gamma s^2(1)) &= m \textstyle\int_0^1 (d_1 s + c d_2 r) s f(z-s, z-r) dx \\ &\le m(d_1 + c d_2) \textstyle\int_0^1 s^2 f(z,z) dx. \end{aligned}$$

By the variational property of the principle eigenvalue, we have

$$\int_0^1 s_x^2 dx + \gamma s^2(1) \ge \lambda_1 \int_0^1 s^2 f(z,z) dx.$$

Hence $(d_1 d_3 \lambda_1 - m(d_1 + c d_2)) \int_0^1 s^2 f(z,z) dx \le 0$, which leads to $s = 0$, a contradiction. A similar result holds if $m \le d_1 n / d_2$. This completes the proof.

Thus if $m \le d_1 d_3 \lambda_1 / (d_1 + c d_2)$ and $n \le d_2 d_3 \lambda_1 / (d_1 + c d_2)$, then the washout solution $(z, z, 0)$ is the unique nontrivial nonnegative solution of (3.1)(3.2).

Theorem 3.3 Suppose $m \ge d_1 d_3 \lambda_1 / (d_1 + c d_2)$ and $n \ge d_2 d_3 \lambda_1 / (d_1 + c d_2)$. Then there exists a positive solution of (3.4)(3.5).

Proof. It is easy to check that (3.4)(3.5) is a quasi-monotone increasing system. Let $(\bar{s}, \bar{r}) = (z, z)$ and $(\underline{s}, \underline{r}) = (\delta\phi, \delta\phi)$, where ϕ is the principle eigenfunction defined by (3.7) and $\delta > 0$ is small enough. Obviously $(\bar{s}, \bar{r}) = (z, z)$ is the upper solution of (3.4)(3.5). Again

$$\begin{aligned} & d_1 \underline{s}_{xx} + m(\frac{d_1 \underline{s} + c d_2 \underline{r}}{d_3}) f(z - \underline{s}, z - \underline{r}) \\ = \;& \delta\phi((\frac{m(d_1 + c d_2)}{d_3} - d_1 \lambda_1) f(z,z) - \frac{m(d_1 + c d_2)}{d_3}(f(z,z) - f(z - \delta\phi, z - \delta\phi))) \\ \ge \;& \delta\phi((\frac{m(d_1 + c d_2)}{d_3} - d_1 \lambda_1) \frac{1}{\gamma + a + b} - \frac{m(d_1 + c d_2)\delta\phi}{d_3(1 + (a+b)(z - \theta\delta\phi))^2}) \quad (0 < \theta < 1) \end{aligned}$$

as long as δ is sufficiently small, we have

$$d_1 \underline{s}_{xx} + m(\frac{d_1 \underline{s} + c d_2 \underline{r}}{d_3}) f(z - \underline{s}, z - \underline{r}) > 0.$$

Similarly we have

$$d_2 \underline{r}_{xx} + n(\frac{d_1 \underline{s} + c d_2 \underline{r}}{d_3}) g(z - \underline{s}, z - \underline{r}) > 0.$$

Thus, for sufficiently small $\delta > 0$, the pair $(\bar{s}, \bar{r})$ and $(\underline{s}, \underline{r})$ are the ordered upper and lower solutions of (3.4)(3.5). From [7], there exists a solution (s, r) satisfies $(\delta\phi, \delta\phi) \le (s, r) \le (z, z)$.

Theorem 3.4. Suppose that either $m > d_3 \lambda_1$, $n \le d_2 d_3 \lambda_1 / (d_1 + c d_2)$ or $m \le d_1 d_3 \lambda_1 / (d_1 + c d_2)$, $n > d_3 \lambda_1 / c$. Then there exists a positive solution of (3.4)(3.5).

Proof. We consider the former case, the other case can be done similarly. Let $(\bar{s},\bar{r}) = (z,z)$ and $(\underline{s},\underline{r}) = (\delta\phi, 0)$, then

$$\begin{aligned} & d_1\underline{s}_{xx} + m(\frac{d_1\underline{s}+cd_2\underline{r}}{d_3})f(z-\underline{s}, z-\underline{r}) \\ & = \delta\phi((\frac{md_1}{d_3} - d_1\lambda_1)f(z,z) - \frac{md_1}{d_3}(f(z,z) - f(z-\delta\phi, z))). \end{aligned}$$

For sufficiently small $\delta > 0$, we note that $(\bar{s},\bar{r})$ and $(\underline{s},\underline{r})$ are the ordered upper and lower solution of (3.4)(3.5). Hence there exists solution (s,r) of (3.4)(3.5) such that $(\delta\phi, 0) \le (s,r) \le (z,z)$. So $s > 0$. It follows that $r > 0$ from lemma 3.2. This completes the proof.

3.3. Bifurcation Theorem

Now, for fixed $n \le d_2d_3\lambda_1/(d_1+cd_2)$, we treat m as a bifurcation parameter to obtain the local bifurcation which corresponds to the positive solution of (3.4)(3.5).

At first, we rewrite (3.4)(3.5) as

$$\begin{aligned} & s_{xx} + m(\frac{d_1s+cd_2r}{d_1d_3})f(z,z) + F_1(s,r) = 0, \quad 0<x<1, \\ & r_{xx} + n(\frac{d_1s+cd_2r}{d_2d_3})g(z,z) + F_2(s,r) = 0, \quad 0<x<1, \end{aligned} \tag{3.8}$$

with the same boundary conditions, where

$$\begin{aligned} F_1(s,r) &= m(\frac{d_1s+cd_2r}{d_1d_3})(f(z-s,z-r) - f(z,z)), \\ F_2(s,r) &= n(\frac{d_1s+cd_2r}{d_2d_3})(g(z-s,z-r) - g(z,z)). \end{aligned}$$

Let K be the inverse operator of $-\frac{d^2}{dx^2}$, then

$$\begin{aligned} & s - mK((\frac{d_1s+cd_2r}{d_1d_3})f(z,z)) - KF_1(s,r) = 0, \quad 0<x<1, \\ & r - nK((\frac{d_1s+cd_2r}{d_2d_3})g(z,z)) - KF_2(s,r) = 0, \quad 0<x<1, \end{aligned}$$

Let $T(m,s,r) = (mK((\frac{d_1s+cd_2r}{d_1d_3})f(z,z)) + KF_1(s,r), nK((\frac{d_1s+cd_2r}{d_2d_3})g(z,z)) + KF_2(s,r))$, and $G(m,s,r) = (s,r) - T(m,s,r)$. Then the zeros of $G(m,s,r)$ are the solutions of (3.4)(3.5).

Let $C_B^1[0,1] = \{u \in C^1[0,1] : u_x(0) = 0, u_x(1) + \gamma u(1) = 0\}$, endowed with the usual norm $\|\cdot\|$, and $X = C_B^1[0,1] \times C_B^1[0,1]$. Then we have the following theorem

Theorem 3.5. Suppose $n \le \dfrac{d_2d_3\lambda_1}{d_1+cd_2}$. Then $(m_0,0,0)$ is a bifurcation point of $G(m,s,r) = 0$, and in the neighborhood of $(m_0,0,0)$, part of the bifurcation branch corresponds to the positive solution of (3.4)(3.5), where $m_0 = d_3\lambda_1 - cn$.

Proof. Let $L(m,0,0) = DG_{(s,r)}(m,0,0)$ is the Frechet derivative of $G(m,s,r)$ with respect to (s,r) at (0,0). Straightforward computation gives

$$L(m_0,0,0)(\omega,\chi) = (\omega - m_0K((\frac{d_1\omega + cd_2\chi}{d_1d_3})f(z,z)), \chi - nK((\frac{d_1\omega + cd_2\chi}{d_2d_3})g(z,z))).$$

Then $L(m_0,0,0)(\omega,\chi) = 0$ leads to

$$\begin{aligned}
&\omega_{xx} + m_0(\frac{d_1\omega + cd_2\chi}{d_1d_3})f(z,z) = 0, && 0 < x < 1,\\
&\chi_{xx} + n(\frac{d_1\omega + cd_2\chi}{d_2d_3})g(z,z) = 0, && 0 < x < 1,\\
&\omega_x(0) = 0, \quad \chi_x(0) = 0, &&\\
&\omega_x(1) + \gamma\omega(1) = 0, \quad \chi_x(1) + \gamma\chi(1) = 0. &&
\end{aligned} \tag{3.9}$$

Noting that $m_0 = d_3\lambda_1 - cn$ and $f(z,z) = g(z,z)$, we have

$$(d_1\omega + cd_2\chi)_{xx} + \lambda_1(d_1\omega + cd_2\chi)f(z,z) = 0,$$

so $d_1\omega + cd_2\chi = \phi$, and putting this into (3.9), we find

$$\omega_{xx} + \frac{m_0}{d_1d_3}f(z,z)\phi = 0, \quad \chi_{xx} + \frac{n}{d_2d_3}g(z,z)\phi = 0.$$

It is easy to show that there exists a unique positive solution (ω_1, χ_1) of the above problem. Moreover $\omega_1 \geq \chi_1$ and $d_1\omega_1 + cd_2\chi_1 = \phi$. Hence the null space of $L(m_0,0,0)$, $N(L(m_0,0,0)) = spans\{(\omega_1,\chi_1)\}$. This, $dimN(L(m_0,0,0)) = 1$. Let $R(L(m_0,0,0))$ be the range of the operator $L(m_0,0,0)$. If $(h_1,h_2) \in R(L(m_0,0,0))$, then there exists $(\Phi,\Psi) \in X$ satisfies

$$\begin{aligned}
&\Phi_{xx} + m_0(\frac{d_1\Phi + cd_2\Psi}{d_1d_3})f(z,z) = h_{1xx}, && 0 < x < 1,\\
&\Psi_{xx} + n(\frac{d_1\Phi + cd_2\Psi}{d_2d_3})g(z,z) = h_{2xx}, && 0 < x < 1,\\
&\Phi_x(0) = 0, \quad \Psi_x(0) = 0, &&\\
&\Phi_x(1) + \gamma\Phi(1) = 0, \quad \Psi_x(1) + \gamma\Psi(1) = 0. &&
\end{aligned}$$

Thus, we find

$$(d_1\Phi + cd_2\Psi)_{xx} + \lambda_1(d_1\Phi + cd_2\Psi)f(z,z) = (d_1h_1 + cd_2h_2)_{xx},$$

multiplying the above equation by ϕ, and integrating over $(0,1)$, shows

$$-\int_0^1 \lambda_1\phi(d_1h_1 + cd_2h_2)dx = 0,$$

which implies $R(L(m_0,0,0)) = \{(h_1,h_2) \in X : \int_0^1 \phi(d_1h_1 + cd_2h_2)dx = 0\}$ and $codimR(L$ $(m_0,0,0)) = 1$.

Now Let $L_1(m_0,0,0) = D^2G_{m(s,r)}(m_0,0,0)$, then

$$L_1(m_0,0,0)(m_0,0,0)(\omega_1,\chi_1) = (-K(\frac{d_1\omega_1 + cd_2\chi_1}{d_1d_3})f(z,z),0).$$

It is easy to see that $L_1(m_0,0,0)(m_0,0,0)(\omega_1,\chi_1) \notin R(L(m_0,0,0))$. According to Theorem1.7 in [10], there exists a $\delta > 0$ and a C^1 function $(m(\tau),\omega(\tau),\chi(\tau)) : (-\tau,\tau) \to R\times X$, such that $m(0) = m_0$, $\omega(0) = 0$, $\chi(0) = 0$ and $(m(\tau),s(\tau),r(\tau)) = (m(\tau),\tau(\omega_1 + \omega(\tau)),\tau(\chi_1 + \chi(\tau)))(|\tau| < \delta)$ satisfies $G(m(\tau),s(\tau),r(\tau)) = 0$. Point on the curve $\{(m(\tau),z - \tau(\omega_1 + \omega(\tau)),z - \tau(\chi_1 + \chi(\tau))) : |\tau| < \delta\}$ with $\tau > 0$ corresponds to the positive solutions of (3.1)(3.2).

3.4. Stability of the Bifurcation Solution

In this section we shall determine the stability of the bifurcation solutions.

Lemma 3.3. 0 is a $i - simple$ eigenvalue of $L(m_0,0,0)$.

Proof. Suppose $L(m_0,0,0) = 0$. From the proof of Theorem 3.5, we have $N(L(m_0,0,0))= spans\{(\omega_1,\chi_1)\}$, $codimR(L(m_0,0,0)) = dimN(L(m_0,0,0)) = 1$. We say $i(\omega_1,\chi_1) \notin R(L(m_0,0,0))$, otherwise

$$\int_0^1 f(z,z)\phi(d_1\omega_1 + cd_2\chi_1)dx = 0,$$

which is impossible. Thus we complete the proof of the lemma.

Let $L(m(\tau),s(\tau),r(\tau))$ be the linearized operator of (5.1) at $(m(\tau),s(\tau),r(\tau))$. Then the corollary 1.13 and Theorem 1.16 in[11] can be applied and we have the following lemma.

Lemma 3.4. There exist C^1 function $m \to (\xi(m),U(m))$, $\tau \to (\eta(\tau),V(\tau))$ defined on the neighborhoods of m_0 and 0, respectively, into $R \times X$, such that $(\xi(m_0),U(m_0)) = (0,(\omega_1,\chi_1)) = (\eta(0),V(0))$ and on these neighborhoods

$$\begin{array}{ll} L(m,0,0)U(m) = \xi(m)U(m), & |m - m_0| << 1, \\ L(m(\tau),s(\tau),r(\tau))V(\tau) = \eta(\tau)v(\tau), & |\tau| << 1 \end{array} \tag{3.10}$$

where $U(m) = (u_1(m),u_2(m))$, $V(\tau) = (v_1(\tau),v_2(\tau))$, and $\xi'(m_0) \neq 0$, $\eta(\tau)$ and $-\tau m'(\tau)\xi'(m_0)$ have the same sign if $\eta(\tau) \neq 0$.

Theorem 3.6 The differential $\xi'(m_0) > 0$.

Proof. By (3.10) we have

$$\begin{array}{l} u_{1xx} + m(\dfrac{d_1u_1 + cd_2u_2}{d_1d_3})f(z,z) = \xi(m)u_1, \quad 0 < x < 1, \\ u_{2xx} + n(\dfrac{d_1u_1 + cd_2u_2}{d_2d_3})g(z,z) = \xi(m)u_2, \\ u_{1x}(0) = 0, \quad u_{2x}(0) = 0, \\ u_{1x}(1) + \gamma u_1(1) = 0, \quad u_{2x}(1) + \gamma u_2(1) = 0. \end{array} \tag{3.11}$$

Clearly,$u_1 \not\equiv 0$, $u_2 \not\equiv 0$. Since $U(m)$ is continuous and $U(m_0) = (\omega_1, \chi_1)$, $u_1(m) > 0$, $u_2(m) > 0$ for $|m - m_0| << 1$. By (3.11) we have

$$(d_1u_1 + cd_2u_2)_{xx} + \frac{m + cn}{d_3}(d_1u_1 + cd_2u_2)f(z, z) = \xi(m)(d_1u_1 + cd_2u_2).$$

Since $d_1u_1 + cd_2u_2 > 0$, it follows that $\xi(m)$ is the principle eigenvalue of $L_2 = \dfrac{d^2}{dx^2} + \dfrac{m + cn}{d_3}f(z, z)$, and increases in m for $|m-m_0| << 1$. Again $\xi^{'}(m_0) \neq 0$, so we must have $\xi^{'}(m_0) > 0$.

Theorem 3.7. The differential of $m(\tau)$ at $\tau = 0$ satisfies

$$m^{'}(0)\int_0^1 \phi^2 f(z, z)dx = \int_0^1 \phi^2 \frac{m_0\omega_1 + cn\chi_1 + (bm_0 - acn)(\omega_1 - \chi)z}{(1 + (a + b)z)^2}dx.$$

Proof. Substitute $(m(\tau), s(\tau), r(\tau))$ into the equation of (3.4), divide by τ, differential with respect to τ, and set $\tau = 0$, this gives

$$\begin{aligned}
&d_1\omega^{'}(0)_{xx} + m^{'}(0)(\frac{d_1\omega_1 + cd_2\chi_1}{d_3})f(z, z) + m_0(\frac{d_1\omega^{'}(0) + cd_2\chi^{'}(0)}{d_3})f(z, z)\\
&+m_0(\frac{d_1\omega_1 + cd_2\chi_1}{d_3})\frac{-\omega_1 + b(\chi_1 - \omega_1)z}{(1 + (a + b)z)^2} = 0\\
&d_2\chi^{'}(0)_{xx} + n(\frac{d_1\omega^{'}(0) + cd_2\chi^{'}(0)}{d_3})f(z, z) + n(\frac{d_1\omega_1 + cd_2\chi_1}{d_3})\frac{-\chi_1 + a(\omega_1 - \chi_1)z}{(1 + (a + b)z)^2} = 0.
\end{aligned}$$

Now, multiplying the first equation by ϕ, adding to the second equation which is multiplied by $c\phi$, integrating over $(0, 1)$ to get

$$\begin{aligned}
&m^{'}(0)\int_0^1 \frac{d_1\omega_1 + cd_2\chi_1}{d_3}\phi f(z, z)dx\\
= &\int_0^1 \frac{d_1\omega_1 + cn\chi_1}{d_3}\phi\frac{m_0\omega_1 + cn\chi_1 + (bm_0 - acn)(\omega_1 - \chi)z}{(1 + (a + b)z)^2}dx
\end{aligned}$$

i.e.

$$m^{'}(0)\int_0^1 \phi^2 f(z, z)dx = \int_0^1 \phi^2 \frac{m_0\omega_1 + cn\chi_1 + (bm_0 - acn)(\omega_1 - \chi)z}{(1 + (a + b)z)^2}dx.$$

Now, we have

Theorem 3.8. Suppose $n \leq \dfrac{d_2d_3\lambda_1}{d_1 + cd_2}$ and $bm_0 \geq acn$. Then the bifurcation solutions defined by Theorem 3.5 are stable for $\tau > 0$.

Proof. From Theorem 3.5 and 3.7, $m^{'}(0) > 0$, $m^{'}(\tau) > 0$ for $|\tau| << 1$. By Lemma 3.4 and Theorem 3.6, we have $\eta(\tau) < 0$ for $\tau > 0$, which completes the proof of Theorem.

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