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BLOW UP OF SOLUTIONS FOR A SEMILINEAR HYPERBOLIC EQUATION

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Abstract. In this paper we consider a semilinear hyperbolic equation with source and damping terms. We will prove a blow up result of solutions for positive initial energy.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. We are concerned with the blow up of solutions of an initial-boundary value problem for a semilinear hyperbolic equation with dissipative terms:

$$u_{tt} + Au - \alpha\Delta u_t + g(u_t) = \beta f(u), \quad x \in \Omega, \quad t \geq 0 \tag{1.1}$$

$$u(x,\ t) = 0, \qquad x \in \partial\Omega, \tag{1.2}$$

$$u(x,\ 0) = u_0(x), u_t(x,\ 0) = u_1(x), \quad x \in \Omega, \tag{1.3}$$

where $\alpha > 0$, $\beta > 0$ and u_0, u_1 are given functions. A is a second order elliptic operator where the coefficients are depended on x and t. f and g are some functions specified later.
In the case $A = -\Delta$, many mathematicians studied the problem $(1.1)-(1.3)$. For $\alpha = 0, g(v) \equiv 0$ (absence of the damping term), the source term $f(u)$, in the case where the initial energy is negative, causes the blow up of solutions (see [1, 8]). In contract, in the absence of the source term ($\beta = 0$), the damping term (with $\alpha = 0$) assures global existence for arbitrary initial data (see [7, 9]). The interaction between the damping and the source terms was considered by Levine [9, 10] in linear damping case ($\alpha = 0, g(v) \cong v$) and polynomial source term of the form $f(u) = |u|^{p-2}u, p > 2$. He showed that the solutions with negative initial energy blow up in finite time. Georgiev and Todorova [5] extended Levine's result to the nonlinear case, where the damping term is given by $|u_t|^{m-2}u_t, m > 2$. Precisely, they showed that the solution continues to exist globally 'in time' if $m \geq p$ and blows up in finite time if $m < p$ and the initial energy is sufficiently negative. Vitillaro [16]

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extended the result in to situation when the damping is nonlinear and the solution has positive initial energy. Recently, Yu [17] studied the same problem of Vittilaro with strongly damping term. He proved that the solution exists globally if $E(t) < d,\ m < p$ and blows up in finite time in unstable set.

G.Li and al [11] considered the Petrovsky equation $u_{tt} + \Delta^2 u - \Delta u_t + |u_t|^{m-2}u_t = |u|^{p-2}u$ and proved the global existence of the solution under conditions without any relation between m and p, and established an exponential decay rate. They also showed that the solution blows up in finite time if $p > m$ and the initial energy is less than the potential well depth.

Messaoudi in [14] studied the following problem:

$$\begin{aligned} & u_{tt} - \Delta u + a(1 + |u_t|^{m-2})u_t = b|u|^{p-2}u, \quad x \in \Omega, \quad t \geq 0 \\ & u(x,t) = 0, \quad x \in \partial\Omega \\ & u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{aligned}$$

where $a, b > 0,\ p, m > 2$. He showed that if the initial energy is negative, then the solutions blow up in finite time.

In this work, we will prove that if the initial energy is positive, then the solution of problem $(1.1) - (1.3)$ blows up in finite time.

2. PRELIMINARIES

In this section we shall give some assumptions and notations which will be used throughout this work.

$H_1)$ The elliptic operator A is defined as follows:

$$A(t)\varphi = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial \varphi}{\partial x_j}\right),$$

where $a_{ij} \in C^1(\overline{\Omega} \times [0,\ \infty)) \quad \forall 1 \leq i, j \leq n$ is symmetric and there exists a constant $a_0 > 0$ such that :

$$\begin{aligned} & a) \ \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq a_0|\xi|^2, \\ & b) \ \sum_{i,j=1}^{n} \left(\frac{\partial}{\partial t}a_{ij}(x,t)\right)\xi_i\xi_j \leq 0, \end{aligned}$$

for all $(x,\ t) \in \overline{\Omega} \times (0,\ \infty)$ and $\xi = (\xi_1 \ldots \xi_n) \in \mathbb{R}^n$.

$H_2)$ We assume that the function $g(v)$ is increasing and $g(v) \in C^0(\mathbb{R}) \cap$

$C^1(\mathbb{R}^*)$. Furthermore, there exist two positive constants k_0 and k_1 such that:

$$a)\ g(v)v \geq k_0|v|^m$$
$$b)\ |g(v)| \leq k_1|v|(1+|v|^{m-2}),$$

for all $v \in \mathbb{R}$ and $2 < m < \infty$.

H_3) The function $f \in C^0(\mathbb{R},\ \mathbb{R}_+)$, with the primitive

$$F(u) = \int_0^u f(t)dt,$$

satisfies

$$a)\ f(s)s \geq pF(s),$$
$$b)\ |F(s)| \leq c_0|s|^p,$$

where $s \in \mathbb{R}$, $\quad c_0 > 0$ and $p > 2$. A typical example of these functions is $f(u) = |u|^{p-2}u$.

Next we introduce some notations, which will be used in the sequel:

$$u(x,t) = u\ ; \quad \frac{\partial u}{\partial t} = u_t\ ; \quad \frac{\partial^2 u}{\partial t^2} = u_{tt};$$
$$(u,v) = \int_\Omega u(x)v(x)dx; \qquad \|u\|_{L^r(\Omega)} = \|u\|_r\,; \quad 1 \leq r \leq \infty,$$

where $L^r(\Omega)$ is the Lebesgue space.

Remark. By using Poincaré's inequality and the Sobolev embedding theorem. Then, there exists a constant C_* depending on $\Omega,\ r$ only such that

$$\forall u \in H_0^1(\Omega), \quad \|u(t)\|_r \leq C_* \|\nabla u(t)\|_2\,, \quad 2 \leq r \leq \frac{2n}{n-2},\ n \geq 3 \tag{2.1}$$

3. Local existence of solutions

To allow for studying the local existence and blow up of solutions, we proceed to obtain a variational formulation of the problem $(1.1)-(1.3)$. By multiplying equation (1.1) by $v \in H_0^1(\Omega)$, integrating over Ω and using integration par parts, it is easy to verify that under the hypothesis (H_1) the problem $(1.1)-(1.3)$ is equivalent to the following variational problem:

$$(u_{tt},v) + a(u,v) + \alpha(\nabla u_t, \nabla v) + (g(u_t),v) = \beta(f(u),v), \quad \forall v \in H_0^1(\Omega),$$

where

$$a(u,v) = (Au,v) = \sum_{i,j=1}^n \int_\Omega a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx. \tag{3.1}$$

By using the hypothesis (H_1), we verify that the bilinear form $a(.,.) : H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \mathbb{R}$ is symmetric and continuous.
On the other hand, from $H_1a)$ for $\xi_i = \dfrac{\partial u}{\partial x_i}$, we get

$$a(u,u) \geq a_0 \int_\Omega \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^2 dx = a_0 \left\| \nabla u \right\|_2^2, \tag{3.2}$$

which implies that $a(.,.)$ is coercive.
Referring to [3] and [5], by using the precedent hypotheses we can demonstrate the following theorem, which confirms the local existence and uniqueness of a weak solution.

Theorem 3.1. *Assume that $H_1a), H_2$ and H_3 hold. Suppose that $m \geq 2,\ 2 \leq p \leq 2\dfrac{n-1}{n-2}$ if $n \geq 3$ and let $u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega)$, then there exists $T > 0$ such that the problem $(1.1) - (1.3)$ has a unique local solution $u(t)$ having the following regularities :*

$$\begin{aligned}
&u \in L^\infty\left([0,T)\,; H_0^1(\Omega)\right), \\
&u_t \in L^\infty\left([0,T)\,; L^2(\Omega)\right) \cap L^m\left(\Omega \times [0,T)\right) \cap L^2\left([0,T)\,; H^1(\Omega)\right).
\end{aligned}$$

4. Blow-up of solutions

In this section, we will establish our main blow-up result concerning the problem $(1.1) - (1.3)$. We set

$$\lambda_0 = \left(\frac{a_0}{c_0 \beta} C_*^{-p} \right)^{\frac{1}{p-2}}, \quad E_0 = a_0 \left(\frac{1}{2} - \frac{1}{p}\right) \lambda_0^2. \tag{4.1}$$

We define the energy function associated to the solution u of the problem $(1.1) - (1.3)$ by

$$E(u(t), u_t(t)) = E(t) = \frac{1}{2} \left\| u_t(t) \right\|_2^2 + \frac{1}{2} a(u(t), u(t)) - \beta \int_\Omega F(u) du, \quad t \geq 0 \tag{4.2}$$

By multiplying equation (1.1) by u_t, integrating over Ω and using integration par parts. Then, under the stated assumptions (H_1b) and (H_2a), we obtain the following result:

Lemma 4.1. *Let $u(x,t)$ be a solution to the problem* $(1.1)-(1.3)$*. Then $E(t)$ is decreasing function for $t>0$ and*

$$\frac{d}{dt}E(t) = -\alpha\left\|\nabla u_t(t)\right\|_2^2 - \int_\Omega g(u_t(t))u_t(t)dx + \tag{4.3}$$
$$+\sum_{i,j=1}^{n}\int_\Omega\left(\frac{\partial}{\partial t}a_{ij}(x,t)\right)\frac{\partial u(t)}{\partial x_i}\frac{\partial u(t)}{\partial x_j}dx.$$

By using arguments similar to those used by Vitillaro [16], we prove the following Lemma, which is very important to obtain the blow-up result.

Lemma 4.2. *Let u be a solution of* $(1.1)-(1.3)$ *with initial data satisfy*

$$E(0) < E_0\ ; \quad \left\|\nabla u_0\right\|_2 > \lambda_0. \tag{4.4}$$

Then there exists a constant $\lambda_1 > \lambda_0$ such that:

$$\left\|\nabla u(t)\right\|_2 > \lambda_1\ ; \quad \left\|u(t)\right\|_p > C_*\lambda_1\ , \qquad \forall t\in[0,\ T]\,. \tag{4.5}$$

Proof. By using (H_3b), from (4.2) it follows

$$E(t) \geq \frac{1}{2}a(u(t),u(t)) - \frac{c_0\beta}{p}\left\|u(t)\right\|_p^p. \tag{4.6}$$

Then, using (2.1) and (3.2) we have

$$E(t) \geq \frac{a_0}{2}\left\|\nabla u(t)\right\|_2^2 - \frac{c_0\beta}{p}C_*^p\left\|\nabla u(t)\right\|_2^p = Q(\left\|\nabla u(t)\right\|_2), \quad t\geq 0$$

then

- $Q(s)$ has a single maximum value $E_0 = Q(\lambda_0)$ at λ_0,
- $Q(s)$ is strictly increasing on $[0,\ \lambda_0)$,
- $Q(s)$ is strictly decreasing on $(\lambda_0,\ \infty)$ and $Q(s)\to-\infty$ as $s\to+\infty$.

Therefore, since $E(0)<E_0$, there exists $\lambda_1>\lambda_0$ such that $Q(\lambda_1)=E(0)$. If we set $\lambda_2 = \left\|\nabla u_0\right\|_2$, then by (4.6) we have $Q(\lambda_2)\leq E(0)=Q(\lambda_1)$, which implies that $\lambda_2\geq\lambda_1$.
To establish $\left\|\nabla u(t)\right\|_2 > \lambda_1$, we suppose by contradiction that $\left\|\nabla u(t_0)\right\|_2 < \lambda_1$, for some $t_0>0$ and by the continuity of $\left\|\nabla u(.)\right\|_2$ we can chose t_0 such that $\left\|\nabla u(t_0)\right\|_2 > \lambda_0$. Again the use of (4.6) leads to

$$E(t_0) \geq Q\left(\left\|\nabla u(t_0)\right\|\right) > Q(\lambda_1) = E(0).$$

This is impossible since $E(t)\leq E(0)$, for all $t\geq 0$.
To prove $\left\|u(t)\right\|_p > C_*\lambda_1$, we exploit (4.2) and (H_3b) to see

$$\frac{1}{2}a(u(t),u(t)) - \frac{c_0\beta}{p}\left\|u(t)\right\|_p^p \leq E(t) \leq E(0).$$

Then

$$\begin{aligned}\frac{c_0\beta}{p}\|u(t)\|_p^p &\geq \frac{a_0}{2}\|\nabla u(t)\|_2^2 - E(0)\\ &\geq \frac{a_0}{2}\lambda_1^2 - Q(\lambda_1) = \frac{c_0\beta}{p}C_*^p\lambda_1^p.\end{aligned}$$

□

Referring to [13], we will show the following theorem, which permit us to confirm that the solution of the problem (1.1) – (1.3) blows up in finite time.

Theorem 4.3. *Suppose that*

$$2 \leq m < p \leq 2\frac{n-1}{n-2}, \quad n \geq 3 \tag{4.7}$$

Then any solution of (1.1) – (1.3)*, with initial data satisfying* (4.4) *blows up at finite time i.e., there exists* $T^* < +\infty$ *such that*

$$\lim_{t\to T^{*-}}\left[\|u(t)\|_p^p + \|\nabla u(t)\|_2^2 + H(t) + \|u_t(t)\|_2^2\right] = +\infty.$$

Proof. By contradiction, we suppose that the solution of the problem (1.1)–(1.3) is global, then for every fixed $T > 0$ there exists a constant C such that

$$\|u(t)\|_p^p + \|\nabla u(t)\|_2^2 + H(t) + \|u_t(t)\|_2^2 \leq C \qquad \forall t \in [0,T]. \tag{4.8}$$

We set

$$H(t) = E_0 - E(t), \quad \forall t \in [0,T]. \tag{4.9}$$

By Lemma 4.1, we deduce that $H'(t) \geq 0$. Thus by (4.4), we obtain

$$H(t) \geq H(0) = E_0 - E(0) > 0. \tag{4.10}$$

From (4.9), (4.2) and (H_3b), we get

$$H(t) \leq E_0 - \frac{a_0}{2}\|\nabla u(t)\|_2^2 + \frac{c_0\beta}{p}\|u(t)\|_p^p.$$

Then, from Lemma 4.2 it follows

$$H(t) \leq E_0 - \frac{a_0}{2}\lambda_0^2 + \frac{c_0\beta}{p}\|u(t)\|_p^p.$$

Hence

$$0 < H(0) \leq H(t) \leq \frac{c_0\beta}{p}\|u(t)\|_p^p, \quad \forall t \in [0,T]. \tag{4.11}$$

For ε small to be chosen later, we then define the following auxiliary function:

$$G(t) = H^{1-\sigma}(t) + \varepsilon\int_\Omega u_t(t)u(t)dx + \frac{\varepsilon\alpha}{2}\|\nabla u(t)\|_2^2, \tag{4.12}$$

where

$$0 < \sigma \leq \min\left\{\frac{p-2}{2p}, \ \frac{p-m}{p(m-1)}\right\}. \tag{4.13}$$

Let us remark that G is a small perturbation of the energy.
By taking the time derivation of (4.12) and using a variational formulation, we obtain that

$$\begin{aligned}\frac{d}{dt}G(t) = (1-\sigma)H^{-\sigma}(t)H_t(t) + \varepsilon\left\|u_t(t)\right\|_2^2 - \varepsilon a(u(t),u(t)) + \\ + \varepsilon\beta\int_\Omega f(u(t))u(t)dx - \varepsilon\int_\Omega g(u_t(t))u(t)dx.\end{aligned} \tag{4.14}$$

By using (4.2), (H_3) and (4.9) from (4.14) we deduce that :

$$\begin{aligned}\frac{d}{dt}G(t) \geq (1-\sigma)H^{-\sigma}(t)H_t(t) + \varepsilon\left(\frac{p}{2}+1\right)\left\|u_t(t)\right\|_2^2 + \varepsilon pH(t) \\ + \varepsilon\left(\frac{p}{2}-1\right)a(u(t),u(t)) - \varepsilon\int_\Omega g(u_t(t))u(t)dx - \varepsilon pE_0.\end{aligned} \tag{4.15}$$

Using the assumption (H_2b), we get

$$\left|\int_\Omega g(u_t(t))u(t)dx\right| \leq k_1\int_\Omega |u_t(t)||u(t)|dx + k_1\int_\Omega |u_t(t)|^{m-1}|u(t)|dx.$$

Then we exploit the following Young's inequality :

$$XY \leq \frac{\delta^r}{r}X^r + \frac{\delta^{-s}}{s}Y^s, \quad X,Y \geq 0, \quad \delta > 0, \quad \frac{1}{r}+\frac{1}{s} = 1,$$

with $r = m$ and $s = \frac{m}{m-1}$ to get

$$k_1\int_\Omega |u_t(t)|^{m-1}|u(t)|dx \leq k_1\frac{\delta^m}{m}\left\|u(t)\right\|_m^m + k_1\frac{m-1}{m}\delta^{-\frac{m-1}{m}}\left\|u_t(t)\right\|_m^m, \tag{4.16}$$

for all positive constant δ.
By using Holder's inequality and (2.1) we get

$$k_1\int_\Omega |u_t(t)||u(t)|dx \leq k_1c(\lambda)C_*^2\left\|\nabla u(t)\right\|_2^2 + k_1c_1(\lambda)\left\|u_t(t)\right\|_2^2, \tag{4.17}$$

where $c(\lambda),\ c_1(\lambda)$ are positive constants.
Inserting (4.16), (4.17) and (3.2) in (4.15), we arrive at

$$\begin{aligned}\frac{d}{dt}G(t) \geq (1-\sigma)H^{-\sigma}(t)H_t(t) + \varepsilon pH(t) - \varepsilon pE_0 - \varepsilon k_1\frac{\delta^m}{m}\left\|u(t)\right\|_m^m \\ - \varepsilon k_1\frac{m-1}{m}\delta^{-\frac{m-1}{m}}\left\|u_t(t)\right\|_m^m + \varepsilon\left(\frac{p}{2}+1-k_1c_1(\lambda)\right)\left\|u_t(t)\right\|_2^2 + \\ + \varepsilon\left(a_0\left(\frac{p}{2}-1\right) - k_1c(\lambda)C_*^2\right)\left\|\nabla u(t)\right\|_2^2.\end{aligned} \tag{4.18}$$

We observe that

$$a_0\left(\frac{p}{2}-1\right)\|\nabla u(t)\|_2^2 - pE_0 = a_0\left(\frac{p}{2}-1\right)\frac{\lambda_1^2-\lambda_0^2}{\lambda_1^2}\|\nabla u(t)\|_2^2 + \\ + a_0\left(\frac{p}{2}-1\right)\lambda_0^2\frac{\|\nabla u(t)\|_2^2}{\lambda_1^2} - pE_0,$$

where λ_1 is given in Lemma 4.2. From (4.5), it follows:

$$a_0\left(\frac{p}{2}-1\right)\|\nabla u(t)\|_2^2 - pE_0 \geq C_1\|\nabla u(t)\|_2^2 + C_2, \tag{4.19}$$

where $C_1 = a_0(\frac{p}{2}-1)\frac{\lambda_1^2-\lambda_0^2}{\lambda_1^2}$, using Lemma 4.2, we have $C_1 > 0$ and by (4.9), we see that $C_2 = a_0(\frac{p}{2}-1)\lambda_0^2 - pE_0 > 0$.
Since $H_t(t) \geq k_0\|u_t\|_m^m$ and by (4.19), we get

$$\frac{d}{dt}G(t) \geq \left((1-\sigma)H^{-\sigma}(t) - \varepsilon\frac{k_1}{k_0}\frac{m-1}{m}\delta^{-\frac{m-1}{m}}\right)H_t(t) + \varepsilon pH(t) + \\ + \varepsilon\left(\frac{p}{2}+1-k_1c_1(\lambda)\right)\|u_t(t)\|_2^2 + \varepsilon\left(C_1 - k_1c(\lambda)C_*^2\right)\|\nabla u(t)\|_2^2 - \\ - \varepsilon k_1\frac{\delta^m}{m}\|u(t)\|_m^m.$$

At this point we choose δ so that $\delta^{-\frac{m-1}{m}} = MH^{-\sigma}(t)$, for M a large constant to be determined later, and substituting in the last inequality, we obtain

$$\frac{d}{dt}G(t) \geq \left((1-\sigma) - \varepsilon\frac{k_1}{k_0}\frac{m-1}{m}M\right)H^{-\sigma}(t) + \varepsilon pH(t)H_t(t) + \\ + \varepsilon\left(\frac{p}{2}+1-k_1c_1(\lambda)\right)\|u_t(t)\|_2^2 + \varepsilon\left(C_1 - k_1c(\lambda)C_*^2\right)\|\nabla u(t)\|_2^2 + \\ - \varepsilon\frac{k_1}{m}M^{1-m}H^{\sigma(m-1)}(t)\|u(t)\|_m^m. \tag{4.20}$$

Since $p > m$, we have

$$\int_\Omega |u(t)|^m dx \leq C_3\left[\int_\Omega |u(t)|^p dx\right]^{\frac{m}{p}},$$

where C_3 is a positive constant depending on Ω only.
We also have from (4.11)

$$H^{\sigma(m-1)}(t)\int_\Omega |u(t)|^m dx \leq C_3\left(\frac{c_0\beta}{p}\right)^{\sigma(m-1)}\left[\int_\Omega |u(t)|^p dx\right]^{\sigma(m-1)+\frac{m}{p}}.$$

Exploiting the following algebraic inequality:

$$z^\tau \leq z+1 \leq \left(1+\frac{1}{d}\right)(z+d), \quad \forall z \geq 0, \quad 0 < \tau \leq 1, \quad d \geq 0,$$

with $z = \|u(t)\|_p^p$, $e = 1 + \dfrac{1}{H(0)}$, $d = H(0)$ and $\tau = \sigma(m-1) + \frac{m}{p}$, then the condition (4.13) implies that $0 < \tau \leq 1$ and therefore,

$$\begin{aligned} \left[\int_\Omega |u(t)|^p dx\right]^{\sigma(m-1)+\frac{m}{p}} &\leq e\left(\|u(t)\|_p^p + H(0)\right) \qquad (4.21) \\ &\leq e\left(\|u(t)\|_p^p + H(t)\right), \quad \forall t \in [0,T]. \end{aligned}$$

Inserting the estimation (4.21) into (4.20) we have

$$\begin{aligned} \frac{d}{dt}G(t) \geq &\left((1-\sigma) - \varepsilon\frac{k_1}{k_0}\frac{m-1}{m}M\right) H^{-\sigma}(t)H_t(t)+ \qquad (4.22) \\ &+ \varepsilon\left(\frac{p}{2} + 1 - k_1c_1(\lambda)\right)\|u_t(t)\|^2 + \varepsilon\left(C_1 - k_1c(\lambda)C_*^2\right)\|\nabla u(t)\|_2^2 + \\ &+ \varepsilon\left[pH(t) - e\frac{k_1}{m}M^{1-m}C_3\left(\frac{c_0\beta}{p}\right)^{\sigma(m-1)}\left(\|u(t)\|_p^p + H(t)\right)\right]. \end{aligned}$$

At this point we choose $\lambda > 0$, (it is the case where $k_1 \max\left(c(\lambda),\ c_1(\lambda)\right) < \min\left(1 + \frac{p}{2},\ \frac{C_1}{C_*^2}\right)$) such that

$$\begin{cases} K_1 = \left(\frac{p}{2} + 1 - k_1c_1(\lambda)\right) > 0, \\ K_2 = \left(C_1 - k_1c(\lambda)C_*^2\right) > 0, \end{cases}$$

and we can choose $M > \left[\left(\frac{1}{c_0\beta} + \frac{1}{p}\right)e\frac{k_1}{m}C_3)\right]^{\frac{1}{m-1}}\left(\frac{c_0\beta}{p}\right)^\sigma$ so large enough so that (4.22) becomes,

$$\begin{aligned} \frac{d}{dt}G(t) \geq &\left((1-\sigma) - \varepsilon\frac{k_1}{k_0}\frac{m-1}{m}M\right) H^{-\sigma}(t)H_t(t)+ \qquad (4.23) \\ &+ K_1\|u_t(t)\|^2 + K_2\|\nabla u(t)\|_2^2 + \varepsilon K_3\left(\|u(t)\|_p^p + H(t)\right). \end{aligned}$$

Once M is fixed, we pick ε small enough such that

$$\begin{cases} (1-\sigma) - \varepsilon\frac{k_1}{k_0}\frac{m-1}{m}M \geq 0, \\ G(0) = H^{1-\sigma}(0) + \varepsilon\int_\Omega u_1u_0dx + \frac{\varepsilon\alpha}{2}\|\nabla u_0\|_2^2 > 0. \end{cases}$$

Then, from (4.23) we deduce that:

$$\frac{d}{dt}G(t) \geq K\varepsilon\left[H(t) + \|\nabla u(t)\|_2^2 + \|u(t)\|_p^p + \|u_t(t)\|_2^2\right], \qquad (4.24)$$

where $K = \min\left(K_1,\ K_2,\ K_3\right)$. Hence $G(t) \geq G(0) > 0$, $\forall t \in [0,T]$.

Now we set $r = \frac{1}{1-\sigma}$, by $(a+b)^r \leq 2^{r-1}(a^r + b^r)$ for all positive a, b and

$r > 1$, we obtain on the other hand from (4.12),

$$G^r(t) \le \left(H^{1-\sigma}(t) + \epsilon \int_\Omega u_t(t)u(t)dx + \frac{\varepsilon\alpha}{2}\left\|\nabla u(t)\right\|_2^2\right)^r \tag{4.25}$$
$$\le C_4\left(H(t) + \left(\int_\Omega u_t(t)u(t)dx\right)^r + \|\nabla u(t)\|)_2^{2r}\right),$$

where $C_4 = 2^{2(r-1)}\max\left\{1,\ \varepsilon^r \max\left\{1,\ (\frac{\alpha}{2})^r\right\}\right\}$.
For $p > 2$ and by using Holder's and Young's inequalities, we obtain

$$\left(\int_\Omega u(t)u_t(t)dx\right)^r \le \|u(t)\|_2^r\,\|u_t(t)\|_2^r \le C_5\left(\|u(t)\|_p^{\mu r} + \|u_t(t)\|_2^{\theta r}\right), \tag{4.26}$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$ and C_5 depending on $\Omega,\ \mu,\ \theta$ only. We take $\theta = 2(1-\sigma)$, to get $\mu r = \frac{2}{1-2\sigma} \le p$ by (4.13).
Therefore (4.26) becomes

$$\left(\int_\Omega u(t)u_t(t)dx\right)^r \le C_5\left(\|u(t)\|_p^{\frac{2}{1-2\sigma}} + \|u_t(t)\|_2^2\right).$$

Again by using (4.13) and (4.21) we deduce

$$\left(\|u(t)\|_p^p\right)^{\frac{2}{(1-2\sigma)p}} \le e\left(\|u(t)\|_p^p + H(t)\right) \le e\left(\|\nabla u(t)\|_2^2 + \|u(t)\|_p^p + H(t)\right),$$

so

$$\left(\int_\Omega u(t)u_t(t)dx\right)^r \le eC_5\left(\|\nabla u(t)\|_2^2 + \|u(t)\|_p^p + H(t) + \|u_t(t)\|_2^2\right). \tag{4.27}$$

From (4.8) and (4.10), we have

$$\|\nabla u(t)\|_2^{2r} \le C^{\frac{1}{1-\sigma}} \le \frac{C^{\frac{1}{1-\sigma}}}{H(0)}H(t). \tag{4.28}$$

It follows from (4.27), (4.28) and (4.25) that

$$G^r(t) \le C_6\left(\|\nabla u(t)\|_2^2 + \|u(t)\|_p^p + H(t) + \|u_t(t)\|_2^2\right), \quad \forall t \in [0,T], \tag{4.29}$$

where $C_6 = C_4\left(1 + eC_5 + \frac{C^{\frac{1}{1-\sigma}}}{H(0)}\right)$. Combining (4.29) and (4.24), we arrive at

$$\frac{d}{dt}G(t) \ge \frac{\varepsilon K}{C_6}G^{\frac{1}{1-\sigma}}(t), \quad \forall t \in [0,T]. \tag{4.30}$$

A simple integration of (4.30) over $(0,\ t)$ then yields

$$G^{\frac{\sigma}{1-\sigma}}(t) \ge \frac{1}{G^{\frac{-\sigma}{1-\sigma}}(0) - K\varepsilon\sigma t/\left[C_6(1-\sigma)\right]}, \quad \forall t \in [0,T]. \tag{4.31}$$

Therefore $G(t)$ blows up in a time

$$T^* \le \frac{C_6(1-\sigma)}{K\varepsilon\sigma G^{\frac{\sigma}{1-\sigma}}(0)},$$

the estimate (4.31) is valid on $[0, T]$ for every fixed $T > 0$, then we can choose T such that $T^* < T$. Furthermore, we get from (4.29) that

$$\lim_{t \to T^{*-}} \|\nabla u(t)\|_2^2 + \|u\|_p^p + H(t) + \|u_t\|_2^2 = +\infty,$$

which is in contradiction with (4.8). Thus, the solution of the problem (1.1)−(1.3) blows up in finite time.

□

Remark. For $E(0) < 0$, we set $H(t) = -E(t)$, instead of (4.9) and use arguments similar to those used in the proof of Theorem 4.3 to deduce that the solution blows up in finite time.

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