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Existence and iteration of monotone positive solutions for third-order nonlocal BVPs involving integral conditions*

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Abstract

This paper is concerned with the existence of monotone positive solution for the following third-order nonlocal boundary value problem $u'''(t)+f(t,u(t),u'(t))=0$, $0<t<1$; $u(0)=0$, $au'(0)-bu''(0)=\alpha[u]$, $cu'(1)+du''(1)=\beta[u]$, where $f\in C([0,1]\times R^+\times R^+,R^+)$, $\alpha[u]=\int_0^1 u(t)dA(t)$ and $\beta[u]=\int_0^1 u(t)dB(t)$ are linear functionals on $C[0,1]$ given by Riemann-Stieltjes integrals. By applying monotone iterative techniques, we not only obtain the existence of monotone positive solution but also establish an iterative scheme for approximating the solution. An example is also included to illustrate the main results.

Keywords: Monotone iterative method; Positive solutions; Nonlocal; Integral conditions
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1 Introduction

Third-order differential equation arises in a variety of different areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three layer beam, electromagnetic waves or gravity driven flows and so on [1].

BVPs with Stieltjes integral boundary condition (BC for short) have been considered recently as both multipoint and integral type BCs are treated in a single framework. For

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more comments on Stieltjes integral BC and its importance, we refer the reader to the papers by Webb and Infante [2, 3, 4] and their other related works. In recent years, third-order nonlocal BVPs have received much attention from many authors, see, for example [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and the references therein. For fourth-order or higher-order nonlocal BVPs, one can refer to [3, 4, 21, 22]. In particular, it should be pointed out that Webb and Infante in [2], gave a unified approach for studying the existence of multiple positive solutions of second-order BVPs subject to various nonlocal BCs. In [3], they extended their method to cover equations of order N with any number up to N of nonlocal BCs in a single theory.

Recently, iterative methods have been successfully employed to prove the existence of positive solutions of nonlinear BVPs for ordinary differential equations, see [23, 24, 25, 26, 27] and the references therein. It is worth mentioning that, Sun et al. [24] obtained the existence of monotone positive solutions for third-order three-point BVPs, the main tools used were monotone iterative techniques. Inspired by the above mentioned excellent works, the aim of this paper is to investigate the existence and iteration of monotone positive solution for the following BVP

$$\begin{cases} u'''(t) + f(t, u(t), u'(t)) = 0, \ 0 < t < 1, \\ u(0) = 0, \\ au'(0) - bu''(0) = \alpha[u], \\ cu'(1) + du''(1) = \beta[u], \end{cases} \tag{1.1}$$

where $f \in C([0,1] \times R^+ \times R^+, R^+)$, $\alpha[u] = \int_0^1 u(t)dA(t)$ and $\beta[u] = \int_0^1 u(t)dB(t)$ are linear functionals on $C[0,1]$ given by Riemann-Stieltjes integrals and a, b, c, d are nonnegative constants with $\rho := ac + ad + bc > 0$. By a positive solution of BVP (1.1), we understand a solution $u(t)$ which is positive on $t \in (0,1)$ and satisfies BVP (1.1). By applying monotone iterative techniques, we construct a successive iterative scheme whose starting point is a zero function, which is very useful and feasible for computational purpose. An example is also included to illustrate the main results.

2 Preliminary lemmas

In this section, the ideas and the method we will adopt, which have been widely used, are due to Webb and Infante in [2, 3].

In our case, the existence of positive solutions of nonlocal BVP (1.1) with two nonlocal boundary terms $\alpha[u]$, $\beta[u]$, can be studied, via a perturbed Hammerstein integral equation of the type

$$u(t) = \gamma(t)\alpha[u] + \delta(t)\beta[u] + \int_0^1 G(t,s) f(s, u(s), u'(s))\, ds =: Tu(t). \tag{2.1}$$

Here $\gamma(t)$, $\delta(t)$ are linearly independent and given by

$$-\gamma'''(t) = 0, \ \gamma(0) = 0, \ a\gamma'(0) - b\gamma''(0) = 1, \ c\gamma'(1) + d\gamma''(1) = 0,$$

$$-\delta'''(t) = 0, \ \delta(0) = 0, \ a\delta'(0) - b\delta''(0) = 0, \ c\delta'(1) + d\delta''(1) = 1,$$

which imply $\gamma(t) = \frac{2ct+2dt-ct^2}{2\rho}$ and $\delta(t) = \frac{at^2+2bt}{2\rho}$, $t \in [0,1]$. A direct calculation shows that for $t \in [\theta, 1]$, $\gamma(t) \geq c_1\|\gamma\|_\infty$ and $\delta(t) \geq c_2\|\delta\|_\infty$ ($\|\cdot\|_\infty$ is the usual supremum norm in $C[0,1]$), where $c_1 = \frac{2c\theta+2d\theta-c\theta^2}{c+2d}$ and $c_2 = \frac{a\theta^2+2b\theta}{a+2b}$; $G(t,s)$ is the Green's function for the corresponding problem with local terms when $\alpha[u]$ and $\beta[u]$ are identically 0.

We first make the following hypotheses on the Green's function:

$(H1)$ The kernel G is measurable, non-negative, and for every $\tau \in [0,1]$ satisfies

$$\lim_{t\to\tau}|G(t,s) - G(\tau,s)| = 0 \quad \text{for}\, s \in [0,1].$$

$(H2)$ There exist a subinterval $[a,b] \subseteq [0,1]$, a measurable function Φ, and a constant $c_3 \in (0,1]$ such that

$$G(t,s) \leq \Phi(s) \quad \text{for}\, t \in [0,1],\ s \in [0,1],$$

$$G(t,s) \geq c_3\Phi(s) \quad \text{for}\, t \in [a,b],\ s \in [0,1].$$

$(H3)$ A, B are functions of bounded variation, and $\mathcal{K}_A(s), \mathcal{K}_B(s) \geq 0$ for $s \in [0,1]$, where

$$\mathcal{K}_A(s) := \int_0^1 G\,(t,s)\, dA(t) \text{ and } \mathcal{K}_B(s) := \int_0^1 G\,(t,s)\, dB(t).$$

In the remainder of this paper, we always assume that $0 \leq \alpha[\gamma], \beta[\delta] < 1$, $\alpha[\delta], \beta[\gamma] \geq 0$ and $D := (1-\alpha[\gamma])(1-\beta[\delta]) - \alpha[\delta]\beta[\gamma] > 0$.

As shown in Theorem 2.3 in [3], if u is a fixed point of T in (2.1), then u is a fixed point of S, which is now given by

$$\begin{aligned} Su(t) \;&:=\; \frac{\gamma(t)}{D}\left((1-\beta[\delta])\int_0^1 \mathcal{K}_A(s) f\left(s,u\left(s\right),u'\left(s\right)\right) ds + \alpha[\delta]\int_0^1 \mathcal{K}_B(s) f\left(s,u\left(s\right),u'\left(s\right)\right) ds\right) \\ &\quad+\frac{\delta(t)}{D}\left(\beta[\gamma]\int_0^1 \mathcal{K}_A(s) f\left(s,u\left(s\right),u'\left(s\right)\right) ds + (1-\alpha[\gamma])\int_0^1 \mathcal{K}_B(s) f\left(s,u\left(s\right),u'\left(s\right)\right) ds\right) \\ &\quad+\int_0^1 G(t,s) f\left(s,u\left(s\right),u'\left(s\right)\right) ds =: \int_0^1 G_S(t,s) f\left(s,u\left(s\right),u'\left(s\right)\right) ds \end{aligned}$$

in our case. The kernel G_S is the Green's function corresponding to the BVP (1.1).

Lemma 2.1 *Let $\rho := ac+ad+bc > 0$. Then the Green's function $G(t,s)$ satisfies $(H1)$, $(H2)$ with $[a,b] = [\theta,1]$, $c_3 = \frac{\rho\int_0^\theta \Phi(\tau)d\tau}{(a+b)(c+d)}$, $0 < \theta < 1$.*

Proof. A direct calculation shows that,

$$G\,(t,s) = \begin{cases} \frac{(at^2+2bt)(c(1-s)+d)}{2\rho} - \frac{(t-s)^2}{2}, & 0 \leq s \leq t \leq 1, \\ \frac{(at^2+2bt)(c(1-s)+d)}{2\rho}, & 0 \leq t \leq s \leq 1. \end{cases}$$

For any fixed $s \in [0,1]$, it is easy to see that

$$G_1(t,s) := \frac{\partial G\,(t,s)}{\partial t} = \frac{1}{\rho}\begin{cases} (b+as)\,(d+c(1-t))\,, & 0 \leq s \leq t \leq 1, \\ (b+at)\,(d+c(1-s))\,, & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.2}$$

which shows that

$$0 \le G_1(t,s) \le \frac{1}{\rho}(b+as)\,(d+c(1-s)) =: \Phi(s), \text{ for } (t,s) \in [0,1]\times[0,1], \tag{2.3}$$

and so,

$$G(t,s) = \int_0^t G_1(\tau,s)\,d\tau \le \int_0^t \Phi(s)d\tau = \Phi(s)t \le \Phi(s), \text{ for } (t,s)\in[0,1]\times[0,1]. \tag{2.4}$$

On the other hand,

$$\frac{G_1(t,s)}{\Phi(s)} = \begin{cases} \frac{(b+as)(d+c(1-t))}{(b+as)(d+c(1-s))} = \frac{(b+at)(d+c(1-t))}{(b+at)(d+c(1-s))} \ge \frac{\rho\Phi(t)}{(a+b)(c+d)}, & 0\le s\le t\le 1, \\ \frac{(b+at)(d+c(1-s))}{(b+as)(d+c(1-s))} = \frac{(b+at)(d+c(1-t))}{(b+as)(d+c(1-t))} \ge \frac{\rho\Phi(t)}{(a+b)(c+d)}, & 0\le t\le s\le 1, \end{cases} \tag{2.5}$$

so,

$$G_1(t,s) \ge \frac{\rho\Phi(t)}{(a+b)(c+d)}\Phi(s), \text{ for } (t,s)\in[0,1]\times[0,1]. \tag{2.6}$$

Thus,

$$G(t,s) = \int_0^t G_1(\tau,s)\,d\tau \ge \int_0^t \frac{\rho\Phi(\tau)}{(a+b)(c+d)}\Phi(s)d\tau \ge \frac{\rho\int_0^\theta \Phi(\tau)d\tau}{(a+b)(c+d)}\Phi(s), \text{ for } (t,s)\in[\theta,1]\times[0,1]. \tag{2.7}$$

□

Lemma 2.2 *$G_S(t,s)$ satisfies $(H1)$, $(H2)$ for a function Φ_1, the same interval $[\theta,1]$, and the constant $c_0 = \min\{c_1,c_2,c_3\}$.*

Proof. Let $\Phi_1(s) := \frac{\|\gamma\|_\infty}{D}((1-\beta[\delta])\mathcal{K}_A(s)+\alpha[\delta]\mathcal{K}_B(s)) + \frac{\|\delta\|_\infty}{D}(\beta[\gamma]\mathcal{K}_A(s)+(1-\alpha[\gamma])\mathcal{K}_B(s)) + \Phi(s)$, $s\in[0,1]$. The proof is same to the Theorem 2.4 in [2], so omitted. □

Moreover, we easily know that

$$0 \le \frac{\partial G_S(t,s)}{\partial t} \le \Phi_2(s), \ t,s\in[0,1]\times[0,1] \tag{2.8}$$

for a function Φ_2, i.e., for $s\in[0,1]$,

$$\Phi_2(s) := \frac{\|\gamma'\|_\infty}{D}[(1-\beta[\delta])\mathcal{K}_A(s)+\alpha[\delta]\mathcal{K}_B(s)] + \frac{\|\delta'\|_\infty}{D}[\beta[\gamma]\mathcal{K}_A(s)+(1-\alpha[\gamma])\mathcal{K}_B(s)] + \Phi(s).$$

We will use the classical Banach space $E = C^1[0,1]$ equipped with the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$, where $\|u\|_\infty$ is the usual supremum norm in $C[0,1]$.

Let

$$P = \{u\in E : u(t)\ge 0\}$$

and let c_0 be same as in Lemma 2.2, then define

$$K = \left\{u\in P : \min_{t\in[\theta,1]} u(t) \ge c_0\|u\|_\infty \text{ and } u'(t)\ge 0, \ t\in[0,1]\right\}.$$

Then it is to verify that P and K are cones in E. Note that this induces an order relation $\preceq$ in E by defining $u\preceq v$ if and only if $v-u\in K$.

Similar to the proofs of lemma 2.6, 2.7 and 2.8 in [2], we can get the following lemmas.

Lemma 2.3 *The maps* $T,\ S : P \to E$ *are compact.*

Lemma 2.4 $T : K \to K$ *and* $S : P \to K$*.*

Lemma 2.5 T *and* S *have the same fixed points (in* K*).*

3 Main results

Now we apply monotone iterative techniques to seek solution of BVP (1.1) as fixed point of the integral operator S.

Theorem 3.1 *Let* $\sigma = \max\left\{\max_{s\in[0,1]} \Phi_1(s), \max_{s\in[0,1]} \Phi_2(s)\right\}$. *Assume that* $f(t,0,0) \not\equiv 0$ *for* $t \in [0,1]$ *and there exists a constant* $r > 0$ *such that*

$$f(t,u_1,v_1) \le f(t,u_2,v_2) \le \frac{r}{\sigma},\quad 0 \le t \le 1,\ 0 \le u_1 \le u_2 \le r,\ 0 \le v_1 \le v_2 \le r. \tag{3.1}$$

If we construct an iterative sequence $v_{n+1} = Sv_n, n = 0,1,2,\ldots,$ *where* $v_0(t) = 0$ *for* $t \in [0,1]$, *then* $\{v_n\}_{n=0}^{\infty}$ *converges to* v^* *in* $C^1[0,1]$, *which is a monotone positive solution of the BVP (1.1) and satisfies*

$$0 < v^*(t) \le r \quad \textit{for } t \in (0,1], \quad 0 \le (v^*)'(t) \le r \quad \textit{for } t \in [0,1].$$

Proof. Let $K_r = \{u \in K : \|u\| < r\}$. We assert that $S : \overline{K_r} \to \overline{K_r}$. In fact, if $u \in \overline{K_r}$, then

$$0 \le u(s) \le \|u\|_\infty \le \|u\| \le r,\ 0 \le u'(s) \le \|u'\|_\infty \le \|u\| \le r,\ \text{for } s \in [0,1],$$

which together with the condition (3.1) and Lemma 2.2 and (2.8) implies that

$$0 \le (Su)(t) = \int_0^1 G_S(t,s) f(s,u(s),u'(s))\,ds \le r,\ t \in [0,1],$$
$$0 \le (Su)'(t) = \int_0^1 \frac{\partial G_S(t,s)}{\partial t} f(s,u(s),u'(s))\,ds \le r,\ t \in [0,1].$$

Hence, we have shown that $S : \overline{K_r} \to \overline{K_r}$.

Now, we assert that $\{v_n\}_{n=0}^{\infty}$ converges to v^* in $C^1[0,1]$, which is a monotone positive solution of the BVP (1.1) and satisfies

$$0 < v^*(t) \le r \ \text{ for } t \in (0,1], \quad 0 \le (v^*)'(t) \le r \ \text{ for } t \in [0,1].$$

In fact, in view of $v_0 \in \overline{K_r}$ and $S : \overline{K_r} \to \overline{K_r}$, we have that $v_n \in \overline{K_r},\ n = 1,2,\ldots$. Since the set $\{v_n\}_{n=0}^{\infty}$ is bounded and T is completely continuous, we know that $\{v_n\}_{n=0}^{\infty}$ is relatively compact.

In what follows, we prove that $\{v_n\}_{n=0}^{\infty}$ is monotone by induction. Firstly, by $v_0 = 0$ and $S : P \to K$, we easily know $v_1 - v_0 \in K$, which shows that $v_0 \preceq v_1$. Next, we assume that $v_{k-1} \preceq v_k$. Then, in view of Lemma 2.2 and (3.1), we have

$$\begin{aligned} 0 \le v_{k+1}(t) - v_k(t) &= \int_0^1 G_S(t,s)\left[f\left(s, v_k(s), v_k'(s)\right) - f\left(s, v_{k-1}(s), v_{k-1}'(s)\right)\right] ds \\ &\le \int_0^1 \Phi_1(s)\left[f\left(s, v_k(s), v_k'(s)\right) - f\left(s, v_{k-1}(s), v_{k-1}'(s)\right)\right] ds, t \in [0,1] \end{aligned}$$

and

$$\begin{aligned} v_{k+1}(t) - v_k(t) &= \int_0^1 G_S(t,s)\left[f\left(s, v_k(s), v_k'(s)\right) - f\left(s, v_{k-1}(s), v_{k-1}'(s)\right)\right] ds \\ &\ge c_0 \int_0^1 \Phi_1(s)\left[f\left(s, v_k(s), v_k'(s)\right) - f\left(s, v_{k-1}(s), v_{k-1}'(s)\right)\right] ds,\ t \in [\theta,1], \end{aligned}$$

which imply that

$$v_{k+1}(t) - v_k(t) \ge c_0 \left\|v_{k+1} - v_k\right\|_\infty,\ t \in [\theta,1]. \tag{3.2}$$

At the same time, by Lemma 2.2, (2.8) and (3.1), we also have

$$v_{k+1}'(t) - v_k'(t) = \int_0^1 \frac{\partial G_S(t,s)}{\partial t}\left[f\left(s, v_k(s), v_k'(s)\right) - f\left(s, v_{k-1}(s), v_{k-1}'(s)\right)\right] ds \ge 0,\ t \in [0,1] \tag{3.3}$$

It follows from (3.2) and (3.3) that $v_{k+1}(t) - v_k(t) \in K$, which shows that $v_k \preceq v_{k+1}$. Thus, we have shown that $v_n \preceq v_{n+1},\ n = 0, 1, 2 \ldots$.

Since $\{v_n\}_{n=0}^{\infty}$ is relatively compact and monotone, there exists a $v^* \in \overline{K_r}$ such that $\|v_n - v^*\| \to 0$ $(n \to \infty)$, which together with the continuity of S and the fact that $v_{n+1} = Sv_n$ implies that $v^* = Sv^*$. Moreover, in view of $f(t,0,0) \not\equiv 0$ for $t \in (0,1)$, we know that the zero function is not a solution of BVP (1.1). Thus, $\|v^*\|_\infty > 0$. So, it follows from $v^* \in \overline{K_r}$ that

$$0 < v^*(t) \le r \ \text{ for } t \in (0,1], \quad 0 \le (v^*)'(t) \le r \ \text{ for } t \in [0,1].$$

□

4 An example

Consider the BVP

$$\begin{cases} u'''(t) + \frac{1}{2}tu + \frac{1}{8}u'^2 + 1 = 0,\ 0 < t < 1, \\ u(0) = 0, \\ u'(0) = \alpha[u], \\ u'(1) = \beta[u], \end{cases} \tag{4.1}$$

where $\alpha[u] = \int_0^1 (1-s)u(s)ds$ and $\beta[u] = \int_0^1 su(s)ds$ are nonlocal BCs of integral type. For this BCs the corresponding $\gamma(t) = \frac{2t-t^2}{2}$ and $\delta(t) = \frac{t^2}{2}$. By simple calculation shows that

$$\alpha[\gamma] = \frac{1}{8},\ \alpha[\delta] = \frac{1}{24},\ \beta[\gamma] = \frac{5}{24},\ \beta[\delta] = \frac{1}{8},\ D = (1-\alpha[\gamma])(1-\beta[\delta]) - \alpha[\delta]\beta[\gamma] = \frac{109}{144},$$

$$\mathcal{K}_A(s) := \int_0^1 G(t,s)(1-t)dt = \frac{s}{8} - \frac{s^2}{4} + \frac{s^3}{6} - \frac{s^4}{24},\ \mathcal{K}_B(s) := \int_0^1 G(t,s)\,tdt = \frac{5s}{24} - \frac{s^2}{4} + \frac{s^4}{24},$$

$$\Phi_1(s) = \frac{265s}{218} - \frac{145s^2}{109} + \frac{13s^3}{109} - \frac{s^4}{218},\ \Phi_2(s) = \frac{156s}{109} - \frac{181s^2}{109} + \frac{26s^3}{109} - \frac{s^4}{109},$$

and $\sigma = \max\left\{\max_{s\in[0,1]}\Phi_1(s), \max_{s\in[0,1]}\Phi_2(s)\right\} \approx 0.3303$. Then all the hypotheses of Theorem 3.1 are fulfilled with $r = 1$. It follows from Theorem 3.1 that the BVP (4.1) has a monotone positive solution v^* satisfying

$$0 < v^*(t) \le 1 \ \text{ for } t \in (0,1], \quad 0 \le (v^*)'(t) \le 1 \ \text{ for } t \in [0,1].$$

Moreover, the iterative scheme is

$$\begin{aligned}
&v_0(t) = 0, \quad t \in [0,1],\\
&v_{n+1}(t) = \int_0^t \left[\frac{2ts - t^2s - s^2}{2} + g(t,s)\right]\left(\frac{1}{2}sv_n(s) + \frac{1}{8}\left(v_n'(s)\right)^2 + 1\right)ds\\
&\quad + \int_t^1 \left[\frac{t^2(1-s)}{2} + g(t,s)\right]\left(\frac{1}{2}sv_n(s) + \frac{1}{8}\left(v_n'(s)\right)^2 + 1\right)ds, \quad t \in [0,1],\ n = 1,2,\dots,\\
&v_{n+1}'(t) = \int_0^t \left[s(1-t) + g_t'(t,s)\right]\left(\frac{1}{2}sv_n(s) + \frac{1}{8}\left(v_n'(s)\right)^2 + 1\right)ds\\
&\quad + \int_t^1 \left[t(1-s) + g_t'(t,s)\right]\left(\frac{1}{2}sv_n(s) + \frac{1}{8}\left(v_n'(s)\right)^2 + 1\right)ds, \quad t \in [0,1],\ n = 1,2,\dots.
\end{aligned}$$

where

$$\begin{aligned}
g(t,s) &= \left(\frac{126t}{109} - \frac{48t^2}{109}\right)\left(\frac{s}{8} - \frac{s^2}{4} + \frac{s^3}{6} - \frac{s^4}{24}\right) + \left(\frac{6t}{109} + \frac{60t^2}{109}\right)\left(\frac{5s}{24} - \frac{s^2}{4} + \frac{s^4}{24}\right),\\
g_t'(t,s) &= \left(\frac{126}{109} - \frac{96t}{109}\right)\left(\frac{s}{8} - \frac{s^2}{4} + \frac{s^3}{6} - \frac{s^4}{24}\right) + \left(\frac{6}{109} + \frac{120t}{109}\right)\left(\frac{5s}{24} - \frac{s^2}{4} + \frac{s^4}{24}\right),
\end{aligned}$$

for $t, s \in [0,1] \times [0,1]$.

The first, second and third terms of the scheme v_n and v_n' are as follows:

$$\begin{aligned}
v_0(t) &= 0,\\
v_1(t) &= \frac{7}{436}t + \frac{142}{545}t^2 - \frac{1}{6}t^3,\\
v_2(t) &= \frac{255406447517}{15664670784000}t + \frac{57295606951}{217564872000}t^2 - \frac{506939}{3041536}t^3\\
&\quad - \frac{497}{5702880}t^4 - \frac{379849}{570288000}t^5 - \frac{71}{130800}t^6 + \frac{1}{4032}t^7,
\end{aligned}$$

$$\begin{aligned}
v_0'(t) &= 0,\\
v_1'(t) &= \frac{7}{436} + \frac{284}{545}t - \frac{1}{2}t^2,\\
v_2'(t) &= \frac{255406447517}{15664670784000} + \frac{57295606951}{108782436000}t - \frac{1520817}{3041536}t^2\\
&\quad - \frac{497}{1425720}t^3 - \frac{379849}{114057600}t^4 - \frac{71}{21800}t^5 + \frac{1}{576}t^6.
\end{aligned}$$

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