Electronic Journal of Qualitative Theory of Differential Equations
2012, No. **34**, 1-12; **http://www.math.u-szeged.hu/ejqtde/**

Almost periodic solutions for Fox production harvesting model with delay

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Abstract. By utilizing the continuation theorem of coincidence degree theory, we shall prove that a Fox production harvesting model with delay has at least one positive almost periodic solution. Some preliminary assertions are provided prior to proving our main theorem. We construct a numerical example along with graphical representations to illustrate feasibility of the theoretical result.

Keywords. Almost periodic solution; Fox production harvesting model; Coincidence degree theory.
AMS subject classification: 34K14.

1 Introduction

Consider the following equation of population dynamics [1, 2]

$$x'(t) = -xF(t,x) + xG(t,x), \quad x'(t) = \frac{\mathrm{d}x}{\mathrm{d}t}, \tag{1}$$

where $x = x(t)$ is the size of the population, $F(t,x)$ is the per–capita harvesting rate and $G(t,x)$ is the per–capita fecundity rate. Let $G(t,x)$ and $F(t,x)$ be defined in the form

$$F(t,x) = \alpha(t) \ \text{ and } \ G(t,x) = \beta(t)\ln^{\gamma}\Big(\frac{K(t)}{x(t)}\Big), \ \gamma > 0$$

then equation (1) becomes

$$x'(t) = -\alpha(t)x(t) + \beta(t)x(t)\ln^{\gamma}\Big(\frac{K(t)}{x(t)}\Big), \tag{2}$$

where $\alpha(t)$ is a variable harvesting rate, $\beta(t)$ is an intrinsic factor and $K(t)$ is a varying environmental carrying capacity. The positive parameter γ is referred to as an interaction parameter [1, 3, 4]. Indeed, if $\gamma > 1$ then intra–specific competition is high whereas if $0 < \gamma < 1$ then the competition is low. For $\gamma = 1$, equation (2) reduces to a classical

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Gompertzian model with harvesting [2, 5]. Equation (2) is called a Fox surplus production model that has been used to build up certain prediction models such as microbial growth model, demographic model and fisheries model. This equation is considered to be an efficient alternative to the well known γ–logistic model. Specifically, Fox model is more appropriate upon describing lower population density; we refer the reader to the papers [1, 3, 4, 6, 7, 8] for more informations.

One of the most important behaviors of solutions which has been a main object of investigations among authors is the periodic behavior of solutions [9, 10, 11, 12, 13, 14, 15, 16]. To consider periodic environmental factors acting on a population model, it is natural to study the model subject to periodic coefficients. Indeed, the assumption of periodicity of the parameters in the model is a way of incorporating the time–dependent variability of the environment (e.g. seasonal effects of weather, food supplies, mating habits and harvesting). On the other hand, upon considering long–term dynamical behavior, it has been found that the periodic parameters often turn out to experience some perturbations that may lead to a changing in character. Thus, the investigation of almost periodic behavior is considered to be in more accordance with reality; see the remarkable monographs [17, 18, 19] for more details. In [20], the author has declared that population models involving delayed argument provide better description for real phenomena. For this reason, we shall consider system (2) in the form

$$x'(t) = -\alpha(t)x(t) + \beta(t)x(t)\ln^{\gamma}\left(\frac{K(t)}{x(\tau(t))}\right), \tag{3}$$

where $\tau(t) < t$ and the parameters are to satisfy certain almost periodic assumptions that will be specified later. System (3) has been rarely investigated in the literature; see for instance the papers [21, 22] where the authors concentrated on studying the existence of periodic solutions, stability, oscillation and the global attractivity of the solutions. It is worth pointing out, however, that all the above mentioned papers are presented under periodic assumptions and to the best of authors' observation no paper has been published regarding the almost periodicity of system (3).

Although it has more widespread applications in real life, the notion of almost periodicity has been less considered among researchers. We mention here some recent works [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38] in which several methods have been used to prove, in particular, the existence of almost periodic solutions. Following this trend, we shall utilize the continuation theorem of coincidence degree theory to prove that system (3) has at least one positive almost periodic solution.

The remaining part of this paper is organized as follows: In Section 2, some preliminary assertions are provided prior to proving our main theorem. In section 3, sufficient conditions are established to investigate the existence of almost periodic solutions of the said system. Section 4 contains a numerical example along with graphical representations to illustrate the feasibility of the theoretical result.

2 Preliminary assertions

Through the change of variables $x(t) = e^{y(t)}$, we observe that (3) reduces to the equation

$$y'(t) = -\alpha(t) + \beta(t) \ln^{\gamma} \left(\frac{K(t)}{e^{y(\tau(t))}} \right). \tag{4}$$

We consider equation (4) together with the initial condition

$$y(t) = \phi(t), \ \ y(0) = y_0, \quad t \in (-\infty, 0). \tag{5}$$

Equations (4) and (5) are considered under the following assumptions:

(I) $\alpha(t), \beta(t) \in C([0, +\infty), [0, +\infty))$ and $K(t) \in C([0, +\infty), (0, +\infty))$;

(II) $\tau(t) \in C([0, +\infty), [0, +\infty))$ with $\tau(t) \leq t$ and $\lim_{t \to \infty} \tau(t) = \infty$;

(III) $\phi(t) \in C((-\infty, 0), [0, +\infty))$ with $\phi(t) \geq 0$ and $y_0 > 0$.

By a solution of (4) and (5) we mean an absolutely continuous function $y(t)$ defined on $\mathbb{R}$ satisfying (4) almost everywhere for $t \geq 0$ and (5). Due to certain biological purposes, we focus our discussion on the positive solutions of (4).

To prove the main results of this paper we shall utilize the continuation theorem of coincidence degree theory [39, 40]. However, before proceeding to the main results we set forth some basic concepts in the framework of this theory.

Let X, Y be normed vector spaces, $L : \mathrm{Dom}L \subset X \to Y$ be a linear mapping and $N : X \to Y$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim\mathrm{Ker}\, L = \mathrm{codim}\mathrm{Im}\, L < +\infty$ and $\mathrm{Im}\, L$ is closed in Y. If L is a Fredholm mapping of index zero and there exists continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}\, P = \mathrm{Ker}\, L$, $\mathrm{Ker}\, Q = \mathrm{Im}\, L = \mathrm{Im}\, (I - Q)$, it follows that the mapping $L|_{\mathrm{Dom}\, L \cap \mathrm{Ker} P} : (I - P)X \to \mathrm{Im}\, L$ is invertible. We denote the inverse of that mapping by K_P. If Ω is an open bounded subset of X, then the mapping N will be called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact. Since $\mathrm{Im}\, Q$ is isomorphic to $\mathrm{Ker}\, L$, there exists an isomorphism $J : \mathrm{Im}\, Q \to \mathrm{Ker}\, L$.

We mention the statement of the continuation theorem.

Theorem 1. [40] *Let $\Omega \subset X$ be an open bounded set and let $N : X \to Y$ be a continuous operator which is $L-$compact on $\overline{\Omega}$. Assume*

(1) *$Ly \neq \lambda Ny$ for every $y \in \partial\Omega \cap \mathrm{Dom}L$ and $\lambda \in (0, 1)$;*

(2) *$QNy \neq 0$ for every $y \in \partial\Omega \cap \mathrm{Ker}L$;*

(3) *The Brouwer degree* $\deg \Big\{ JQN, \, \Omega \cap \mathrm{Ker}L, \, 0 \Big\} \neq 0$.

Then $Ly = Ny$ has at least one solution in $\mathrm{Dom}L \cap \overline{\Omega}$.

Let $AP(\mathbb{R})$ denote the set of all real valued almost periodic functions on $\mathbb{R}$. For $f \in AP(\mathbb{R})$ we denote by

$$\Lambda(f) = \left\{ \tilde{\lambda} \in \mathbb{R} : \lim_{T\to\infty} \frac{1}{T} \int_0^T f(s) e^{-i\tilde{\lambda}s} \,\mathrm{d}s \neq 0 \right\}$$

and

$$(\mathrm{mod}\ f) = \left\{ \sum_{j=1}^{m} n_j \tilde{\lambda}_j : n_j \in \mathbb{Z}, m \in \mathbb{N},\ \tilde{\lambda}_j \in \Lambda(f),\ j = 1, 2, \ldots, m \right\}$$

the set of Fourier exponents and the module of f, respectively. Let $K(f, \varepsilon, S)$ denote the set of $\varepsilon-$almost periods for f with respect to $S \subset C((-\infty, 0], \mathbb{R})$ and $l(\varepsilon)$ denote the length of the inclusion interval.

Definition 1. [18] *$y(t) \in C(\mathbb{R}, \mathbb{R})$ is said to be almost periodic on $\mathbb{R}$ if for any $\varepsilon > 0$ the set $K(y, \varepsilon) = \{\delta : |y(t + \delta) - y(t)| < \varepsilon,\ \forall\, t \in \mathbb{R}\}$ is relatively dense, that is, for any $\varepsilon > 0$ it is possible to find a real number $l(\varepsilon) > 0$ for any interval with length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $|y(t + \delta) - y(t)| < \varepsilon$ for any $t \in \mathbb{R}$.*

Throughout the rest of the paper we assume the following condition

$$\text{(H)}\ \ \alpha(t), \beta(t), K(t) \in AP(\mathbb{R}) \ \text{ and } \ \inf_{t\in[0,\infty)} \ln K(t) > \tfrac{m(\alpha(t))}{\gamma m(\beta(t))},\ m(\beta(t)) \neq 0,$$

where $m(f) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(s)\mathrm{d}s$.

In our case, we set

$$\mathbb{X} = \mathbb{Y} = V_1 \oplus V_2,$$

where

$$V_1 = \left\{ y \in AP(\mathbb{R}) : (\mathrm{mod}\ y(t)) \subset (\mathrm{mod}\ F),\ \forall\ \mu \in \Lambda(y(t)) \text{ satisfying } |\mu| > c \right\}$$

and

$$V_2 = \{y(t) \equiv k,\ k \in \mathbb{R}\},$$

where

$$F = F(t, \phi) = -\alpha(t) + \beta(t) \ln^{\gamma} \left(\frac{K(t)}{e^{\phi(\tau(0))}} \right),\ \phi \in C((-\infty, 0], \mathbb{R})$$

and c is a given positive constant.

Remark 1. *If f is $T-$periodic function, then $\int^t f(s)\,\mathrm{d}s$ is $T-$periodic if and only if $m(f) = 0$. However, if $f(t) \in AP(\mathbb{R})$, then f does not necessarily have an almost periodic primitive though $m(f) = 0$. This is why we do not choose the space V_1 as usual. Stated another way, the choice of the space $\overline{V}_1 = \{y(t) \in AP(\mathbb{R}) : m(y(t)) = 0\}$ is not appropriate for our approach.*

Define the norm

$$\|y\| = \sup_{t\in\mathbb{R}} |y(t)|, \quad y \in \mathbb{X} \text{ (or } \ \mathbb{Y}).$$

Lemma 1. *$\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces equipped with the norm $\|\cdot\|$.*

Proof. If $\{y_n\} \subset V_1$ and y_n converges to y_0, then it is easy to show that $y_0 \in AP(\mathbb{R})$ with $\pmod{y_0} \subset \pmod{F}$. Indeed, for all $|\tilde{\lambda}| \leq \alpha$ we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T y_n(s) e^{-i\tilde{\lambda}s}\, \mathrm{d}s = 0.$$

Thus

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T y_0(s) e^{-i\tilde{\lambda}s}\, \mathrm{d}s = 0,$$

which implies that $y_0 \in V_1$. One can easily see that V_1 is a Banach space endowed with the norm $\|\cdot\|$. The same can be concluded for the spaces $\mathbb{X}$ and $\mathbb{Y}$. The proof is complete. $\square$

Lemma 2. *Let $L : \mathbb{X} \to \mathbb{Y}$ such that $Ly = -\alpha(t) + \beta(t) \ln^\gamma \left(\frac{K(t)}{e^{y(\tau(t))}} \right)$ where $Ly = y' = \frac{\mathrm{d}y}{\mathrm{d}t}$. Then L is a Fredholm mapping of index zero.*

Proof. It is obvious that L is a linear operator and $\mathrm{Ker}L = V_2$. It remains to prove that $\mathrm{Im}\, L = V_1$. Suppose that $\phi(t) \in \mathrm{Im}\, L \subset \mathbb{Y}$. Then, there exists $\phi_1 \in V_1$ and $\phi_2 \in V_2$ such that

$$\phi = \phi_1 + \phi_2.$$

From the definitions of $\phi(t)$ and $\phi_1(t)$, one can deduce that $\int^t \phi(s)\, \mathrm{d}s$ and $\int^t \phi_1(s)\, \mathrm{d}s$ are almost periodic functions and thus $\phi_2(t) \equiv 0$ which implies that $\phi(t) \in V_1$. This tells that

$$\mathrm{Im}\, L \subset V_1.$$

On the other hand, if $\varphi(t) \in V_1 \backslash \{0\}$ then we have $\int_0^t \varphi(s)\, \mathrm{d}s \in AP(\mathbb{R})$. Indeed, if $\tilde{\lambda} \neq 0$ then we obtain

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Big[\int_0^t \varphi(s)\, \mathrm{d}s \Big] e^{-i\tilde{\lambda}t}\, \mathrm{d}t = \frac{1}{i\tilde{\lambda}} \lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi(t) e^{-i\tilde{\lambda}t}\, \mathrm{d}t.$$

It follows that

$$\Lambda \Big[\int_0^t \varphi(s)\, \mathrm{d}s - m\Big(\int_0^t \varphi(s)\, \mathrm{d}s \Big) \Big] = \Lambda(\varphi(t)).$$

Thus

$$\int_0^t \varphi(s)\, \mathrm{d}s - m\Big(\int_0^t \varphi(s)\, \mathrm{d}s \Big) \in V_1 \subset \mathbb{X}.$$

We note that $\int_0^t \varphi(s)\, \mathrm{d}s - m(\int_0^t \varphi(s)\, \mathrm{d}s)$ is the primitive of $\varphi(t)$ in $\mathbb{X}$, therefore we have $\varphi(t) \in \mathrm{Im}\, L$. Hence, we deduce that

$$V_1 \subset \mathrm{Im}\, L$$

which completes the proof of our claim. Therefore, $\mathrm{Im}\, L = V_1$.

Furthermore, one can easily show that $\mathrm{Im}\, L$ is closed in $\mathbb{Y}$ and

$$\mathrm{dimKer}L = 1 = \mathrm{codimIm}L.$$

Therefore, L is a Fredholm mapping of index zero. $\square$

Lemma 3. *Let $N : \mathbb{X} \to \mathbb{Y}$, $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ such that*

$$Ny = -\alpha(t) + \beta(t) \ln^{\gamma} \Big(\frac{K(t)}{e^{y(\tau(t))}} \Big), \quad y \in \mathbb{X}$$

and

$$Py = m(y), \;\; y \in \mathbb{X}, \quad Qz = m(z), \quad z \in \mathbb{Y}.$$

Then, N is $L-$compact on $\overline{\Omega}$ (Ω is an open and bounded subset of $\mathbb{X}$).

Proof. The projections P and Q are continuous such that

$$\operatorname{Im} P = \operatorname{Ker} L \;\text{ and }\; \operatorname{Im} L = \operatorname{Ker} Q.$$

It is clear that

$$(I - Q)V_2 = \{0\} \quad \text{and} \quad (I - Q)V_1 = V_1.$$

Therefore

$$\operatorname{Im}(I - Q) = V_1 = \operatorname{Im} L.$$

In view of

$$\operatorname{Im} P = \operatorname{Ker} L \quad \text{and} \quad \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$$

we can conclude that the generalized inverse (of L) $K_p : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ exists and is given by

$$K_p(z) = \int_0^t z(s)\, \mathrm{d}s - m\Big(\int_0^t z(s)\, \mathrm{d}s \Big).$$

Thus

$$QNy = m\Big(-\alpha(t) + \beta(t) \ln^{\gamma} \frac{K(t)}{e^{y(\tau(t))}} \Big)$$

and

$$K_p(I - Q)Ny = f[y(t)] - Qf[y(t)],$$

where $f[y]$ is defined by

$$f[y(t)] = \int_0^t \Big[Ny(s) - QNy(s) \Big] \mathrm{d}s.$$

The integral form of the terms of both QN and $(I - Q)N$ imply that they are continuous. We claim that K_p is also continuous. By our hypothesis, for any $\varepsilon < 1$ and any compact set $S \subset C([-\sigma, 0], \mathbb{R})$, let $l(\varepsilon, S)$ be the inclusion interval of $K(F, \varepsilon, S)$. Suppose that $\{z_n(t)\} \subset \operatorname{Im} L = V_1$ and $z_n(t)$ uniformly converge to $z_0(t)$. Because $\int_0^t z_n(s)\, \mathrm{d}s \in \mathbb{Y}(n = 0, 1, 2, 3, \ldots)$, there exists ρ $(0 < \rho < \varepsilon)$ such that $K(F, \rho, S) \subset K(\int_0^t z_n(s)\, \mathrm{d}s, \varepsilon)$. Let $l(\rho, S)$ be the inclusion interval of $K(F, \rho, S)$ and

$$l = \max\{l(\rho, S), l(\varepsilon, S)\}.$$

It is easy to see that l is the inclusion interval of both $K(F,\varepsilon,S)$ and $K(F,\rho,S)$. Hence, for all $t\notin[0,l]$, there exists $\mu_t\in K(F,\rho,S)\subset K(\int_0^t z_n(s)\,\mathrm{d}s,\varepsilon)$ such that $t+\mu_t\in[0,l]$. Therefore, by the definition of almost periodic functions we observe that

$$\begin{aligned}\Big\|\int_0^t z_n(s)\,\mathrm{d}s\Big\| &= \sup_{t\in\mathbb{R}}\Big|\int_0^t z_n(s)\,\mathrm{d}s\Big| \le \sup_{t\in[0,l]}\Big|\int_0^t z_n(s)\,\mathrm{d}s\Big| \\ &+ \sup_{t\notin[0,l]}\Big|\Big(\int_0^t z_n(s)\,\mathrm{d}s-\int_0^{t+\mu_t} z_n(s)\,\mathrm{d}s\Big)+\int_0^{t+\mu_t} z_n(s)\,\mathrm{d}s\Big| \\ &\le 2\sup_{t\in[0,l]}\Big|\int_0^t z_n(s)\,\mathrm{d}s\Big|+\sup_{t\notin[0,l]}\Big|\Big(\int_0^t z_n(s)\,\mathrm{d}s-\int_0^{t+\mu_t} z_n(s)\,\mathrm{d}s\Big)\Big| \\ &\le 2\int_0^l |z_n(s)|\,\mathrm{d}s+\varepsilon. \end{aligned}\tag{6}$$

By applying (6), we conclude that $\int_0^t z(s)\,\mathrm{d}s$ $(z\in\operatorname{Im}L)$ is continuous and consequently K_p and $K_p(I-Q)Ny$ are also continuous.

From (6), we also have $\int_0^t z(s)\,\mathrm{d}s$ and $K_p(I-Q)Ny$ are uniformly bounded in $\overline{\Omega}$. In addition, it is not difficult to verify that $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)Ny$ is equicontinuous in $\overline{\Omega}$. Hence by the Arzela–Ascoli theorem, we can immediately conclude that $K_p(I-Q)N(\overline{\Omega})$ is compact. Thus N is $L-$compact on $\overline{\Omega}$. □

3 The main result

One immediate observation is that if equation (4) has one almost periodic solution y, then $x=e^y$ is a positive almost periodic solution of (3). Therefore, in the proof of Theorem 2 it suffices to show that (4) has one almost periodic solution.

Theorem 2. *Let condition* (H) *holds. Then, equation (4) has at least one positive almost periodic solution.*

Proof. In order to use the continuation theorem of coincidence degree theory, we set the Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ the same as those in Lemma 1 and the mappings L,N,P,Q the same as those defined in Lemma 2 and Lemma 3, respectively. Thus, we can deduce that L is a Fredholm mapping of index zero and N is a continuous operator which is L–compact on $\overline{\Omega}$. It remains to search for an appropriate open and bounded subset Ω.

Corresponding to the operator equation

$$Ly=\lambda Ny,\quad \lambda\in(0,1)$$

we may write

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t}=\lambda\Big[-\alpha(t)+\beta(t)\ln^{\gamma}\Big(\frac{K(t)}{e^{y(\tau(t))}}\Big)\Big].\tag{7}$$

Let $y=y(t)\in\mathbb{X}$ be an arbitrary solution of (7) for a certain $\lambda\in(0,1)$. Define

$$y(\eta)=\sup_{t\in\mathbb{R}}y(t)\ \text{ and }\ y(\zeta)=\inf_{t\in\mathbb{R}}y(t).$$

In view of (7), we have

$$m(\alpha(t)) = m\Big(\beta(t)\ln^{\gamma}\Big(\frac{K(t)}{e^{y(\tau(t))}}\Big)\Big).$$

It follows that

$$y(\tau(t)) = \ln K(t) - \frac{1}{\gamma}\frac{m(\alpha(t))}{m(\beta(t))}.$$

Therefore, we may write

$$y(\zeta) \le \ln K(t) - \frac{1}{\gamma}\frac{m(\alpha(t))}{m(\beta(t))}. \tag{8}$$

Similarly, we can get

$$y(\eta) \ge \ln K(t) - \frac{1}{\gamma}\frac{m(\alpha(t))}{m(\beta(t))}. \tag{9}$$

By inequalities (8) and (9), we can find that there exists $t_0 \in \mathbb{R}$ such that

$$|y(t_0)| \le M_1,$$

where

$$M_1 = \Big|\ln K(t) - \frac{1}{\gamma}\frac{m(\alpha(t))}{m(\beta(t))}\Big|.$$

By virtue of (6), we get

$$\|y(t)\| \le |y(t_0)| + \sup_{t\in\mathbb{R}}\Big|\int_{t_0}^{t} y'(s)ds\Big| \le M_1 + 2\sup_{t\in[t_0,t_0+l]}\Big|\int_{t_0}^{t} y'(s)ds\Big| + \varepsilon$$

or

$$\|y(t)\| \le M_1 + 2\int_{t_0}^{t_0+l} |y'(s)|ds + 1. \tag{10}$$

Choose a point $\nu - t_0 \in [l, 2l] \cap K(F, \rho, S)$ where ρ $(0 < \rho < \varepsilon)$ satisfies $K(F, \rho, S) \subset K(z, \varepsilon)$. Integrating (7) from t_0 to ν, we get

$$\begin{aligned} \lambda\int_{t_0}^{\nu}\Big[\beta(s)\ln^{\gamma}\Big(\frac{K(s)}{e^{y(\tau(s))}}\Big)\Big]\,\mathrm{d}s &= \lambda\int_{t_0}^{\nu}|\alpha(s)|\,\mathrm{d}s + \int_{t_0}^{\nu} y'(s)\,\mathrm{d}s \\ &\le \lambda\int_{t_0}^{\nu}|\alpha(s)|\,\mathrm{d}s + \varepsilon. \end{aligned} \tag{11}$$

However, from (7) and (11), we obtain

$$\begin{aligned} \int_{t_0}^{\nu}|y'(s)|\,\mathrm{d}s &\le \lambda\int_{t_0}^{\nu}|\alpha(s)|\,\mathrm{d}s + \lambda\int_{t_0}^{\nu}\Big[\beta(s)\ln^{\gamma}\Big(\frac{K(s)}{e^{y(\tau(s))}}\Big)\Big]\,\mathrm{d}s \\ &\le 2\int_{t_0}^{\nu}|\alpha(s)|\,\mathrm{d}s + \varepsilon \le 2\int_{t_0}^{\nu}|\alpha(s)|\,\mathrm{d}s + 1. \end{aligned} \tag{12}$$

Substituting back in (10) and for $\nu \ge t_0 + l$, we have

$$\|y(t)\| \le M_2,$$

where

$$M_2 = M_1 + 4\int_0^{\nu} |\alpha(s)|\,\mathrm{d}s + 3.$$

Let $M = M_1 + M_2$. Obviously, it is independent of λ. Take

$$\Omega = \{y \in \mathbb{X} : \|y\| < M\}.$$

It is clear that Ω satisfies assumption (1) of Theorem 1. If $y \in \partial\Omega \cap \mathrm{Ker}L$, then y is a constant with $\|y\| = M$. It follows that

$$QNy = m\Big(- \alpha(t) + \beta(t)\ln^{\gamma}\Big(\frac{K(t)}{e^{y(\tau(t))}}\Big)\Big) \neq 0,$$

which implies that assumption (2) of Theorem 1 is satisfied. The isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$ is defined by $J(z) = z$ for $z \in \mathbb{R}$. Thus, $JQNy \neq 0$. In order to compute the Brouwer degree, we consider the homotopy

$$H(y,s) = -sy + (1-s)JQNy,\ 0 \le s \le 1.$$

For any $y \in \partial\Omega \cap \mathrm{Ker}L$, $s \in [0,1]$, we have $H(y,s) \neq 0$. By the homotopic invariance of topological degree, we get

$$\deg\Big\{JQN,\ \Omega \cap \mathrm{Ker}L,\ 0\Big\} = \deg\Big\{-y,\ \Omega \cap \mathrm{Ker}L,\ 0\Big\} \neq 0.$$

Therefore, assumption (3) of Theorem 1 holds. Hence, $Ly = Ny$ has at least one solution in $\mathrm{Dom}L \cap \overline{\Omega}$. In other words, equation (4) has at least one positive almost periodic solution $y(t)$. The proof is complete. □

4 A Numerical example

Here we give an example that illustrates the almost periodic behavior of Fox production harvesting model with delay.

Example 1. *Let* $\alpha(t) = e^{-\pi}(2 + \sin\sqrt{3}\,t)$, $\beta(t) = e^{-\pi}(3 + \cos 2\,t)$ *and* $K(t) = 3 + \sin\,t$. *Then, equation (4) becomes*

$$y'(t) = -e^{-\pi}(2 + \sin\sqrt{3}\,t) + e^{-\pi}(3 + \cos 2\,t)\ln^{\gamma}\Big(\frac{3 + \sin t}{e^{y(t-\tau)}}\Big). \tag{13}$$

One can easily realize that $\inf_{t\in[0,\infty)} \ln(3 + \sin\,t) = \ln 2$, $m(\alpha(t)) = 2e^{-\pi}$ *and* $m(\beta(t)) = 3e^{-\pi} \neq 0$. *Thus, it is clear that*

$$\ln 2 > \frac{2}{3\gamma} \tag{14}$$

and therefore condition (H) is satisfied when $\gamma > \frac{2}{3ln2} \approx 0.96$. *Therefore, by the consequence of Theorem 1, equation (13) has at least one positive almost periodic solution* $y(t)$. *The following graphs illustrate the almost periodic behavior for equation (13).*

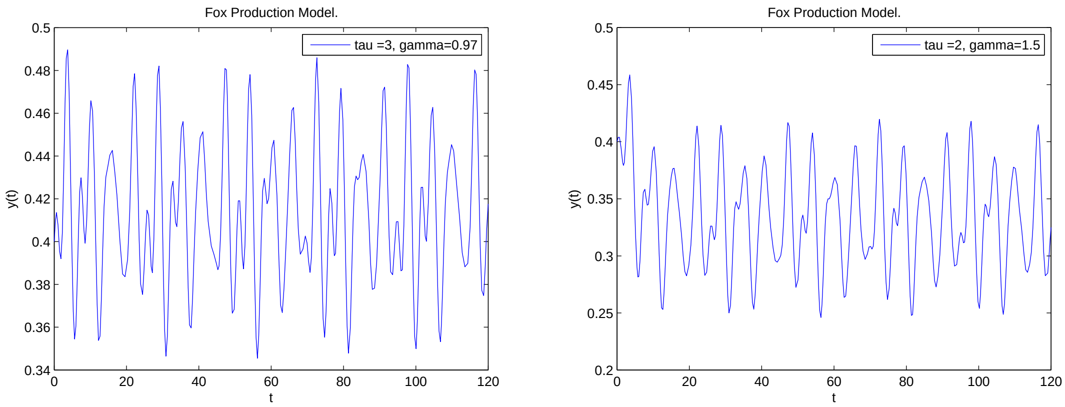

Figure 1: A Matlab simulation shows that the solution $y(t)$ of (13) converges to an almost periodic solution.

Acknowledment

The second author (S. Obaidat) extends his appreciation to the Deanship of Scientific Research at King Saud University for funding the work through the research group project No. RGP–VPP–117.

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(Received December 19, 2011)