Alternative Analysis Generated by a Differential Equation

Z. Z. Khukhunashvili, V. Z. Khukhunashvili
Department of Mathematics and Computer Science,
Tougaloo College, Tougaloo, MS 39149
Department of Mathematics, Tbilisi State University,
Tbilisi, Georgia

September, 2002

Abstract

It was shown in [1] that for a wide class of differential equations there exist infinitely many binary laws of addition of solutions such that every binary law has its conjugate. From this set of operations we extract commutative algebraic object that is a pair of two alternative to each other fields with common identity elements.

The goal of the present paper is to detect those mathematical constructions that are related to the existence of alternative fields dictated by differential equations. With this in mind we investigate differential and integral calculus based on the commutative algebra that is generated by a given differential equation. It turns out that along with the standard differential and integral calculus there always exists an isomorphic alternative calculus. Moreover, every system of differential equations generates its own calculus that is isomorphic (or homomorphic) to the standard one. The given system written in its own calculus appears to be linear.

It is also shown that there always exist two alternative to each other geometries, and matrix algebra has its alternative isomorphic to the classical one.

§1. Double Field

In [1] we showed that every quasilinear system of equations has a corresponding system of linear homogeneous equations, and their algebro-geometric structures are isomorphic (a discrete fiber of the space of solutions of a nonlinear equation is considered as a single element). The linear system is diagonalizable.

1. Under these circumstances, we can simplify the problem and study those algebraic operations that arise in linear equations. We consider one-dimensional linear equation in the complex plane:

$$\frac{dw}{dt} = Aw, \tag{1.1}$$

where A - is a given complex number, and t - a real variable. By W we denote the space of solutions of (1.1). We concentrate on commutative groups. Assume that w_1, w_2 are nonzero elements of W. In [1] it was shown that in the space of solutions there always exist two alternative to each other groups with binary operations:

$$w_1 \dot{+} w_2 = w_1 + w_2 \tag{1.2}$$

and

$$w_1 \ddot{+} w_2 = \left(w_1^{-1} + w_2^{-1}\right)^{-1}. \tag{1.3}$$

In the future (1.3) will be called an *alternative summation.* The mutually conjugate identity elements are $w = 0$ and $w = \infty$. The following is easily derived:

$$w \dot{+} 0 = 0, \quad w \dot{+} \infty = \infty, \tag{1.4}$$

and

$$w \ddot{+} \infty = w, \quad w \ddot{+} 0 = 0. \tag{1.5}$$

In these groups w and $-w$ are opposite elements.

From (1.4) and (1.5) the identity elements $w = 0$ and $w = \infty$ unite the alternative commutative groups (1.2) and (1.3) into a single algebraic object. It is important to note that $w = 0$ and $w = \infty$ play equal roles in the obtained algebra.

From (1.1) we see that if w is a solution, then aw is also a solution where a is an arbitrary complex number. The distributive property can be obtained from (1.1-4):

$$\begin{aligned} a\left(w_1 \dot{+} w_2\right) &= aw_1 \dot{+} aw_2, \\ a\left(w_1 \ddot{+} w_2\right) &= aw_1 \ddot{+} aw_2. \end{aligned} \tag{1.6}$$

So we have arrived at the commutative algebra acting in W.

2. Now let us put aside the equation (1.1) for a moment, and think of W as a set of all complex numbers plus a point at infinity. We introduce operations (1.2-3) on W. Elements 0 and ∞ satisfy (1.4-5), and we also assume that the standard complex multiplication in W together with the distributive property is valid:

$$\begin{aligned} w_1\left(w_2 \dot{+} w_3\right) &= w_1 w_2 \dot{+} w_1 w_3, \\ w_1\left(w_2 \ddot{+} w_3\right) &= w_1 w_2 \ddot{+} w_1 w_3. \end{aligned} \tag{1.7}$$

If operation (1.3) is removed, then W becomes the standard complex field. If instead operation (1.2) is removed , then W will be isomorphic to the complex field. We call these two objects *alternative fields.*

Therefore (1.1) generates an algebraic object consisting of two alternative fields that act (alternatively) in W. We call this object a *double field.*

The existence of the double field is not an exclusive property of (1.1), and it will be shown below that double fields arise for a wide set of differential equations.

3. Within the frames of this work it would be superfluous to discuss an importance of differential equations for the description of physical phenomena. We only point out that an equation carries a law of motion of a described phenomenon. But then we have to admit that algebraic properties of the differential equation reflect the properties of the phenomenon itself at least within the precision limits of the mathematical model. It follows that the differential and integral calculus that arise in continuous processes should be constructed on the base of a double field in order for the mathematical picture to be complete and agreeable to the laws of motion of the phenomenon.

§2. Differential Calculus

Our task is to construct differential calculus based on algebra (1.2-7) without leaving its frameworks. To do that, we consider a set L of sufficiently smooth complex-valued functions defined in some neighbourhood of a given point in the set R of real numbers. We restrict to functions that in the neighbourhood may have only isolated zeros and singularities.

1. We know that the derivative of a function $f(t) \in L$, $t,t_0 \in R$, is defined as:

$$\lim_{t \to t_0} \frac{1}{t - t_0} \left(f(t) - f(t_0) \right) = f'(t_0). \tag{2.1}$$

Note that the function under the limit sign is formed using algebraic operations (1.2) and (1.7).

2. By analogy with (2.1), using (1.3), we can consider the difference $F(\hat{t}) \mathbin{\ddot{-}} F(\hat{t}_0)$, where $F(\hat{t}) \in L$ is a given function in the neighbourhood of $\hat{t}_0 \in R$. This difference can be written as:

$$F(\hat{t}) \mathbin{\ddot{-}} F(\hat{t}_0) = \frac{F(\hat{t}) \cdot F(\hat{t}_0)}{F(\hat{t}_0) - F(\hat{t})}.$$

Applying the mean theorem, we obtain:

$$\lim_{\hat{t} \to \hat{t}_0} \frac{F(\hat{t}) \mathbin{\ddot{-}} F(\hat{t}_0)}{\frac{1}{\hat{t}} \mathbin{\ddot{-}} \frac{1}{\hat{t}_0}} = \frac{1}{\left(\frac{1}{F(\hat{t})} \right)'_{\hat{t} = \hat{t}_0}}. \tag{2.2}$$

Here we used operations (1.3) and (1.7).

3. To simplify the presentation and fix the notation, we introduce the following

Definition 2.1 *If two magnitudes A and B satisfy $A \cdot B = 1$, then we call them algebraically conjugate or a-conjugate magnitudes.*

The relations

$$t \cdot \hat{t} = 1,$$
$$f(t) \cdot \hat{f}(\hat{t}) = 1. \tag{2.3}$$

imply that t, $\hat{t}$ and $f(t)$, $\hat{f}(\hat{t})$ by definition are a-conjugate magnitudes.

In (2.2) let us put

$$F(\hat{t}) = f(t), \tag{2.4}$$

where t and $\hat{t}$ are related as (2.3). But since $f(t)$ and $\hat{f}(\hat{t})$ are a-conjugates, we derive:

$$\hat{f}(\hat{t}) = \frac{1}{f(t)} = \frac{1}{F(\hat{t})}. \tag{2.5}$$

The following is clear

$$\frac{1}{\hat{t}} \ddot{-} \frac{1}{\hat{t}_0} = t \ddot{-} t_0. \tag{2.6}$$

Taking into account (2.4-6) it immediately follows from (2.2) that

$$\lim_{t \to t_0} \frac{1}{t \ddot{-} t_0} \left(f(t) \ddot{-} f(t_0) \right) = \frac{1}{\hat{f}'(\hat{t}_0)}. \tag{2.7}$$

We denote the limit of the left-hand side of this equality as

$$f^{(!)}(t_0) = \lim_{t \to t_0} \frac{1}{t \ddot{-} t_0} \left(f(t) \ddot{-} f(t_0) \right). \tag{2.8}$$

The conclusion is that the function $f(t)$ at the same point t has two alternative derivatives (2.1) and (2.8). The term "alternative" is a consequence of algebra (1.2-7).

Definition 2.2 *In the future (2.1) will be called a standard derivative and (2.8) an alternative derivative.*

From (2.7-8) it immediately follows that:

$$f^{(!)}(t) \cdot \hat{f}'(\hat{t}) = 1. \tag{2.9}$$

In other words, if $f(t)$ and $\hat{f}(\hat{t})$ a-conjugate functions, i.e. satisfy (2.3), then the corresponding derivatives are also a-conjugate functions.

4. Let us study properties of alternative derivatives.

a) Let $f(t) = c = const$. From (2.9) it follows that:

$$c^{(!)} = \frac{1}{\hat{c}'} = \frac{1}{0} = \infty, \tag{2.10}$$

where $c \cdot \hat{c} = 1$.

b) For the function $c \cdot f(t)$ we have:

$$(c \cdot f(t))^{(!)} = \frac{1}{\left(\hat{c} \cdot \hat{f}(\hat{t})\right)'} = \frac{1}{\hat{c} \cdot \hat{f}'(\hat{t})} = c \cdot f^{(!)}(t),$$

i.e.

$$(c \cdot f(t))^{(!)} = c \cdot f^{(!)}(t), \tag{2.11}$$

where $c \cdot \hat{c} = 1$.

c) If $f(t)$, $\hat{f}(\hat{t})$ and $g(t)$, $\hat{g}(\hat{t})$ are a-conjugate functions respectively, then $f(t)\ddot{+}g(t)$ and $\hat{f}(\hat{t})\dot{+}\hat{g}(\hat{t})$ are also a-conjugate. Obviously

$$f(t)\ddot{+}g(t) = \frac{1}{\frac{1}{f(t)} + \frac{1}{g(t)}} = \frac{1}{\hat{f}(\hat{t}) + \hat{g}(\hat{t})}.$$

Based on (2.9), we can obtain:

$$\big(f(t)\ddot{+}g(t)\big)^{(!)} = f^{(!)}(t)\ddot{+}g^{(!)}(t). \tag{2.12}$$

It is worth mentioning that the following equalities do not hold:

$$(f(t) + g(t))^{(!)} = f^{(!)}(t) + g^{(!)}(t)$$

and

$$\big(f(t)\ddot{+}g(t)\big)^{'} = f^{'}(t)\ddot{+}g^{'}(t).$$

d)

$$(f(t) \cdot g(t))^{(!)} = f^{(!)}(t) \cdot g(t)\ddot{+}f(t) \cdot g^{(!)}(t). \tag{2.13}$$

e)

$$\left(\frac{f(t)}{g(t)}\right)^{(!)} = \frac{f^{(!)}(t) \cdot g(t)\ddot{-}f(t) \cdot g^{(!)}(t)}{g^2(t)}. \tag{2.14}$$

f) Taking into account (2.9) one can write the alternative derivative for the composite function:

$$(f\left(\varphi\left(t\right)\right))^{(!)} = \frac{1}{\left(\hat{f}\left(\hat{\varphi}\left(\hat{t}\right)\right)\right)^{'}} = \frac{1}{\hat{f}^{'}\left(\hat{\varphi}\left(\hat{t}\right)\right)\hat{\varphi}^{'}\left(\hat{t}\right)\cdot} = f^{(!)}\left(\varphi\left(t\right)\right) \cdot \varphi^{(!)}\left(t\right),$$

i.e.

$$(f\left(\varphi\left(t\right)\right))^{(!)} = f^{(!)}\left(\varphi\left(t\right)\right) \cdot \varphi^{(!)}\left(t\right). \tag{2.15}$$

Example 2.1

a)

$$(t^{\alpha})^{(!)} = \frac{1}{\alpha\hat{t}^{\alpha-1}} = \frac{1}{\alpha}t^{\alpha-1}.$$

Then

$$(t^{\alpha})^{(!)} = \frac{1}{\alpha}t^{\alpha-1}.$$

b)

$$\left(a^{t}\right)^{(!)} = \frac{1}{\left(a^{-\frac{1}{t}}\right)^{'}} = \frac{\hat{t}^{2}}{a^{-\frac{1}{\hat{t}}}\ln a} = \frac{a^{t}}{t^{2}\ln a},$$

so

$$\left(a^{t}\right)^{(!)} = \frac{a^{t}}{t^{2}\ln a}.$$

In particular $a = e$, so then:

$$\left(e^t\right)^{(!)} = \frac{1}{t^2} e^t.$$

5. Using relations (2.9), it is easy to find higher order derivatives:

$$(f(t))^{(!n)} = \frac{1}{\hat{f}^{(n)}\left(\hat{t}\right)}. \tag{2.16}$$

6. We define the differentiability of a function in a standard way. Function $f(t)$ is called alternatively differentiable at a point t_0, if the increment of the function

$$\hat{\Delta} f(t_0) = f(t) \ddot{-} f(t_0) \tag{2.17}$$

in some neighbourhood of this point can be presented in the form:

$$\hat{\Delta} f(t_0) = A\hat{\Delta} t \ddot{+} \omega\left(\hat{\Delta} t\right)\hat{\Delta} t, \tag{2.18}$$

where

$$\hat{\Delta} t = t \ddot{-} t_0. \tag{2.19}$$

In (2.18) the term $\omega\left(\hat{\Delta} t\right)$ is infinitely large when $\hat{\Delta} t \to \infty$. If we divide (2.18) by $\hat{\Delta} t$ and take a limit, we will have:

$$\lim_{\hat{\Delta} t \to \infty} \frac{\hat{\Delta} f(t_0)}{\hat{\Delta} t} = \lim_{\hat{\Delta} t \to \infty}\left(A \ddot{+} \omega\left(\hat{\Delta} t\right)\right) = A.$$

Thus from this and (2.8) relations we conclude:

$$A = f^{(!)}(t_0) \tag{2.20}$$

Now we define a differential of a function.

Definition 2.3 *An alternative differential of a function is the first term in the right hand side of (2.18), and it is denoted by*

$$\hat{d} f(t) = f^{(!)}(t)\, \hat{d} t. \tag{2.21}$$

From this equality it follows that:

$$\frac{\hat{d} f(t)}{\hat{d} t} = f^{(!)}(t). \tag{2.22}$$

Obviously

$$\Delta \hat{t} = \hat{t} - \hat{t}_0 = \frac{1}{t} - \frac{1}{t_0} = \frac{1}{t \ddot{-} t_0} = \frac{1}{\hat{\Delta} t}. \tag{2.23}$$

So we can write:

$$\hat{d} t = \frac{1}{d\hat{t}}.$$

From(2.9) and (2.21) we have:

$$\hat{d}t \cdot d\hat{t} = 1,$$
$$\hat{d}f(t) \cdot d\hat{f}\left(\hat{t}\right) = 1. \tag{2.24}$$

7. Recall the Taylor formula

$$f(t) = f(t_0) + \frac{f^{'}(t_0)}{1!}(t - t_0) + \cdots + \frac{f^{(n)}(t_0)}{n!}(t - t_0)^n + R_n, \tag{2.25}$$

where the remainder can be written as:

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(t - t_0)^{n+1}. \tag{2.26}$$

Using (2.3) and (2.25), we obtain:

$$f(t) = 1/(\hat{f}\left(\hat{t}_0\right) + \frac{\hat{f}^{'}\left(\hat{t}_0\right)}{1!}\left(\hat{t} - \hat{t}_0\right) + \cdots + \frac{\hat{f}^{(n)}\left(\hat{t}_0\right)}{n!}\left(\hat{t} - \hat{t}_0\right)^n + \frac{\hat{f}^{(n+1)}\left(\hat{\xi}\right)}{(n+1)!}\left(\hat{t} - \hat{t}_0\right)^{n+1}).$$

From (2.16) and (2.23), it follows that

$$f(t) = 1/(\frac{1}{f(t_0)} + \frac{1}{1!}\frac{1}{f^{(!)}(t_0)}\frac{1}{\left(t \ddot{-} t_0\right)} + \cdots + \frac{1}{n!}\frac{1}{f^{(!n)}(t_0)}\frac{1}{\left(t \ddot{-} t_0\right)^n} + \frac{1}{n!}\frac{1}{f^{(!(n+1))}(\xi)}\frac{1}{\left(t \ddot{-} t_0\right)^{n+1}}).$$

Or

$$f(t) = f(t_0) \ddot{+} 1! \cdot f^{(!)}(t_0) \cdot \left(t \ddot{-} t_0\right) \ddot{+} \cdots \ddot{+} n! \cdot f^{(!n)}(t_0) \cdot \left(t \ddot{-} t_0\right)^n \ddot{+} (n+1)! \cdot f^{(!(n+1))}(\xi) \cdot \left(t \ddot{-} t_0\right)^{n+1}, \tag{2.27}$$

where ξ and $\hat{\xi}$ are related as $\xi \cdot \hat{\xi} = 1$.

Thus we conclude that for any smooth function $f(t)$ at point t_0, there is an alternative expansion (2.27) along with the standard Taylor expansion (2.25).

Example 2.2 *Suppose*

$$f(t) = t^2. \tag{2.28}$$

We can expand this function at point $t_0 = 1$ according to the alternative Taylor formula (2.27). From example a) it follows that:

$$\left(t^2\right)^{(!)}|_{t=1} = \frac{1}{2}t|_{t=1} = \frac{1}{2},$$

$$\left(t^2\right)^{(!2)}|_{t=1} = \frac{1}{2}, \tag{2.29}$$

$$\left(t^2\right)^{(!n)}|_{t=1} = \infty, \text{ for } n \geqslant 3.$$

But then from (2.27) and (1.5) we have:

$$f(t) = 1 \ddot{+} \frac{1}{2}\left(t \ddot{-} 1\right) \ddot{+} \frac{2!}{2}\left(t \ddot{-} 1\right)^2. \tag{2.30}$$

Easy transformations show that the right hand side identically coincides with t^2*, that agrees with (2.28). If we restrict ourselves to (2.30) terms, we obtain*

$$y = 1 \ddot{+} \frac{1}{2}\left(t \ddot{-} 1\right),$$

or

$$y = \frac{t}{2-t}. \tag{2.31}$$

8. The conducted study shows that the construction of the standard differential calculus is restricted to the field (1.2), (1.4), (1.7_1) of the double field (1.2-7), whereas the one of the alternative differential calculus - to the field (1.3), (1.5), (1.7_2). The identity element 0 plays an essential role in the standard calculus also by announcing its conjugate identity ∞ a singular number. In the alternative calculus 0 and ∞ exchange their roles.

We saw that both standard and alternative calculus have the same properties. Therefore if we consider them as algebraic objects, then there should be an isomorphic relation between them. Some kind of relativity of calculus arises.

§3. Integral Calculus

1. Let a continuous function $f(t)$ be defined on $[a, b]$. Perform a partition of $[a, b]$ into n parts: $a = t_0 < t_1 < \cdots < t_n = b$. Recall that the usual Cauchy integral on $[a, b]$ is defined as a limit of the partial sum

$$\sum_{k=1}^{n} f(\xi_k)\,\Delta t_k, \tag{3.1}$$

when $\max\limits_k |\Delta t_k| \to 0$, where $\xi_k \in [t_{k-1}, t_k]$, and

$$\Delta t_k = t_k - t_{k-1}. \tag{3.2}$$

So

$$\lim_{\max\limits_k |\Delta t_k| \to 0} \sum_{k=1}^{n} f(\xi_k)\,\Delta t_k = \int_a^b f(t)\,dt. \tag{3.3}$$

2. Let us form the alternative partial sum based on (1.3):

$$\ddot{\sum_{k=1}^{n}} f(\xi_k)\,\hat{\Delta} t_k, \tag{3.4}$$

where

$$\hat{\Delta} t_k = t_k \overset{\cdot\cdot}{-} t_{k-1}. \tag{3.5}$$

If the limit of (3.4) when $\min\limits_k \left|\hat{\Delta} t_k\right| \to \infty$ exists, we call it an *alternative integral of* $f(t)$ *on* $[a, b]$*:*

$$\lim_{\min\limits_k \left|\hat{\Delta} t_k\right| \to \infty} \mathop{\overset{\bullet\,\bullet}{\sum}}\limits_{k=1}^{n} f(\xi_k)\, \hat{\Delta} t_k = \mathop{\overset{\bullet\,\bullet}{\int}}\limits_a^b f(t)\, \hat{d}t. \tag{3.6}$$

3. Rewrite (3.4) as follows:

$$\mathop{\overset{\bullet\,\bullet}{\sum}}\limits_{k=1}^{n} f(\xi_k)\, \hat{\Delta} t_k = \left(\sum_{k=1}^{n} \frac{1}{f(\xi_k)} \frac{1}{\hat{\Delta} t_k} \right)^{-1}.$$

Using (2.3) and (2.23) we obtain:

$$\mathop{\overset{\bullet\,\bullet}{\sum}}\limits_{k=1}^{n} f(\xi_k)\, \hat{\Delta} t_k = \left(\sum_{k=1}^{n} \hat{f}\left(\hat{\xi}_k\right) \Delta \hat{t}_k \right)^{-1}.$$

Obviously, $\min\limits_k \left|\hat{\Delta} t_k\right| \to \infty$ implies $\max\limits_k \left|\Delta \hat{t}_k\right| \to 0$. But then

$$\mathop{\overset{\bullet\,\bullet}{\int}}\limits_a^b f(t)\, \hat{d}t = \left(\int_{\hat{a}}^{\hat{b}} \hat{f}\left(\hat{t}\right) d\hat{t} \right)^{-1}, \tag{3.7}$$

where $a \cdot \hat{a} = 1$, $b \cdot \hat{b} = 1$. From this we immediately conclude that integrals $\mathop{\overset{\bullet\,\bullet}{\int}}\limits_a^b f(t)\, \hat{d}t$ and $\int\limits_{\hat{a}}^{\hat{b}} \hat{f}\left(\hat{t}\right) d\hat{t}$ are a-conjugate numbers.

4. Consider the function:

$$F(t) = \mathop{\overset{\bullet\,\bullet}{\int}}\limits_a^t f(\tau)\, \hat{d}\tau, \tag{3.8}$$

and introduce

$$\hat{F}\left(\hat{t}\right) = \int_{\hat{a}}^{\hat{t}} \hat{f}(\tau)\, d\tau, \tag{3.9}$$

where $t \cdot \hat{t} = 1$. Then it follows from (3.7) that

$$F(t) \cdot \hat{F}\left(\hat{t}\right) = 1. \tag{3.10}$$

Let us take the alternative derivative of (3.8). Using (2.9), we obtain:

$$F^{(!)}(t) = \left(\overset{\bullet\bullet}{\int\limits_a^t} f(\tau)\, \hat{d}\tau \right)^{(!)} = \frac{1}{\hat{F}'\left(\hat{t}\right)} = \frac{1}{\hat{f}'\left(\hat{t}\right)}.$$

But since $f(t) \cdot \hat{f}\left(\hat{t}\right) = 1$, finally we have

$$F^{(!)}(t) = \left(\overset{\bullet\bullet}{\int\limits_a^t} f(\tau)\, \hat{d}\tau \right)^{(!)} = f(t). \tag{3.11}$$

So we showed that $F(t)$ is an alternative antiderivative of $f(t)$.

5. From (3.9) it follows that

$$\int\limits_{\hat{a}}^{\hat{b}} \hat{f}(\tau)\, d\tau = \hat{F}\left(\hat{b}\right) - \hat{F}(\hat{a}). \tag{3.12}$$

But then from (3.7) and (3.10) we have:

$$\overset{\bullet\bullet}{\int\limits_a^b} f(\tau)\, \hat{d}\tau = \left(\hat{F}\left(\hat{b}\right) - \hat{F}(\hat{a})\right)^{-1} = \left[(F(b))^{-1} - (F(a))^{-1}\right]^{-1}.$$

Or finally

$$\overset{\bullet\bullet}{\int\limits_a^b} f(\tau)\, \hat{d}\tau = F(b) \ddot{-} F(a). \tag{3.13}$$

6. Let us study properties of the alternative integral:

a) From (3.7) it immediately follows that

$$\overset{\bullet\bullet}{\int\limits_a^b} c \cdot f(t)\, \hat{d}t = c \cdot \overset{\bullet\bullet}{\int\limits_a^b} f(t)\, \hat{d}t. \tag{3.14}$$

b) Let $f(t) = f_1(t) \ddot{+} f_2(t)$. Using equality

$$f_1(t) \ddot{+} f_2(t) = \frac{1}{\frac{1}{f_1(t)} + \frac{1}{f_2(t)}} = \frac{1}{\hat{f}_1(\hat{t}) + \hat{f}_2(\hat{t})},$$

from (3.7) we derive:

$$\int\limits_{a}^{\overset{b}{\bullet\bullet}} \left[f_1(t)\,\ddot{+}f_2(t)\right]\hat{d}t = \left(\int\limits_{\hat{a}}^{\hat{b}}\left[\hat{f}_1(\hat{t})+\hat{f}_2(\hat{t})\right]d\hat{t}\right)^{-1} = \left(\int\limits_{\hat{a}}^{\hat{b}}\hat{f}_1(\hat{t})d\hat{t}+\int\limits_{\hat{a}}^{\hat{b}}\hat{f}_2(\hat{t})d\hat{t}\right)^{-1} =$$

$$\left[\left(\int\limits_{a}^{\overset{b}{\bullet\bullet}} f_1(t)\hat{d}t\right)^{-1}+\left(\int\limits_{a}^{\overset{b}{\bullet\bullet}} f_2(t)\hat{d}t\right)^{-1}\right]^{-1} = \int\limits_{a}^{\overset{b}{\bullet\bullet}} f_1(t)\hat{d}t\,\ddot{+}\int\limits_{a}^{\overset{b}{\bullet\bullet}} f_2(t)\hat{d}t.$$

Or

$$\int\limits_{a}^{\overset{b}{\bullet\bullet}} \left[f_1(t)\,\ddot{+}f_2(t)\right]\hat{d}t = \int\limits_{a}^{\overset{b}{\bullet\bullet}} f_1(t)\hat{d}t\,\ddot{+}\int\limits_{a}^{\overset{b}{\bullet\bullet}} f_2(t)\hat{d}t. \tag{3.15}$$

7. Consider

$$\int\limits_{a}^{\overset{b}{\bullet\bullet}} f(t)\cdot g^{(!)}(t)\,\hat{d}t.$$

Using (2.3), (2.9) and (3.7), we obtain:

$$\int\limits_{a}^{\overset{b}{\bullet\bullet}} f(t)\cdot g^{(!)}(t)\,\hat{d}t = \left(\int\limits_{\hat{a}}^{\hat{b}}\hat{f}\left(\hat{t}\right)\cdot\hat{g}'\left(\hat{t}\right)d\hat{t}\right)^{-1}.$$

Integrating the right hand side by parts:

$$\int\limits_{a}^{\overset{b}{\bullet\bullet}} f(t)\cdot g^{(!)}(t)\,\hat{d}t = \left(\hat{f}\left(\hat{b}\right)\cdot\hat{g}\left(\hat{b}\right)-\hat{f}(\hat{a})\cdot\hat{g}(\hat{a})-\int\limits_{\hat{a}}^{\hat{b}}\hat{f}'\left(\hat{t}\right)\cdot\hat{g}\left(\hat{t}\right)d\hat{t}\right)^{-1},$$

or equivalently:

$$\int\limits_{a}^{\overset{b}{\bullet\bullet}} f(t)\cdot g^{(!)}(t)\,\hat{d}t = f(b)\cdot g(b)\,\ddot{-}f(a)\cdot g(a)\,\ddot{-}\int\limits_{a}^{\overset{b}{\bullet\bullet}} f^{(!)}(t)\cdot g(t)\,\hat{d}t. \tag{3.16}$$

In particular if we put $f(t)\equiv 1$ in (3.16), then taking into account $f^{(!)}(t)=\infty$, it is easy to see that

$$\int\limits_{a}^{\overset{b}{\bullet\bullet}} g^{(!)}(t)\,\hat{d}t = g(b)\,\ddot{-}g(a). \tag{3.17}$$

8. Let $t = \varphi(\tau)$, where $a = \varphi(\alpha)$, $b = \varphi(\beta)$. $\varphi(\tau)$ is differentiable and monotone function. It is easy to see that

$$\overset{\bullet\bullet}{\int_a^b} f(t)\,\hat{d}t = \overset{\bullet\bullet}{\int_\alpha^\beta} f(\varphi(\tau))\,\varphi^{(!)}(\tau)\,\hat{d}\tau. \tag{3.18}$$

9. If function $f(t)$ is continuous on $[a,b]$, then the following takes place

$$\overset{\bullet\bullet}{\int_a^b} f(t)\,\hat{d}t = f(c)\left(b \ddot{-} a\right), \tag{3.19}$$

where $c \in [a,b]$.

Thus we constructed the alternative integral calculus based on algebra (1.3), (1.5) and (1.7). This calculus is isomorphic to the standard integral calculus.

§4. Alternative Matrix Algebra

1. Consider a sufficiently smooth function of several variables

$$v = f\left(x^1, \ldots, x^N\right) \tag{4.1}$$

defined on R^N. We introduce a-conjugate (to f) function $\hat{f}\left(\hat{x}^1, \ldots, \hat{x}^N\right)$, i.e. such that:

$$\begin{gathered} x^1 \cdot \hat{x}^1 = 1, \ldots, x^N \cdot \hat{x}^N = 1, \\ f\left(x^1, \ldots, x^N\right) \cdot \hat{f}\left(\hat{x}^1, \ldots, \hat{x}^N\right) = 1. \end{gathered} \tag{4.2}$$

2. To introduce the formula of the alternative total differential of $f\left(x^1, \ldots, x^N\right)$ we let $M = \left(x^1, \ldots, x^N\right)$ and $M_0 = \left(x_0^1, \ldots, x_0^N\right)$ be points in R^N. The alternative increment is

$$\hat{\Delta} v = f(x) \ddot{-} f(x_0).$$

Then from (4.2) we obtain:

$$\hat{\Delta} v = \left(\hat{f}(\hat{x}) - \hat{f}(\hat{x}_0)\right)^{-1} = \left[\sum_{k=1}^{N} \frac{\partial \hat{f}(\hat{x}_0)}{\partial \hat{x}^k}\left(\hat{x}^k - \hat{x}_0^k\right) + \omega(\hat{x} - \hat{x}_0)\right]^{-1},$$

where $\omega \to 0$, whenever $\hat{x} \to \hat{x}_0$, and there is a summation from 1 to N along the index k. Using results of § 2, it is easy to show that (as in (2.9)) the following takes place

$$\frac{\hat{\partial} f(x_0)}{\hat{\partial} x^1} \cdot \frac{\partial \hat{f}(\hat{x}_0)}{\partial \hat{x}^1} = 1, \ldots, \frac{\hat{\partial} f(x_0)}{\hat{\partial} x^N} \cdot \frac{\partial \hat{f}(\hat{x}_0)}{\partial \hat{x}^N} = 1. \tag{4.3}$$

Then

$$\hat{\Delta} v=\frac{\hat{\partial} f\left(x_0\right)}{\hat{\partial} x^1} \hat{\Delta} x^1 \ddot{+} \ldots \ddot{+} \frac{\hat{\partial} f\left(x_0\right)}{\hat{\partial} x^N} \hat{\Delta} x^N \ddot{+} \frac{1}{\omega\left(\hat{\Delta} x\right)}, \tag{4.4}$$

where

$$\hat{\Delta} x^k=x^k \ddot{-} x_0^k, \qquad \Delta \hat{x}^k=\hat{x}^k-\hat{x}_0^k.$$

Definition 4.1 *First N terms in (4.4) are called the alternative total differential of the function (4.1):*

$$\hat{d} f=\frac{\hat{\partial} f}{\hat{\partial} x^{\hat{k}}} \hat{d} x^{\hat{k}}, \tag{4.5}$$

where the alternative summation takes place along the repeated indices $\hat{k}$.

In the future we assume that the usual summation is performed along the repeated indices. If these indices have the sign $\hat{}$, then the alternative summation is performed instead.

3. Let us now consider N functions of N variables:

$$y^k=f^k\left(x^1, \ldots, x^N\right), \tag{4.6}$$

$$(k=1, \ldots, N) .$$

The alternative differentials are

$$\hat{d} y^k=\frac{\hat{\partial} f^k(x)}{\hat{\partial} x^{\hat{\imath}}} \hat{d} x^{\hat{\imath}}, \tag{4.7}$$

$$(k=1, \ldots, N) .$$

Let

$$x^i=\varphi^i\left(t^1, \ldots, t^N\right), \tag{4.8}$$

$$(i=1, \ldots, N) .$$

Then obviously (4.7) will be

$$\hat{d} y^k=\frac{\hat{\partial} f^k}{\hat{\partial} x^{\hat{\imath}}} \cdot \frac{\hat{\partial} \varphi^{\hat{\imath}}}{\hat{\partial} t^{\hat{\jmath}}} \hat{d} t^{\hat{\jmath}}. \tag{4.9}$$

It follows that

$$\frac{\hat{\partial} f^k}{\hat{\partial} t^j}=\frac{\hat{\partial} f^k}{\hat{\partial} x^{\hat{\imath}}} \cdot \frac{\hat{\partial} \varphi^{\hat{\imath}}}{\hat{\partial} t^j}. \tag{4.10}$$

4. Consider matrices A, B, C with matrix elements $\frac{\hat{\partial} f^k}{\hat{\partial} x^i}$, $\frac{\hat{\partial} \varphi^k}{\hat{\partial} t^i}$, $\frac{\hat{\partial} f^k}{\hat{\partial} t^i}$ respectively. Then (4.10) will become

$$C=A \hat{\cdot} B, \tag{4.11}$$

where matrix elements c_i^k are computed as follows:

$$c_i^k=a_{\hat{\jmath}}^k \cdot b_i^{\hat{\jmath}} \quad\left(=a_1^k \cdot b_i^1 \ddot{+} \cdots \ddot{+} a_N^k \cdot b_i^N\right), \tag{4.12}$$

and we call (4.12) the *alternative matrix multiplication.*

5. To investigate matrix algebra based on operations (1.3), (1.7), we determine how the matrices based on the alternative algebra are added.

Let

$$Q^k(x) = f^k(x) \ddot{+} g^k(x), \qquad k = (1, \dots, N), \tag{4.13}$$

where

$$f^k(x), g^k(x) \in L^N, \ x \in R^N.$$

Since $\hat{d}Q^k(x) = \hat{d}f^k(x) \ddot{+} \hat{d}g^k(x)$, the expanded form can be written as:

$$\frac{\hat{\partial} Q^k(x)}{\hat{\partial} x^{\hat{\imath}}} \hat{d}x^{\hat{\imath}} = \frac{\hat{\partial} f^k(x)}{\hat{\partial} x^{\hat{\imath}}} \hat{d}x^{\hat{\imath}} \ddot{+} \frac{\hat{\partial} g^k(x)}{\hat{\partial} x^{\hat{\imath}}} \hat{d}x^{\hat{\imath}}, \qquad k = (1, \dots, N). \tag{4.14}$$

In the right hand side of (4.14) we can factor $\hat{d}x^{\hat{\imath}}$. Because $x^1, \dots, x^N$ are independent variables, their differentials are also independent. Then we can set the coefficients of $\hat{d}x^{\hat{\imath}}$ in the left and right hand sides equal to each other.

Fixing the notation

$$A_i^k = \frac{\hat{\partial} f^k(x)}{\hat{\partial} x^i}, \ B_i^k = \frac{\hat{\partial} g^k(x)}{\hat{\partial} x^i}, \ C_i^k = \frac{\hat{\partial} Q(x)^k}{\hat{\partial} x^i}, \tag{4.15}$$

we obtain:

$$C_i^k = A_i^k \ddot{+} B_i^k,$$

or in matrix form

$$C = A \ddot{+} B. \tag{4.16}$$

To prove the distributive law

$$A \hat{\cdot} \left(B \ddot{+} C\right) = A \hat{\cdot} B \ddot{+} A \hat{\cdot} C,$$

for matrices A, B, C, let

$$Q = Q(p(x)), \tag{4.17}$$

where

$$Q(p(x)), \ f(x), \ g(x) \in L^N, \quad p(x) = f(x) \ddot{+} g(x), \quad x \in R^N.$$

Using $\hat{d}p^i = \hat{d}f^i(x) \ddot{+} \hat{d}g^i(x)$ and differentiating (4.17), we have:

$$\hat{d}Q^k = \frac{\hat{\partial} Q^k}{\hat{\partial} p^{\hat{\imath}}} \hat{d}\left(f^{\hat{\imath}} \ddot{+} g^{\hat{\imath}}\right) = \frac{\hat{\partial} Q^k}{\hat{\partial} p^{\hat{\imath}}} \left(\hat{d}f^{\hat{\imath}} \ddot{+} \hat{d}g^{\hat{\imath}}\right), \qquad k = (1, \dots, N). \tag{4.18}$$

i.e.

$$\hat{d}Q^k = \frac{\hat{\partial} Q^k}{\hat{\partial} p^{\hat{\imath}}} \hat{d} f^{\hat{\imath}} \ddot{+} \frac{\hat{\partial} Q^k}{\hat{\partial} p^{\hat{\imath}}} \hat{d} g^{\hat{\imath}}, \tag{4.19}$$
$$k = (1, \dots, N).$$

Express $\hat{d} f^{\hat{\imath}}$ and $\hat{d} g^{\hat{\imath}}$ via $\hat{d} x^{\hat{\imath}}$. Then from (4.18) and (4.19) we obtain

$$\frac{\hat{\partial} Q^k}{\hat{\partial} p^{\hat{\imath}}} \left(\frac{\hat{\partial} f^{\hat{\imath}}}{\hat{\partial} x_{\hat{\jmath}}} \ddot{+} \frac{\hat{\partial} g^{\hat{\imath}}}{\hat{\partial} x_{\hat{\jmath}}} \right) \hat{d} x_{\hat{\jmath}} = \left(\frac{\hat{\partial} Q^k}{\hat{\partial} p^{\hat{\imath}}} \frac{\hat{\partial} f^{\hat{\imath}}}{\hat{\partial} x_{\hat{\jmath}}} \ddot{+} \frac{\hat{\partial} Q^k}{\hat{\partial} p^{\hat{\imath}}} \frac{\hat{\partial} g^{\hat{\imath}}}{\hat{\partial} x_{\hat{\jmath}}} \right) \hat{d} x_{\hat{\jmath}}.$$

From independence of $x^1, \dots, x^N$ and from

$$A^i_j = \frac{\hat{\partial} f^i}{\hat{\partial} x_j},\ B^i_j = \frac{\hat{\partial} g^i}{\hat{\partial} x_j},\ C^k_i = \frac{\hat{\partial} Q^k}{\hat{\partial} p^i},$$

we derive:

$$C^k_{\hat{\imath}} \hat{\cdot} \left(A^{\hat{\imath}}_j \ddot{+} B^{\hat{\imath}}_j \right) = C^k_{\hat{\imath}} \hat{\cdot} A^{\hat{\imath}}_j \ddot{+} C^k_{\hat{\imath}} \hat{\cdot} B^{\hat{\imath}}_j,$$

or in matrix form:

$$C \hat{\cdot} \left(A \ddot{+} B \right) = C \hat{\cdot} A \ddot{+} C \hat{\cdot} B.$$

6. We now introduce the identity matrix. Let

$$A \hat{\cdot} \hat{d} x = \hat{d} x,$$

or in the expanded form:

$$a^k_1 \hat{d} x^1 \ddot{+} \dots \ddot{+} a^k_N \hat{d} x^N = \hat{d} x^k. \tag{4.20}$$

If $\hat{x}^k$ are conjugates, $\hat{x}^k = \frac{1}{x^k},\ (k = 1, ...N)$,then (4.20) will have the form:

$$\left(\frac{1}{a^k_1} d\hat{x}^1 + \dots + \frac{1}{a^k_N} d\hat{x}^N \right)^{-1} = \frac{1}{d\hat{x}^k},$$

or

$$\frac{1}{a^k_1} d\hat{x}^1 + \dots + \frac{1}{a^k_N} d\hat{x}^N = d\hat{x}^k.$$

This equality for derivatives $d\hat{x}^k$ will be satisfied if

$$\frac{1}{a^k_i} = \delta^k_i = \{ \begin{matrix} 1,\ k = i, \\ 0, k \neq i\ . \end{matrix}$$

Thus the a-conjugate identity matrix is:

$$a^k_i = \hat{\delta}^k_i = \{ \begin{matrix} 1,\ k = i, \\ \infty, k \neq i\ . \end{matrix} \tag{4.21}$$

We denote this matrix by $\hat{\mathbf{1}}$. It is easy to verify that:

$$\hat{\mathbf{1}}\hat{\cdot}A = A\hat{\cdot}\hat{\mathbf{1}} = A. \tag{4.22}$$

Definition 4.2 *If the matrices A and B satisfy*

$$A\hat{\cdot}B = \hat{\mathbf{1}}, \tag{4.23}$$

then we call them alternatively inverse matrices.

Therefore the conjugate commutative group (1.3) and (1.5) generates the alternative matrix algebra isomorphic to the standard one.

§5. Alternative Geometry

1. Consider N-dimensional space R^N with zero curvature. Let x^ν be coordinates. As we know the change of coordinates is performed by the linear transformation:

$$\bar{x}^\nu = a^\nu_\tau x^\tau + a^\nu, \tag{5.1}$$

where the matrix A $(= a^\nu_\tau)$ is not singular.

Choose two points $M \in R^N$ and $M_0 \in R^N$ with coordinates x^ν and x^ν_0 respectively. The distance between these points is defined as:

$$ds^2 = \mathring{g}_{\nu\tau} dx^\nu dx^\tau, \tag{5.2}$$

where

$$dx^\nu = x^\nu \dot{-} x^\nu_0 = x^\nu - x^\nu_0, \tag{5.3}$$

and $\mathring{g}_{\nu\tau}$ is a metric tensor. Since R^N is of zero curvature, metric tensor $\mathring{g}_{\nu\tau}$ is constant.

2. From (5.3) and §4, we can write:

$$d\bar{x}^\nu = a^\nu_\tau dx^\tau. \tag{5.4}$$

In order for the tranformations (5.1) and (5.4) to preserve the metric (5.2), the matrix of coordinate change A has to be orthogonal:

$$\mathring{g}_{\nu\tau} a^\nu_\sigma a^\tau_\mu = \mathring{g}_{\sigma\mu}. \tag{5.5}$$

The set of all orthogonal matrices forms a noncommutative group.

3. We define the differential dx^ν in the form of (5.3). But a differential equation dictates the existence of the alternative algebra (1.3). Thus along with dx^ν there exist the alternative differentials:

$$\hat{d}x^\nu = x^\nu \ddot{-} x^\nu_0. \tag{5.6}$$

As it follows from the above in this case the transformations (5.1) and (5.6) will not be compatible.

To reach the compatibility of (5.6) with the coordinate change, we introduce a-conjugate coordinates $\hat{x}^\nu$ that are related to x^ν as follows:

$$\hat{x}^\nu = \frac{1}{x^\nu}. \tag{5.7}$$

Then from (5.1) we find:

$$\bar{\hat{x}}^\nu = \hat{a}^\nu_{\hat{\sigma}} \hat{x}^{\hat{\sigma}} \ddot{+} \hat{a}^\nu, \tag{5.8}$$

where

$$\hat{a}^\nu_\sigma = \frac{1}{a^\nu_\sigma}, \;\; \hat{a}^\nu = \frac{1}{a^\nu}. \tag{5.9}$$

Recall that the alternative summation is performed along the indices $\hat{\sigma}$.

4. It follows from (5.7) that:

$$\hat{d}\hat{x}^\nu = \frac{1}{dx^\nu}. \tag{5.10}$$

Then the metric (5.2) will be:

$$ds^2 = \mathring{g}_{\nu\tau} \frac{1}{\hat{d}\hat{x}^\nu} \frac{1}{\hat{d}\hat{x}^\tau},$$

or

$$\hat{d}\hat{s}^2 = \overset{o}{\hat{g}}_{\hat{\nu}\hat{\tau}} \hat{d}\hat{x}^{\hat{\nu}} \hat{d}\hat{x}^{\hat{\tau}}, \tag{5.11}$$

where

$$\overset{o}{\hat{g}}_{\nu\tau} = \frac{1}{\mathring{g}_{\nu\tau}}. \tag{5.12}$$

As for $\hat{d}\hat{s}$ and ds, they are related by the equality

$$\hat{d}\hat{s} \cdot ds = 1. \tag{5.13}$$

Let us go back to subsection **1**. Let x^ν be coordinates in R^N. Two geometries arise depending on how we define the differentials of coordinates x^ν.

a) If $dx^\nu = x^\nu \dot{-} x^\nu_0 = x^\nu - x^\nu_0$, the metric has the form (5.2), and the group of coordinate changes becomes (5.1) with imposed orthogonality condition (5.5).

b) If $\hat{d}x^\nu = x^\nu \ddot{-} x^\nu_0$, then the metric becomes:

$$\hat{d}s^2 = \overset{o}{\hat{g}}_{\hat{\nu}\hat{\tau}} \hat{d}x^{\hat{\nu}} \hat{d}x^{\hat{\tau}}, \tag{5.14}$$

where $\mathring{g}_{\nu\tau}$ and $\overset{o}{\hat{g}}_{\nu\tau}$are related to each other by (5.12). It should be mentioned that (5.2) and (5.14) are written in the same coordinate system. In this case the coordinate change will be:

$$\bar{x}^\nu = b^\nu_{\hat{\sigma}} x^{\hat{\sigma}} \ddot{+} b^\nu. \tag{5.15}$$

The orthogonality condition becomes:

$$\overset{o}{\hat{g}}_{\hat{\nu}\hat{\tau}} b^{\hat{\nu}}_\sigma b^{\hat{\tau}}_\mu = \overset{o}{\hat{g}}_{\sigma\mu}. \tag{5.16}$$

Thus differential equations point out the existence of two alternative and isomorphic geometries in the space. We will call them the *alternative geometries.*

§6. Double field in nonlinear equations

We will construct an algebra generated by a system of nonlinear equations and study the properties of mathematical objects defined and built on the base of this algebra.

Let the following system of N complex autonomous differential equations be defined:

$$\frac{du^k}{dt} = F^k\left(u^1, \dots, u^N\right), \qquad (6.1)$$
$$(k=1,\dots,N).$$

Like in [1], we assume that $F^k\left(u^1, \dots, u^N\right)$ are defined and smooth everywhere in their domains, and $F(u)$ as a vector field can have only isolated zeros and infinities. t is an independent real variable.

In [1] it was shown that the space J of solutions of (6.1) is a discrete fiber bundle [2]. The base space is W - space of solutions of the system of linear equations (1.13) [1]. The projection is the mapping $\exp\varphi : J \to W$, where $\varphi(u)$ are characteristic functions [1] of (6.1). $\varphi(u)$ is short for $\varphi^k\left(u^1, \dots, u^N\right)$, $(k = 1, \dots, N)$. Inverse mapping $\varphi^{-1}(\ln)$ generates the discrete fibers of the space J.

We introduce a smooth dicrete fiber manifold P, elements of which are arbitrary, sufficiently smooth functions of the real independent variable t. Let the base space be a smooth N-dimensional manifold B^N, the projection be the same mapping as in the discrete fiber bundle J, i.e. $\exp\varphi : P \to B^N$, and the structure group that acts in discrete fibers of P be the same group D that acts in discrete fibers J [1].

1. First we extend the binary operations (1.9) [1] and (1.11) [1] to the elements of P as follows:

$$f_1 \underset{\varphi}{\dot{+}} f_2 = \varphi^{-1}\left(\ln\left[\exp\varphi(f_1) \dot{+} \exp\varphi(f_2)\right]\right), \qquad (6.2)$$

$$f_1 \underset{\varphi}{\ddot{+}} f_2 = \varphi^{-1}\left(\ln\left[\exp\varphi(f_1) \ddot{+} \exp\varphi(f_2)\right]\right), \qquad (6.3)$$

where $f_1, f_2 \in P$. In the right hand sides of (6.2-3) operations "$\dot{+}$" and "$\ddot{+}$" are the same as (1.2-3).

Note that in (6.2-3) as well as further we use the compact notation. For example, (6.2) in an expanded form is:

$$\left(f_1 \underset{\varphi}{\dot{+}} f_2\right)^k = \left(\varphi^{-1}\right)^k (\ln\left[\exp\varphi^1\left(f_1^1, \dots, f_1^N\right) \dot{+} \exp\varphi^1\left(f_2^1, \dots, f_2^N\right)\right], \dots,$$
$$\ln\left[\exp\varphi^N\left(f_1^1, \dots, f_1^N\right) \dot{+} \exp\varphi^N\left(f_2^1, \dots, f_2^N\right)\right]),$$
$$k = (1, \dots, N).$$

The identity and conjugate identity elements in P are e and h [1]:

$$f \underset{\varphi}{\dot{+}} e = f, \quad f \underset{\varphi}{\dot{+}} h = h, \qquad (6.4)$$
$$f \underset{\varphi}{\ddot{+}} e = e, \quad f \underset{\varphi}{\ddot{+}} h = f,$$

where f is an element of P. Recall that e and h are singular points of the vector field $F(u)$.

In [1] every $u \in J$ has its inverse in algebra (1.9) [1], (1.11) [1]. We extend the notion of inverse for all elements of manifold P and introduce the following

Definition 6.1 *If the following equalities hold*

$$f \underset{\varphi}{\dot{+}} f^{\tilde{}} = e, \quad f \underset{\varphi}{\ddot{+}} f^{\tilde{}} = h, \tag{6.5}$$

then $f^{\tilde{}} \in P$ is called a φ-inverse of $f \in P$ in algebra (6.2-4).

Let us study some aspects of the inverse of $f^{\tilde{}}$. Based on (1.18) [1], the following immediately takes place:

$$\begin{aligned} \varphi(e) &= -\infty, \\ \varphi(h) &= +\infty. \end{aligned} \tag{6.6}$$

From (6.2) and (6.5) we have that:

$$\exp\varphi(e) = \exp\varphi(f) + \exp\varphi\left(f^{\tilde{}}\right).$$

Taking into account (6.6), we can write:

$$\exp\varphi\left(f^{\tilde{}}\right) = -\exp\varphi(f) = \exp(\varphi(f) + i\pi).$$

Then

$$f^{\tilde{}} = \varphi^{-1}(\varphi(f) + i\pi). \tag{6.7}$$

Same way we can conclude (6.7) for (6.3).

In the future instead of $f_1 \underset{\varphi}{\dot{+}} f_2^{\tilde{}}$ we use the notation:

$$f_1 \underset{\varphi}{\dot{-}} f_2.$$

Then (6.7) implies:

$$f_1 \underset{\varphi}{\dot{-}} f_2 = \varphi^{-1}\left(\ln\left[\exp\varphi(f_1) \dot{-} \exp\varphi(f_2)\right]\right), \tag{6.8}$$

$$f_1 \underset{\varphi}{\ddot{-}} f_2 = \varphi^{-1}\left(\ln\left[\exp\varphi(f_1) \ddot{-} \exp\varphi(f_2)\right]\right). \tag{6.9}$$

2. Let us go back to (1.13) [1], where matrix b is diagonal. Then if $w(t) = \left(w^1(t), \ldots, w^N(t)\right)$ is a solution of (1.13) [1], then

$$\mathring{c} \cdot w(t) = \left(\mathring{c}^1 \cdot w^1(t), \ldots, \mathring{c}^N \cdot w^N(t)\right),$$

where $\mathring{c}^k$, $(k = 1, \ldots, N)$ are arbitrary constants ($\mathring{c} = \left(\mathring{c}^1, \ldots, \mathring{c}^N\right)$ can be thought as elements of B^N), is also a solution of the same equation. But because of $\varphi^{-1}(\ln) : W \to J$ we conclude that $\varphi^{-1}(\ln(\mathring{c} \cdot w)) \in J$.

Since $\mathring{c} \in B^N$, clearly

$$c = \varphi^{-1}\left(\ln\left(\mathring{c}\right)\right) \tag{6.10}$$

will be an element of P. Using (6.10), the solution $\varphi^{-1}\left(\ln\left(\mathring{c}\cdot w\right)\right) \in J$ has the form: $\varphi^{-1}\left(\ln\left(\exp\left[\varphi\left(c\right)+\varphi\left(u\right)\right]\right)\right)$. For simplicity of notation we introduce

$$c \odot u = \varphi^{-1}\left(\varphi\left(c\right)+\varphi\left(u\right)\right), \tag{6.11}$$

where c, u, $c \odot u$ are elements of P. Then the operation

$$f_1 \odot f_2 = \varphi^{-1}\left(\varphi\left(f_1\right)+\varphi\left(f_2\right)\right), \tag{6.12}$$

will be called a φ - *product in* P.

3. φ-product has the following properties.

a) From (6.12) it follows that it is commutative

$$f_1 \odot f_2 = f_2 \odot f_1.$$

b) To prove associativity:

$$f_1 \odot \left(f_2 \odot f_3\right) = \varphi^{-1}\left(\varphi\left(f_1\right)+\varphi\left(f_2 \odot f_3\right)\right) = \varphi^{-1}\left(\varphi\left(f_1\right)+\varphi\left(f_2\right)+\varphi\left(f_3\right)\right) =$$
$$\varphi^{-1}\left(\varphi\left(f_1 \odot f_2\right)+\varphi\left(f_3\right)\right) = \left(f_1 \odot f_2\right) \odot f_3.$$

c) Distributivity:

$$g \odot \left(f_1 \underset{\varphi}{\dot{+}} f_2\right) = g \odot f_1 \underset{\varphi}{\dot{+}} g \odot f_2 \tag{6.13}$$

where $f_1,\ f_2,\ f_3 \in P$.

To prove (6.13) we use (6.2), (6.12):

$$\begin{gathered} g \odot \left(f_1 \underset{\varphi}{\dot{+}} f_2\right) = \varphi^{-1}\left(\varphi\left(g\right)+\varphi\left(f_1 \underset{\varphi}{\dot{+}} f_2\right)\right) = \\ \varphi^{-1}\left(\ln\exp\varphi\left(g\right)+\ln\left(\exp\varphi\left(f_1\right)\dot{+}\exp\varphi\left(f_2\right)\right)\right) = \\ \varphi^{-1}\left(\ln\left[\exp\left(\varphi\left(g\right)+\varphi\left(f_1\right)\right)\dot{+}\exp\left(\varphi\left(g\right)+\varphi\left(f_2\right)\right)\right]\right) = \\ \varphi^{-1}\left(\ln\left[\exp\varphi\left(g \odot f_1\right)\dot{+}\exp\varphi\left(g \odot f_2\right)\right]\right) = \\ g \odot f_1 \underset{\varphi}{\dot{+}} g \odot f_2. \end{gathered}$$

Same way can be proved that

$$g \odot \left(f_1 \underset{\varphi}{\ddot{+}} f_2\right) = g \odot f_1 \underset{\varphi}{\ddot{+}} g \odot f_2 \tag{6.14}$$

d) The following are easy to check:

$$\begin{aligned} e \odot f &= e, \\ h \odot f &= h, \end{aligned} \tag{6.15}$$

where $e, h \in P$ are the identity and conjugate identity elements, and $f \in P$. The first equality, for example, follows from (6.6) and (6.12), so we have

$$e \odot f = \varphi^{-1} \left(\varphi \left(e\right) + \varphi \left(f\right)\right) = \varphi^{-1} \left(-\infty\right) = e.$$

4. Let us study the identity element of the operation (6.12). Consider the φ-product of two elements from P:

$$E \odot f = \varphi^{-1} \left(\varphi \left(E\right) + \varphi \left(f\right)\right).$$

If we demand

$$\varphi \left(E\right) = 0,$$

then for any $f \in P$:

$$E \odot f = \varphi^{-1} \left(\varphi \left(f\right)\right) = f. \tag{6.16}$$

This means if there exists a solution of $\varphi \left(E\right) = 0$, then $E \in P$ plays the role of the unit element of the φ-product (6.12).

5. To study the inverse operation of φ-product, let us consider elements $f,\ f^{-} \in P$, that satisfy the following:

$$f \odot f^{-} = E. \tag{6.17}$$

Using $\varphi \left(E\right) = 0$ and (6.12) we have:

$$\varphi \left(f\right) + \varphi \left(f^{-}\right) = 0. \tag{6.18}$$

Of course, here we assume that element $E \in P$ exists, and equation (6.18) is solvable for $f^{-} \in P$. Then from (6.18) we find:

$$f^{-} = \varphi^{-1} \left(-\varphi \left(f\right)\right). \tag{6.19}$$

In the future the element $f^{-} \in P$ will be called *φ-inverse element for f*.

We fix the notation for the φ-product of elements $f,\ g^{-} \in P$:

$$_{(\varphi)}\frac{f}{g} \equiv f \odot g^{-} = \varphi^{-1} \left(\varphi \left(f\right) + \varphi \left(g^{-}\right)\right).$$

Since $\varphi \left(g^{-}\right) = -\varphi \left(g\right)$, finally obtain:

$$_{(\varphi)}\frac{f}{g} = \varphi^{-1} \left(\varphi \left(f\right) - \varphi \left(g\right)\right). \tag{6.20}$$

The operation (6.20) is inverse of (6.12). We call it *φ-relation for elements f and g in P*.

By analogy with (6.15) it is easy to prove the following property:

$$_{(\varphi)}\frac{f}{e} = h, \quad _{(\varphi)}\frac{f}{h} = e. \tag{6.21}$$

6. In §2 a-conjugate functions arose in the construction of the differential calculus satisfying conditions (2.3). By the same token we can introduce a-conjugate functions in the discrete fiber manifold P.

Definition 6.2 *If* $f(t)$, $\overset{\varphi}{\hat{f}}(\hat{t}) \in P$ *satisfy*

$$t \cdot \hat{t} = 1, \tag{6.22}$$
$$f(t) \odot \overset{\varphi}{\hat{f}}(\hat{t}) = E,$$

where $E \in P$ *is the unit element of* φ*-product, then* $f(t)$ *and* $\overset{\varphi}{\hat{f}}(\hat{t})$ *will be called a-conjugate functions in* P.

From (6.18) we can write:

$$\varphi(f(t)) + \varphi\left(\overset{\varphi}{\hat{f}}(\hat{t})\right) = 0. \tag{6.23}$$

The existence of $\overset{\varphi}{\hat{f}}(\hat{t}) \in P$ for given $f(t) \in P$ follows from (6.19).

Let $g(t)$, $\hat{g}(\hat{t}) \in B^N$ be projections of functions $f(t)$, $\overset{\varphi}{\hat{f}}(\hat{t}) \in P$ respectively, i.e.

$$g(t) = \exp\varphi(f(t)), \qquad \hat{g}(\hat{t}) = \exp\varphi\left(\overset{\varphi}{\hat{f}}(\hat{t})\right). \tag{6.24}$$

Let us form a product of $g(t) \cdot \hat{g}(\hat{t})$. In the coordinate notation

$$g^k(t) \cdot \hat{g}^k(\hat{t}) = \exp\varphi^k(f(t)) \cdot \exp\varphi^k\left(\overset{\varphi}{\hat{f}}(\hat{t})\right),$$
$$(k = 1, \ldots, N),$$

where no summation is performed along the index k. From (6.23) it follows that:

$$\begin{gathered} t \cdot \hat{t} = 1, \\ g^k(t) \cdot \hat{g}^k(\hat{t}) = 1, \\ (k = 1, \ldots, N). \end{gathered} \tag{6.25}$$

Thus we conclude that if $f(t)$, $\overset{\varphi}{\hat{f}}(\hat{t})$ are a-conjugate functions in P, then their projections $g(t)$, $\hat{g}(\hat{t})$ on the base space B^N are a-conjugate functions in the sense of §2.

Example 6.1 *For the equation* $\dfrac{du}{dt} = \sin u$ *the characteristic function is [1]:*

$$\varphi(u) \equiv \ln tg\frac{u}{2} = t + c. \tag{6.26}$$

In [1] we have shown that the alternative additive operations (6.2-3) are:

$$f_1 \underset{\varphi}{\dot{+}} f_2 = 2arctg\left(tg\frac{f_1}{2} + tg\frac{f_2}{2}\right) + 2\pi m,$$

$$f_1 \underset{\varphi}{\ddot{+}} f_2 = 2arcctg\left(ctg\frac{f_1}{2} + ctg\frac{f_2}{2}\right) + 2\pi m,$$

with the identity and conjugate identity elements

$$e = 2\pi m, \ \ h = \pi + 2\pi m.$$

From (6.12) we can find the φ-product:

$$f_1 \odot f_2 = 2arctg\left(tg\frac{f_1}{2} \cdot tg\frac{f_2}{2}\right) + 2\pi m.$$

The unit element is

$$E = \frac{\pi}{2} + 2\pi m.$$

For the φ-relation we have

$$_{(\varphi)}\frac{f_1}{f_2} = 2arctg\left(tg\frac{f_1}{2} \cdot ctg\frac{f_2}{2}\right) + 2\pi m,$$

and the φ-inverse element f^- of f is:

$$f^- = 2arctg\left(ctg\frac{f}{2}\right) + 2\pi m.$$

Thus a differential equation via its characteristic function φ introduces the discrete fiber bundle. Moreover, it generates an algebraically closed commutative object that acts in the discrete fiber manifold P.

To decipher this result let us return to the mapping $\varphi^{-1}(\ln) : B^N \to P$. Recall that it was found in [1], and it mapped the space W of solutions of linear equations (1.13) [1] to the space J of solutions of (6.1). But (1.13) [1] is a diagonalizable system of N equations. Then it follows from §1 that the double field arises for every component. In other words the base space B^N contains N double fields and each of them, generally speaking, acts independently from the other on the corresponding coordinates of B^N. The mapping $\varphi^{-1}(\ln)$ maps these double fields into the manifold P. The algebraic object which arises and acts in P is homomorphic (isomorphic) to the double fields. We leave open the question about irreducible representations. One of the representations of the double field is described in subsections 1-5. In the future we call it *φ-representation of the double field* or *φ-double field.*

We already saw that the characteristic function φ takes infinite value at the identity elements e and h. Many key topological properties of the space of solutions of (6.1) seem to hide in the neighborhoods of these elements. However at this point we allow ourselves not to consider them but stress out that these properties are clearly no less important for the completeness of the theory than algebraic ones.

§7. Differential calculus based on φ-double field

1. Based on (6.8) the increment of function $f(t) \in P$ is:

$$\overset{\varphi}{\Delta} f = f(t) \underset{\varphi}{\dot{-}} f(t_0) = \varphi^{-1}\left(\ln\left[\exp\varphi(f(t)) \dot{-} \exp\varphi(f(t_0))\right]\right). \tag{7.1}$$

The mean value theorem implies

$$\exp\varphi^k(f(t)) - \exp\varphi^k(f(t_0)) = \frac{d}{dt}\exp\varphi^k(f(\xi))\,\Delta t, \tag{7.2}$$
$$(k = 1, \ldots, N),$$

where $\Delta t = t - t_0$, $\xi \in (t_0, t)$. By substituting (7.2) in (7.1) we find:

$$\overset{\varphi}{\Delta} f = \varphi^{-1}\left(\ln\left[\frac{d}{dt}\exp\varphi(f(\xi))\,\Delta t\right]\right). \tag{7.3}$$

Let $T \in B^N$ be an element that has all its coordinates equal to the independent variable t. Let

$$\tau = \varphi^{-1}(\ln T), \tag{7.4}$$

where $\tau \in P$, $T = (t, \ldots, t) \in B^N$. Then $\overset{\varphi}{\Delta}\tau = \tau \underset{\varphi}{\dot{-}} \tau_0$. From(7.1) and (7.4) we derive:

$$\overset{\varphi}{\Delta}\tau = \varphi^{-1}(\ln \Delta T) = \varphi^{-1}(\ln\Delta t, \ldots, \ln\Delta t). \tag{7.5}$$

Clearly $\overset{\varphi}{\Delta}\tau \in P$. After simple calculations, (6.20) together with (7.3) and (7.5) implies:

$$_{(\varphi)}\frac{\overset{\varphi}{\Delta} f}{\overset{\varphi}{\Delta}\tau} = \varphi^{-1}\left(\ln\left[\frac{d}{dt}\exp\varphi(f(\xi))\right]\right). \tag{7.6}$$

Since the given fucntions are sufficiently smooth, the limit of the right hand side of (7.6), when $t \to t_0$, exists. Let us denote it by

$$f_t^{(\varphi)}(t_0) = \lim_{t\to t_0} {}_{(\varphi)}\frac{\overset{\varphi}{\Delta} f}{\overset{\varphi}{\Delta}\tau},$$

or finally:

$$f_t^{(\varphi)}(t) = \varphi^{-1}\left(\ln\left[\frac{d}{dt}\exp\varphi(f(t))\right]\right). \tag{7.7}$$

In the future we will call (7.7) the *φ-derivative of the function $f(t) \in P$* based on the algebraic operation (6.2).

2. It is easy to prove the analog of the mean value theorem in algebra (1.3):

$$F(t) \ddot{-} F(t_0) = F^{(!)}(\xi)\left(t \ddot{-} t_0\right), \tag{7.8}$$

where $\xi \in (t_0, t)$.

To study the derivative based on the algebraic operation (6.3), we form the increment of function $f(t) \in P$ using (6.9):

$$\overset{\varphi}{\hat{\Delta}} f = f(t) \underset{\varphi}{\ddot{-}} f(t_0) = \varphi^{-1}\left(\ln\left[\exp\varphi(f(t)) \ddot{-} \exp\varphi(f(t_0))\right]\right).$$

From (7.8) we have:

$$\overset{\varphi}{\hat{\Delta}} f = \varphi^{-1}\left(\ln\left[\frac{\hat{d}}{\hat{d}t}\exp\varphi(f(\xi))\,\hat{\Delta}t\right]\right), \tag{7.9}$$

where $\frac{\hat{d}}{\hat{d}t}$ is the operator of the alternative derivative (2.22), and $\hat{\Delta}t = t\ddot{-}t_0$.

Taking into account (6.9), the increment for (7.4) will have the form:

$$\overset{\varphi}{\hat{\Delta}}\tau = \tau\underset{\varphi}{\ddot{-}}\tau_0 = \varphi^{-1}\left(\ln\hat{\Delta}T\right) = \varphi^{-1}\left(\ln\hat{\Delta}t,\ldots,\ln\hat{\Delta}t\right). \tag{7.10}$$

Taking φ-relation of (7.9) and (7.10), we obtain:

$$_{(\varphi)}\frac{\overset{\varphi}{\hat{\Delta}} f}{\overset{\varphi}{\hat{\Delta}}\tau} = \varphi^{-1}\left(\ln\left[\frac{\hat{d}}{\hat{d}t}\exp\varphi(f(\xi))\right]\right).$$

Then the limit, when $t \to t_0$, is:

$$f_t^{(!\varphi)}(t) = \varphi^{-1}\left(\ln\left[\frac{\hat{d}}{\hat{d}t}\exp\varphi(f(t))\right]\right). \tag{7.11}$$

In the future we will call (7.11) an *alternative φ-derivative.*

3. Let us study some properties of φ-derivatives.

a) Let $c = const. \in P$, e - the identity element of φ-double field. Then

$$(c)^{(\varphi)} = e. \tag{7.12}$$

It follows from (7.7) that

$$(c)_t^{(\varphi)} = \varphi^{-1}\left(\ln\left[\frac{d}{dt}\exp\varphi(c)\right]\right) = \varphi^{-1}(\ln 0) = \varphi^{-1}(-\infty) = e.$$

b) If $f_1,\ f_2 \in P$,

$$\left(f_1\underset{\varphi}{\dot{+}}f_2\right)^{(\varphi)} = f_1^{(\varphi)}\underset{\varphi}{\dot{+}}f_2^{(\varphi)}. \tag{7.13}$$

By the definition of φ-derivative

$$\left(f_1\underset{\varphi}{\dot{+}}f_2\right)^{(\varphi)} = \varphi^{-1}\left(\ln\left[\frac{d}{dt}\exp\varphi\left(f_1\underset{\varphi}{\dot{+}}f_2\right)\right]\right),$$

and, using (6.2) in the right hand side, we obtain

$$\left(f_1 \dot{\underset{\varphi}{+}} f_2\right)^{(\varphi)} = \varphi^{-1}\left(\ln \frac{d}{dt}\left[\exp\varphi\left(f_1\right) \dot{+} \exp\varphi\left(f_2\right)\right]\right). \tag{7.14}$$

On the other hand, again from (6.2) we have:

$$f_1^{(\varphi)} \dot{\underset{\varphi}{+}} f_2^{(\varphi)} = \varphi^{-1}\left(\ln\left[\exp\varphi\left(f_1^{(\varphi)}\right) \dot{+} \exp\varphi\left(f_2^{(\varphi)}\right)\right]\right).$$

Then, by definition (7.7) for elements $f_1^{(\varphi)}$ and $f_2^{(\varphi)}$ in the right hand side we can write

$$f_1^{(\varphi)} \dot{\underset{\varphi}{+}} f_2^{(\varphi)} = \varphi^{-1}\left(\ln\left[\frac{d}{dt}\exp\varphi\left(f_1\right) \dot{+} \frac{d}{dt}\exp\varphi\left(f_2\right)\right]\right). \tag{7.15}$$

Comparing (7.14) and (7.15), we arrive at (7.13).

The same way we can show that the following properties are true:

c)

$$\left(f_1 \odot f_2\right)^{(\varphi)} = f_1^{(\varphi)} \odot f_2 \dot{\underset{\varphi}{+}} f_1 \odot f_2^{(\varphi)}, \tag{7.16}$$

d)

$$\left({}_{(\varphi)}\frac{f_1}{f_2}\right)^{(\varphi)} = {}_{(\varphi)}\frac{f_1^{(\varphi)} \odot f_2 \dot{\underset{\varphi}{-}} f_1 \odot f_2^{(\varphi)}}{f_2 \odot f_2} \tag{7.17}$$

The alternative φ-derivative has the analogous properties:

a)

$$(c)^{(!\varphi)} = h, \quad c = const.,$$

b)

$$\left(f_1 \ddot{\underset{\varphi}{+}} f_2\right)^{(!\varphi)} = f_1^{(!\varphi)} \ddot{\underset{\varphi}{+}} f_2^{(!\varphi)},$$

c)

$$\left(f_1 \odot f_2\right)^{(!\varphi)} = f_1^{(!\varphi)} \odot f_2 \ddot{\underset{\varphi}{+}} f_1 \odot f_2^{(!\varphi)},$$

d)

$$\left({}_{(\varphi)}\frac{f_1}{f_2}\right)^{(!\varphi)} = {}_{(\varphi)}\frac{f_1^{(!\varphi)} \odot f_2 \ddot{\underset{\varphi}{-}} f_1 \odot f_2^{(!\varphi)}}{f_2 \odot f_2}.$$

4. It is easy to show that the φ-derivative and alternative φ-derivative of higher order have the form:

$$f^{(n\varphi)} = \varphi^{-1}\left(\ln \frac{d^n}{dt^n}\exp\varphi\left(f\right)\right),$$

$$f^{(!n\varphi)} = \varphi^{-1}\left(\ln \frac{\hat{d}^n}{\hat{d}t^n}\exp\varphi\left(f\right)\right).$$

5. Let us study the existence of φ-differential. First we prove that if $g_1, g_2 \in B^N$, then

$$\varphi^{-1}\left(\ln\left[g_1 \dot{+} g_2\right]\right) = \varphi^{-1}\left(\ln g_1\right) \underset{\varphi}{\dot{+}} \varphi^{-1}\left(\ln g_2\right). \tag{7.18}$$

From (6.2) for the right hand side we have:

$$\varphi^{-1}\left(\ln g_1\right) \underset{\varphi}{\dot{+}} \varphi^{-1}\left(\ln g_2\right) =$$
$$\varphi^{-1}\left(\ln\left[\exp\varphi\left(\varphi^{-1}\left(\ln g_1\right)\right) \dot{+} \exp\varphi\left(\varphi^{-1}\left(\ln g_2\right)\right)\right]\right) =$$
$$\varphi^{-1}\left(\ln\left[g_1 \dot{+} g_2\right]\right).$$

We form the increment (7.1) for the function $f(t) \in P$. Since the functions under consideration are sufficiently smooth, and if $\Delta T = (\Delta t, \ldots, \Delta t) \in B^N$, $\omega(\Delta t) = \left(\omega^1(\Delta t), \ldots, \omega^N(\Delta t)\right) \in B^N$, and $\omega^k(\Delta t)$, $(k = 1, \ldots, N)$ are infinitely small with respect to Δt, we can write:

$$\overset{\varphi}{\Delta} f = \varphi^{-1}\left(\ln\left[\frac{d}{dt}\exp\varphi\left(f\left(t_0\right)\right)\Delta T \dot{+} \omega\left(\Delta t\right)\Delta T\right]\right). \tag{7.19}$$

Then from (7.18) for (7.19) it follows that:

$$\overset{\varphi}{\Delta} f = \varphi^{-1}\left(\ln\left[\frac{d}{dt}\exp\varphi\left(f\left(t_0\right)\right)\Delta T\right]\right) \underset{\varphi}{\dot{+}} \varphi^{-1}\left(\ln\left[\omega\left(\Delta t\right)\Delta T\right]\right) = \tag{7.20}$$
$$\varphi^{-1}\left(\ln\frac{d}{dt}\exp\varphi\left(f\left(t_0\right)\right) \dot{+} \ln\Delta T\right) \underset{\varphi}{\dot{+}} \varphi^{-1}\left(\ln\omega\left(\Delta t\right) + \ln\Delta T\right).$$

Taking into account (7.5), (7.7), the equality

$$\ln\omega\left(\Delta t\right) = \varphi\left(\varphi^{-1}\left(\ln\omega\left(\Delta t\right)\right)\right),$$

and using (6.12), (7.20) will have the form:

$$\overset{\varphi}{\Delta} f = f_t^{(\varphi)}\left(t_0\right) \odot \overset{\varphi}{\Delta}\tau \underset{\varphi}{\dot{+}} \varphi^{-1}\left(\ln\omega\left(\Delta t\right)\right) \odot \overset{\varphi}{\Delta}\tau. \tag{7.21}$$

As one would expect, when $\Delta t \to 0$, $\overset{\varphi}{\Delta}\tau \to e$ and $\varphi^{-1}\left(\ln\omega\left(\Delta t\right)\right) \to e$.

Definition 7.1 *The first term of (7.21) by analogy with the standard differential calculus will be called the φ-differential of the function $f(t) \in P$ and be denoted by $d_\varphi f$:*

$$d_\varphi f = f_t^{(\varphi)} \odot d_\varphi \tau, \tag{7.22}$$

where $d_\varphi \tau = \varphi^{-1}\left(\ln dt, \ldots, \ln dt\right) \in P$.

If $\hat{d}_\varphi \tau = \varphi^{-1}\left(\ln\hat{d}t, \ldots, \ln\hat{d}t\right) \in P$,the alternative φ-differential will be

$$\hat{d}_\varphi f = f_t^{(!\varphi)} \odot \hat{d}_\varphi \tau. \tag{7.23}$$

Since $d_\varphi \tau,\ \hat{d}_\varphi \tau \in P$, the equalities (7.22-23) can be written as:

$$_{(\varphi)}\frac{d_\varphi f(t)}{d_\varphi \tau} = f_t^{(\varphi)}(t)\,,\ \ _{(\varphi)}\frac{\hat{d}_\varphi f(t)}{\hat{d}_\varphi \tau} = f_t^{(!\varphi)}(t)\,. \tag{7.24}$$

6. Let $f(t)$, $\overset{\varphi}{\hat{f}}\left(\hat{t}\right) \in P$ be a-conjugate functions (in the sense of § 6). Then (6.22-25) are true.

From (6.24) we can write the alternative φ-derivative (7.11) as follows:

$$f_t^{(!\varphi)}(t) = \varphi^{-1}\left(\ln\left[\frac{\hat{d}}{\hat{d}t}g(t)\right]\right).$$

But because of (6.25), from (2.9) we obtain:

$$f_t^{(!\varphi)}(t) = \varphi^{-1}\left(\ln\left[\frac{1}{\frac{d}{d\hat{t}}\hat{g}\left(\hat{t}\right)}\right]\right). \tag{7.25}$$

Then (7.7) implies

$$\ln\left[\frac{1}{\frac{d}{d\hat{t}}\hat{g}\left(\hat{t}\right)}\right] = -\varphi\left(\left(\overset{\varphi}{\hat{f}}\left(\hat{t}\right)\right)_{\hat{t}}^{(\varphi)}\right). \tag{7.26}$$

Equalities (7.25-26) immediately lead to

$$\varphi\left(f_t^{(!\varphi)}(t)\right) + \varphi\left(\left(\overset{\varphi}{\hat{f}}\left(\hat{t}\right)\right)_{\hat{t}}^{(\varphi)}\right) = 0,$$

which is the same as

$$f_t^{(!\varphi)}(t) \odot \left(\overset{\varphi}{\hat{f}}\left(\hat{t}\right)\right)_{\hat{t}}^{(\varphi)} = E. \tag{7.27}$$

Thus by analogy with §2, we conclude that if $f(t)$ and $\hat{f}\left(\hat{t}\right)$ are a-conjugate elements of manifold P, then their alternative derivatives are also a-conjugate elements in P.

As in (2.24), we can show that the following is true:

$$\hat{d}t \cdot d\hat{t} = 1 \tag{7.28}$$

$$\hat{d}_\varphi f(t) \odot d_\varphi \overset{\varphi}{\hat{f}}\left(\hat{t}\right) = E$$

7. Let us return to (7.7) that computes the φ-derivative. From (7.7) it follows that the given function $f(t) \in P$ is projected on the base space B^N:

$$g(t) = \exp\left[\varphi\left(f(t)\right)\right],$$

where $g(t) \in B^N$. But since B^N arose from the space of solutions of equation (1.13) [1] which is a collection of N equations (1.1), we can differentiate $g(t)$ by the rule described in § 2. This takes place in (7.7). Next from (7.7) it follows that the obtained derivative $g'(t) \in B^N$ is mapped to the fiber manifold P under the mapping $\varphi^{-1}(\ln)$:

$$g'(t) \overset{\varphi^{-1}(\ln)}{\to} f_t^{(\varphi)}(t) \in P.$$

By analogy the alternative φ-derivative can be found from (7.11).

Therefore if we consider φ-differential calculus with its properties as an algebraic object, the mapping $\exp(\varphi)$ establishes homomorphic relations between this object and the standard and alternative differential calculus.

8. What will the equation (6.1) look like if written in φ-differential calculus that it generates? To see this, we consider the characteristic function [1]:

$$\varphi(u) = bt + c, \tag{7.29}$$

where $b, c \in B^N$ are constants. Recall that $u \in P$ is a solution of (6.1).

If we differentiate (7.29) with respect to t and substitute the result into (7.7), we obtain:

$$u_t^{(\varphi)} = \varphi^{-1}\left(\ln \frac{d}{dt} \exp \varphi(u)\right) = \varphi^{-1}\left(\ln\left[b \exp \varphi(u)\right]\right). \tag{7.30}$$

Since $b \in B^N$, it can be represented as

$$b = \exp \varphi(B),$$

where $B \in P$. Then using (6.12) and (7.24), from (7.30) one finally derives:

$$_{(\varphi)}\frac{d_\varphi u}{d_\varphi \tau} = B \odot u \tag{7.31}$$

9. Equation (6.1) is written in the standard differential calculus that arises from one-dimensional linear equation (1.1). Written in its own algebra the equation (6.1) has the form (7.31).

Let us return to τ ((7.4)) that arises during the construction of the differential calculus in the fiber manifold P. Looking at (7.21-23) we conclude that the increment of the function $f \in P$ is proportional to the increment of the variable $\tau \in P$. The rate of change of function f is equal to the relation of differentials of f and τ. Therefore if t in (6.1) plays the role of time, then probably it will not be very far from the truth to interpret the multi-component object τ as *inner times* of the phenomenon described by the equation (6.1). As for t, it is an *inner time* for the equation (1.1) and an *outer time* for (6.1).

It has to be noted that along with t there is the a-conjugate time $\hat{t}$, satisfying the equality (2.3). By analogy there exists the conjugate multi-component object $\hat{\tau}$, satisfying the following:

$$\tau \odot \hat{\tau} = E,$$

where $E \in P$ is the unit element of the φ-product.

Example 7.1

a) *Consider the equation:*

$$\frac{du}{dt} = u\,(1-u)\,.$$

The characteristic function is:

$$\varphi\,(u) = \ln\frac{u}{1-u} = t + c.$$

Compute the φ-product of $u\,(t)$:

$$_{(\varphi)}\frac{d_\varphi u}{d_\varphi \tau} = \varphi^{-1}\left(\ln\frac{d}{dt}\exp\varphi\,(u)\right) =$$

$$\varphi^{-1}\left(\ln\left(\exp\varphi\,(u)\,\frac{d\varphi}{dt}\right)\right) = \varphi^{-1}\left(\varphi\,(u)\right) = u.$$

Finally we get:

$$_{(\varphi)}\frac{d_\varphi u}{d_\varphi \tau} = u.$$

From (7.4) we find the inner time

$$\tau = \frac{t}{1+t}, \qquad \hat{\tau} = \frac{\hat{t}}{1+\hat{t}},$$

where $t \cdot \hat{t} = 1$.

b) *Let*

$$\frac{du}{dt} = \sin u.$$

The characteristic function is (6.26). Then

$$_{(\varphi)}\frac{d_\varphi u}{d_\varphi \tau} = u.$$

The inner times are:

$$\begin{aligned} \tau &= 2arctg\ t + 2\pi m, \\ \hat{\tau} &= 2arctg\ \hat{t} + 2\pi m, \end{aligned}$$

where $t \cdot \hat{t} = 1$. From this equality it follows that every leaf has its own inner time.

Summary

The found complexes of the alternative integral and differential calculus, their representations, the rise of multi-component inner times, the existence of two geometries that are alternative to each other, are the direct consequence of the algebraic properties of differential equations. We can hardly avoid taking them into account without proper justification and violation of the complete mathematical picture.

1 Acknowledgements

Authors express their deep gratitude to Zaur Khukhunashvili, who constantly inspired and supported them by giving valuable suggestions and guidance.

References

1. Z.V. Khukhunashvili, Z.Z. Khukhunashvili, Algebraic Structure of Space and Field, EJQTDE, 2001, #6.

2. Dubrovin B.A., Novikov S.P., Fomenko A.T., Modern Geometry Methods and Applications (translated from Russian), New York, Spinger-Verlag, 1981.