Electronic Journal of Qualitative Theory of Differential Equations
2012, No. **23**, 1-28; **http://www.math.u-szeged.hu/ejqtde/**

Triple Positive Solutions of BVP for Second Order ODE with One Dimensional Laplacian on the Half Line

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Abstract. By applying Leggett-Williams fixed point theorem in a suitably constructed cone, we obtain the existence of at least three bounded positive solutions for a boundary value problem on the half line. Our result improves and complements some of the work in the literature.

Keywords: Second order differential equation on a half line, non-homogeneous boundary value problem, Leggett-Williams fixed point theorem.

AMS Subject Classification: 34B10, 34B15, 35B10.

1 Introduction

Recently there is increasing interest in the existence of positive solutions of boundary value problems (BVP) for differential equations on the half lines, see the references [1–9, 12–38]. It is observed that fixed point theorems have been useful in establishing the existence of positive solutions. To apply a fixed point theorem, one needs a Banach space, a cone, and a completely continuous operator. A brief survey of the Banach spaces, cones and operators used in the literature is given below.

The Banach spaces used in the literature include

- $C[0,\infty) = \{x : [0,\infty) \to \mathbb{R} : x \text{ is continuous on } [0,\infty) \text{ and } \lim_{t\to\infty} x(t) \text{ exists}\}$ with the norm $\|x\|_0 = \sup_{t\in[0,\infty)} |x(t)|$ (see [6, 13]);
- $C^1[0,\infty) = \{x : [0,\infty) \to \mathbb{R} : x, x' \text{ are continuous on } [0,\infty) \text{ and } \lim_{t\to\infty} x(t),\ \lim_{t\to\infty} x'(t) \text{ exist}\}$ with the norm $\|x\|_1 = \max\left\{\sup_{t\in[0,\infty)} |x(t)|,\ \sup_{t\in[0,\infty)} |x'(t)|\right\}$ (see [12, 15]);
- $C^1[0,\infty) = \left\{x : [0,\infty) \to \mathbb{R} : x, x' \text{ are continuous on } [0,\infty) \text{ and } \lim_{t\to\infty} \frac{x(t)}{1+t},\ \lim_{t\to\infty} x'(t) \text{ exist}\right\}$ with the norm $\|x\| = \max\left\{\sup_{t\in[0,\infty)} \frac{|x(t)|}{1+t},\ \sup_{t\in[0,\infty)} |x'(t)|\right\}$ (see [14, 16, 36]);
- weighted Banach spaces (with weights $u, v : [0,\infty) \to (0,\infty)$) such as
  - $C_u[0,\infty) = \left\{x : [0,\infty) \to \mathbb{R} : x \text{ is continuous on } [0,\infty) \text{ and } \lim_{t\to\infty} \frac{x(t)}{1+u(t)} \text{ exists}\right\}$ with the norm $\|x\|_u = \sup_{t\in[0,\infty)} \frac{|x(t)|}{1+u(t)}$ (see [17, 18, 19, 20, 26, 31, 32]); and

*Supported by Natural Science Foundation of Guangdong province (No.S2011010001900)

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– $C^1_{u,v}[0,\infty) = \{x : [0,\infty) \to \mathbb{R} : x, x' \text{ are continuous on } [0,\infty) \text{ and } \lim_{t\to\infty} \frac{x(t)}{1+u(t)}, \lim_{t\to\infty} v(t)x'(t) \text{ exist}\}$ with the norm $\|x\|_{u,v} = \max\{\sup_{t\in[0,\infty)} \frac{|x(t)|}{1+u(t)}, \sup_{t\in[0,\infty)} |v(t)x'(t)|\}$ (see [33, 34, 35]).

For the construction of a suitable cone, in the literature two methods have been used, one is by using the Green's function of the corresponding boundary value problem [9, 13, 29, 30], while the other is by using the concavity property of the solutions [12, 14, 17, 18, 20].

To define an appropriate operator, in [9, 13, 29] the nonlinear operator is defined by using the Green's function, whereas in [12, 14, 17, 18, 20, 30] the boundary condition $x'(\infty) = 0$ is instrumental in transforming the boundary value problem into an integral equation which leads to the definition of the nonlinear operator.

On the fixed point theorems used in the literature, in [21, 32] the method of upper and lower solutions or Tychonoff fixed point theorem has been used to establish the existence of bounded solutions of boundary value problems for second order differential equations on the half line. Yan [31] has applied the Leray-Schauder theorem and the fixed point index theory to establish the existence of multiple nonnegative unbounded solutions of the boundary value problem

$$\begin{cases} \dfrac{1}{p(t)}[p(t)x'(t)]' + f(t, x(t)) = 0, \quad t \in (0,\infty), \\ x(0) = a \geq 0, \\ \lim\limits_{t\to\infty} p(t)x'(t) = b \geq 0. \end{cases}$$

Motivated by the above mentioned papers, in this paper we consider the following non-homogeneous boundary value problem for the differential equation on the half line whose boundary conditions are of integral form

$$\begin{cases} [p(t)\phi(x'(t))]' + f(t, x(t)) = 0, \quad t \in (0,\infty), \\ x(0) = \displaystyle\int_0^\infty g(s)x(s)ds + a, \\ \lim\limits_{t\to\infty} \phi^{-1}(p(t))x'(t) = b. \end{cases} \tag{1.1}$$

Note that here we do not have the boundary condition $x'(\infty) = 0$ as in [12, 14, 17, 18, 20, 30]. In (1.1) it is assumed that $a, b \geq 0$, $g : [0,\infty) \to [0,\infty)$ is continuous with $\int_0^\infty g(s)ds < 1$, $f : (0,\infty) \times [0,\infty) \to [0,\infty)$, $p : [0,\infty) \to (0,\infty)$ is continuous (may be singular at $t = 0$), and $\phi(x) = |x|^{q-2}x$ with $q > 1$ is called *one dimensional Laplacian*. The inverse function of ϕ is $\phi^{-1}(x) = |x|^{q'-2}x$ where $1/q + 1/q' = 1$. We say $x : [0,\infty) \to (0,\infty)$ is a *positive solution* of (1.1) if $x \in C^1[0,\infty)$, $[p\phi(x')]' \in L^1(0,\infty)$ and x satisfies (1.1). We shall establish existence results for at least three bounded positive solutions of (1.1) by applying the Leggett-Williams fixed point theorem. In our derivation, the Banach space involved is motivated by [31], but the cone needed has to be very technically constructed – this is so since the boundary value problem involves the nonlinear operator $[p\phi(x')]'$ and the possible solutions are not concave if $p \not\equiv 1$, hence the cone cannot be constructed by using the concavity of x or even the Green's function.

We also consider the following boundary value problem

$$\begin{cases} [\Phi(x'(t))]' + f(t, x(t)) = 0, \quad t \in (0, \infty), \\ x(0) = \int_0^\infty g(s)x(s)ds, \\ \lim_{t\to\infty} x'(t) = 0 \end{cases} \tag{1.2}$$

where $g : [0, \infty) \to [0, \infty)$ is continuous with $\int_0^\infty g(s)ds < 1$, $f : (0, \infty) \times \mathbb{R} \to [0, \infty)$, and $\Phi : \mathbb{R} \to \mathbb{R}$ is a *pseudo sup-multiplicative function* (see Definition 2.2). Note that the one dimensional Laplacian ϕ is a special case of a pseudo sup-multiplicative function. We say $x : [0, \infty) \to (0, \infty)$ is a *positive solution* of (1.2) if $x \in C^1[0, \infty)$, $[\Phi(x')]'$ satisfies $\int_0^\infty \Phi^{-1}\left(\int_s^\infty [\Phi(x')]'(u)du\right) ds < \infty$ and x satisfies (1.2). We shall establish sufficient conditions for the existence of bounded positive solutions of (1.2) by using Schauder fixed point theorem.

Our results improve and complement the work of [5–9, 12–14, 17–20, 22–33, 37, 38]. The paper is organized as follows. Section 2 contains some background definitions and the Leggett-Williams fixed point theorem. The results for (1.1) and (1.2) are given in sections 3 and 4 respectively. Finally, in section 5 we present some examples to illustrate the results obtained.

2 Preliminaries

In this section, we present some background definitions and results.

Definition 2.1. The function $f : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is called a *Carathéodory function* if

(i) for each $u \in \mathbb{R}$, $t \mapsto f(t, u)$ is measurable on $(0, \infty)$;

(ii) for *a.e.* $t \in (0, \infty)$, $u \mapsto f(t, u)$ is continuous on $\mathbb{R}$;

(iii) for each $r > 0$, there exists $B_r \in L^1(0, \infty)$ satisfying $B_r(t) > 0$, $t \in (0, \infty)$ and $\int_0^\infty B_r(s)ds < \infty$ such that $|u| \le r$ implies

$$|f(t, u)| \le B_r(t), \quad a.e.\ t \in (0, \infty).$$

Definition 2.2. An odd homeomorphism Φ of the real line $\mathbb{R}$ onto itself is called a *pseudo sup-multiplicative function* if there exists a homeomorphism ω of $[0, \infty)$ onto itself which *supports* Φ in the sense that for all $v_1, v_2 \ge 0$ we have

$$\Phi(v_1v_2) \ge \omega(v_1)\Phi(v_2).$$

ω is called the *supporting function* of Φ. (Note that in [10] pseudo sup-multiplicative function is also known as *sup-multiplicative-like* function.)

Remark 2.1. Note that any sup-multiplicative function is a pseudo sup-multiplicative function. Also any function of the form

$$\Phi(u) := \sum_{j=0}^{k} c_j|u|^j u, \quad u \in \mathbb{R}$$

is pseudo sup-multiplicative, provided that $c_j \geq 0$. Here a supporting function is defined by $\omega(u) := \min\{u^{k+1}, u\}$, $u \geq 0$.

Remark 2.2. It is clear that a pseudo sup-multiplicative function Φ and any corresponding supporting function ω are increasing functions vanishing at zero, moreover their inverses Φ^{-1} and ν respectively are increasing and for all $v_1, v_2 \geq 0$ we have

$$\Phi^{-1}(v_1 v_2) \leq \nu(v_1)\Phi^{-1}(v_2).$$

Let X be a real Banach space. The nonempty convex closed subset P of X is called a *cone* in X if (i) $ax \in P$ and $x + y \in P$ for all $x, y \in P$ and $a \geq 0$; (ii) $x \in X$ and $-x \in X$ imply $x = 0$. A map $\psi : P \to [0, \infty)$ is a *nonnegative continuous concave* (*convex*) *functional map* provided ψ is nonnegative, continuous and satisfies

$$\psi(tx + (1-t)y) \geq \ (\leq)\ t\psi(x) + (1-t)\psi(y) \quad \text{for all } x, y \in P,\ t \in [0,1].$$

An operator $T : X \to X$ is *completely continuous* if it is continuous and maps bounded sets into pre-compact sets.

Let ψ be a nonnegative functional on a cone P of a real Banach space X. We define the sets

$$P_r = \{y \in P : \ \|y\| < r\},$$
$$P(\psi; a, b) = \{y \in P : \ a \leq \psi(y),\ \|y\| < b\}.$$

Theorem 2.1. [11] (Leggett-Williams Fixed-Point Theorem) Let $A < B < D < C$ be positive numbers, $T : \overline{P}_C \to \overline{P}_C$ be a completely continuous operator, and ψ be a nonnegative continuous concave functional on P such that $\psi(y) \leq \|y\|$ for all $y \in \overline{P}_C$. Suppose that

(E1) $\{y \in P(\psi; B, D) \mid \psi(y) > B\} \neq \emptyset$ and $\psi(Ty) > B$ for $y \in P(\psi; B, D)$;

(E2) $\|Ty\| < A$ for $y \in P$ with $\|y\| \leq A$;

(E3) $\psi(Ty) > B$ for $y \in P(\psi; B, C)$ with $\|Ty\| > D$.

Then T has at least three fixed points y_1, y_2 and y_3 such that $\|y_1\| < A$, $\psi(y_2) > B$ and $\|y_3\| > A$ with $\psi(y_3) < B$.

3 Bounded Positive Solutions of BVP (1.1)

In this section we shall establish the existence of at least three bounded positive solutions of BVP (1.1). For easy referencing, we list the conditions needed as follows:

(A1) $p : [0, \infty) \to (0, \infty)$ is continuous and satisfies

$$\int_0^\infty \phi^{-1}\left(\frac{1}{p(s)}\right) ds < \infty, \quad \int_0^\infty g(s) \int_0^s \phi^{-1}\left(\frac{1}{p(u)}\right) du ds < \infty;$$

(A2) $f:(0,\infty)\times[0,\infty)\to[0,\infty)$ is a Carathéodory function with $f(t,0)\not\equiv 0$ on each subinterval of $[0,\infty)$.

We denote

$$\tau=\tau(t)=\int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right)ds.$$

Choose k (>1) large enough such that

$$\int_0^{\frac{1}{k}} \phi^{-1}\left(\frac{1}{p(s)}\right)ds=\tau\left(\frac{1}{k}\right)<1.$$

Let

$$\mu=\int_0^{\frac{1}{k}} \phi^{-1}\left(\frac{1}{p(s)}\right)ds\ \frac{1}{1+\int_0^{\infty}\phi^{-1}\left(\frac{1}{p(s)}\right)ds}.$$

It is clear that

$$0<\mu<\int_0^{\frac{1}{k}} \phi^{-1}\left(\frac{1}{p(s)}\right)ds\ \frac{1}{1+\int_0^{1/k}\phi^{-1}\left(\frac{1}{p(s)}\right)ds}<1.$$

Let the Banach space

$$X=\left\{x\in C^0[0,\infty):\ \text{there exists the limt }\lim_{t\to\infty}x(t)\right\} \tag{3.1}$$

be equipped with the norm

$$\|x\|=\sup_{t\in[0,\infty)}|x(t)|\ \text{ for } x\in X. \tag{3.2}$$

Define the cone P in X by

$$P=\left\{x\in X: x(t)\geq 0 \text{ on } [0,\infty),\ x(t) \text{ is non-decresing on } [0,\infty),\ \min_{t\in[1/k,k]}x(t)\geq\mu\sup_{t\in[0,\infty)}x(t)\right\}. \tag{3.3}$$

Define the functional $\psi:P\to\mathbb{R}$ by

$$\psi(y)=\min_{t\in[1/k,k]}y(t),\quad y\in P. \tag{3.4}$$

It is easy to see that ψ is a nonnegative continuous concave functional on P such that $\psi(y)\leq\|y\|$ for all $y\in P$.

Now, to study (1.1), for $x\in X$ we consider the following boundary value problem

$$\begin{cases}[p(t)\phi(y'(t))]'+f(t,x(t))=0, & t\in(0,\infty),\\ y(0)=\displaystyle\int_0^{\infty}g(s)y(s)ds+a,\\ \displaystyle\lim_{t\to\infty}\phi^{-1}(p(t))y'(t)=b.\end{cases} \tag{3.5}$$

Lemma 3.1. Suppose that (A1) and (A2) hold and y is a solution of (3.5) for $x\in X$. Then, y can be expressed as

$$\begin{aligned}y(t) &= \frac{1}{1-\int_0^{\infty}g(s)ds}\int_0^{\infty}g(t)\int_0^t\phi^{-1}\left(\frac{1}{p(s)}\phi(b)+\frac{1}{p(s)}\int_s^{\infty}f(u,x(u))du\right)dsdt\\ &\quad+\int_0^t\phi^{-1}\left(\frac{1}{p(s)}\phi(b)+\frac{1}{p(s)}\int_s^{\infty}f(u,x(u))du\right)ds+\frac{a}{1-\int_0^{\infty}g(s)ds}.\end{aligned}$$

Proof. Since $x \in X$ is a solution of BVP (3.5) and f is a Carathéodory function, we get

$$\int_0^\infty f(s, x(s))ds < \infty.$$

Hence

$$y'(t) = \phi^{-1}\left(\frac{1}{p(t)}\phi(b) + \frac{1}{p(t)}\int_t^\infty f(u, x(u))du\right), \quad t \geq 0.$$

Integrating gives

$$y(t) = y(0) + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)}\int_s^\infty f(u, x(u))du\right)ds, \ t \geq 0. \tag{3.6}$$

The boundary conditions in (3.5) imply that

$$y(0) = y(0)\int_0^\infty g(s)ds + \int_0^\infty g(t)\int_0^t \phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)}\int_s^\infty f(u, x(u))du\right)dsdt + a.$$

It follows that

$$y(0) = \frac{\int_0^\infty g(t)\int_0^t \phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)}\int_s^\infty f(u, x(u))du\right)dsdt + a}{1 - \int_0^\infty g(s)ds}. \tag{3.7}$$

Substituting (3.7) into (3.6) completes the proof. □

Lemma 3.2. Suppose that (A1) and (A2) hold and y is a solution of (3.5) for $x \in X$. Then $y'(t) \geq 0$ for all $t \in [0, \infty)$, $y(t) > 0$ for all $t \in (0, \infty)$ and $y(t)$ is concave with respect to τ on $[0, \infty)$, where

$$\tau = \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right)ds.$$

Proof. First, we shall prove that y' is positive on $[0, \infty)$. Since y is a solution of (3.5), (A2) implies that $[p(t)\phi(y'(t))]' \leq 0$ for all $t \in [0, \infty)$. Then

$$\phi(b) - p(t)\phi(y'(t)) \leq 0, \quad t \in [0, \infty).$$

Since $b \geq 0$, we have $p(t)\phi(y'(t)) \geq 0$. Thus $y'(t) \geq 0$ for all $t \in [0, \infty)$.

Next, we shall prove that $y(t) \geq 0$ for $t \in [0, \infty)$. Since $y'(t) \geq 0$ for all $t \in [0, \infty)$, it suffices to show that $y(0) \geq 0$. The boundary conditions in (3.5) imply that

$$y(0) = \int_0^\infty g(s)y(s)ds + a \geq y(0)\int_0^\infty g(s)ds.$$

Since $\int_0^\infty g(s)ds < 1$, we get $y(0) \geq 0$. Hence, $y(t) \geq 0$ for $t \in [0, \infty)$. It follows from (A2) that $y(t) > 0$ for all $t \in (0, \infty)$.

Finally, we shall prove that y is concave with respect to τ on $[0, \infty)$. From (A1) we have $\int_0^\infty \phi^{-1}\left(\frac{1}{p(s)}\right)ds < \infty$. So $\tau \in C\left([0, \infty), \left[0, \int_0^\infty \phi^{-1}\left(\frac{1}{p(s)}\right)ds\right)\right)$ and

$$\frac{d\tau}{dt} = \phi^{-1}\left(\frac{1}{p(t)}\right) > 0.$$

Thus

$$\frac{dy}{dt} = \frac{dy}{d\tau}\frac{d\tau}{dt} = \frac{dy}{d\tau}\phi^{-1}\left(\frac{1}{p(t)}\right). \tag{3.8}$$

It follows that

$$\frac{dy}{d\tau} = \frac{dy}{dt}\frac{1}{\phi^{-1}\left(\frac{1}{p(t)}\right)} \geq 0.$$

Moreover, since

$$p(t)\phi\left(\frac{dy}{dt}\right) = \phi\left(\frac{dy}{d\tau}\right),$$

we get

$$\left[p(t)\phi\left(\frac{dy}{dt}\right)\right]' = \phi'\left(\frac{dy}{d\tau}\right)\frac{d^2y}{d\tau^2}\frac{d\tau}{dt}.$$

So

$$\frac{d^2y}{d\tau^2} = \frac{\left[p(t)\phi\left(\frac{dy}{dt}\right)\right]'}{\phi'\left(\frac{dy}{d\tau}\right)\frac{d\tau}{dt}}.$$

Since $[p(t)\phi(y'(t))]' \leq 0$, $\phi'(y) > 0$ $(y > 0)$ and $\frac{d\tau}{dt} > 0$, we obtain $\frac{d^2y}{d\tau^2} \leq 0$. Hence $y(t)$ is concave with respect to τ on $[0, \infty)$. The proof is complete. □

Define the nonlinear operator $T : P \to X$ by

$$\begin{aligned}(Tx)(t) &= \frac{1}{1-\int_0^\infty g(s)ds}\int_0^\infty g(t)\int_0^t \phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)}\int_s^\infty f(u,x(u))du\right)dsdt \\ &\quad + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)}\int_s^\infty f(u,x(u))du\right)ds + \frac{a}{1-\int_0^\infty g(s)ds}.\end{aligned} \tag{3.9}$$

Lemma 3.3. Suppose that (A1) and (A2) hold. We have the following:

(i) For $x \in P$, Tx satisfies

$$\begin{cases} [p(t)\phi((Tx)'(t))]' + f\,(t, x(t)) = 0, \quad t \in (0,\infty), \\ (Tx)(0) = \displaystyle\int_0^\infty g(s)(Tx)(s)ds + a, \\ \displaystyle\lim_{t\to\infty} \phi^{-1}(p(t))(Tx)'(t) = b; \end{cases} \tag{3.10}$$

(ii) $Tx \in P$ for each $x \in P$;

(iii) x is a bounded positive solution of BVP (1.1) if and only if x is a solution of the operator equation $x = Tx$ in P.

Proof. The proofs of (i) and (iii) follow from the definition of T and are omitted.

To show (ii), we note from (i) that Tx is a solution of (3.5). Then, Lemma 3.2 implies that $(Tx)(t) \geq 0$ and $(Tx)'(t) \geq 0$ for all $t \in [0,\infty)$, and $(Tx)(t)$ is concave with respect to $\tau = \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right)ds$. To complete the proof of $TP \subseteq P$, it suffices to prove that

$$\min_{t\in[1/k,k]}(Tx)(t) \geq \mu \sup_{t\in[0,\infty)}(Tx)(t). \tag{3.11}$$

Since $x \in X$ and f is a Carathéodory function, there exist $r > 0$ and $B_r \in L^1(0,\infty)$ such that $\|x\| < r$ and $0 \le f(t, x(t)) \le B_r(t)$ for $t \in [0,\infty)$. Then,

$$\begin{aligned}
(Tx)(t) &= \frac{1}{1-\int_0^\infty g(s)ds}\int_0^\infty g(t)\int_0^t \phi^{-1}\left(\frac{1}{p(s)}\phi(b)+\frac{1}{p(s)}\int_s^\infty f(u,x(u))du\right)dsdt \\
&\quad + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\phi(b)+\frac{1}{p(s)}\int_s^\infty f(u,x(u))du\right)ds + \frac{a}{1-\int_0^\infty g(s)ds} \\
&\le \frac{1}{1-\int_0^\infty g(s)ds}\int_0^\infty g(t)\int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right)dsdt\ \phi^{-1}\left(\phi(b)+\int_0^\infty B_r(u)du\right) \\
&\quad + \int_0^\infty \phi^{-1}\left(\frac{1}{p(s)}\right)ds\ \phi^{-1}\left(\phi(b)+\int_0^\infty B_r(u)du\right) + \frac{a}{1-\int_0^\infty g(s)ds} \\
&< \infty.
\end{aligned}$$

So $\sup_{t\in[0,\infty)}(Tx)(t)$ exists. We shall consider two cases.

First, suppose $(Tx)(t)$ achieves its maximum at $\sigma \in [0,\infty)$. Noting that

$$\tau(t) = \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right)ds,$$

and the inverse function of $\tau = \tau(t)$ is denoted by $t = t(\tau)$, one sees for $t \in [1/k, k]$ that

$$\begin{aligned}
(Tx)(t) &\ge (Tx)(1/k) \\
&= (Tx)\left(t(\tau(1/k))\right) \\
&= (Tx)\left(t\left(\frac{1-\tau(1/k)+\tau(\sigma)}{1+\tau(\sigma)}\frac{\tau(1/k)}{1-\tau(1/k)+\tau(\sigma)}+\frac{\tau(1/k)}{1+\tau(\sigma)}\tau(\sigma)\right)\right).
\end{aligned}$$

Noting that $\tau(1/k) < 1$ and $(Tx)(t)$ is concave with respect to τ, we find for $t \in [1/k, k]$,

$$\begin{aligned}
(Tx)(t) &\ge \frac{1-\tau(1/k)+\tau(\sigma)}{1+\tau(\sigma)}(Tx)\left(t\left(\frac{\tau(1/k)}{1-\tau(1/k)+\tau(\sigma)}\right)\right)+\frac{\tau(1/k)}{1+\tau(\sigma)}(Tx)\left(t\left(\tau(\sigma)\right)\right) \\
&\ge \frac{\tau(1/k)}{1+\tau(\sigma)}(Tx)\left(t\left(\tau(\sigma)\right)\right) \\
&= \int_0^{\frac{1}{k}} \phi^{-1}\left(\frac{1}{p(s)}\right)ds\ \frac{1}{1+\tau(\sigma)}(Tx)(\sigma) \\
&\ge \int_0^{\frac{1}{k}} \phi^{-1}\left(\frac{1}{p(s)}\right)ds\ \frac{1}{1+\int_0^\infty \phi^{-1}\left(\frac{1}{p(s)}\right)ds}\ \sup_{t\in[0,\infty)}(Tx)(t) \\
&= \mu \sup_{t\in[0,\infty)}(Tx)(t).
\end{aligned}$$

Next, suppose $(Tx)(t)$ achieves its maximum at ∞. Choose $\sigma' \in [0, \infty)$. Similar to the above discussion, we get for $t \in [1/k, k]$ that

$$(Tx)(t) \geq \mu(Tx)(\sigma').$$

Let $\sigma' \to \infty$, we get for $t \in [1/k, k]$ that

$$(Tx)(t) \geq \mu \sup_{t \in [0,\infty)} (Tx)(t).$$

We have shown that (3.11) holds in both cases. Hence $Tx \in P$. $\square$

Lemma 3.4. $T : P \to P$ is completely continuous.

Proof. It is easy to verify that $T : P \to P$ is well defined. We shall prove that T is continuous and maps bounded sets into pre-compact sets.

Let $x_n \to x_0$ as $n \to \infty$ in P, then there exists r_0 such that $\sup_{n \geq 0} \|x_n\| < r_0$. Set

$$B_{r_0}(t) = \sup_{|u| \in [0, r_0]} f(t, u).$$

Then, we have

$$\int_0^\infty |f(s, x_n(s)) - f(s, x_0(s))| ds \leq 2 \int_0^\infty B_{r_0}(s) ds.$$

Therefore by the Lebesgue dominated convergence theorem, we obtain

$$\int_t^\infty f(u, x_n(u)) du \to \int_t^\infty f(u, x_0(u)) du \ \text{ uniformly as } n \to \infty.$$

For any $\epsilon > 0$, since, for all n,

$$\phi(b) + \int_s^\infty f(u, x_n(u)) du \leq \phi(b) + \int_0^\infty B_{r_0}(u) du \equiv r,$$

and ϕ^{-1} is uniformly continuous on $[0, r]$, we see that there exists $\delta > 0$ such that $x, y \in [0, r]$ and $|x - y| < \delta$ implies

$$\left|\phi^{-1}(x) - \phi^{-1}(y)\right| < \epsilon.$$

So for this $\delta > 0$, there exists $N > 0$ such that

$$\left| b + \int_t^\infty f(u, x_n(u)) du - \left(b + \int_t^\infty f(u, x_0(u)) du \right) \right| < \delta, \quad n > N, \ t \in [0, \infty).$$

Then for $n > N$, we have

$$\left| \phi^{-1} \left(b + \int_t^\infty f(u, x_n(u)) du \right) - \phi^{-1} \left(b + \int_t^\infty f(u, x_0(u)) du \right) \right| < \epsilon.$$

Thus, we get for $t \in [0,\infty)$ and $n > N$ that

$$\begin{aligned}
0 &\leq |[(Tx_n)-(Tx_0)](t)| \\
&= \left| \frac{1}{1-\int_0^\infty g(s)ds} \int_0^\infty g(t) \int_0^t \left[\phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)} \int_s^\infty f(u,x_n(u))du \right) \right. \right. \\
&\qquad \left. - \phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)} \int_s^\infty f(u,x_0(u))du \right) \right] dsdt \\
&\quad + \int_0^t \left[\phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)} \int_s^\infty f(u,x_n(u))du \right) \right. \\
&\qquad \left. \left. - \phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)} \int_s^\infty f(u,x_0(u))du \right) \right] ds \right| \\
&\leq \frac{1}{1-\int_0^\infty g(s)ds} \int_0^\infty g(t) \int_0^t \left| \phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)} \int_s^\infty f(u,x_n(u))du \right) \right. \\
&\qquad \left. - \phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)} \int_s^\infty f(u,x_0(u))du \right) \right| dsdt \\
&\quad + \int_0^t \left| \phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)} \int_s^\infty f(u,x_n(u))du \right) \right. \\
&\qquad \left. - \phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)} \int_s^\infty f(u,x_0(u))du \right) \right| ds \\
&\leq \frac{1}{1-\int_0^\infty g(s)ds} \int_0^\infty g(t) \int_0^t \phi^{-1}\left(\frac{1}{p(s)} \right) \epsilon dsdt \; + \; \int_0^t \phi^{-1}\left(\frac{1}{p(s)} \right) \epsilon ds \\
&\leq \left[\frac{1}{1-\int_0^\infty g(s)ds} \int_0^\infty g(t) \int_0^t \phi^{-1}\left(\frac{1}{p(s)} \right) dsdt + \int_0^\infty \phi^{-1}\left(\frac{1}{p(s)} \right) ds \right] \epsilon.
\end{aligned}$$

It follows that

$$\|(Tx_n)-(Tx_0)\| \to 0$$

uniformly as $n \to \infty$. So, T is continuous.

Let Ω be any bounded subset of P. First, we shall prove that $T\Omega$ is bounded. Since Ω is bounded, there exists $r > 0$ such that $\|x\| \leq r$ for all $x \in \Omega$. Denote

$$B_r(t) = \sup_{|u| \in [0,r]} f(t,u).$$

Obviously, we have

$$\begin{aligned}
0 &\leq (Tx)(t) \\
&= \frac{1}{1-\int_0^\infty g(s)ds}\int_0^\infty g(t)\int_0^t \phi^{-1}\left(\frac{1}{p(s)}\phi(b)+\frac{1}{p(s)}\int_s^\infty f(u,x(u))du\right)dsdt \\
&\quad +\int_0^t \phi^{-1}\left(\frac{1}{p(s)}\phi(b)+\frac{1}{p(s)}\int_s^\infty f(u,x(u))du\right)ds+\frac{a}{1-\int_0^\infty g(s)ds} \\
&\leq \frac{1}{1-\int_0^\infty g(s)ds}\int_0^\infty g(t)\int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right)dsdt\ \phi^{-1}\left(\phi(b)+\int_0^\infty B_r(u)du\right) \\
&\quad +\int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right)ds\ \phi^{-1}\left(\phi(b)+\int_0^\infty B_r(u)du\right)+\frac{a}{1-\int_0^\infty g(s)ds} \\
&\leq \frac{1}{1-\int_0^\infty g(s)ds}\int_0^\infty g(t)\int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right)dsdt\ \phi^{-1}\left(\phi(b)+\int_0^\infty B_r(u)du\right) \\
&\quad +\int_0^\infty \phi^{-1}\left(\frac{1}{p(s)}\right)ds\ \phi^{-1}\left(\phi(b)+\int_0^\infty B_r(u)du\right)+\frac{a}{1-\int_0^\infty g(s)ds} \\
&< \infty.
\end{aligned}$$

So $T\Omega$ is bounded.

Next, since

$$(Tx)'(t)=\phi^{-1}\left(\frac{1}{p(t)}\right)\phi^{-1}\left(\phi(b)+\int_t^\infty f(u,x(u))du\right),$$

we find for any $N\in(0,\infty)$ and $t_1,t_2\in[0,N]$,

$$\begin{aligned}
|(Tx)(t_1)-(Tx)(t_2)| &\leq \left|\int_{t_1}^{t_2}(Tx)'(s)ds\right| \\
&= \left|\int_{t_1}^{t_2}\phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(\phi(b)+\int_s^\infty f(u,x(u))du\right)ds\right| \\
&\leq \left|\int_{t_1}^{t_2}\phi^{-1}\left(\frac{1}{p(s)}\right)ds\right|\phi^{-1}\left(\phi(b)+\int_0^\infty B_r(u)du\right) \\
&\to 0 \text{ uniformly as } t_1\to t_2
\end{aligned}$$

for all $x\in\Omega$. So $\{(Tx)(t):x\in\Omega\}$ is equicontinuous on any compact interval of $[0,\infty)$.

Finally, we shall prove that $\{(Tx)(t):x\in\Omega\}$ is equiconvergent at infinity. Since

$$0\leq\phi(b)+\int_t^\infty f(u,x(u))du\leq\phi(b)+\int_0^\infty B_r(s)ds\equiv r,$$

and $\int_0^\infty \phi^{-1}\left(\frac{1}{p(s)}\right)ds<\infty$, we know there exists $N>0$ such that

$$\left|\int_{t_1}^{t_2}\phi^{-1}\left(\frac{1}{p(s)}\right)ds\right|<\frac{\epsilon}{\phi^{-1}(r)},\quad t_1,t_2>N.$$

It follows that

$$\begin{aligned} |(Tx)(t_1) - (Tx)(t_2)| &\leq \left| \int_{t_1}^{t_2} \phi^{-1}\left(\frac{1}{p(s)}\right) ds \right| \phi^{-1}\left(\phi(b) + \int_0^\infty B_r(u)du\right) \\ &< \frac{\epsilon}{\phi^{-1}(r)}\, \phi^{-1}(r) \;=\; \epsilon, \quad t_1, t_2 > N. \end{aligned}$$

So $\{(Tx)(t) : x \in \Omega\}$ is equiconvergent at infinity. By using Theorem 2.5 in [23], we obtain that $\{(Tx)(t) : x \in \Omega\}$ is pre-compact. Hence, $T : P \to P$ is completely continuous. □

For positive numbers $e_1,\ e_2$, and C, let $M,\ M_1$ and L be defined by

$$M = C\left[\phi\left(\frac{\left(1 - \int_0^\infty g(s)ds\right) C - a}{\int_0^\infty g(t) \int_0^t \frac{1}{\phi^{-1}(p(s))} ds dt + \left(1 - \int_0^\infty g(s)ds\right) \int_0^\infty \frac{1}{\phi^{-1}(p(s))} ds}\right) - \phi(b)\right]^{-1}, \tag{3.12}$$

$$M_1 = e_1\left[\phi\left(\frac{\left(1 - \int_0^\infty g(s)ds\right) e_1 - a}{\int_0^\infty g(t) \int_0^t \frac{1}{\phi^{-1}(p(s))} ds dt + \left(1 - \int_0^\infty g(s)ds\right) \int_0^\infty \frac{1}{\phi^{-1}(p(s))} ds}\right) - \phi(b)\right]^{-1}, \tag{3.13}$$

and

$$L = \mu(k-1)e_2\left[\phi\left(\frac{\mu(1+k)e_2\left(1 - \int_0^\infty g(s)ds\right) - a}{\left(1 - \int_0^\infty g(s)ds\right) \int_0^{1/k} \frac{1}{\phi^{-1}(p(s))} ds}\right) - \phi(b)\right]^{-1}. \tag{3.14}$$

Theorem 3.1. Suppose that (A1) and (A2) hold and there exist constants $e_1,\ e_2$ and C such that

$$0 < e_1 < \mu(1+k)e_2 < (1+k)e_2 < C, \qquad LC > M\mu(1+k)e_2 > 0$$

and

(C1) $f(t,x) \leq \frac{C}{M(1+t)^2}$ for $t \in (0,\infty)$ and $x \in [0, C]$;

(C2) $f(t,x) \leq \frac{e_1}{M_1(1+t)^2}$ for $t \in (0,\infty)$ and $x \in [0, e_1]$;

(C3) $f(t,x) \geq \frac{\mu(1+k)e_2}{L(1+t)^2}$ for $t \in [1/k, k]$ and $x \in [\mu(1+k)e_2, (1+k)e_2]$.

Then, BVP (1.1) has at least three bounded positive solutions x_1, x_2 and x_3 satisfying

$$\sup_{t\in[0,\infty)} x_1(t) < e_1, \quad \min_{t\in[1/k,k]} x_2(t) > \mu(1+k)e_2$$

and

$$\sup_{t\in[0,\infty)} x_3(t) > e_1, \quad \min_{t\in[1/k,k]} x_3(t) < \mu(1+k)e_2.$$

Proof. We shall apply Theorem 2.1 with T, P and ψ defined in (3.9), (3.3) and (3.4) respectively. To recap, a fixed point of T is a solution of (1.1) (Lemma 3.3), $T : P \to P$ is completely continuous (Lemma 3.4), and ψ is a nonnegative continuous concave functional on the cone P with $\psi(y) \leq \|y\|$ for all $y \in P$. Further, corresponding to Theorem 2.1, we choose

$$D = (1+k)e_2, \quad B = \mu(1+k)e_2, \quad A = e_1.$$

Then $0 < A < B < D < C$. We divide the remainder of the proof into four steps.

Step 1. We shall prove that $T(\overline{P_C}) \subset \overline{P_C}$. Let $x \in \overline{P_C}$, then $\|x\| \leq C$, so

$$0 \leq x(t) \leq C,\ t \in [0, \infty).$$

It follows from (C1) that

$$f(t, x(t)) \leq \frac{C}{M(1+t)^2},\ t \in [0, \infty).$$

We find

$$\begin{aligned}
\|Tx\| &= \sup_{t\in[0,\infty)} (Tx)(t) \\
&= \frac{\int_0^\infty g(t) \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)}\int_s^\infty f(u, x(u))du\right) dsdt}{1 - \int_0^\infty g(s)ds} \\
&\quad + \int_0^\infty \phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)}\int_s^\infty f(u, x(u))du\right) ds + \frac{a}{1 - \int_0^\infty g(s)ds} \\
&\leq \frac{\int_0^\infty g(t) \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right) dsdt\ \phi^{-1}\left(\phi(b) + \int_0^\infty f(u, x(u))du\right)}{1 - \int_0^\infty g(s)ds} \\
&\quad + \int_0^\infty \phi^{-1}\left(\frac{1}{p(s)}\right) ds\ \phi^{-1}\left(\phi(b) + \int_0^\infty f(u, x(u))du\right) + \frac{a}{1 - \int_0^\infty g(s)ds} \\
&\leq \frac{\int_0^\infty g(t) \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right) dsdt\ \phi^{-1}\left(\phi(b) + \int_0^\infty \frac{C}{M(1+u)^2}du\right)}{1 - \int_0^\infty g(s)ds} \\
&\quad + \int_0^\infty \phi^{-1}\left(\frac{1}{p(s)}\right) ds\ \phi^{-1}\left(\phi(b) + \int_0^\infty \frac{C}{M(1+u)^2}du\right) + \frac{a}{1 - \int_0^\infty g(s)ds} \\
&= \left[\frac{\int_0^\infty g(t) \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right) dsdt}{1 - \int_0^\infty g(s)ds} + \int_0^\infty \phi^{-1}\left(\frac{1}{p(s)}\right) ds\right] \phi^{-1}\left(\phi(b) + \frac{C}{M}\right) \\
&\quad + \frac{a}{1 - \int_0^\infty g(s)ds} \\
&= C
\end{aligned}$$

where the last equality follows from the definition of M in (3.12). Hence, $Tx \in \overline{P_C}$. This shows that $T(\overline{P_C}) \subset \overline{P_C}$.

Step 2. We shall show that (E1) of Theorem 2.1 holds, i.e.,

$$\{y \in P(\psi; B, D) \mid \psi(y) > B\} = \{y \in P(\psi; \mu(1+k)e_2, (1+k)e_2) \mid \psi(y) > \mu(1+k)e_2\} \neq \emptyset$$

and $\psi(Ty) > B = \mu(1+k)e_2$ for $y \in P(\psi; \mu(1+k)e_2, (1+k)e_2)$.

To prove that $\{y \in P(\psi; \mu(1+k)e_2, (1+k)e_2) \mid \psi(y) > \mu(1+k)e_2\} \neq \emptyset$, we choose $\lambda > 0$ and let

$$y_0(t) = \begin{cases} \lambda - k^2\lambda\left(t - \dfrac{1}{k}\right)^2, & t \in \left[0, \dfrac{1}{k}\right], \\ \lambda, & t \geq \dfrac{1}{k}. \end{cases}$$

It is easy to see that

$$\min_{t\in[1/k,k]} y_0(t) = \lambda,$$

and

$$\sup_{t\in[0,\infty)} y_0(t) \leq \lambda.$$

Since $\mu < 1$, we get $\min_{t\in[1/k,k]} y_0(t) \geq \mu \sup_{t\in[0,\infty)} y_0(t)$. It is easy to see that $y_0 \in \{y \in P(\psi; B, D) \mid \psi(y) > B\}$ if $\lambda \in (B, D)$.

Next, let $y \in P(\psi; \mu(1+k)e_2, (1+k)e_2)$, then $\psi(y) \geq \mu(1+k)e_2$ and $\|y\| \leq (1+k)e_2$. So

$$\min_{t\in[1/k,k]} y(t) \geq \mu(1+k)e_2, \quad \sup_{t\in[0,\infty)} y(t) \leq (1+k)e_2.$$

Hence,

$$\mu(1+k)e_2 \leq y(t) \leq (1+k)e_2, \ t \in [1/k, k].$$

It follows from (C3) that

$$f(t, y(t)) \geq \frac{\mu(1+k)e_2}{L(1+t)^2}, \ t \in [1/k, k].$$

We find

$$\begin{aligned}
\psi(Ty) &= \min_{t\in[1/k,k]} (Ty)(t) \\
&= (Ty)\left(\frac{1}{k}\right) \\
&= \frac{1}{1 - \int_0^\infty g(s)ds} \int_0^\infty g(t) \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)}\int_s^\infty f(u, y(u))du\right) dsdt \\
&\quad + \int_0^{1/k} \phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)}\int_s^\infty f(u, y(u))du\right) ds + \frac{a}{1 - \int_0^\infty g(s)ds} \\
&\geq \int_0^{1/k} \phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)}\int_{1/k}^k f(u, y(u))du\right) ds + \frac{a}{1 - \int_0^\infty g(s)ds} \\
&\geq \int_0^{1/k} \phi^{-1}\left(\frac{1}{p(s)}\right) ds \ \phi^{-1}\left(\phi(b) + \int_{1/k}^k \frac{\mu(1+k)e_2}{L(1+u)^2}du\right) + \frac{a}{1 - \int_0^\infty g(s)ds} \\
&= \int_0^{1/k} \phi^{-1}\left(\frac{1}{p(s)}\right) ds \ \phi^{-1}\left(\phi(b) + \frac{\mu(k-1)e_2}{L}\right) + \frac{a}{1 - \int_0^\infty g(s)ds} \\
&= B
\end{aligned}$$

where the last equality follows from the definition of L in (3.14). This completes the proof of step 2.

Step 3. We shall prove that (E2) of Theorem 2.1 holds, i.e., $\|Ty\| < A$ for $y \in P$ with $\|y\| \leq A$. Let $y \in P$ with $\|y\| \leq A = e_1$, then

$$\sup_{t\in[0,\infty)} y(t) \leq e_1.$$

It follows from (C2) that

$$f(t, y(t)) \leq \frac{e_1}{M_1(1+t)^2},\ t \in [0,\infty).$$

We find

$$\begin{aligned}
\|Ty\| &= \sup_{t\in[0,\infty)} (Ty)(t) \\
&= \sup_{t\in[0,\infty)} \frac{\int_0^\infty g(t) \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)}\int_s^\infty f(u,y(u))du\right) dsdt}{1-\int_0^\infty g(s)ds} \\
&\quad + \int_0^\infty \phi^{-1}\left(\frac{1}{p(s)}\phi(b) + \frac{1}{p(s)}\int_s^\infty f(u,y(u))du\right) ds + \frac{a}{1-\int_0^\infty g(s)ds} \\
&< \frac{\int_0^\infty g(t) \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right) dsdt\ \phi^{-1}\left(\phi(b) + \frac{e_1}{M_1}\right)}{1-\int_0^\infty g(s)ds} \\
&\quad + \int_0^\infty \phi^{-1}\left(\frac{1}{p(s)}\right) ds\ \phi^{-1}\left(\phi(b) + \frac{e_1}{M_1}\right) + \frac{a}{1-\int_0^\infty g(s)ds} \\
&= e_1
\end{aligned}$$

where the last equality follows from the definition of M_1 in (3.13). Thus, $\|Ty\| < e_1$ for $y \in P$ with $\|y\| \leq e_1$. This completes the proof of step 3.

Step 4. We shall show that (E3) of Theorem 2.1 holds, i.e., $\psi(Ty) > B$ for $y \in P(\psi; B, C)$ with $\|Ty\| > D$. Let $y \in P(\psi; B, C) = P(\psi; \mu(1+k)e_2, C)$ with $\|Ty\| > D = (1+k)e_2$, then

$$\sup_{t\in[0,\infty)} (Ty)(t) \geq (1+k)e_2 \quad \text{and} \quad \|y\| = \sup_{t\in[0,\infty)} y(t) \leq C.$$

Noting $Ty \in P$, we get

$$\psi(Ty) = \min_{t\in[1/k,k]} (Ty)(t) \geq \mu \sup_{t\in[0,\infty)} (Ty)(t) \geq \mu(1+k)e_2 = B.$$

This completes the proof of step 4.

We have shown that all the conditions of Theorem 2.1 are satisfied. Hence, by Theorem 2.1 the operator T has three fixed points x_1, x_2 and $x_3 \in \overline{P_C}$ such that

$$\|x_1\| < A,\ \ \psi(x_2) > B,\ \ \|x_3\| > A \text{ with } \psi(x_3) < B,$$

i.e., x_1, x_2 and x_3 satisfy

$$\sup_{t\in[0,\infty)} x_1(t) < e_1, \quad \min_{t\in[1/k,k]} x_2(t) > \mu(1+k)e_2 \tag{3.15}$$

and

$$\sup_{t\in[0,\infty)} x_3(t) > e_1, \quad \min_{t\in[1/k,k]} x_3(t) < \mu(1+k)e_2. \tag{3.16}$$

Hence, BVP (1.1) has at least three positive solutions $x_1,\ x_2$ and x_3 satisfying (3.15) and (3.16). It is easy to see that $x_1,\ x_2$ and x_3 are bounded positive solutions since $x_1, x_2, x_3 \in \overline{P_c}$ imply that

$$\sup_{t\in[0,\infty)} x_i(t) \le C, \ \ i = 1, 2, 3.$$

The proof is complete. □

Let

$$\psi(t) = \begin{cases} \frac{1}{\sqrt{t}}, & t \in (0,1), \\ \frac{1}{t^2}, & t \ge 1. \end{cases}$$

For positive numbers $e_1,\ e_2$, and C, let $M,\ M_1$ and L be defined by

$$M = 3C\left[\phi\left(\frac{\left(1-\int_0^\infty g(s)ds\right)C - a}{\int_0^\infty g(t)\int_0^t \frac{1}{\phi^{-1}(p(s))}dsdt + \left(1-\int_0^\infty g(s)ds\right)\int_0^\infty \frac{1}{\phi^{-1}(p(s))}ds}\right) - \phi(b)\right]^{-1}, \tag{3.17}$$

$$M_1 = 3e_1\left[\phi\left(\frac{\left(1-\int_0^\infty g(s)ds\right)e_1 - a}{\int_0^\infty g(t)\int_0^t \frac{1}{\phi^{-1}(p(s))}dsdt + \left(1-\int_0^\infty g(s)ds\right)\int_0^\infty \frac{1}{\phi^{-1}(p(s))}ds}\right) - \phi(b)\right]^{-1}, \tag{3.18}$$

and

$$L = \mu\left(3 - \frac{1}{k} - \sqrt{\frac{1}{k}}\right)e_2\left[\phi\left(\frac{\mu(1+k)e_2\left(1-\int_0^\infty g(s)ds\right) - a}{\left(1-\int_0^\infty g(s)ds\right)\int_0^{1/k}\frac{1}{\phi^{-1}(p(s))}ds}\right) - \phi(b)\right]^{-1}. \tag{3.19}$$

Theorem 3.2. Suppose that (A1) and (A2) hold and there exist constants $e_1,\ e_2$ and C such that

$$0 < e_1 < \mu(1+k)e_2 < (1+k)e_2 < C, \qquad LC > M\mu(1+k)e_2 > 0$$

and

(D1) $f(t,x) \le \frac{C\psi(t)}{M}$ for $t \in [0,\infty)$ and $x \in [0,C]$;

(D2) $f(t,x) \le \frac{e_1\psi(t)}{M_1}$ for $t \in [0,\infty)$ and $x \in [0,e_1]$;

(D3) $f(t,x) \ge \frac{\mu(1+k)e_2\psi(t)}{L}$ for $t \in [1/k,k]$ and $x \in [\mu(1+k)e_2, (1+k)e_2]$.

Then, BVP (1.1) has at least three bounded positive solutions x_1, x_2 and x_3 satisfying

$$\sup_{t\in[0,\infty)} x_1(t) < e_1, \quad \min_{t\in[1/k,k]} x_2(t) > \mu(1+k)e_2$$

and

$$\sup_{t\in[0,\infty)} x_3(t) > e_1, \quad \min_{t\in[1/k,k]} x_3(t) < \mu(1+k)e_2.$$

Proof. The proof is similar to that of Theorem 3.1 and is omitted. □

Remark 3.1. In (C1) and (C2) of Theorem 3.1, it is easy to see that f is bounded on $(0,\infty)\times[0,C]$ or $(0,\infty)\times[0,e_1]$. However, in (D1) and (D2) of Theorem 3.2, f may be unbounded on $(0,\infty)\times[0,C]$ or $(0,\infty)\times[0,e_1]$, i.e., f may be singular at $t=0$ since $\psi(t)$ is singular at $t=0$.

4 Bounded Positive Solutions of BVP (1.2)

In this section we shall establish the existence of at least one bounded positive solution of BVP (1.2).

Remark 4.1. In [18, 19, 20, 37], the authors study the existence of multiple positive solutions (such that $\sup_{t\in[0,\infty)} \frac{x(t)}{1+t} < \infty$) of the multi-point boundary value problem for differential equation on the half line

$$\begin{cases} [\phi(x'(t))]' + f(t,x(t)) = 0, \quad t\in(0,\infty), \\ x(0) = \sum_{i=1}^{m} a_i x(\xi_i), \\ \lim_{t\to\infty} x'(t) = 0. \end{cases}$$

Here, ϕ is an *increasing homeomorphism and positive homomorphism* satisfying:

(i) $\phi(x) \le \phi(y)$ for all $x \le y$;

(ii) ϕ is a continuous bijection with $\phi(0)=0$ and its inverse function is also continuous;

(iii) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in [0,\infty)$ or for all $x, y \in \mathbb{R}$.

Note that if ϕ satisfies (i)–(iii) and is differentiable at $t=1$ with $\phi'(1) > 0$, one actually gets $\phi(x) = |x|^{q-2}x$ for some $q > 1$, i.e., ϕ is an one dimensional Laplacian. In fact, we have

$$\begin{aligned} \phi'(x) &= \lim_{\Delta x\to 0} \frac{\phi(x+\Delta x)-\phi(x)}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\phi\left(x\left(1+\frac{\Delta x}{x}\right)\right)-\phi(x)}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\phi(x)\,\phi\left(1+\frac{\Delta x}{x}\right)-\phi(x)\phi(1)}{\Delta x} \;=\; \frac{\phi(x)}{x}\phi'(1). \end{aligned}$$

So $|\phi(x)| = |x|^{\phi'(1)}$. Since $\phi'(1) > 0$, it is easy to see that there exists $q > 1$ such that $\phi(x) = |x|^{q-2}x$, i.e., ϕ is an one dimensional Laplacian. Hence, it will be interesting to consider a more general ϕ which is an increasing homeomorphism, indeed the Φ in (1.2) is the more general case.

For easy referencing, we list the conditions needed as follows:

(A3) $f(t,0) \not\equiv 0$ on each subinterval of $[0,\infty)$, $t \to f(t,u)$ is measurable and $u \to f(t,u)$ is continuous, and for each $r > 0$ there exists $B_r \in L^1(0,\infty)$ such that

$$f(t,x) \le B_r(t), \text{ for all } t \in (0,\infty) \text{ and } |x| \le r, \int_0^\infty \Phi^{-1}\left(\int_s^\infty B_r(u)du\right) ds < \infty;$$

(A4) $\Phi : \mathbb{R} \to \mathbb{R}$ is a pseudo sup-multiplicative function with supporting function w (the inverse of w is ν), Φ maps $[0,\infty)$ into $[0,\infty)$, and there exists a constant $\mu > 0$ such that $|\Phi^{-1}(x) - \Phi^{-1}(y)| \le \mu\Phi^{-1}(|x-y|)$ for all $x, y \ge 0$;

(A5)$_\sigma$ there exist positive number $\sigma > 0$ and positive functions ψ_i $(i = 1, 2)$ such that

$$\int_0^\infty \Phi^{-1}\left(\int_s^\infty \psi_i(u)du\right) ds < \infty \quad (i = 1, 2)$$

and

$$|f(t,x) - \psi_1(t)| \le \psi_2(t)\Phi(|x|^\sigma), \quad t \in (0,\infty), \ x \in \mathbb{R}.$$

Theorem 4.1. Suppose that (A3) and (A4) hold. Then, BVP (1.2) has at least one bounded positive solution if (A5)$_\sigma$ holds for

(i) $\sigma > 1$ and

$$\frac{\|\psi_0\|^{1-\sigma}(\sigma-1)^{\sigma-1}}{\sigma^\sigma} \ge \frac{\mu\int_0^\infty \nu\left(\int_s^\infty \psi_2(u)du\right) ds}{1 - \int_0^\infty g(s)ds}$$

where

$$\begin{aligned}\psi_0(t) &= \frac{1}{1-\int_0^\infty g(s)ds}\int_0^\infty g(t)\int_0^t \Phi^{-1}\left(\int_s^\infty \psi_1(u)du\right) dsdt \\ &\quad + \int_0^t \Phi^{-1}\left(\int_s^\infty \psi_1(u)du\right) ds, \quad \text{or}\end{aligned}$$

(ii) $\sigma \in (0,1)$, or

(iii) $\sigma = 1$ and

$$\frac{\mu\int_0^\infty \nu\left(\int_s^\infty \psi_2(u)du\right) ds}{1 - \int_0^\infty g(s)ds} < 1.$$

Proof. Let the Banach space X and its norm be defined as in (3.1) and (3.2). Define the nonlinear operator T by

$$\begin{aligned}(Tx)(t) &= \frac{1}{1-\int_0^\infty g(s)ds}\int_0^\infty g(t)\int_0^t \Phi^{-1}\left(\int_s^\infty f(u,x(u))du\right) dsdt \\ &\quad + \int_0^t \Phi^{-1}\left(\int_s^\infty f(u,x(u))du\right) ds.\end{aligned} \tag{4.1}$$

We have the following (the proofs are similar to those of Lemmas 3.3 and 3.4):

(i) For $x \in X$, Tx satisfies

$$\begin{cases} [\Phi((Tx)'(t))]' + f\,(t, x(t)) = 0, \ \ t \in (0, \infty), \\ (Tx)(0) = \int_0^\infty g(s)(Tx)(s)ds, \\ \lim\limits_{t\to\infty} (Tx)'(t) = 0; \end{cases}$$

(ii) $T : X \to X$ is well defined;

(iii) x is a bounded positive solution of BVP (1.2) if and only if x is a solution of the operator equation $x = Tx$ in X;

(iv) $T : X \to X$ is completely continuous.

Let

$$\psi_0(t) = \frac{1}{1 - \int_0^\infty g(s)ds} \int_0^\infty g(t) \int_0^t \Phi^{-1}\left(\int_s^\infty \psi_1(u)du\right) dsdt + \int_0^t \Phi^{-1}\left(\int_s^\infty \psi_1(u)du\right) ds.$$

It is easy to show that $\psi_0 \in X$. Let $r > 0$ and define $M_r = \{x \in X : \|x - \psi_0\| \le r\}$.

Since Φ is a pseudo sup-multiplicative function, from Remark 2.2 we have

$$\Phi^{-1}(v_1 v_2) \le \nu(v_1)\Phi^{-1}(v_2) \text{ for all } v_1, v_2 \ge 0.$$

For $x \in M_r$, using (A4) and $(A5)_\sigma$ we find

$$\begin{aligned} \|Tx - \psi_0\| &= \sup_{t\in[0,\infty)} \left| \frac{1}{1 - \int_0^\infty g(s)ds} \int_0^\infty g(t) \int_0^t \Phi^{-1}\left(\int_s^\infty f(u, x(u))du\right) dsdt \right. \\ &\quad + \int_0^t \Phi^{-1}\left(\int_s^\infty f(u, x(u))du\right) ds \\ &\quad - \frac{1}{1 - \int_0^\infty g(s)ds} \int_0^\infty g(t) \int_0^t \Phi^{-1}\left(\int_s^\infty \psi_1(u)du\right) dsdt \\ &\quad \left. - \int_0^t \Phi^{-1}\left(\int_s^\infty \psi_1(u)du\right) ds \right| \\ &= \sup_{t\in[0,\infty)} \left| \frac{1}{1 - \int_0^\infty g(s)ds} \int_0^\infty g(t) \int_0^t \left[\Phi^{-1}\left(\int_s^\infty f(u, x(u))du\right) - \Phi^{-1}\left(\int_s^\infty \psi_1(u)du\right)\right] dsdt \right. \\ &\quad \left. + \int_0^t \left[\Phi^{-1}\left(\int_s^\infty f(u, x(u))du\right) - \Phi^{-1}\left(\int_s^\infty \psi_1(u)du\right)\right] ds \right| \\ &\le \frac{1}{1 - \int_0^\infty g(s)ds} \int_0^\infty g(t) \int_0^\infty \left|\Phi^{-1}\left(\int_s^\infty f(u, x(u))du\right) - \Phi^{-1}\left(\int_s^\infty \psi_1(u)du\right)\right| dsdt \\ &\quad + \int_0^\infty \left|\Phi^{-1}\left(\int_s^\infty f(u, x(u))du\right) - \Phi^{-1}\left(\int_s^\infty \psi_1(u)du\right)\right| ds \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{1-\int_0^\infty g(s)ds}\int_0^\infty g(t)\int_0^\infty \mu\Phi^{-1}\left(\int_s^\infty |f(u,x(u))-\psi_1(u)|du\right)dsdt\\
&\quad+\int_0^\infty \mu\Phi^{-1}\left(\int_s^\infty |f(u,x(u))-\psi_1(u)|du\right)ds\\
&\leq \frac{\mu}{1-\int_0^\infty g(s)ds}\int_0^\infty g(t)\int_0^\infty \Phi^{-1}\left(\int_s^\infty \psi_2(u)\Phi(|x(u)|^\sigma)du\right)dsdt\\
&\quad+\mu\int_0^\infty \Phi^{-1}\left(\int_s^\infty \psi_2(u)\Phi(|x(u)|^\sigma)du\right)ds\\
&\leq \frac{\mu\int_0^\infty g(t)dt}{1-\int_0^\infty g(s)ds}\int_0^\infty \Phi^{-1}\left(\int_s^\infty \psi_2(u)\Phi(\|x\|^\sigma)du\right)ds\\
&\quad+\mu\int_0^\infty \Phi^{-1}\left(\int_s^\infty \psi_2(u)\Phi(\|x\|^\sigma)du\right)ds\\
&\leq \frac{\mu\int_0^\infty g(t)dt}{1-\int_0^\infty g(s)ds}\int_0^\infty \nu\left(\int_s^\infty \psi_2(u)du\right)ds\ \|x\|^\sigma+\mu\int_0^\infty \nu\left(\int_s^\infty \psi_2(u)du\right)ds\ \|x\|^\sigma\\
&= \frac{\mu\int_0^\infty \nu\left(\int_s^\infty \psi_2(u)du\right)ds}{1-\int_0^\infty g(s)ds}\|x\|^\sigma\\
&\leq \frac{\mu\int_0^\infty \nu\left(\int_s^\infty \psi_2(u)du\right)ds}{1-\int_0^\infty g(s)ds}\left(\|x-\psi_0\|+\|\psi_0\|\right)^\sigma\\
&\leq \frac{\mu\int_0^\infty \nu\left(\int_s^\infty \psi_2(u)du\right)ds}{1-\int_0^\infty g(s)ds}\left(r+\|\psi_0\|\right)^\sigma.
\end{aligned}
$$

Case (i). $\sigma>1$. Let $r=r_0=\frac{\|\psi_0\|}{\sigma-1}$. By assumption,

$$\frac{r_0}{(r_0+\|\psi_0\|)^\sigma}=\frac{\|\psi_0\|^{1-\sigma}(\sigma-1)^{\sigma-1}}{\sigma^\sigma}\geq\frac{\mu\int_0^\infty \nu\left(\int_s^\infty \psi_2(u)du\right)ds}{1-\int_0^\infty g(s)ds}.$$

Then, for $x\in M_{r_0}$ we have

$$\|Tx-\psi_0\|\leq\frac{\mu\int_0^\infty \nu\left(\int_s^\infty \psi_2(u)du\right)ds}{1-\int_0^\infty g(s)ds}\left(r_0+\|\psi_0\|\right)^\sigma\leq r_0.$$

Hence, we have a bounded subset $M_{r_0}\subseteq X$ such that $T(M_{r_0})\subseteq M_{r_0}$. Then, Schauder fixed point theorem implies that T has a fixed point $x\in M_{r_0}$. Hence, x is a bounded solution of BVP (1.2).

Case (ii). $\sigma\in(0,1)$. Choose $r>0$ sufficiently large such that

$$\frac{\mu\int_0^\infty \nu\left(\int_s^\infty \psi_2(u)du\right)ds}{1-\int_0^\infty g(s)ds}\left(r+\|\psi_0\|\right)^\sigma\leq r.$$

Then, for $x\in M_r$ we have

$$\|Tx-\psi_0\|\leq\frac{\mu\int_0^\infty \nu\left(\int_s^\infty \psi_2(u)du\right)ds}{1-\int_0^\infty g(s)ds}\left(r+\|\psi_0\|\right)^\sigma\leq r.$$

So $T(M_r) \subseteq M_r$ and Schauder fixed point theorem implies that T has a fixed point $x \in M_r$. This x is a bounded solution of BVP (1.2).

Case (iii). $\sigma = 1$. We choose

$$r \geq \frac{\frac{\mu \int_0^\infty \nu\left(\int_s^\infty \psi_2(u)du\right)ds}{1-\int_0^\infty g(s)ds}\|\psi_0\|}{1-\frac{\mu \int_0^\infty \nu\left(\int_s^\infty \psi_2(u)du\right)ds}{1-\int_0^\infty g(s)ds}}.$$

Then, for $x \in M_r$ we have

$$\|Tx - \psi_0\| \leq \frac{\mu \int_0^\infty \nu\left(\int_s^\infty \psi_2(u)du\right) ds}{1-\int_0^\infty g(s)ds}\left(r + \|\psi_0\|\right) \leq r.$$

Hence, as in earlier cases we conclude that T has a fixed point $x \in M_r$, which is a bounded solution of BVP (1.2).

Now, we shall prove that x is a *positive* solution of BVP (1.2). Since x satisfies (1.2), then

$$[\Phi(x'(t))]' + f(t, x(t)) = 0, \quad x(0) = \int_0^\infty g(s)x(s)ds, \quad \lim_{t\to\infty} x'(t) = 0.$$

By $f : (0,\infty)\times\mathbb{R} \to [0,\infty)$ and the definition of Φ, we see that x' is decreasing on $(0,\infty)$. Together with $\lim_{t\to\infty} x'(t) = 0$ we can see that $x'(t) \geq 0$ for all $t \in (0,\infty)$. Hence

$$x(0) = \int_0^\infty g(s)x(s)ds \geq x(0)\int_0^\infty g(s)ds.$$

It follows that $x(0) \geq 0$ since $\int_0^\infty g(s)ds < 1$. So $x(t) \geq 0$ for all $t \in [0,\infty)$. If there exists $t_0 > 0$ such that $x(t_0) = 0$, together with the increasing property on x, then $x(t) \equiv 0$ on $[0, t_0]$. Since Φ is odd, then $\Phi(0) = 0$. Hence $0 = [\Phi(x'(t))]' = -f(t,0) = 0$ on $[0,t_0]$. This contradicts (A3). Thus x is a positive solution of BVP (1.2). The proof is complete. □

5 Examples

To illustrate the usefulness of our main results, we present some examples that our results can readily apply, whereas the known results in the literature are not applicable.

Example 5.1. Consider the following boundary value problem

$$\begin{cases} \left[\, e^t[x'(t)]^3\right]' + f(t, x(t)) = 0, \ \ t \in (0,\infty), \\ x(0) = \dfrac{1}{2}\displaystyle\int_0^\infty e^{-s}x(s)ds + 2, \\ \lim\limits_{t\to\infty} e^{t/3}\, x'(t) = 1, \end{cases} \tag{5.1}$$

where f is defined by

$$
\begin{aligned}
f(t,x) &= \frac{t}{10^{29}(1+t)^3}+\frac{1}{(1+t)^2}f_0\left(x\right),\\
f_0(x) &= \frac{1}{2}\frac{48^3-27^3}{27^3},\ \ x\in[0,10],\\
f_0(x) &= \frac{1}{2}\frac{48^3-27^3}{27^3}\\
&\quad+\frac{x-10}{100-10}\left[\frac{404\left[303\left(1-e^{-\frac{1}{300}}\right)\times10^4-16\right]^3-404\times12^3\left(1-e^{-\frac{1}{300}}\right)^3}{594\left(1-e^{-\frac{1}{300}}\right)^3}\right.\\
&\quad\left.+\frac{8^3(102\times10^4-4)^3-48^3}{2\times27^3}\right],\ \ x\in[10,100],\\
f_0(x) &= \frac{404\left[303\left(1-e^{-\frac{1}{300}}\right)\times10^4-16\right]^3-404\times12^3\left(1-e^{-\frac{1}{300}}\right)^3}{594\left(1-e^{-\frac{1}{300}}\right)^3}\\
&\quad+\frac{8^3(102\times10^4-4)^3-27^3}{2\times27^3},\ \ x\in[100,102\times10^4],\\
f_0(x) &= \left[\frac{404\left[303\left(1-e^{-\frac{1}{300}}\right)\times10^4-16\right]^3-404\times12^3\left(1-e^{-\frac{1}{300}}\right)^3}{594\left(1-e^{-\frac{1}{300}}\right)^3}\right.\\
&\quad\left.+\frac{8^3(102\times10^4-4)^3-48^3}{2\times27^3}\right]e^{x-102\times10^4},\ \ x\geq102\times10^4.
\end{aligned}
$$

Corresponding to BVP (1.1), we have $\phi(x)=x^3$, $p(t)=e^t$, $g(t)=\frac{1}{2}e^{-t}$, $a=2$ and $b=1$. Then, $\phi^{-1}(x)=x^{\frac{1}{3}}$. It is easy to see that (A1) and (A2) hold.

Choose $k=100$, $e_1=10$, $e_2=10000$ and $C=102\times10^4$. By direct computation we obtain from (3.12)–(3.14)

$$
\begin{aligned}
\mu &= \int_0^{\frac{1}{k}}\phi^{-1}\left(\frac{1}{p(s)}\right)ds\ \frac{1}{1+\int_0^{\infty}\phi^{-1}\left(\frac{1}{p(s)}\right)ds}=\frac{3}{4}\left(1-e^{-\frac{1}{300}}\right),\\
M &= \frac{27^3\times102\times10^4}{8^3(102\times10^4-4)^3-27^3},\\
M_1 &= \frac{27^3\times10}{48^3-27^3},\\
L &= \frac{297\left(1-e^{-\frac{1}{300}}\right)^4\times10^4}{4\left[303\left(1-e^{-\frac{1}{300}}\right)\times10^4-16\right]^3-4\times12^3\left(1-e^{-\frac{1}{300}}\right)^3}
\end{aligned}
$$

and

$$0 < e_1 < \mu(1+k)e_2 < (1+k)e_2 < C, \qquad LC > M\mu(1+k)e_2 > 0.$$

On the other hand, $D = (1+k)e_2,\ B = \mu(1+k)e_2,\ A = e_1$ and we see from the definition of f that

- $f(t,x) \le \frac{8^3(102\times10^4-4)^3-27^3}{27^3}\frac{1}{(1+t)^2}$ for $t \ge 0$ and $x \in [0, 102\times10^4]$;
- $f(t,x) \le \frac{48^3-27^3}{27^3}\frac{1}{(1+t)^2}$ for $t \ge 0$ and $x \in [0,10]$;
- $f(t,x) \ge \frac{404\left[303\left(1-e^{-\frac{1}{300}}\right)\times10^4-16\right]^3-404\times12^3\left(1-e^{-\frac{1}{300}}\right)^3}{297\left(1-e^{-\frac{1}{300}}\right)^3}\frac{1}{(1+t)^2}$ for $t \in [0.01,100]$ and $x \in [100, 1010000]$.

It is easy to see that (C1)–(C3) hold. Hence, Theorem 3.1 implies that BVP (5.1) has at least three bounded positive solutions x_1, x_2 and x_3 such that

$$\sup_{t\in[0,\infty)} x_1(t) < 10, \quad \min_{t\in[0.01,100]} x_2(t) > 2520.80$$

and

$$\sup_{t\in[0,\infty)} x_3(t) > 10, \quad \min_{[0.01,100]} x_3(t) < 2520.80.$$

Remark 5.1. It is easy to see that Example 5.1 cannot be covered by the theorems in [5–9, 12–14, 17–20, 22–33, 37, 38]. Further, it is evident from Example 5.1 that (i) there is a large number of functions that satisfy the conditions of Theorem 3.1, and (ii) the conditions of Theorem 3.1 are easy to check.

Example 5.2. Consider the following boundary value problem

$$\begin{cases} \left[\frac{1}{k^3}|x'(t)|^2x'(t) + \frac{3}{k^6}|x'(t)|^5x'(t)\right]' + \psi_1(t) \\ \qquad\qquad + \lambda e^{-2t}\left[\frac{1}{k^3}|x(t)|^{3\sigma} + \frac{3}{k^6}|x(t)|^{6\sigma}\right] = 0, \quad t \in (0,\infty), \\ x(0) = \frac{1}{2}\int_0^\infty e^{-s}x(s)ds, \\ \lim_{t\to\infty} x'(t) = 0, \end{cases} \tag{5.2}$$

where $k > 0$, $\sigma > 0$ and $\lambda > 2$ are constants, and ψ_1 is nonnegative and satisfies

$$\int_0^\infty \Phi^{-1}\left(\int_s^\infty \psi_1(u)du\right)ds < \infty.$$

Corresponding to BVP (1.2), we have $\Phi(x) = \frac{1}{k^3}x^3 + \frac{3}{k^6}|x|^5x$, $f(t,x) = \psi_1(t) + \lambda e^{-2t}\Phi(|x|^\sigma)$, $g(t) = \frac{1}{2}e^{-t}$. It is easy to see that Φ is a pseudo sup-multiplicative function and the supporting function of Φ is $\omega(x) = \min\{x^6, x\}$ for $x \ge 0$.

It is easy to show that (A3) holds and $(A5)_\sigma$ holds with $\psi_2(t) = \lambda e^{-2t}$ and $\psi_1(t)$ given in (5.2).

The inverse function of Φ is

$$\Phi^{-1}(x) = \begin{cases} k\sqrt[3]{\frac{-1+\sqrt{1+12x}}{6}}, & x \geq 0, \\ -k\sqrt[3]{\frac{-1+\sqrt{1-12x}}{6}}, & x \leq 0. \end{cases}$$

Since

$$\frac{-1+\sqrt{1+12xy}}{-1+\sqrt{1+12y}} = x\frac{1+\sqrt{1+12y}}{1+\sqrt{1+12xy}} \leq \begin{cases} x, x \geq 1, \\ \sqrt{x}, x \in [0,1], \end{cases}$$

the supporting function of Φ^{-1} is

$$\nu(x) = \begin{cases} \sqrt[3]{x}, x \geq 1, \\ \sqrt[6]{x}, x \in [0,1]. \end{cases} \tag{5.3}$$

It is well known that $a^p - b^p \leq (a-b)^p$ for all $p \in (0,1]$ and $a \geq b \geq 0$. Then, for $x, y \geq 0$, without loss of generality $x \geq y$, we have

$$\begin{aligned} |\Phi^{-1}(x) - \Phi^{-1}(y)| &= \Phi^{-1}(x) - \Phi^{-1}(y) \\ &= k\sqrt[3]{\frac{-1+\sqrt{1+12x}}{6}} - k\sqrt[3]{\frac{-1+\sqrt{1+12y}}{6}} \\ &\leq k\sqrt[3]{\frac{\sqrt{1+12x}-\sqrt{1+12y}}{6}} \\ &\leq k\sqrt[3]{\frac{\sqrt{12(x-y)}}{6}} \\ &\leq \sqrt[6]{2}k\sqrt[3]{\frac{-1+\sqrt{1+12(x-y)}}{6}} \\ &= \sqrt[6]{2}\Phi^{-1}(|x-y|). \end{aligned}$$

Hence, (A4) holds with ν defined by (5.3) and $\mu = \sqrt[6]{2}$.

Note that

$$\begin{aligned} \frac{\mu\int_0^\infty \nu\left(\int_s^\infty \psi_2(u)du\right)ds}{1-\int_0^\infty g(s)ds} &= \frac{\sqrt[6]{2}k\int_0^\infty \nu\left(\int_s^\infty \lambda e^{-2u}du\right)ds}{1-\frac{1}{2}} \\ &= 2\sqrt[6]{2}k\int_0^\infty \nu\left(\frac{\lambda}{2}e^{-2s}\right)ds \\ &= 2\sqrt[6]{2}k\left[\int_0^{\frac{1}{2}\ln\frac{\lambda}{2}} \sqrt[3]{\frac{\lambda}{2}}e^{-\frac{2}{3}s}ds + \int_{\frac{1}{2}\ln\frac{\lambda}{2}}^\infty \sqrt[6]{\frac{\lambda}{2}}e^{-\frac{1}{3}s}ds\right] \\ &= 3\sqrt[6]{2}k\left(\sqrt[3]{\frac{\lambda}{2}}+1\right). \end{aligned}$$

Also, we have

$$\psi_0(t) = \frac{1}{1-\int_0^\infty g(s)ds}\int_0^\infty g(t)\int_0^t \Phi^{-1}\left(\int_s^\infty \psi_1(u)du\right)dsdt + \int_0^\infty \Phi^{-1}\left(\int_s^\infty \psi_1(u)du\right)ds.$$

By Theorem 4.1, we conclude that BVP (5.2) has at least one positive solution if

(i) $\sigma > 1$ and

$$\|\psi_0\|^{1-\sigma} \geq 3\sqrt[6]{2}k\left[\sqrt[3]{\frac{\lambda}{2}}+1\right]\frac{\sigma^\sigma}{(\sigma-1)^{\sigma-1}}$$

or

(ii) $\sigma \in (0,1)$, or

(iii) $\sigma = 1$ and $3\sqrt[6]{2}k\left[\sqrt[3]{\frac{\lambda}{2}}+1\right] < 1$.

Remark 5.2. Suppose $\psi_1(t) = e^{-3t}$ in Example 5.2. Then, we have

$$\begin{aligned}
\psi_0(t) &= \frac{1}{1-\int_0^\infty g(s)ds}\int_0^\infty g(t)\int_0^t \Phi^{-1}\left(\int_s^\infty \psi_1(u)du\right)dsdt \\
&\quad + \int_0^\infty \Phi^{-1}\left(\int_s^\infty \psi_1(u)du\right)ds \\
&= \frac{1}{1-\int_0^\infty g(s)ds}\int_0^\infty g(t)\int_0^t \Phi^{-1}\left(\int_s^\infty e^{-3u}du\right)dsdt \\
&\quad + \int_0^\infty \Phi^{-1}\left(\int_s^\infty e^{-3u}du\right)ds \\
&= \frac{1}{1-\int_0^\infty g(s)ds}\int_0^\infty g(t)\int_0^t \Phi^{-1}\left(\frac{1}{3}e^{-3s}\right)dsdt \\
&\quad + \int_0^\infty \Phi^{-1}\left(\frac{1}{3}e^{-3s}\right)ds \\
&= k\frac{1}{1-\int_0^\infty g(s)ds}\int_0^\infty g(t)\int_0^t \sqrt[3]{\frac{-1+\sqrt{1+4e^{-3s}}}{6}}dsdt \\
&\quad + k\int_0^\infty \sqrt[3]{\frac{-1+\sqrt{1+4e^{-3s}}}{6}}ds.
\end{aligned}$$

Since

$$\sqrt[3]{\frac{-1+\sqrt{1+4e^{-3s}}}{6}} = \sqrt[3]{\frac{4e^{-3s}}{6(1+\sqrt{1+4e^{-3s}})}} \leq \sqrt[3]{\frac{e^{-3s}}{3}} = \frac{e^{-s}}{\sqrt[3]{3}},$$

we get

$$\|\psi_0\| \leq 2k\int_0^\infty g(t)\int_0^\infty \frac{e^{-s}}{\sqrt[3]{3}}dsdt + k\int_0^\infty \frac{e^{-s}}{\sqrt[3]{3}}ds = \frac{3k}{2\sqrt[3]{3}}.$$

Hence, from Theorem 4.1 we conclude that BVP (5.2) (when $\psi_1(t) = e^{-3t}$) has at least one positive solution if $\sigma > 1$ and

$$3\sqrt[6]{2}k\left[\sqrt[3]{\frac{\lambda}{2}}+1\right] \leq \left(\frac{3k}{2\sqrt[3]{3}}\right)^{1-\sigma}\frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}.$$

Acknowledgement. The authors would like to thank the referee for the comments which helped to improve the paper.

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(Received January 28, 2012)