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On solutions of neutral stochastic delay Volterra equations with singular kernels*

Xiaotai Wu[1], Litan Yan[2]

1. Department of Mathematics, Anhui Polytechnic University

Beijing Middle Road, Wuhu, Anhui 241000, P.R. China

Corresponding author, Email: aaxtwu@gmail.com

2. Department of Mathematics Donghua University

2999 North Renmin Road, Songjiang, Shanghai 201620, P.R. China

Abstract

In this paper, existence, uniqueness and continuity of the adapted solutions for neutral stochastic delay Volterra equations with singular kernels are discussed. In addition, continuous dependence on the initial date is also investigated. Finally, stochastic Volterra equation with the kernel of fractional Brownian motion is studied to illustrate the effectiveness of our results.

Keywords: Neutral stochastic delay Volterra equations, Singular kernel, Fractional Brownian motion.

MSC2010:Primary 60H10, Secondary 60G51, 60J73

1. Introduction

This paper is concerned with solutions of neutral stochastic delay Volterra equations (NSDVE) driven by Poisson random measure as follows:

$$\begin{aligned} X(t) - D(X_t) =& \xi(0) - D(\xi) + \int_0^t f(t,s,X_s)ds + \int_0^t g(t,s,X_s)dB(s) \\ &+ \int_0^t \int_{|y|<c} H(t,s,X_{s-},y)\widetilde{N}(ds,dy), \end{aligned} \tag{1.1}$$

where $B(t), t \geq 0$ is a standard Brownian motion and $\widetilde{N}(dt,dl)$ is the compensated Poisson random measure; the mappings $D : C([-\tau,0];\mathbb{R}^d) \to \mathbb{R}^d$, $f : \mathbb{R}^+ \times \mathbb{R}^+ \times C([-\tau,0];\mathbb{R}^d) \to \mathbb{R}^d$, $g : \mathbb{R}^+ \times \mathbb{R}^+ \times C([-\tau,0];\mathbb{R}^d) \to \mathbb{R}^{d\times m}$

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and $H:\mathbb{R}^+\times\mathbb{R}^+\times C([-\tau,0];\mathbb{R}^d)\times\mathbb{R}^d\to\mathbb{R}^d$ are all Borel-measurable functions; $c\in(0,+\infty]$ is the maximum allowable jump size.

Stochastic Volterra equation(SVE) was first studied by Berger and Mizel ([1], [2]) for equations:

$$X(t)=x+\int_0^t f(t,s,X(s))ds+\int_0^t g(t,s,X(s))dB(s). \tag{1.2}$$

Such equations arise in many applications such as mathematical finance, biology. etc. During the past 30 years, the theory of SVE has been developed in a variety directions. Lots of the well-known results are concerned with Eq.(1.2) with regular kernels. In particular, Protter [3] studied SVE driven by a general semimartingale and resolved a conjecture of Berger and Mizel. Using the Skorohod integral, Pardoux and Protter [4] investigated SVE with anticipating coefficients. Recently, Some results of backward stochastic volterra equations were obtained (see e.g. [5], [6], [7], [8]), which can be used for discussing mathematical finance and stochastic optimal control.

On the other hand, there are also some papers which consider Eq.(1.2) with the singular kernel. One can see Cochran et al. [9], Decreusefond [10], Wang [11], Zhang [12], [13] and the references therein. Wang [11] proved that there exists a unique continuous adapted solution to SVE with singular kernels. Zhang [12] established the existence-uniqueness and large deviation estimate for SVE in 2-smooth Banach spaces, and Zhang in [13] studied the numerical solutions and the large deviation principles of Freidlin-Wentzell's type for SVE with singular kernels.

Stochastic differential equations with delay have been widely used in many branches of science and industry (see e.g. [14], [15]), and neutral type stochastic delay differential equations have been intensively studied in recent years(see e.g. [15], [16]). However, few work has been done on the NSDVE with singular kernels. In this paper, we prove the existence, uniqueness and continuity of the adapted solutions to NSDVE with singular kernels. The continuous dependence of solutions on the initial data is also investigated. Moreover, NSDVE with the kernel of fractional Brownian motion is given to illustrate the effectiveness of our results, where the kernel of fractional Brownian motion is a singular kernel, for it may take the infinity at points $s=0$ and $s=t$.

The paper is organized as follows. In Section 2, we give the preliminaries, and devote Section 3 to deal with the existence and uniqueness result. The path continuity of the solution is obtained in Section 4. The continuous dependence of solution on the initial data is presented in section 5. Finally, NSDVE with the kernel of fractional Brownian motion is studied to illustrate the obtained results in section 6.

2. Preliminaries

Throughout this paper, we let $(\Omega,\mathcal{F},P)$ be a complete probability space with some filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e., the filtration is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $|x|$ be the Euclidean norm in $x\in\mathbb{R}^d$. Let $\tau>0$, $\mathbb{R}^+=[0,+\infty)$ and $C([-\tau,0];\mathbb{R}^d)$ be the family of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^d$ with norm $||\varphi|| = \sup_{-\tau\leq\theta\leq 0}|\varphi(\theta)|$. Denote by $L^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^d)$ the family of $\mathcal{F}_t$-measurable, $C([-\tau,0];\mathbb{R}^d)$-valued random variables $\xi = \{\xi(s), -\tau \leq s \leq 0\}$ such that $E||\xi||^p = \sup_{-\tau\leq s\leq 0} E|\xi(s)|^p < +\infty$. For Eq.(1.1), the initial data $X(0) = \xi(0) \in L^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^d)$.

If $X(t)$ is an $\mathbb{R}^d$-valued stochastic process on $t \in [-\tau, \infty)$, we let $X_t = \{X(t+\theta) : -\tau \leq \theta \leq 0\}$. Let $B = (B(t), t \geq 0)$ be an m-dimensional standard $\mathcal{F}_t$-adapted Brownian motion and N be an independent $\mathcal{F}_t$-adapted Poisson random measure defined on $\mathbb{R}^+ \times (\mathbb{R}^d - \{0\})$ with compensator $\widetilde{N}$ and intensity measure ν, where $\widetilde{N}(dt,dy) := N(dt,dy) - \nu(dy)dt$. Let L^p_T be the family of $X(t)$ such that $\int_0^T E|X^n(t)|^p dt < \infty$. For simplicity, we denote by $a \vee b = \max\{a,b\}$.

In this paper, we make the following assumptions:

(A.1) For some $p > 2$, there exist two functions $G(.)$ and $K(t,s)$, such that for $s,t \in [0,T]$,

$$\begin{aligned}&|f(t,s,x_1) - f(t,s,x_2)|^2 \vee |g(t,s,x_1) - g(t,s,x_2)|^2 \vee \int_{|y|<c} |H(t,s,x_1,y) - H(t,s,x_2,y)|^2 \nu(dy)\\ =&K(t,s)G^{\frac{2}{p}}(|x_1 - x_2|^p), \ \ x_1, x_2 \in \mathbb{R}^d\end{aligned}$$

where $G(.)$ is a concave continuous and nondecreasing function from $\mathbb{R}^+ \mapsto \mathbb{R}^+$, $G(0) = 0$ and $\int_{0+} \frac{1}{G(s)} ds = +\infty$; $K(t,s)$ is a positive function on $\mathbb{R}^+ \times \mathbb{R}^+$.

(A.2) There exists a positive constant K_1 such that

$$\int_0^t |f(t,s,0)|^2 ds \vee \int_0^t |g(t,s,0)|^2 ds \vee \int_0^t \int_{|y|<c} |H(t,s,0,y)|^2 \nu(dy) ds \vee \int_0^t K^{\frac{p}{p-2}}(t,s) ds \leq K_1.$$

(A.3) Assuming that there exists a positive number $\kappa < 1$, such that for $x_1, x_2 \in \mathbb{R}^d$,

$$E|D(x_1) - D(x_2)| \leq \kappa E|x_1 - x_2|$$

and $D(0) = 0$.

Lemma 2.1. ([17]) *For $p \geq 1, x_1, x_2 \in \mathbb{R}^d$ and $\kappa \in (0,1)$, we have*

$$|x_1 + x_2|^p \leq \frac{|x_1|^p}{\kappa^{p-1}} + \frac{|x_2|^p}{(1-\kappa)^{p-1}}.$$

Lemma 2.2. *(Bihari inequality) Let $G : \mathbb{R}^+ \to \mathbb{R}^+$ be a concave continuous and nondecreasing function such that $G(r) > 0$ for $r > 0$. If $u(t), v(t)$ are strictly positive functions on $\mathbb{R}^+$ such that*

$$u(t) \leq u_0 + \int_0^t v(s) G(u(s)) ds,$$

then

$$u(t) \leq \rho^{-1}(\rho(u_0) + \int_0^t v(s) ds),$$

for all such $t \in [0,T]$ that

$$\rho(u_0) + \int_0^t v(s) ds \in Dom(\rho^{-1}),$$

where $\rho(r) = \int_1^r \frac{ds}{G(s)}, r \geq 0$ and ρ^{-1} is the inverse function of ρ. In particular, $u_0 = 0$ and $\int_{0+} \frac{ds}{G(s)} = \infty$, then $u(t) = 0$ for all $0 \leq t \leq T$.

3. Existence and uniqueness of the solution

Theorem 3.1. *Assume conditions (A.1-A.3) hold. Then there exists a unique progressively measurable process* $\{X(t), 0 \le t \le T\}$ *satisfying Eq.(1.1).*

Proof. Let $X^0(t) = \xi(0), X_t^0 = \xi$, for $t \in [0,T]$. Define the following Picard sequence:

$$\begin{aligned} X^n(t) - D(X_t^n) =& \xi(0) - D(\xi) + \int_0^t f(t,s,X_s^{n-1})ds + \int_0^t g(t,s,X_s^{n-1})dB(s) \\ &+ \int_0^t \int_{|y|<c} H(t,s,X_{s-}^{n-1},y)\widetilde{N}(ds,dy), \end{aligned} \tag{3.1}$$

where $t \in [0,T]$, $n = 1,2,\cdots$. Set $\tilde{X}^n(t) = X^n(t) - D(X_t^n)$ and $\tilde{X}^0(0) = \xi(0) - D(\xi)$, we have

$$\begin{aligned} E|\tilde{X}^n(t)|^p \le& E|\tilde{X}^0(0) + \int_0^t f(t,s,0)ds + \int_0^t g(t,s,0)dB(s) + \int_0^t \int_{|y|<c} H(t,s,0,y)\widetilde{N}(ds,dy) \\ &+ \int_0^t [f(t,s,X_s^{n-1}) - f(t,s,0)]ds + \int_0^t [g(t,s,X_s^{n-1}) - g(t,s,0)]dB(s) \\ &+ \int_0^t \int_{|y|<c} [H(t,s,X_{s-}^{n-1},y) - H(t,s,0,y)]\widetilde{N}(ds,dy)|^p, \end{aligned}$$

which yields

$$\begin{aligned} E|\tilde{X}^n(t)|^p \le& 3^{p-1}[E|\tilde{X}^n(0)|^p + E|\int_0^t f(t,s,0)ds + \int_0^t g(t,s,0)dB(s) + \int_0^t \int_{|y|<c} H(t,s,0,y)\widetilde{N}(ds,dy)|^p \\ &+ E|\int_0^t [f(t,s,X_s^{n-1}) - f(t,s,0)]ds + \int_0^t [g(t,s,X_s^{n-1}) - g(t,s,0)]dB(s) \\ &+ \int_0^t \int_{|y|<c} [H(t,s,X_{s-}^{n-1},y) - H(t,s,0,y)]\widetilde{N}(ds,dy)|^p] \\ =: & 3^{p-1}(I_1 + I_2 + I_3). \end{aligned} \tag{3.2}$$

According to (A.1), $I_2 \le 3^p K_1^{\frac{p}{2}}(T+1)$. Noticing that $E|\xi(0) - D(\xi(t))| \le (1+\kappa)E||\xi||$, we have $I_1 \le (1+\kappa)^p E||\xi||^p$.

By (A.1),(A.2) and Hölder's inequlity, we obtain

$$\begin{aligned} I_3 \le& 3^{p-1}[TE(\int_0^t |f(t,s,X_s^{n-1}) - f(t,s,0)|^2 ds)^{\frac{p}{2}} + E(\int_0^t |g(t,s,X_s^{n-1}) - g(t,s,0)|^2 ds)^{\frac{p}{2}} \\ &+ E(\int_0^t \int_{|y|<c} |H(t,s,X_{s-}^{n-1},y) - H(t,s,0,y)|^2 \nu(dy)ds)^{\frac{p}{2}}] \\ \le& 3^{p-1}2(T+1)E[\int_0^t K(t,s)G^{\frac{2}{p}}(|X_s^{n-1}|^p)ds]^{\frac{p}{2}} + 3^{p-1}E[\int_0^t K(t,s)G^{\frac{2}{p}}(|X_{s-}^{n-1}|^p)ds]^{\frac{p}{2}} \\ \le& 3^{p-1}[\int_0^t K^{\frac{p}{p-2}}(t,s)ds]^{\frac{p-2}{2}}[2(T+1)\int_0^t G(E|X_s^{n-1}|^p)ds + \int_0^t G(E|X_{s-}^{n-1}|^p)ds] \\ \le& 3^{p-1}K_1^{\frac{p-2}{2}}[2(T+1)\int_0^t G(E|X_s^{n-1}|^p)ds + \int_0^t G(E|X_{s-}^{n-1}|^p)ds]. \end{aligned}$$

Substituting the above inequalities of I_1, I_2 and I_3 into (3.2) implies

$$E|\tilde{X}^n(t)|^p \leq C_{3,1} + C_{3,2}\int_0^t G(E|X_s^{n-1}|^p)ds + C_{3,2}\int_0^t G(E|X_{s-}^{n-1}|^p)ds,$$

where $C_{3,1} = 3^{p-1}[(1+\kappa)^p E||\xi||^p + 3^p K_1^{\frac{p}{2}}(T+1)], C_{3,2} = 3^{2p-2}2(T+1)K_1^{\frac{p-2}{2}}$. For $G(u)$ is a positive concave function, there exists a positive constant a, such that $G(u) \leq a(1+u)$. Hence

$$\begin{aligned} E|\tilde{X}^n(t)|^p \leq & C_{3,1} + C_{3,2}\int_0^t a(1+E|X_s^{n-1}|^p)ds + C_{3,2}\int_0^t a(1+E|X_{s-}^{n-1}|^p)ds \\ \leq & C_{3,1} + 2aC_{3,2}T + aC_{3,2}\int_0^t E|X_s^{n-1}|^p ds + aC_{3,2}\int_0^t E|X_{s-}^{n-1}|^p ds. \end{aligned} \tag{3.3}$$

By Lemma 2.1, we derive

$$\begin{aligned} E|X^n(t)|^p \leq & \frac{1}{(1-\kappa)^{p-1}}E|\tilde{X}^n(t)|^p + \frac{1}{\kappa^{p-1}}E|D(X_t^n)|^p \leq \frac{1}{(1-\kappa)^{p-1}}E|\tilde{X}^n(t)|^p + \kappa E|X_t^n|^p \\ \leq & \frac{1}{(1-\kappa)^{p-1}}E|\tilde{X}^n(t)|^p + \kappa E||\xi||^p + \kappa \sup_{0\leq s\leq t} E|X^n(s)|^p. \end{aligned}$$

It follows that

$$\sup_{0\leq s\leq t} E|X^n(s)|^p \leq \frac{\kappa}{1-\kappa}E||\xi||^p + \frac{1}{(1-\kappa)^p}\sup_{0\leq s\leq t} E|\tilde{X}^n(s)|^p. \tag{3.4}$$

Substituting (3.3) into (3.4), we get

$$\begin{aligned} \sup_{0\leq s\leq t} E|X^n(s)|^p \leq & C_{3,3} + \frac{aC_{3,2}}{(1-\kappa)^p}\int_0^t E|X^{n-1}(s+\theta)|^p ds + \frac{aC_{3,2}}{(1-\kappa)^p}\int_0^t E|X^{n-1}(s+\theta-)|^p ds \\ \leq & C_{3,3} + C_{3,4}\int_0^t (E||\xi||^p + \sup_{0\leq r\leq s} E|X^{n-1}(r)|^p)ds \\ \leq & C_{3,3} + TC_{3,4}E||\xi||^p + C_{3,4}\int_0^t \sup_{0\leq r\leq s} E|X^{n-1}(r)|^p ds, \end{aligned}$$

where $C_{3,3} = \frac{\kappa}{1-\kappa}E||\xi||^p + \frac{1}{(1-\kappa)^p}(C_{3,1} + 2aC_{3,2}T), C_{3,4} = \frac{2aC_{3,2}}{(1-\kappa)^p}$. Therefore,

$$\begin{aligned} \max_{1\leq n\leq k}\sup_{0\leq s\leq t} E|X^n(s)|^p \leq & C_{3,3} + TC_{3,4}E||\xi||^p + C_{3,4}\int_0^t (E||\xi||^p + \max_{1\leq n\leq k}\sup_{0\leq r\leq s} E|X^n(r)|^p)ds \\ \leq & C_{3,3} + 2TC_{3,4}E||\xi||^p + C_{3,4}\int_0^t (\max_{1\leq n\leq k}\sup_{0\leq r\leq s} E|X^n(r)|^p)ds. \end{aligned}$$

So we can apply the Gronwall inequality to get the inequality

$$\max_{1\leq n\leq k}\sup_{0\leq s\leq t} E|X^n(s)|^p \leq (C_{3,3} + 2TC_{3,4}E||\xi||^p)e^{C_{3,4}T}.$$

Since k is arbitrary, this leads to the inequality

$$\sup_n \sup_{t\in[0,T]} E|X^n(t)|^p < +\infty. \tag{3.5}$$

Consequently, we know that $X^n(t) \in L_T^p$ for each $n \in N$. In the following, we will prove the existence and uniqueness of the solution to Eq.(1.1), we first study the existence.

Existence. Let

$$\begin{aligned} I_4 = &\int_0^t [f(t,s,X_s^{n-1}) - f(t,s,X_s^{m-1})]ds + \int_0^t [g(t,s,X_s^{n-1}) - g(t,s,X_s^{m-1})]dB(s) \\ &+ \int_0^t \int_{|y|<c} [H(t,s,X_{s-}^{n-1},y) - H(t,s,X_{s-}^{m-1},y)]\widetilde{N}(ds,dy). \end{aligned}$$

Using Lemma 2.1 and (3.1), we have

$$\begin{aligned} E|X^n(t) - X^m(t)|^p \le &\frac{1}{\kappa^{p-1}} E|D(X_t^n) - D(X_t^m)|^p + \frac{1}{(1-\kappa)^{p-1}} E|I_4|^p \\ \le &\kappa \sup_{0\le s\le t} E|X^n(s) - X^m(s)|^p + \frac{1}{(1-\kappa)^{p-1}} E|I_4|^p. \end{aligned} \tag{3.6}$$

According to (A.1) and Hölder's inequality,

$$\begin{aligned} E|I_4|^p \le &3^{p-1}[TE(\int_0^t |f(t,s,X_s^{n-1}) - f(t,s,X_s^{m-1})|^2 ds)^{\frac{p}{2}} + E(\int_0^t |g(t,s,X_s^{n-1}) - g(t,s,X_s^{m-1})|^2 ds)^{\frac{p}{2}} \\ &+ E(\int_0^t \int_{|y|<c} |H(t,s,X_{s-}^{n-1},y) - H(t,s,X_{s-}^{m-1},y)|^2 \nu(dy)ds)^{\frac{p}{2}}] \\ \le &3^{p-1}\{2(T+1)E[\int_0^t K(t,s)G^{\frac{2}{p}}(|E|X_s^{n-1} - X_s^{m-1}|^p)ds]^{\frac{p}{2}} + E[\int_0^t K(t,s)G^{\frac{2}{p}}(|E|X_{s-}^{n-1} - X_{s-}^{m-1}|^p)ds]^{\frac{p}{2}}\} \\ \le &3^{p-1}(\int_0^t K^{\frac{p}{p-2}}(t,s)ds)^{\frac{p-2}{2}}[2(T+1)\int_0^t G(E|X_s^{n-1} - X_s^{m-1}|^p)ds + \int_0^t G(E|X_{s-}^{n-1} - X_{s-}^{m-1}|^p)ds] \\ \le &3^p(T+1)K_1^{\frac{p-2}{2}} \int_0^t G(\sup_{0\le r\le s} E|X^{n-1}(r) - X^{m-1}(r)|^p)ds. \end{aligned} \tag{3.7}$$

Combining this with (3.6), we see that

$$E|X^n(t) - X^m(t)|^p \le \kappa \sup_{0\le s\le t} E|X^n(s) - X^m(s)|^p + C_{3,5}\int_0^t G(\sup_{0\le s\le r} E|X^{n-1}(r) - X^{m-1}(r)|^p)ds,$$

where $C_{3,5} = \frac{3^p(T+1)}{(1-\kappa)^{p-1}} K_1^{\frac{p-2}{2}}$. Consequently,

$$\sup_{0\le s\le t} E|X^n(s) - X^m(s)|^p \le \frac{C_{3,5}}{(1-\kappa)} \int_0^t G(\sup_{0\le r\le s} E|X^{n-1}(r) - X^{m-1}(r)|^p)ds.$$

Set $h(t) = \limsup_{n,m\to\infty} E|X^n(t) - X^m(t)|^p$. Thus, by Fatou's Lemma,

$$\sup_{0\le s\le t} E|h(s)|^p \le \frac{C_{3,5}}{(1-\kappa)} \int_0^t G(\sup_{0\le r\le s} E|h(r)|^p)ds.$$

According to Lemma 2.2, we have $h(t) \equiv 0$, for any $t \in [0,T]$. This means that $\{X^{(n)}, n \in N\}$ is a cauchy sequence in L_T^p, hence there is an $X \in L_T^p$, such that

$$\lim_{n\to\infty} \sup_{0\le s\le T} E|X^n(s) - X(s)|^p = 0.$$

Moreover, Letting $n \to \infty$, we can get for all $t \in [0, T]$,

$$
\begin{aligned}
&E|\int_0^t (f(t,s,X_s^{n-1}) - f(t,s,X_s))ds|^p \\
\le & TE(\int_0^t |f(t,s,X^{n-1}(s)) - f(t,s,X(s))|^2 ds)^{\frac{p}{2}} \\
\le & TE(\int_0^t K(t,s)G^{\frac{2}{p}}(|X^{n-1}(s) - X(s)|^p)ds)^{\frac{p}{2}} \\
\le & T(\int_0^t K^{\frac{p}{p-2}}(t,s)ds)^{\frac{p-2}{2}} \int_0^t G(E|X^{n-1}(s) - X(s)|^p)ds \to 0.
\end{aligned}
$$

Similarly, as $n \to \infty$, we can obtain

$$
\begin{aligned}
&E|\int_0^t [g(t,s,X_s^{n-1}) - g(t,s,X_s)]dB(s)|^p \to 0, \\
&E|\int_0^t \int_{|y|<c} [H(t,s,X_{s-}^{n-1},y) - H(t,s,X_{s-},y)]\widetilde{N}(ds,dy)|^p \to 0.
\end{aligned}
$$

Taking limits on both sides of (3.1) gives the existence.

Uniqueness. Let $X(t)$ and $\overline{X}(t)$ be two solutions of Eq.(1.1). Set

$$
\begin{aligned}
I_5 = &\int_0^t [f(t,s,X_s) - f(t,s,\overline{X}_s)]ds + \int_0^t [g(t,s,X_s) - g(t,s,\overline{X}_s)]dB(s) \\
&+ \int_0^t \int_{|y|<c} [H(t,s,X_{s-},y) - H(t,s,\overline{X}_{s-},y)]\widetilde{N}(ds,dy).
\end{aligned}
$$

By virtue of Lemma 2.1, we get

$$
\begin{aligned}
E|X(t) - \overline{X}(t)|^p = & E|D(X_t) - D(\overline{X}_t) + I_5|^p \le \frac{1}{\kappa^{p-1}} \sup_{0\le s\le t} E|D(X_t) - D(\overline{X}_t)|^p + \frac{1}{(1-\kappa)^{p-1}} E|I_5|^p \\
\le & \kappa \sup_{0\le s\le t} E|X(s) - \overline{X}(s)|^p + \frac{1}{(1-\kappa)^{p-1}} E|I_5|^p. \qquad (3.8)
\end{aligned}
$$

By a similar argument as (3.7), we derive

$$
E|I_5|^p \le 3^p(T+1)K_1^{\frac{p-2}{2}} \int_0^t G(\sup_{0\le r\le s} E|X(r) - \overline{X}(r)|^p)ds.
$$

Substituting this into (3.8) gives

$$
\sup_{0\le s\le t} E|X(s) - \overline{X}(s)|^p \le \kappa \sup_{0\le s\le t} E|X(s) - \overline{X}(s)| + \frac{3^p(T+1)}{(1-\kappa)^{p-1}} K_1^{\frac{p-2}{2}} \int_0^t G(\sup_{0\le r\le s} E|X(r) - \overline{X}(r)|^p)ds,
$$

which implies that

$$
\sup_{0\le s\le t} E|X(s) - \overline{X}(s)|^p \le \frac{3^p(T+1)}{(1-\kappa)^p} K_1^{\frac{p-2}{2}} \int_0^t G(\sup_{0\le r\le s} E|X(r) - \overline{X}(r)|^p)ds. \qquad (3.9)
$$

Combining (3.9) with the Bihari inequality leads to

$$
E|X(t) - \overline{X}(t)|^p = 0.
$$

The uniqueness has been proved. This completes the proof. □

4. Path continuity of the solution

In this section, in addition to the assumptions (A.1) and (A.3), we also assume that:

(A.4) For all $t, t', s \in [0, T]$ and $x \in \mathbb{R}^d$

$$
\begin{aligned}
&|f(t,s,x) - f(t',s,x)|^2 \vee |g(t,s,x) - g(t',s,x)|^2 \vee \int_{|y|<c} |H(t,s,x,y) - H(t',s,x,y)|^2 \nu(dy) \\
\leq &F(t,t',s)(1 + |x|^2),
\end{aligned}
$$

and for $\gamma > 0$, there exists a positive constant K_2, such that

$$\int_0^t F(t,t',s)ds \leq K_2 |t - t'|^\gamma.$$

(A.5) There exists a positive constant K_3, such that

$$\int_0^t |f(t,s,0)|^u ds \vee \int_0^t |g(t,s,0)|^u ds \vee \int_0^t \int_{|y|<c} |H(t,s,0,y)|^2 \nu(dy) ds \vee \int_0^t K^{\frac{p}{p-2}}(t,s) ds \leq K_3,$$

where $1 < u \leq p$.

If we denote by $\tilde{X}(t) = X(t) - D(X_t)$. Then from Eq.(1.1) we get

$$\tilde{X}(t) - \tilde{X}(t') = J_1 + J_2, \;\; t' < t, \tag{4.1}$$

where

$$J_1 = \int_{t'}^t f(t,s,X_s)ds + \int_{t'}^t g(t,s,X_s)dB(s) + \int_{t'}^t \int_{|y|<c} H(t,s,X_{s-},y) \widetilde{N}(ds,dy)$$

and

$$
\begin{aligned}
J_2 = &\int_0^{t'} [f(t,s,X_s) - f(t',s,X_s)]ds + \int_0^{t'} [g(t,s,X_s) - g(t',s,X_s)]dB(s) \\
&+ \int_0^{t'} \int_{|y|<c} [H(t,s,X_{s-},y) - H(t',s,X_{s-},y)] \widetilde{N}(ds,dy).
\end{aligned}
$$

In what follows, we will study the path continuity of the solutions. Firstly, we give an useful Lemma.

Lemma 4.1. *Under assumptions (A.1) and (A.3-A.5), there exist two positive constants C_1 and $\lambda \in (0,1]$, such that*

$$E|\tilde{X}(t) - \tilde{X}(t')|^p \leq C_1 |t - t'|^\lambda, \;\; t > t'.$$

Proof. By (4.1), we have

$$E|\tilde{X}(t) - \tilde{X}(t')|^p \leq 2^{p-1}(E|J_1|^p + E|J_2|^p). \tag{4.2}$$

In the following, we will consider $E|J_1|^p$ and $E|J_2|^p$, respectively. For $E|J_1|^p$,

$$\begin{aligned}E|J_1|^p \leq & 2^{p-1}[E|\int_{t'}^t f(t,s,0)ds + \int_{t'}^t g(t,s,0)dB(s) + \int_{t'}^t\int_{|y|<c} H(t,s,0,y)\widetilde{N}(ds,dy)|^p \\ & + E|\int_{t'}^t [f(t,s,X_s) - f(t,s,0)]ds + \int_{t'}^t [g(t,s,X_s) - g(t,s,0)]dB(s) \\ & + \int_{t'}^t\int_{|y|<c}[H(t,s,X_{s-},y) - H(t,s,0,y)]\widetilde{N}(ds,dy)|^p] \\ =: & 2^{p-1}(E|J_{11}|^p + E|J_{12}|^p). \end{aligned} \tag{4.3}$$

Letting $\alpha > 1$, $\theta = \frac{\alpha}{\alpha-1}$ and $p \geq 2\alpha$, then by Hölder's inequality, we have

$$\begin{aligned}E|J_{11}|^p \leq & 3^{p-1}[TE(\int_{t'}^t |f(t,s,0)|^2 ds)^{\frac{p}{2}} + E(\int_{t'}^t |g(t,s,0)|^2 ds)^{\frac{p}{2}} + E(\int_{t'}^t\int_{|y|<c} |H(t,s,0,y)|^2\nu(dy)ds)^{\frac{p}{2}}] \\ \leq & 3^{p-1}(T+1)[(\int_{t'}^t |f(t,s,0)|^{2\alpha}ds)^{\frac{p}{2\alpha}} + (\int_{t'}^t |g(t,s,0)|^{2\alpha}ds)^{\frac{p}{2\alpha}} \\ & + (\int_{t'}^t[\int_{|y|<c}|H(t,s,0,y)|^2\nu(dy)]^\alpha ds)^{\frac{p}{2\alpha}}]|t-t'|^{\frac{p}{2\theta}} \\ \leq & C_{4,1}|t-t'|^{\frac{p}{2\theta}}, \end{aligned} \tag{4.4}$$

where $C_{4,1} = 3^{p-1}(2K_3^{\frac{p}{2\alpha}} + K_3^{\frac{p}{2}}T^{\frac{p}{2\alpha}})(T+1)$. Using (A.1), we derive

$$\begin{aligned}E|J_{12}|^p \leq & 3^{p-1}\{E|\int_{t'}^t [f(t,s,X_s) - f(t,s,0)]ds|^p + E|\int_{t'}^t [g(t,s,X_s) - g(t,s,0)]dB(s)|^p \\ & + E|\int_{t'}^t\int_{|y|<c}[H(t,s,X_{s-},y) - H(t,s,0,y)]\widetilde{N}(ds,dy)|^p\} \\ \leq & 3^{p-1}[TE(\int_{t'}^t |f(t,s,X_s) - f(t,s,0)|^2 ds)^{\frac{p}{2}} + E(\int_{t'}^t |g(t,s,X_s) - g(t,s,0)|^2 ds)^{\frac{p}{2}} \\ & + E(\int_{t'}^t\int_{|y|<c}|H(t,s,X_{s-},y) - H(t,s,0,y)|^2\nu(dy)ds)^{\frac{p}{2}}] \\ \leq & 3^{p-1}2(T+1)E[\int_{t'}^t K(t,s)G^{\frac{2}{p}}(|X_s|^p)ds]^{\frac{p}{2}} + 3^{p-1}E[\int_{t'}^t K(t,s)G^{\frac{2}{p}}(|X_{s-}|^p)ds]^{\frac{p}{2}}. \end{aligned}$$

Thanks to the extended Minkowski's inequality (see [18], Corollary 1.30), we obtain

$$\{E[\int_{t'}^t K(t,s)G^{\frac{2}{p}}(|X_s|^p)ds]^{\frac{p}{2}}\}^{\frac{2}{p}} \leq \int_{t'}^t K(t,s)[EG(|X_s|^p)]^{\frac{2}{p}}ds.$$

Therefore

$$
\begin{aligned}
E|J_{12}|^p \leq & 3^{p-1}2(T+1)(\int_{t'}^{t} K(t,s)[EG(|X_s|^p)]^{\frac{2}{p}}ds)^{\frac{p}{2}} + 3^{p-1}(\int_{t'}^{t} K(t,s)[EG(|X_{s-}|^p)]^{\frac{2}{p}}ds)^{\frac{p}{2}} \\
\leq & 3^p a(T+1)[\int_{t'}^{t} K(t,s)(1+E||\xi||^p + \sup_{0\leq s\leq T} E|X(s)|^p)^{\frac{2}{p}}ds]^{\frac{p}{2}} \\
\leq & 3^p a(T+1)(1+E||\xi||^p + \sup_{0\leq s\leq T} E|X(s)|^p)(\int_{t'}^{t} K(t,s)ds)^{\frac{p}{2}} \\
\leq & 3^p a(T+1)(1+E||\xi||^p + \sup_{0\leq s\leq T} E|X(s)|^p)(\int_{t'}^{t} K^{\frac{p}{p-2}}(t,s)ds)^{\frac{p-2}{2}}|t-t'| \\
\leq & C_{4,2}|t-t'|, \qquad (4.5)
\end{aligned}
$$

where $C_{4,2} = 3^p a(T+1)(1+E||\xi||^p + \sup_{0\leq s\leq T} E|X(s)|^p)K_3^{\frac{p-2}{2}}$. Substituting (4.4) and (4.5) into (4.3) implies

$$
E|J_1|^p \leq 2^{p-1}[C_{4,1}|t-t'|^{\frac{p}{2\theta}} + C_{4,2}|t-t'|]. \qquad (4.6)
$$

For $E|J_2|^p$, it follows from (A.4) that

$$
\begin{aligned}
E|J_2|^p \leq & 3^{p-1}[TE(\int_0^{t'} |f(t,s,X_s) - f(t',s,X_s)|^2 ds)^{\frac{p}{2}} + E(\int_0^{t'} |g(t,s,X_s) - g(t',s,X_s)|^2 ds)^{\frac{p}{2}} \\
& + E(\int_0^{t'}\int_{|y|<c} |H(t,s,X_{s-},y) - H(t',s,X_{s-},y)|^2\nu(dy)ds)^{\frac{p}{2}}] \\
\leq & 3^{p-1}2(T+1)E[\int_0^{t'} F(t',t,s)(1+|X_s|^2)ds]^{\frac{p}{2}} + 3^{p-1}E[\int_0^{t'} F(t',t,s)(1+|X_{s-}|^2)ds]^{\frac{p}{2}}. \qquad (4.7)
\end{aligned}
$$

Similarly, by the extended Minkowski's inequality, we have

$$
\begin{aligned}
E(\int_0^{t'} F(t',t,s)(1+|X_{s-}|^2)ds)^{\frac{p}{2}} \leq & (\int_0^{t'} F(t',t,s)[E(1+|X_{s-}|^2)^{\frac{p}{2}}]^{\frac{2}{p}}ds)^{\frac{p}{2}} \\
\leq & (\int_0^{t'} F(t',t,s)[2^{\frac{p-2}{2}}(1+E|X_{s-}|^p)]^{\frac{2}{p}}ds)^{\frac{p}{2}} \\
\leq & 2^{\frac{p(p-2)}{4}}(1+E||\xi||^p + \sup_{0\leq s\leq T} E|X(s)|^p)(\int_0^{t'} F(t',t,s)ds)^{\frac{p}{2}}. \qquad (4.8)
\end{aligned}
$$

Substituting (4.8) into (4.7) gives

$$
E|J_2|^p \leq C_{4,3}|t-t'|^{\frac{\gamma p}{2}}, \qquad (4.9)
$$

where $C_{4,3} = 3^p(T+1)2^{\frac{p(p-2)}{4}}(1+E||\xi||^p + \sup_{0\leq s\leq T} E|X(s)|^p)$. By (4.2), (4.6) and (4.9), we derive that there exists a positive constant C_1 such that

$$
E|\tilde{X}(t) - \tilde{X}(t')|^p \leq C_1|t-t'|^{\lambda},
$$

where $\lambda = \min\{\frac{p}{2\theta}, \frac{\gamma p}{2}, 1\}$. This completes the proof. □

To get the path continuity of the solution for Eq.(1.1), we need an additional assumption:

(A.6) Let $\xi : [-\tau, 0] \to \mathbb{R}^d$ be a Lipschtiz continuous function satisfying

$$|\xi(t) - \xi(s)| \le K_4|t - s|, \forall -\tau \le t, s \le 0, \tag{4.10}$$

where K_4 is a positive constant.

Theorem 4.1. *Under assumptions (A.1) and (A.3-A.6), there exists a positive constant C_2, such that, for $0 \le t' < t \le T$ and $t - t' \le \tau$,*

$$E|X(t) - X(t')|^p \le C_2|t - t'|^\lambda,$$

where λ is defined in Lemma 4.1.

Proof. Let $\Delta = t - t'$. By Lemma 2.1 and 4.1, we obtain

$$\begin{aligned} E|X(t) - X(t')|^p =& \frac{E|D(X_t) - D(X_{t'})|^p}{\kappa^{p-1}} + \frac{E|X(t) - D(X_t) - X(t') + D(X_{t'})|^p}{(1-\kappa)^{p-1}} \\ =& \frac{E|D(X_t) - D(X_{t'})|^p}{\kappa^{p-1}} + \frac{E|\tilde{X}(t) - \tilde{X}(t')|^p}{(1-\kappa)^{p-1}} \\ \le& \kappa E|X_t - X_{t'}|^p + \frac{C_1\Delta^\lambda}{(1-\kappa)^{p-1}}. \end{aligned} \tag{4.11}$$

Obviously,

$$\begin{aligned} E|X_t - X_{t'}|^p \le& \sup_{-\tau\le\theta\le0} E|X(t+\theta) - X(t'+\theta)|^p \le \sup_{-\tau\le t'\le T-\Delta} E|X(t'+\Delta) - X(t')|^p \\ \le& \sup_{-\tau\le t'\le0} E|X(t'+\Delta) - X(t')|^p + \sup_{0\le t'\le T-\Delta} E|X(t'+\Delta) - X(t')|^p. \end{aligned} \tag{4.12}$$

Substituting (4.12) into (4.11), we get

$$\sup_{0\le t'\le T-\Delta} E|X(t) - X(t')|^p \le \frac{\kappa}{1-\kappa} \sup_{-\tau\le t'\le0} E|X(t'+\Delta) - X(t')|^p + \frac{C_1\Delta^\lambda}{(1-\kappa)^p}. \tag{4.13}$$

Note that

$$E|X(t'+\Delta) - X(t')|^p \le 2^{p-1}E|X(t'+\Delta) - X(0)|^p + 2^{p-1}E|X(0) - X(t')|^p \tag{4.14}$$

and

$$\begin{aligned} \sup_{-\tau\le t'\le0} E|X(t'+\Delta) - X(t')|^p \le& \sup_{-\tau\le t'\le-\Delta} E|X(t'+\Delta) - X(t')|^p + \sup_{-\Delta\le t'\le0} E|X(t'+\Delta) - X(t')|^p \\ \le& K_4\Delta + \sup_{-\Delta\le t'\le0} E|X(t'+\Delta) - X(t')|^p. \end{aligned} \tag{4.15}$$

Thus, by (4.14) and (4.15), we derive

$$\sup_{-\tau\le t'\le0} E|X(t'+\Delta) - X(t')|^p \le (2^{p-1}+1)K_4\Delta + 2^{p-1} \sup_{-\Delta\le t'\le0} E|X(t'+\Delta) - X(0)|^p. \tag{4.16}$$

Substituting (4.16) into (4.13), we see that

$$\begin{aligned}\sup_{0\le t'\le T-\Delta} E|X(t)-X(t')|^p \le C_{4,4} \sup_{-\Delta\le t'\le 0} E|X(t'+\Delta)-X(0)|^p + C_{4,5}\Delta^\lambda \\ = C_{4,4}\sup_{0\le s\le \Delta} E|X(s)-X(0)|^p + C_{4,5}\Delta^\lambda,\end{aligned} \tag{4.17}$$

where $C_{4,4}=\frac{2^{p-1}\kappa}{1-\kappa}, C_{4,5}=\frac{\kappa}{1-\kappa}(2^{p-1}+1)K_4\Delta^{1-\lambda}+\frac{C_1}{(1-\kappa)^p}$. By a similar argument as (4.11), we can get

$$E|X(s)-X(0)|^p \le \kappa \sup_{-\tau\le\theta\le 0} E|X(s+\theta)-X(\theta)|^p + \frac{C_1\Delta^\lambda}{(1-\kappa)^{p-1}}. \tag{4.18}$$

For $0\le s\le\Delta$, we obtain

$$\begin{aligned}\sup_{-\tau\le\theta\le 0} E|X(s+\theta)-X(\theta)|^p \le \sup_{-\tau\le\theta\le -s} E|X(s+\theta)-X(\theta)|^p + \sup_{-s\le\theta\le 0} E|X(s+\theta)-X(\theta)|^p \\ \le K_4\Delta + \sup_{-s\le\theta\le 0} E|X(s+\theta)-X(\theta)|^p\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}&\sup_{-s\le\theta\le 0} E|X(s+\theta)-X(\theta)|^p \\ \le &\frac{1}{\sqrt{\kappa}}\sup_{-s\le\theta\le 0} E|X(s+\theta)-X(0)|^p + \frac{1}{1-\kappa^{\frac{1}{2(p-1)}}}\sup_{-s\le\theta\le 0} E|X(0)-X(\theta)|^p \\ \le &\frac{1}{\sqrt{\kappa}}\sup_{-s\le\theta\le 0} E|X(s+\theta)-X(0)|^p + \frac{1}{1-\kappa^{\frac{1}{2(p-1)}}}K_4\Delta \\ \le &\frac{1}{\sqrt{\kappa}}\sup_{0\le s\le \Delta} E|X(s)-X(0)|^p + \frac{1}{1-\kappa^{\frac{1}{2(p-1)}}}K_4\Delta.\end{aligned} \tag{4.20}$$

Substituting the inequalities (4.19) and (4.20) into (4.18) gives

$$\sup_{0\le s\le\Delta} E|X(s)-X(0)|^p \le \sqrt{\kappa}\sup_{0\le s\le\Delta} E|X(s)-X(0)|^p + C_{4,6}\Delta^\lambda,$$

where $C_{4,6}=\frac{C_1}{(1-\kappa)^{p-1}}+\kappa K_4\Delta^{1-\lambda}+\frac{\kappa}{1-\kappa^{\frac{1}{2(p-1)}}}K_4\Delta^{1-\lambda}$. Therefore

$$\sup_{0\le s\le\Delta} E|X(s)-X(0)|^p \le \frac{C_{4,6}}{1-\sqrt{\kappa}}\Delta^\lambda. \tag{4.21}$$

Using (4.17) and (4.21) leads to

$$E|X(t)-X(t')|^p \le \sup_{0\le t'\le T-\Delta} E|X(t)-X(t')|^p \le (\frac{C_{4,4}C_{4,6}}{1-\sqrt{\kappa}}+C_{4,5})\Delta^\lambda.$$

Letting $C_2=\frac{C_{4,4}C_{4,6}}{1-\sqrt{\kappa}}+C_{4,5}$, then the result follows. □

5 Continuous dependence of solutions on the initial value

In this section, we will give the continuous dependence of solutions on the initial value.

Theorem 5.1. *Let $X_\xi(t), Y_\zeta(t)$ be two solutions of NSDVE Eq.(1.1) with initial value $\xi = \{\xi(t), -\tau \le t \le 0\}$ and $\zeta = \{\zeta(t), -\tau \le t \le 0\}$, respectively. If the assumptions (A.1-A.3) hold. Then for $\forall \varepsilon > 0$, there exists a $\delta > 0$ such that*

$$E|X_\xi(t) - Y_\zeta(t)|^p < \varepsilon, \quad when \quad E||\xi - \zeta||^p < \delta.$$

Proof. Note that $X_\xi(t), Y_\zeta(t)$ are two solutions of NSDVE Eq.(1.1) with initial value ξ and ζ, respectively. We have

$$\begin{aligned} X(t) - D(X_t) =& \xi(0) - D(\xi) + \int_0^t f(t,s,X_s)ds + \int_0^t g(t,s,X_s)dB(s) \\ &+ \int_0^t \int_{|y|<c} H(t,s,X_{s-},y)\widetilde{N}(ds,dy) \end{aligned}$$

and

$$\begin{aligned} Y(t) - D(Y_t) =& \zeta(0) - D(\zeta) + \int_0^t f(t,s,Y_s)ds + \int_0^t g(t,s,Y_s)dB(s) \\ &+ \int_0^t \int_{|y|<c} H(t,s,Y_{s-},y)\widetilde{N}(ds,dy). \end{aligned}$$

Hence,

$$X(t) - Y(t) = \xi(0) - \zeta(0) + D(X_t) - D(Y_t) - [D(\xi) - D(\zeta)] + L(t),$$

where

$$\begin{aligned} L(t) =& \int_0^t [f(t,s,X_s) - f(t,s,Y_s)]ds + \int_0^t [g(t,s,X_s) - g(t,s,Y_s)]dB(s) \\ &+ \int_0^t \int_{|y|<c} [H(t,s,X_{s-},y) - H(t,s,Y_{s-},y)]\widetilde{N}(ds,dy). \end{aligned}$$

By Lemma 2.1, we derive

$$\begin{aligned} & E|X(t) - Y(t)|^p \\ \le& \frac{1}{\kappa^{p-1}} E|D(X_t) - D(Y_t)|^p + \frac{1}{(1-\kappa)^{p-1}} E|\xi(0) - \zeta(0) - [D(\xi(t)) - D(\zeta(t))] + L(t)|^p \\ \le& \kappa E||\xi - \zeta||^p + \kappa \sup_{0\le s\le t} E|X(s) - Y(s)|^p + \frac{1}{(1-\kappa)^{p-1}} E|\xi(0) - \zeta(0) - [D(\xi(t)) - D(\zeta(t))] + L(t)|^p, \end{aligned}$$

which implies

$$\begin{aligned} \sup_{0\le s\le t} E|X(s) - Y(s)|^p \le& \frac{\kappa}{1-\kappa} E||\xi - \zeta||^p + \frac{1}{(1-\kappa)^p} \sup_{0\le s\le t} E|\xi(0) - \zeta(0) - [D(\xi(t)) - D(\zeta(t))] + L(s)|^p \\ \le& C_{5,1} E||\xi - \zeta||^p + \frac{3^{p-1}}{(1-\kappa)^p} \sup_{0\le s\le t} E|L(s)|^p, \end{aligned}$$

where $C_{5,1}=\frac{\kappa}{1-\kappa}+\frac{3^{p-1}(\kappa^p+1)}{(1-\kappa)^p}$. Notice that

$$\begin{aligned}\sup_{0\le s\le t}E|L(s)|^p \le & 3^{p-1}[TE(\int_0^t|f(t,s,X_s)-f(t,s,Y_s)|^2ds)^{\frac{p}{2}}+E(\int_0^t|g(t,s,X_s)-g(t,s,Y_s)|^2ds)^{\frac{p}{2}}\\ &+E(\int_0^t\int_{|y|<c}|H(t,s,X_{s-},y)-H(t,s,Y_{s-},y)|^2\nu(dy)ds)^{\frac{p}{2}}]\\ \le & 3^{p-1}2(T+1)E[\int_0^t K(t,s)G^{\frac{2}{p}}(E|X_s-Y_s|^p)ds]^{\frac{p}{2}}+3^{p-1}E[\int_0^t K(t,s)G^{\frac{2}{p}}(E|X_{s-}-Y_{s-}|^p)ds]^{\frac{p}{2}}\\ \le & 3^{p-1}(\int_0^t K^{\frac{p}{p-2}}(t,s)ds)^{\frac{p-2}{2}}[2(T+1)\int_0^t G(E|X_s-Y_s|^p)ds+\int_0^t G(E|X_{s-}-Y_{s-}|^p)ds]\\ \le & 3^{p-1}K_1^{\frac{p-2}{2}}[2(T+1)\int_0^t G(E|X_s-Y_s|^p)ds+\int_0^t G(E|X_{s-}-Y_{s-}|^p)ds]\\ \le & 3^p(T+1)K_1^{\frac{p-2}{2}}\int_0^t G(E||\xi-\zeta||^p)ds+3^p(T+1)K_1^{\frac{p-2}{2}}\int_0^t G(\sup_{0\le r\le s}E|X(r)-Y(r)|^p)ds.\end{aligned}$$

We derive

$$\begin{aligned}&\sup_{0\le s\le t}E|X(s)-Y(s)|^p\\ \le & C_{5,1}E||\xi-\zeta||^p+\frac{3^{2p-1}(T+1)K_1^{\frac{p-2}{2}}}{(1-\kappa)^p}TG(E||\xi-\zeta||^p)+\frac{3^{2p-1}(T+1)K_1^{\frac{p-2}{2}}}{(1-\kappa)^p}\int_0^t G(\sup_{0\le r\le s}E|X(r)-Y(r)|^p)ds.\end{aligned}$$

Let $u_0=C_{5,1}E||\xi-\zeta||^p+\frac{3^{2p-1}(T+1)K_1^{\frac{p-2}{2}}}{(1-\kappa)^p}TG(E||\xi-\zeta||^p), u(t)=\sup_{0\le s\le t}E|X(s)-Y(s)|^p$ and $v(t)=\frac{3^{2p-1}(T+1)K_1^{\frac{p-2}{2}}}{(1-\kappa)^p}$. By Lemma 2.2, we obtain $u(t)\le\rho^{-1}(\rho(u_0)+Tv(t))$. For $G(s)$ is a continuous nondecreasing positive function, ρ,ρ^{-1} are nondecreasing functions. In fact, $\int_{0+}\frac{1}{G(s)}ds=+\infty$, it follows that , for $\forall\varepsilon>0$, there exists an ε_1 and $\varepsilon_1<\varepsilon$ such that $\int_{\varepsilon_1}^{\epsilon}\frac{1}{G(s)}ds\ge Tv(t)$. Therefore, we can find a δ such that, when $E||\xi-\zeta||^p\le\delta$, $u_0\le\epsilon_1$. Then

$$u(t)\le\rho^{-1}(\rho(\varepsilon_1)+\int_{\varepsilon_1}^{\varepsilon}\frac{ds}{G(s)})=\rho^{-1}(\rho(\varepsilon))=\varepsilon.$$

This completes the proof. □

Remark 5.1. *Here we state some examples of function $G(.)$ satisfying the assumptions in Theorem 5.1. Let $\delta\in(0,1)$ be sufficiently small. Define*

$$G_1(x)=\begin{cases}x\log\frac{1}{x}, & 0\le x\le\delta\ ;\\ \delta\log\frac{1}{\delta}+G_1'(\delta-)(x-\delta), & x>\delta.\end{cases}$$

$$G_2(x)=\begin{cases}x\log\frac{1}{x}\log\log\frac{1}{x}, & 0\le x\le\delta\ ;\\ \delta\log\frac{1}{\delta}\log\log\frac{1}{\delta}+G_2'(\delta-)(x-\delta), & x>\delta.\end{cases}$$

where G' denotes the derivative of function G. G_1 and G_2 are both concave functions satisfying $\int_{0^+}\frac{dx}{G_i(x)}=+\infty, i=1,2$.

6 NSDVE with fractional Brownian motion kernel

We can represent a fractional Brownian motion(FBM) over a finite interval by:

$$B^{(H)}(t) := \int_0^t K_H(t,s)dB(s), t \geq 0,$$

where $B(s)$ is a standard Brownian motion, and $K_H(t,s)$ is called the kernel of FBM with $H \in (0,1)$. For $H > \frac{1}{2}$,

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

where $c_H = [\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}]^{\frac{1}{2}}$, $t > s$ and β is the Beta function. For $H < \frac{1}{2}$,

$$K_H(t,s) = b_H[(\frac{t}{s})^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{3}{2}} du],$$

where $b_H = [\frac{2H}{1-2H}\beta(1-2H, H+\frac{1}{2})]^{\frac{1}{2}}$, $t > s$. More properties of FBM can be found in [19], [20] and [21].

We consider the following NSDVE with the kernel K_H of FBM:

$$\begin{aligned} X(t) - D(X_t) =& \xi(0) - D(\xi) + \int_0^t K_H(t,s) f(s, X_s) ds + \int_0^t K_H(t,s) g(s, X_s) dB(s) \\ &+ \int_0^t \int_{|y|<c} K_H(t,s) H(s, X_{s-}, y) \widetilde{N}(ds, dy). \end{aligned} \tag{6.1}$$

We make the following assumptions:

(A.7) For all $x_1, x_2 \in \mathbb{R}^d$ and $s, t \in [0,T]$,

$$|f(s,x_1) - f(s,x_2)|^2 \vee |g(s,x_1) - g(s,x_2)|^2 \vee \int_{|y|<c} |H(s,x_1,y) - H(s,x_2,y)|^2 \nu(dy) \leq G^{\frac{2}{p}}(|x_1 - x_2|^p),$$

where function $G(.)$ is defined as in (A.1).

(A.8) For all $s \in [0,T]$, there exists a positive constant K_5, such that

$$f(s,0) \vee g(s,0) \vee \int_{|y|<c} H(s,0,y)\nu(dy) \leq K_5.$$

Theorem 6.1. *Assume that one of the following conditions hold:*

(i) Let $p \in (\frac{1}{1-H}, \frac{2}{2H-1})$, $H \in (\frac{1}{2}, \frac{3}{4})$ *and assumptions (A.3),(A6-A.8) hold;*

(ii) Let $p \in (\frac{1}{H}, \frac{2}{1-2H})$, $H \in (\frac{1}{4}, \frac{1}{2})$ *and assumptions (A.3),(A.6-A.8) hold.*

Then there exists a unique continuous progressively measurable process $\{X(t), 0 \leq t \leq T\}$ *satisfying Eq.(6.1).*

Proof. In the proof, we will use the letter C to denote an unimportant constant, whose value only depend on the subscripts and may change for one place to another. Assumption (A.7) implies (A.1), because of

$$\begin{aligned} &|K_H(t,s)(f(s,x_1) - f(s,x_2))|^2 \vee |K_H(t,s)(g(s,x_1) - g(s,x_2))|^2 \\ &\vee \int_{|y|<c} |K_H(t,s)(H(s,x_1,y) - H(s,x_2,y))|^2 \nu(dy) \\ =& |K_H(t,s)|^2 G^{\frac{2}{p}}(E(|x_1 - x_2|^p)). \end{aligned}$$

Using (A.7) and (A.8), we derive

$$\begin{aligned}
&|K_H(t,s)f(s,x) - K_H(t',s)f(s,x)|^2 \\
=&|f(s,x)|^2|K_H(t,s) - K_H(t',s)|^2 \\
=&2(|f(s,x) - f(s,0)|^2 + |f(s,0)|^2)|K_H(t,s) - K_H(t',s)|^2 \\
\leq&2|K_H(t,s) - K_H(t',s)|^2(G^{\frac{2}{p}}(|x|^p) + |f(s,0)|^2) \\
\leq&2|K_H(t,s) - K_H(t',s)|^2[C_p(1+|x|^p)^{\frac{2}{p}} + |f(s,0)|^2] \\
\leq&C_p|K_H(t,s) - K_H(t',s)|^2(1+|x|^2).
\end{aligned}$$

Similarly, we can get

$$\begin{aligned}
&|K_H(t,s)g(s,x) - K_H(t',s)g(s,x)|^2 \vee \int_{|y|<c} |K_H(t,s)H(s,x,y) - K_H(t',s)H(s,x,y)|^2\nu(dy) \\
\leq&C_p|K_H(t,s) - K_H(t',s)|^2(1+|x|^2).
\end{aligned}$$

Thanks to

$$\int_0^t |K_H(t,s) - K_H(t',s)|^2 ds \leq |t-t'|^{2H}.$$

Therefore, assumption (A.4) holds. By Theorem 3.1 and 4.1, if we can prove that assumption (A.5) holds, then the results follow.

For $H > \frac{1}{2}$,

$$K_H(t,s) = c_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du \leq C_H t^{H-\frac{1}{2}}s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}du \leq C_{T,H}s^{\frac{1}{2}-H}(t-s)^{H-\frac{1}{2}},$$

which implies that, for $p > \frac{1}{1-H}$,

$$\int_0^t K_H^{\frac{2p}{p-2}}(t,s)ds \leq C_{T,H}\int_0^t (s^{\frac{1}{2}-H}(t-s)^{H-\frac{1}{2}})^{\frac{2p}{p-2}}ds \leq C_{T,H}\int_0^t (s^{\frac{1}{2}-H})^{\frac{2p}{p-2}}ds < \infty.$$

Obviously, when $p < \frac{2}{2H-1}$,

$$\begin{aligned}
&\int_0^t |K_H(t,s)f(s,0)|^p ds \vee \int_0^t |K_H(t,s)g(s,0)|^p ds \vee \int_0^t\int_{|y|<c} |K_H(t,s)H(s,0,y)|^p\nu(dy)ds \\
\leq&C_{p,T}\int_0^t |K_H(t,s)|^p ds \leq C_{p,T}\int_0^t s^{(\frac{1}{2}-H)p}ds < \infty.
\end{aligned}$$

It leads to

$$\int_0^t |K_H(t,s)f(s,0)|^u ds \vee \int_0^t |K_H(t,s)g(s,0)|^u ds \vee \int_0^t\int_{|y|<c} |K_H(t,s)H(s,0,y)|^2\nu(dy)ds < \infty, \quad (6.2)$$

where $1 \leq u \leq p$. When $H \in (\frac{1}{2}, \frac{3}{4})$, we have $2 < \frac{1}{1-H} < \frac{2}{2H-1}$. Thus, (A.5) holds. So by Theorem 3.1 and 4.1, the result (i) follows.

For $H < \frac{1}{2}$, by the boundness of FBM's kernel(see [21], Theorem 3.2), we have

$$K_H(t,s) \le C_H s^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}}.$$

Therefore, when $p > \frac{1}{H}$, we obtain

$$\begin{aligned}\int_0^t K_H^{\frac{2p}{p-2}}(t,s)ds \le & C_T \int_0^t (s^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}})^{\frac{2p}{p-2}} ds \le C_{T,H}\int_0^1 x^{\frac{(2H-1)p}{p-2}}(1-x)^{\frac{(2H-1)p}{p-2}}dx \\ = & C_{T,H}\beta(\frac{2Hp-2}{p-2}, \frac{2Hp-2}{p-2}) < \infty,\end{aligned}$$

and when $p < \frac{2}{1-2H}$, we get

$$\begin{aligned}&\int_0^t |K_H(t,s)f(s,0)|^p ds \vee \int_0^t |K_H(t,s)g(s,0)|^p ds \vee \int_0^t \int_{|y|<c} |K_H(t,s)H(s,0,y)|^p \nu(dy)ds \\ \le & C_T \int_0^t |K_H(t,s)|^p ds \le C_T \int_0^t |s^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}}|^p ds \\ = & C_{T,H}\beta((H-\frac{1}{2})p+1, (H-\frac{1}{2})p+1) < \infty. \qquad (6.3)\end{aligned}$$

According to (6.3), we know (6.2) holds. Note that $H \in (\frac{1}{4}, \frac{1}{2})$ follows $2 < \frac{1}{H} < \frac{2}{1-2H}$. We see that (A.5) holds. Therefore result (ii) follows. This completes the proof. □

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