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# Cahn-Hilliard Equation with Terms of Lower Order and Non-constant Mobility

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**Abstract.** In this paper, we study the global existence of classical solutions for the Cahn-Hilliard equation with terms of lower order and non-constant mobility. Based on the Schauder type estimates, under some assumptions on the mobility and terms of lower order, we establish the global existence of classical solutions.
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## 1 Introduction

In this paper, we investigate the Cahn-Hilliard equation with terms of lower order

$$\frac{\partial u}{\partial t} + \operatorname{div}\left[m(u)(k\nabla\Delta u - \nabla A(u))\right] + g(u) = 0, \tag{1}$$

on a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary, where $k$ is a positive constant. On the basis of physical consideration, we discuss the following boundary value conditions

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial \Delta u}{\partial n}\right|_{\partial\Omega} = 0, \tag{2}$$

which corresponds to zero flux boundary value condition and the natural boundary value condition, where $n$ is the unit normal vector to $\partial\Omega$.

The initial value condition is supplemented as

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{3}$$

The equation (1) was introduced to study several diffusive processes, such as phase separation in binary alloys, growth and dispersal in population, see for example [1], [2]. Here $u(x,t)$ denotes the concentration of one of two phases in a system which is undergoing phase separation. The term $g(u)$ is the nonlinear source which is introduced to study how the phase transition affected by the steady fluid flow [3], [4]. In particular, the two dimensional case can be used as a mathematical model describing the lubrication for thin viscous films and spreading droplets over a solid surface as well as the flow of a thin neck of fluid in a Hele-Shaw cell, see [5]–[8].

During the past years, many authors have paid much attention to the Cahn–Hilliard equation with concentration dependent mobility

$$\frac{\partial u}{\partial t} + \operatorname{div}\left[m(u)(k\nabla\Delta u - \nabla A(u))\right] = 0,\ k > 0, \tag{4}$$

see [9]–[15]. However, only a few papers devoted to the Cahn-Hilliard equation with terms of lower order. It was G.Grün [16] who first studied the equation (1) with degenerate mobility for a special case, namely, $A(u) = -u$. He proved the existence of weak solutions.

In this paper, we consider the general case of such equations with the mobility being allowed to be concentration dependent, and with general nonlinear terms of lower order. The main purpose is to establish the global existence of classical solutions under much general assumptions. The main result is as follows

**Theorem 1** *Assume that $m(s) \in C^1(R)$, $A(s) \in C^2(R)$ and*

$$(H1) \quad H'(u) = A(u), \ \ H(u) = \frac{1}{4}(u^2-1)^2, \ \ g'(s) \ge 0, \ \ |g(s)A(s)| \le M_3 H(s),$$

$$(H2) \qquad m(s) \ge M_1, \quad |m'(s)|^2 \le M_2 m(s), \quad g^2(s) \le m(s)(M_4 + H(s)),$$

*where $M_1, M_2, M_3, M_4$ are positive constants. Assume also that the initial datum is smooth with appropriate compatibility conditions. Then the problem (1), (2) (3) admits a unique classical solution with small initial energy $F(u_0) = \int_\Omega (\frac{k}{2}|\nabla u_0|^2 + H(u_0))dx$.*

To prove the theorem, the basic a priori estimates are the $L^2$ norm estimates on $u$ and $\nabla u$. For the usual Cahn-Hilliard equation (4) the two estimates can be easily obtained, since the problem (4), (2), (3) has two important properties:

(I) the conservation of mass, namely

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx;$$

(II) there exists a Lyapunov functional

$$F[u] = \int_\Omega (\frac{k}{2}|\nabla u|^2 + H(u))dx,$$

which is decreasing in time.

However, we do not have any of these properties for the problem (1), (2), (3) of the Cahn-Hilliard equation with terms of lower order. This means that we should find a new approach to establish the required estimates on $\|u\|_{L^2(\Omega)}$ and $\|\nabla u\|_{L^2(\Omega)}$. Our approach is based on uniform Schauder type estimates for local in time solutions. To this purpose, we require some delicate local integral estimates rather than the global energy estimates used in the discussion for the Cahn-Hilliard equation with constant mobility.

This paper is constructed as follows. We first present a key step for the a priori estimates on the Hölder norm of solutions in Section 2, and then give the proof of our main theorems subsequently in Section 3.

# 2 Hölder Estimates

As an important step, in this section, we give the Hölder norm estimate on the local in time solutions.

**Proposition 1** *Assume that (H1)–(H2) holds, and $u$ is a smooth solution of the problem (1), (2), (3) with small initial energy $F(u_0)$. Then there exists a constant $C$ depending only on the known quantities, such that for any $(x_1,t_1),(x_2,t_2)\in Q_T$ and some $0<\alpha<1$,*

$$|u(x_1,t_1)-u(x_2,t_2)|\le C(|t_1-t_2|^{\alpha/4}+|x_1-x_2|^{\alpha}).$$

*Proof.* Let $z=k\Delta u-A(u)$. Multiplying both sides of the equation (1) by $z$ and then integrating the resulting relation with respect to $x$ over $\Omega$, we have

$$\int_\Omega \frac{\partial u}{\partial t}(k\Delta u-A(u))dx+\int_\Omega \nabla\cdot(m(u)\nabla z)zdx \\ +\int_\Omega g(u)zdx=0.$$

After integrating by parts, and using the boundary value conditions,

$$\frac{d}{dt}\int_\Omega\left(\frac{k}{2}(\nabla u)^2+H(u)\right)dx+\int_\Omega m(u)|\nabla z|^2dx \\ -\int_\Omega g(u)zdx=0. \tag{5}$$

That is

$$\frac{d}{dt}\int_\Omega\left(\frac{k}{2}(\nabla u)^2+H(u)\right)dx+\int_\Omega m(u)|\nabla z|^2dx \\ +\int_\Omega kg'(u)|\nabla u|^2dx+\int_\Omega g(u)A(u)dx=0. \tag{6}$$

From the assumptions of (H1)–(H2), we obtain

$$\frac{d}{dt}\int_\Omega(k(\nabla u)^2+2H(u))dx+2\int_\Omega m(u)|\nabla z|^2dx \\ \le C\int_\Omega H(u)dx.$$

The Gronwall inequality implies that

$$\int_\Omega|\nabla u|^2dx\le CF(u_0),\quad 0\le t\le T, \tag{7}$$

$$\int_\Omega u^4dx\le CF(u_0),\quad 0\le t\le T. \tag{8}$$

Again multiplying both sides of the equation (1) by $\Delta^2u$ and integrating the resulting relation with respect to $x$ over $\Omega$, we have

$$\int_\Omega \frac{\partial u}{\partial t}\Delta^2udx+\int_\Omega \nabla\cdot[m(u)(k\nabla\Delta u-\nabla A(u))]\Delta^2udx \\ +\int_\Omega g(u)\Delta^2udx=0.$$

Integrating by parts, and using the boundary value conditions, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta u|^2dx+\int_\Omega km(u)(\Delta^2u)^2dx+\int_\Omega km'(u)\nabla u\cdot\nabla\Delta u\Delta^2udx\\&-\int_\Omega m(u)A'(u)\Delta u\Delta^2udx-\int_\Omega m'(u)A'(u)|\nabla u|^2\Delta^2udx\\&-\int_\Omega m(u)A''(u)|\nabla u|^2\Delta^2udx+\int_\Omega g(u)\Delta^2udx=0.\end{aligned}$$

The Hölder inequality yields

$$\begin{aligned}&\left|\int_\Omega m'(u)\nabla u\nabla\Delta u\Delta^2udx\right|\\\leq\;&\frac{1}{2}\int_\Omega m(u)(\Delta^2u)^2dx+\frac{1}{2}\int_\Omega\frac{|m'(u)|^2}{m(u)}|\nabla u|^2|\nabla\Delta u|^2dx\\\leq\;&\frac{1}{2}\int_\Omega m(u)(\Delta^2u)^2dx+\frac{M_2}{2}\int_\Omega|\nabla u|^2|\nabla\Delta u|^2dx\\\leq\;&\frac{1}{2}\int_\Omega m(u)(\Delta^2u)^2dx+\frac{M_2}{2}\left(\int_\Omega|\nabla u|^8dx\right)^{1/4}\left(\int_\Omega|\nabla\Delta u|^{8/3}dx\right)^{3/4}.\end{aligned}$$

It follows by using the Cagliardo-Nirenberg inequalities (noticing that we consider only the two dimensional case)

$$\left(\int_\Omega|\nabla u|^8dx\right)^{1/8}\leq C_0\left(\int_\Omega|\Delta^2u|^2dx\right)^{1/8}\left(\int_\Omega|\nabla u|^2dx\right)^{3/8},$$

$$\left(\int_\Omega|\nabla\Delta u|^{8/3}dx\right)^{3/8}\leq C_1\left(\int_\Omega|\Delta^2u|^2dx\right)^{3/8}\left(\int_\Omega|\nabla u|^2dx\right)^{1/8}+\int_\Omega|\nabla u|^2dx,$$

and (8) as $F(u_0)$ small enough

$$\begin{aligned}&\left|\int_\Omega m'(u)\nabla u\nabla\Delta u\Delta^2udx\right|\\\leq\;&\frac{1}{2}\int_\Omega m(u)(\Delta^2u)^2dx+\frac{M_2}{2}C_0^2C_1^2\left(\int_\Omega|\Delta^2u|^2dx\right)\left(\int_\Omega|\nabla u|^2dx\right)+C_2\\\leq\;&\frac{1}{2}\int_\Omega m(u)(\Delta^2u)^2dx+CF(u_0)\left(\int_\Omega|\Delta^2u|^2dx\right)+C_2\\\leq\;&\frac{5}{8}\int_\Omega m(u)(\Delta^2u)^2dx+C_2.\end{aligned}$$

By the assumption (H2), we have $m(u)\leq C(u^2+1)$. Then, using Cauchy's inequality again, we have

$$\begin{aligned}&\left|\int_\Omega m(u)A'(u)\Delta u\Delta^2udx\right|\\\leq\;&\frac{1}{32}\int_\Omega m(u)(\Delta^2u)^2dx+C\int_\Omega m(u)|A'(u)|^2|\Delta u|^2dx\\\leq\;&\frac{1}{32}\int_\Omega m(u)(\Delta^2u)^2dx+C\int_\Omega(u^2+1)(u^4+1)(\Delta u)^2dx,\end{aligned}$$

and hence

$$\begin{aligned}
&\Big|\int_\Omega m'(u)A'(u)|\nabla u|^2\Delta^2 u dx\Big|\\
\le\quad & \frac{1}{32}\int_\Omega m(u)(\Delta^2 u)^2dx + C\int_\Omega \frac{|m'(u)|^2|A'(u)|^2}{m(u)}|\nabla u|^4 dx\\
\le\quad & \frac{k}{32}\int_\Omega m(u)(\Delta^2 u)^2dx + C\int_\Omega |A'(u)|^2|\nabla u|^4 dx,
\end{aligned}$$

$$\begin{aligned}
&\Big|\int_\Omega m(u)A''(u)|\nabla u|^2\Delta^2 u dx\Big|\\
\le\quad & \frac{1}{32}\int_\Omega m(u)(\Delta^2 u)^2dx + C\int_\Omega m(u)|A''(u)|^2|\nabla u|^4 dx\\
\le\quad & \frac{k}{32}\int_\Omega m(u)(\Delta^2 u)^2dx + C\int_\Omega (u^2+1)u^2(\nabla u)^4 dx.
\end{aligned}$$

From $(H2)$, we have again

$$\begin{aligned}
&\Big|\int_\Omega g(u)\Delta^2 u dx\Big|\\
\le\quad & \frac{1}{32}\int_\Omega m(u)(\Delta^2 u)^2dx + C\int_\Omega \frac{(g(u))^2}{m(u)}dx.\\
\le\quad & \frac{1}{32}\int_\Omega m(u)(\Delta^2 u)^2dx + C\int_\Omega H(u)dx.
\end{aligned}$$

The Nirenberg inequality and (8) yield

$$\begin{aligned}
\sup_{x\in\Omega}|u| &\le C\left(\int_\Omega |\Delta^2 u|^2dx\right)^{1/14}\left(\int_\Omega |u|^4dx\right)^{3/14} + C_2\left(\int_\Omega |u|^4dx\right)^{1/4}\\
&\le C\left(\int_\Omega |\Delta^2 u|^2dx\right)^{1/14} + C.
\end{aligned}$$

Using Cagliardo-Nirenberg inequality and (7), we obtain

$$\begin{aligned}
\int_\Omega |\nabla u|^4dx &\le C\left(\int_\Omega |\nabla u|^2dx\right)^{1/4}\left(\left(\int_\Omega |\Delta u|^2dx\right)^{1/4} + \left(\int_\Omega |\nabla u|^2dx\right)^{1/4}\right)\\
&\le C_1\left(\int_\Omega |\Delta u|^2dx\right)^{1/4} + C_2.
\end{aligned}$$

We notice that

$$\int_\Omega (\Delta u)^2dx \le C\left(\int_\Omega |\Delta^2 u|^2dx\right)^{1/3}.$$

Summing up, we have

$$\frac{d}{dt}\int_\Omega (\Delta u)^2dx + C_1\int_\Omega (\Delta^2 u)^2dx \le C_2.$$

By Gronwall's inequality, we have

$$\int_\Omega (\Delta u)^2 dx \le C, \quad 0 < t < T, \tag{9}$$

and

$$\iint_{Q_T} (\Delta^2 u)^2 dx \le C. \tag{10}$$

The desired estimate follows from (9) and the equation (1) immediately. The proof is complete.

## 3 The Proof of the Main Results

We are now in a position to show the main theorems. Owing to the Hölder norm estimates, the remaining proof can be transformed into the a priori estimates for a linear problem.

In fact, we can rewrite the equation (1) into the following form

$$\frac{\partial u}{\partial t} + \operatorname{div}\Big[a(t,x)\nabla\Delta u\Big] = \operatorname{div} \vec{F} - g(u(x,t)), \tag{11}$$

where

$$a(t,x) = km(u(t,x)), \qquad \vec{F} = m(u(t,x))\nabla A(u(t,x)).$$

We may think of $a(t,x)$ and $\vec{F}(t,x)$ as known functions and consider the reducing linear equation (11). Since $u$ is locally Hölder continuous, we see that $a(t,x)$ is locally Hölder continuous too. Without loss of generality, we may assume that $a(t,x)$ and $\vec{F}(t,x)$ are sufficiently smooth, otherwise we replace them by their approximation functions.

The crucial step is to establish the estimates on the Hölder norm of $\nabla u$. Let $(t_0, x_0) \in (0,T) \times \Omega$ be fixed and define

$$\varphi(\rho) = \iint_{S_\rho} \Big( |\nabla u - (\nabla u)_\rho|^2 + \rho^4 |\nabla\Delta u|^2 \Big)\, dtdx, \quad (\rho > 0)$$

where

$$S_\rho = (t_0 - \rho^4, t_0 + \rho^4) \times B_\rho(x_0), \quad (\nabla u)_\rho = \frac{1}{|S_\rho|} \iint_{S_\rho} \nabla u \, dtdx$$

and $B_\rho(x_0)$ is the ball centred at $x_0$ with radius $\rho$.

Let $u$ be the solution of the problem (11),(2),(3). We split $u$ on $S_R$ into $u = u_1 + u_2$, where $u_1$ is the solution of the problem

$$\frac{\partial u_1}{\partial t} + a(t_0, x_0)\Delta^2 u_1 = 0, \quad (t,x) \in S_R \tag{12}$$

$$\frac{\partial u_1}{\partial n} = \frac{\partial u}{\partial n}, \quad \frac{\partial \Delta u_1}{\partial n} = \frac{\partial \Delta u}{\partial n}, \quad (t,x) \in (t_0 - R^4, t_0 + R^4) \times \partial B_R(x_0) \tag{13}$$

$$u_1 = u, \quad t = t_0 - R^4, \;\; x \in B_R(x_0), \tag{14}$$

and $u_2$ solves the problem

$$\frac{\partial u_2}{\partial t} + a(t_0, x_0)\Delta^2 u_2 = \nabla \cdot \Big[(a(t_0, x_0) - a(t, x))\nabla\Delta u\Big] + \nabla \cdot \vec{F} - g(u(x, t)), \tag{15}$$

$$\frac{\partial u_2}{\partial n} = 0, \quad \frac{\partial \Delta u_2}{\partial n} = 0, \quad (t, x) \in (t_0 - R^4, t_0 + R^4) \times \partial B_R(x_0), \tag{16}$$

$$u_2 = 0, \quad t = t_0 - R^4, \;\; x \in B_R(x_0). \tag{17}$$

By classical linear theory, the above decomposition is uniquely determined by $u$.

We need the following lemmas.

**Lemma 1** Assume that

$$|a(t, x) - a(t_0, x_0)| \le a_\sigma \Big(|t - t_0|^{\sigma/4} + |x - x_0|^\sigma\Big), t \in (t_0 - R^4, t_0 + R^4), \quad x \in B_R(x_0).$$

Then

$$\begin{aligned} & \sup_{(t_0 - R^4, t_0 + R^4)} \int_{B_R(x_0)} |\nabla u_2(t, x)|^2 \, dx + \iint_{S_R} (\nabla\Delta u_2)^2 \, dt dx \\ \le \; & C R^{2\sigma} \iint_{S_R} (\nabla\Delta u)^2 \, dt dx + C \Big(1 + \sup_{S_R} |\vec{F}|^2\Big) R^6. \end{aligned}$$

*Proof.* Multiply the equation (15) by $\Delta u_2$ and integrate the resulting relation over $(t_0 - R^4, t) \times B_R(x_0)$, integrating by parts, we have

$$\begin{aligned} & \frac{1}{2} \int_{B_R(x_0)} |\nabla u_2(t, x)|^2 \, dx + a(t_0, x_0) \int_{t_0 - R^4}^t ds \int_{B_R(x_0)} (\nabla\Delta u_2)^2 \, dx \\ = \; & \int_{t_0 - R^4}^t ds \int_{B_R(x_0)} [a(t_0, x_0) - a(t, x)] \nabla\Delta u \nabla\Delta u_2 \, dx \\ + \; & \int_{t_0 - R^4}^t ds \int_{B_R(x_0)} \vec{F} \, \nabla\Delta u_2 \, dx + \int_{t_0 - R^4}^t ds \int_{B_R(x_0)} g(u(x, t))\Delta u_2 \, dx, \end{aligned}$$

Cauchy's inequality and Poincaré inequality thus yields the desired conclusion and the proof is complete.

**Lemma 2** For $\lambda \in (6, 7)$,

$$\varphi(\rho) \le C_\lambda \Big(1 + \sup_{S_{R_0}} |\vec{F}|\Big) \rho^\lambda, \qquad \rho \le R_0 = \min\Big(\mathrm{dist}(x_0, \partial\Omega), t_0^{1/4}\Big),$$

where $C_\lambda$ depends on $\lambda$, $R_0$ and the known quantities.

*Proof.* Using Lemma 1 and similar to the proof of Lemma 2.5 of [11], there is no essential and new idea for the details of the proof. Here we omit the details.

*Proof of Theorem 1.* Since we are concerned with classical solutions, the uniqueness is quite easy by using the standard arguments, and we omit the details. For the existence, using Proposition 1 and Lemma 2, we have

$$\frac{|\nabla u(t_1,x_1)-\nabla u(t_2,x_2)|}{|t_1-t_2|^{(\lambda-6)/8}+|x_1-x_2|^{(\lambda-6)/2}} \le C\left(1+\sup|\vec{F}|\right) \le C\left(1+\sup|\nabla u|\right).$$

By the interpolation inequality, we thus obtain

$$|\nabla u(t_1,x_1)-\nabla u(t_2,x_2)| \le C\left(|t_1-t_2|^{(\lambda-6)/8}+|x_1-x_2|^{(\lambda-6)/2}\right).$$

The conclusion follows immediately from the classical theory, since we can transform the equation (1) into the form

$$\frac{\partial u}{\partial t}+a_1(t,x)\Delta^2 u+\overline{B_1}(t,x)\nabla\Delta u+a_2(t,x)\Delta u+\overline{B_2}(t,x)\nabla u+g(u(t,x))=0,$$

where the Hölder norms on

$$a_1(t,x)=km(u(t,x)),\quad \overline{B_1}(t,x)=km'(u(t,x))\nabla u(t,x),$$

$$a_2(t,x)=-m(u(t,x))A'(u(t,x)),\quad \overline{B_2}(t,x)=-\nabla(m(u(t,x))A'(u(t,x)))$$

have been estimated in the above discussion. The proof is complete.

# References


[1] A. Novick-Cohen and L. A. Segel, Nonlinear aspects of the Cahn-Hilliard equation, Physica D 10(1984), 277–298.

[2] D. S. Cohen, and J. D. Murry, A generalized diffusion model for growth and dispersal in a population, J. Math. Biology, 12(1981), 237–249.

[3] R. S. Laugesen and M. C. Pugh, Linear stability of steady states for thin film and Cahn-Hilliard type equations., Arch. Ration. Mech. Anal. 154(1)(2000), 3-51.

[4] G. I. Sivashinsky, On cellular instability in the solidification of a dilute binary alloy, Phys. D 8(1983), 243-248.

[5] A. B. Tayler, Mathematical Models in Applied Mechanics, Clarendon, Oxford, 1986.

[6] A. L. Bertozzi, Symmetric singularity formation in lubrication-type equations for interface motion, SIAM J. Applied Math., 56(1996), 681–714.

[7] P. Constaintin, T. F. Dupont, R. E. Goldstein, L. P. Kadanoff, M. J. Shelley, and Zhou Shumin, Droplet breakup in a model of the Hele-Shaw cell, Phys. Rev. E, 47(1993), 4169–4181.

[8] T. F. Dupont, R. E. Goldstein, L. P. Kadanoff, and Zhou Shumin, Finite-time singularity formation in Hele-Shaw systems, Phys. Rev. E., 47(1993), 4182–4196.

[9] J. Yin, On the Cahn-Hilliard equation with nonlinear principal part, Journal of Partial Differential Equations, 7(1)(1994), 77–96.

[10] C. M. Elliott and H. Garcke, On the Cahn-Hilliard equation with degenerate mobility, SIAM J. Math. Anal., 27(2)(1996), 404–423.

[11] J. Yin and C. Liu , Regularity of Solutions of The Cahn–Hilliard equation with conceneration dependent mobility, Nonlinear Analysis, TMA, 45(5)(2001) 543-554.

[12] M. Giaquinta and E. Giusti, Partial regularity for the solutions to nonlinear parabolic systems, Ibid., 97(1973), 235–266.

[13] H. Garcke and A. Novick-Cohen, A singular limit for a system of degenerate Cahn-Hilliard equations, Adv. Diff. Eqs., 5(4-6)(2000), 401–434.

[14] J. W. Barrett and J. F. Blowey, Finite element approximation of the Cahn-Hilliard equation with concentration dependent mobility, Math. Comp., 68(226)(1999), 487–517.

[15] C. M. Elliott and S. M. Zheng, On the Cahn-Hilliard equation, Arch. Rat. Mech. Anal., 96(4)(1986),339-357.

[16] G. Grün, Degenerate parabolic differential equations of fourth order and a plasticity model with nonlocal hardening, Z. Anal. Anwendungen, 14(1995), 541–574.