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Existence and continuous dependence of mild solutions for fractional abstract differential equations with infinite delay

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Abstract

In this paper, we prove the existence, uniqueness, and continuous dependence of the mild solutions for a class of fractional abstract differential equations with infinite delay. The results are obtained by using the Krasnoselskii's fixed point theorem and the theory of resolvent operators for integral equations.

Key words: Fractional differential equation; Mild solution; Infinite delay; Phase space; Resolvent operator
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1. Introduction

Motivated by the fact that abstract functional differential equations with infinite delay arise in many areas of applied mathematics, this type of equations has received much attention in recent years [1]. The works of Bahuguna, Pandey and Ujlayan [2], Liang, Xiao and Casteren [3], Liang and Xiao [4], Henríquez and Lizama [1], Baghli and Benchohra [5], and references therein, provide a basic theory for abstract functional differential or integrodifferential equations with infinite delay. They established theorems about the existence, uniqueness, or existence of periodic solutions.

It is worth pointing out that all of the aforementioned problems have been restricted to integer-order differential equations. Recent investigations in physics, engineering, biological sciences and other fields have demonstrated

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that the dynamics of many systems are described more accurately using fractional differential equations [6-9]. There have been some attempts on fractional evolution equations and their optimal control or controllability [10-15]. Also, the problem of the existence of a mild solution for abstract differential equations with fractional derivatives has started to draw some initial research attention leading to inspiring results, see e.g. [16-18]. In [16-17], Zhou and Jiao introduced an appropriate concept for mild solution and established the criteria on existence and uniqueness of mild solutions to nonlocal Cauchy problem of fractional evolution equations by considering an integral equation which was given in terms of probability density and semigroup. Furthermore, Wang, Wei and Zhou [19] studied the fractional finite time delay evolution systems and optimal controls in infinite-dimensional spaces. They obtained some sufficient conditions for the existence, uniqueness and continuous dependence of mild solutions of these control systems. Then, Wang, Zhou and Medved [20] derived the solvability and optimal controls of a class of fractional integrodifferential evolution systems with infinite delay in Banach spaces. In [20], existence and continuous dependence of mild solutions were investigated by utilizing the techniques of a priori estimation and extension of step by steps.

Particularly, Hernández, O'Regan and Balachandran in [18] considered a different approach to treating a general class of abstract fractional differential equations

$$\begin{cases} D^{\alpha}(x(t)+g(t,x(t))) = Ax(t)+f(t,x(t)), & t\in[0,a], \\ x(0)=x_0 \end{cases} \tag{1.1}$$

where $0<\alpha<1$. The authors of [18] gave the definition of the mild solution to the problem (1.1) by using the well developed theory of resolvent operators for integral equations and studied the existence of this class of solutions.

Motivated by the approach of [18], we consider the following fractional abstract differential equations with infinite delay

$$\begin{cases} D^{\alpha}[x(t)-g(t,x_t)] = Ax(t)+f(t,x_t,\int_0^t\gamma(t,s,x_s)ds), & t\in[0,a], \\ x(t)=\phi(t), & t\in(-\infty,0], \end{cases} \tag{1.2}$$

where $a>0$, $0<\alpha<1$, A is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operators $T(t)_{t\geq 0}$, on a Banach space E with norm $\|.\|$; the functions f,g,γ will be given later. Let $x_t(.)$ denote $x_t(\theta)=x(t+\theta),\theta\in(-\infty,0]$, and the derivative D^{α} is understood here in the Caputo sense.

First, we apply the Krasnoselskii's fixed point theorem and the theory of resolvent operators for integral equations to establish our existence result. Then, we derive the uniqueness of mild solutions by using the Banach contraction principle. Furthermore, the continuous dependence of the mild solutions on the initial condition is investigated.

2. Preliminaries

First, we give some notations needed to establish our results.

We assume that E is a Banach space with norm $\|.\|$; A is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operators. $D(A)$ is the domain of A endowed with the graph norm $\|x\|_{D(A)} = \|x\|+\|Ax\|$. We denote by $L(Z,W)$ the space of bounded linear operators from Z to W endowed with the operator norm denoted by $\|.\|_{L(Z,W)}$. Especially when $Z = W$, we write simply $L(Z)$ and $\|.\|_{L(Z)}$. Let $J \subset R$ and denote $C(J,E)$ to be the Banach space of continuous functions form J into E with the norm $\|x\|_{C(J,E)} = \sup_{t\in J} \|x(t)\|$, where $x \in C(J,E)$.

Next, consider the abstract integral equation:

$$x(t) = f(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{Ax(s)}{(t-s)^{1-\alpha}} \mathrm{ds}, \quad t \in [0,a] \tag{2.1}$$

where $f \in C([0,a],E)$.

We introduce some basic definitions, properties and lemmas which are used throughout this paper.

Definition 2.1. [18, 21] *A function $x \in C([0,b],E)$ is called a mild solution of the integral equation* (2.1) *on* $[0,b]$ *if* $\int_0^t (t-s)^{\alpha-1}x(s)\mathrm{ds} \in D(A)$ *for all* $t \in [0,b]$ *and*

$$x(t) = f(t) + \frac{1}{\Gamma(\alpha)} A \int_0^t \frac{x(s)}{(t-s)^{1-\alpha}} \mathrm{ds}, \quad \forall t \in [0,b].$$

Definition 2.2. [18, 21] *A one parameter family of bounded linear operators $S(t)_{t\geq 0}$ on E is called a resolvent operator for* (2.1) *if the following conditions are satisfied:*

$(S1)$ $S(.)x \in C([0,\infty),E)$ and $S(0)x = x$ for all $x \in E$;

$(S2)$ $S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and every

$t \geq 0$;
$(S3)$ *for every* $x \in D(A)$ *and* $t \geq 0$,

$$S(t)x = x + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{AS(s)}{(t-s)^{1-\alpha}} x\mathrm{d}s.$$

Lemma 2.1. [18, 21] *If the resolvent* $S(t)_{t\geq 0}$ *for* (2.1) *is analytic, then*
i) for each $x \in D(A)$ *there is* $\varphi \in L^1_{loc}([0,\infty), R^+)$ *such that*

$$\|S'(t)x\| \leq \varphi_A(t)\|x\|_{D(A)} \quad a.e. \quad on \quad R^+, for \quad each \quad x \in D(A);$$

ii) if $f \in C([0,b], D(A))$ *for some* $0 \leq b \leq a$, *then* $x : [0,b] \longrightarrow E$ *defined by*

$$x(t) = f(t) + \int_0^t S'(t-s)f(s)\mathrm{d}s, \quad t \in [0,b]$$

is a mild solution of (2.1) *on* $[0,b]$.

Lemma 2.2. [18] *Assume* $S(t)$ *is compact for all* $t > 0$. *Then* $S'(t)$ *is compact for all* $t > 0$ *and the inclusion map* $i_c : D(A) \longrightarrow E$ *is compact.*

Throughout the study, we assume that the phase space $(\mathcal{B}, \|.\|_{\mathcal{B}})$ is a seminormed linear space consisting of functions from $(-\infty, 0]$ into E satisfying the following assumptions (cf. Hale and Kato in [22]).
$(A1)$ If $x : (-\infty, a] \to E$ is continuous on $J = [0,a]$ and $x_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold:
(i) $x_t \in \mathcal{B}$,
(ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$,
(iii) $\|x_t\|_{\mathcal{B}} \leq K_1(t)\sup_{0\leq s\leq t}\|x(s)\| + K_2(t)\|x_0\|_{\mathcal{B}}$,
where $H \geq 0$ is a constant, $K_1 : [0,+\infty) \to [0,+\infty)$ is continuous, $K_2 : [0,+\infty) \to [0,+\infty)$ is locally bounded, and H, K_1, K_2 are independent of $x(.)$. Denote $M_1 = \sup_{t\in J} K_1(t), M_2 = \sup_{t\in J} K_2(t)$.
$(A2)$ For the function $x(.)$ in $(A1)$, x_t is a $\mathcal{B}$-valued continuous function on J.
$(A3)$ The space $\mathcal{B}$ is complete.

Before giving the definition of mild solution of (1.2) , we rewrite (1.2) in the equivalent integral equation:

$$\begin{cases} x(t) = \phi(0) - g(0,\phi) + g(t,x_t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x_s, \int_0^s \gamma(s,\tau,x_\tau)\mathrm{d}\tau)\mathrm{d}s \\ \qquad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Ax(s)\mathrm{d}s, \qquad t \in [0,a], \\ x(t) = \phi(t), \qquad t \in (-\infty, 0]. \end{cases}$$

Let Ω be set defined by

$$\Omega = \{x : (-\infty, a] \to E : x|_{(-\infty,0]} \in \mathcal{B}, x|_J \in C(J, E)\},$$

where $J = [0, a]$.

The concept of a mild solution of (1.2) can now be stated.

Definition 2.3. *A function $x \in \Omega$ is said to be a mild solution of* (1.2) *on* $[0, a]$ *if* $\int_0^t (t-s)^{\alpha-1} x(s)\mathrm{ds} \in D(A), \quad t \in [0, a]$ *and*

$$\begin{cases} x(t) = G(t) + F(t) + \frac{1}{\Gamma(\alpha)} A \int_0^t (t-s)^{\alpha-1} x(s)\mathrm{ds}, & t \in [0, a], \\ x(t) = \phi(t) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , & t \in (-\infty, 0], \end{cases} \tag{2.2}$$

where

$$G(t) = \phi(0) - g(0, \phi) + g(t, x_t),$$

$$F(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s, \int_0^s \gamma(s, \tau, x_\tau)\mathrm{d}\tau)\mathrm{ds}.$$

The key tool in our approach is the following fixed-point theorem.

Lemma 2.3. *(Krasnoselskii's Fixed Point Theorem) Let $\mathcal{Q}$ be a bounded closed and convex subset of E and let F_1, F_2 be maps of $\mathcal{Q}$ into E such that $F_1x + F_2y \in \mathcal{Q}$ for every pair $x, y \in \mathcal{Q}$. If F_1 is a contraction and F_2 is completely continuous, then the equation $F_1x + F_2x = x$ has a solution on $\mathcal{Q}$.*

3. Main results

In this section, we present and prove our main results.

First, we list some assumptions:

(H1) The resolvent operator $S(t)_{t\geq 0}$ is compact.

(H2) $g : [0, a] \times \mathcal{B} \longrightarrow D(A)$ is a continuous function and there exist a constant K_g and a function $L_g \in C([0, a], R^+)$ such that

$$\|g(t, x_t)\|_{D(A)} \leq K_g(\|x_t\|_{\mathcal{B}} + 1),$$

$$\|g(t, x_t) - g(t, y_t)\|_{D(A)} \leq L_g(t)\|x_t - y_t\|_{\mathcal{B}}$$

for all $t \in [0, a]$, and every $x, y \in \Omega$.

(H3) $\gamma : D = \{(t, s) \in [0, a] \times [0, a] : s \leq t\} \times \mathcal{B} \longrightarrow \mathcal{B}$ and there exist a constant $K_\gamma > 0$ and a function $L_\gamma \in C([0,a], R^+)$ such that

$$\|\gamma(t,s,x_s)\|_{\mathcal{B}} \le K_\gamma(\|x_s\|_{\mathcal{B}} + 1),$$

$$\|\gamma(t,s,x_s) - \gamma(t,s,y_s)\|_{\mathcal{B}} \le L_\gamma(t)\|x_s - y_s\|_{\mathcal{B}}$$

for all $t \in [0,a]$ and every $x, y \in \Omega$.

(H4) $f : [0,a] \times \mathcal{B} \times \mathcal{B} \longrightarrow D(A)$ is a continuous function and there exists a constant $K_f > 0$ such that

$$\|f(t,x_t,y_t)\|_{D(A)} \le K_f(\|x_t\|_{\mathcal{B}} + \|y_t\|_{\mathcal{B}} + 1)$$

for all $t \in [0,a]$, and every $x, y \in \Omega$.

(H5) There exists a function $L_f \in C([0,a], R^+)$ such that

$$\|f(t,x_t,w_t) - f(t,y_t,u_t)\|_{D(A)} \le L_f(t)(\|x_t - y_t\|_{\mathcal{B}} + \|w_t - u_t\|_{\mathcal{B}})$$

for all $t \in [0,a]$, $x, y, w, u \in \Omega$.

3.1. Existence of solutions

We now establish our existence result.

Theorem 3.1. *Assume $\alpha \in (0,1)$ and the following conditions are satisfied:*

(i) $\phi(0) \in D(A)$ and there exists a constant K_ϕ such that $\|\phi(0)\|_{D(A)} \le K_\phi\|\phi\|_{\mathcal{B}}$;

(ii) $(H_1) \sim (H_4)$ are satisfied;

(iii) $K_g M_1 < 1$ and $L_g(0)M_1 < 1$.

Then the Cauchy problem (1.2) has a mild solution on $(-\infty, b]$ for some $0 < b \le a$.

Proof. Since $L_g(0)M_1 < 1$, $\quad K_g M_1 < 1$, $|L_g|_{C([0,c],R^+)}M_1 \to L_g(0)M_1$ and $\|\varphi_A\|_{L^1([0,c],R^+)} \to 0$ as $c \to 0$, there exists $0 < b \le a$ such that

$$(1 + b\|\varphi_A\|_{L^1([0,b],R^+)})[K_g M_1 + \frac{K_f b^\alpha}{\alpha\Gamma(\alpha)}(K_\gamma b + 1)M_1] < 1, \tag{3.1}$$

$$(1 + b\|\varphi_A\|_{L^1([0,b],R^+)})|L_g|_{C([0,b],R^+)}M_1 < 1. \tag{3.2}$$

For any positive constant k, $x \in \Omega$ and $\|x\|_{C(J,E)} \le k$, we introduce the map $\Gamma : \Omega \to \Omega$

$$(\Gamma x)(t) = \begin{cases} G(t) + F(t) + \frac{1}{\Gamma(\alpha)}\int_0^t S'(t-s)[G(s) + F(s)]\mathrm{d}s, & t \in [0,b], \\ \phi(t) & , \quad t \in (-\infty, 0]. \end{cases}$$

Now we prove that Γ is well defined.

From the assumptions $(i), (H2) \sim (H4)$, we obtain for $t \in [0, b]$

$$\begin{aligned} \|G(t)\|_{D(A)} &\leq K_\phi \|\phi\|_{\mathcal{B}} + K_g(\|\phi\|_{\mathcal{B}} + 1) + K_g(\|x_t\|_{\mathcal{B}} + 1) \\ &\leq (K_\phi + K_g + K_g M_2)\|\phi\|_{\mathcal{B}} + K_g(M_1 k + 2) \end{aligned}$$

and

$$\begin{aligned} \|F(t)\|_{D(A)} &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x_s, \int_0^s \gamma(s, \tau, x_\tau)\mathrm{d}\tau)\|_{D(A)} \mathrm{ds} \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_f(\|x_s\|_{\mathcal{B}} + \|\int_0^s \gamma(s, \tau, x_\tau)\mathrm{d}\tau\|_{\mathcal{B}} + 1) ds \\ &\leq \frac{b^\alpha K_f}{\alpha\Gamma(\alpha)}(K_\gamma b + 1)(M_1 k + M_2\|\phi\|_{\mathcal{B}} + 1). \end{aligned}$$

Thus from Lemma 2.1, we have

$$\begin{aligned} &\int_0^t \|S'(t-s)[G(s) + F(s)]\|\mathrm{ds} \leq \int_0^t \varphi_A(t-s)\|G(s) + F(s)\|_{D(A)}\mathrm{ds} \\ \leq\ & [(K_\phi + K_g + K_g M_2)\|\phi\|_{\mathcal{B}} + K_g(M_1 k + 2) \\ +\ & \frac{b^\alpha K_f}{\alpha\Gamma(\alpha)}(K_\gamma b + 1)(M_1 k + M_2\|\phi\|_{\mathcal{B}} + 1)] b \|\varphi_A\|_{L^1([0,b], R^+)}, \end{aligned}$$

which implies that the function $s \to S'(t-s)[G(s) + F(s)]$ is integrable on $[0, t]$ for all $t \in [0, b]$. Since $G(.)$ and $F(.)$ are continuous, it shows Γ is well defined.

Let us introduce the map $y(.) : (-\infty, b] \to E$ by

$$y(t) = \begin{cases} \phi(0) + \int_0^t S'(t-s)\phi(0)\mathrm{ds}, & t \in [0, b], \\ \phi(t) \qquad\qquad\qquad\qquad, & t \in (-\infty, 0]. \end{cases}$$

It is clear that $y_0 = \phi$.

For each $z \in C([0, b], E)$ with $z(0) = 0$, let $\hat{z}$ be defined by

$$\hat{z}(t) = \begin{cases} z(t), & t \in [0, b], \\ 0 \quad, & t \in (-\infty, 0]. \end{cases}$$

If $x(.)$ satisfies the integral equation

$$x(t) = G(t) + F(t) + \int_0^t S'(t-s)[G(s) + F(s)]\mathrm{ds},$$

we can decompose $x(.)$ as $x(t) = \hat{z}(t) + y(t), 0 \leq t \leq b$, which implies $x_t = \hat{z}_t + y_t$ for every $0 \leq t \leq b$ and the function $z(.)$ satisfies

$$z(t) = G_z(t) + F_z(t) + \int_0^t S'(t-s)[G_z(s) + F_z(s)]\mathrm{ds}, \tag{3.3}$$

where $G_z(t) = g(t, \hat{z}_t + y_t) - g(0, \phi)$,
$F_z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \hat{z}_s + y_s, \int_0^s \gamma(s, \tau, \hat{z}_\tau + y_\tau)\mathrm{d}\tau)\mathrm{ds}$.

Denote $\Omega_0 = \{z \in \Omega : z_0 = 0\}$ and let $\|.\|_{\Omega_0}$ be the seminorm in Ω_0 defined by

$$\|z\|_{\Omega_0} = \|z_0\|_{\mathcal{B}} + \sup_{0 \leq t \leq b} |z(t)| = \sup_{0 \leq t \leq b} |z(t)|, z \in \Omega_0.$$

Then $(\Omega_0, \|.\|_{\Omega_0})$ is a Banach space.

To bring out the results, we define the operator $P : \Omega_0 \to \Omega_0$ by

$$(Pz)(t) = G_z(t) + F_z(t) + \int_0^t S'(t-s)[G_z(s) + F_z(s)]\mathrm{ds}, \quad t \in [0, b].$$

That the operator Γ has a fixed point is equivalent to P has a fixed point, and so we turn to proving that P has a fixed point.

For any positive constant k, let $z \in \mathbb{B}_k = \{z \in \Omega_0, \|z\|_{\Omega_0} \leq k\}$. We can decompose P as $P = P_1 + P_2$, where

$$(P_1 z)(t) = G_z(t) + \int_0^t S'(t-s) G_z(s)\mathrm{ds},$$

$$(P_2 z)(t) = F_z(t) + \int_0^t S'(t-s) F_z(s)\mathrm{ds}.$$

From (3.1) we know $(1 + b\|\varphi_A\|_{L^1([0,b],R^+)})[K_g M_1 + \frac{K_f b^\alpha}{\alpha\Gamma(\alpha)}(K_\gamma b + 1)M_1] < 1$. Let

$$k_0 = \frac{b\|\varphi_A\| + 1}{1 - [K_g M_1 + \frac{K_f b^\alpha}{\alpha\Gamma(\alpha)}(K_\gamma b + 1)M_1](b\|\varphi_A\| + 1)}$$

$$\cdot \{[K_g + \frac{K_f b^\alpha}{\alpha\Gamma(\alpha)}(K_\gamma b + 1)][M_1 K_\phi \|\phi\|_{\mathcal{B}}(b\|\varphi_A\| + 1) + M_2\|\phi\|_{\mathcal{B}} + 1] + K_g(\|\phi\|_{\mathcal{B}} + 1)\}.$$

We will show the operator equation $z = P_1 z + P_2 z$ has a solution in $\mathbb{B}_{k_0}$. Our proof will be divided into three steps.

Step 1. $P_1 z_1 + P_2 z_2 \in B_{k_0}$ for any $z_1, z_2 \in \mathbb{B}_{k_0}$.

Introducing the notation $C_0 = M_1 k_0 + M_1 K_\phi \|\phi\|_{\mathcal{B}}(1 + b\|\varphi_A\|_{L^1([0,b],R^+)}) + M_2\|\phi\|_{\mathcal{B}}$, $C_1 = K_g(C_0+1) + K_g(\|\phi\|_{\mathcal{B}}+1)$, $C_2 = \frac{b^\alpha K_f}{\alpha\Gamma(\alpha)}(K_\gamma b+1)(C_0+1)$, $C_3 = K_f(K_\gamma b+1)(C_0+1)$.

Obviously $(P_1 z_1)(t)$ and $(P_2 z_2)(t)$ are continuous in $t \in [0,b]$. For any $z \in \mathbb{B}_{k_0}$

$$\begin{aligned}
\|\hat{z}_t + y_t\|_{\mathcal{B}} &\leq M_1 k_0 + M_1[\|\phi(0)\|_{D(A)} + \int_0^t \|S'(t-s)\phi(0)\|\mathrm{ds}] + M_2\|\phi\|_{\mathcal{B}} \\
&\leq M_1 k_0 + M_1 K_\phi\|\phi\|_{\mathcal{B}}(1 + b\|\varphi_A\|_{L^1([0,b],R^+)}) + M_2\|\phi\|_{\mathcal{B}} \\
&= C_0.
\end{aligned}$$

The assumptions (H2) and (H4) imply

$$\begin{aligned}
\|G_{z_1}(t)\|_{D(A)} &\leq K_g(\|\hat{z}_{1t} + y_t\|_{\mathcal{B}} + 1) + K_g(\|\phi\|_{\mathcal{B}} + 1) \\
&\leq K_g(C_0+1) + K_g(\|\phi\|_{\mathcal{B}}+1) = C_1
\end{aligned}$$

and

$$\begin{aligned}
&\|F_{z_2}(t)\|_{D(A)} \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s, \hat{z}_{2s} + y_s, \int_0^s \gamma(s,\tau,\hat{z}_{2\tau} + y_\tau)\mathrm{d}\tau)\|_{D(A)}\mathrm{ds} \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}K_f[\|\hat{z}_{2s} + y_s\|_{\mathcal{B}} + \int_0^s \|\gamma(s,\tau,\hat{z}_{2\tau} + y_\tau)\|_{\mathcal{B}}\mathrm{d}\tau + 1]\mathrm{ds} \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}K_f[\|\hat{z}_{2s} + y_s\|_{\mathcal{B}} + K_\gamma b(\|\hat{z}_{2s} + y_s\|_{\mathcal{B}} + 1) + 1]\mathrm{ds} \\
\leq\ & \frac{b^\alpha K_f}{\alpha\Gamma(\alpha)}(K_\gamma b+1)(C_0+1) = C_2.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\|(P_1 z_1)(t) + (P_2 z_2)(t)\| \\
=\ & \|G_{z_1}(t) + \int_0^t S'(t-s)G_{z_1}(s)\mathrm{ds} + F_{z_2}(t) + \int_0^t S'(t-s)F_{z_2}(s)\mathrm{ds}\| \\
\leq\ & C_1 + C_1 b\|\varphi_A\|_{L^1([0,b],R^+)} + C_2 + C_2 b\|\varphi_A\|_{L^1([0,b],R^+)} \\
\leq\ & (C_1 + C_2)(1 + b\|\varphi_A\|_{L^1([0,b],R^+)}) \\
=\ & k_0,
\end{aligned}$$

which implies $\|(P_1 z_1) + (P_2 z_2)\|_{\Omega_0} \leq k_0$. That is $(P_1 z_1) + (P_2 z_2) \in \mathbb{B}_{k_0}$.

Step 2. P_1 is a contraction on $\mathbb{B}_{k_0}$.

For any $z_1, z_2 \in \mathbb{B}_{k_0}$ and $t \in [0,b]$, we get by using the assumptions (H2)and (A1)

$$\begin{aligned} \|G_{z_1}(t) - G_{z_2}(t)\|_{D(A)} &\le |L_g|_{C([0,b],R^+)} \|\hat{z}_{1t} - \hat{z}_{2t}\|_{\mathcal{B}} \\ &\le |L_g|_{C([0,b],R^+)} M_1 \|z_1 - z_2\|_{\Omega_0}. \end{aligned}$$

Therefore

$$\begin{aligned} & \|(P_1 z_1)(t) - (P_1 z_2)(t)\| \\ \le\ & \|G_{z_1}(t) - G_{z_2}(t)\|_{D(A)} + \int_0^t \|S'(t-s)[G_{z_1}(s) - G_{z_2}(s)]\| \mathrm{ds} \\ \le\ & |L_g|_{C([0,b],R^+)} M_1 \|z_1 - z_2\|_{\Omega_0} + b\|\varphi_A\|_{L^1([0,b],R^+)} |L_g|_{C([0,b],R^+)} M_1 \|z_1 - z_2\|_{\Omega_0} \\ \le\ & |L_g|_{C([0,b],R^+)} M_1 (b\|\varphi_A\|_{L^1([0,b],R^+)} + 1) \|z_1 - z_2\|_{\Omega_0}. \end{aligned}$$

That is

$$\|(P_1 z_1) - (P_1 z_2)\|_{\Omega_0} \le |L_g|_{C([0,b],R^+)} M_1 (\|\varphi_A\|_{L^1([0,b],R^+)} + 1) \|z_1 - z_2\|_{\Omega_0}.$$

According to (3.2) we see P_1 is a contraction map on $\mathbb{B}_{k_0}$.

Step 3. P_2 is a completely continuous operator.

At first, we prove that P_2 is continuous on $\mathbb{B}_{k_0}$. Let $\{z^n\} \subseteq \mathbb{B}_{k_0}$ with $z^n \to z$ on $\mathbb{B}_{k_0}$. Obviously $\hat{z}_t^n \to \hat{z}_t$ as $n \to \infty$ for $t \in [0,b]$. We have by (H_3) and $(A1)$

$$\begin{aligned} & \| \int_0^s \gamma(s,\tau,\hat{z}_\tau^n + y_\tau)\mathrm{d}\tau - \int_0^s \gamma(s,\tau,\hat{z}_\tau + y_\tau)\mathrm{d}\tau \|_{\mathcal{B}} \\ \le\ & \int_0^s \|\gamma(s,\tau,\hat{z}_\tau^n + y_\tau) - \gamma(s,\tau,\hat{z}_\tau + y_\tau)\|_{\mathcal{B}} \mathrm{d}\tau \\ \le\ & \int_0^s |L_\gamma|_{C([0,s],R^+)} \|\hat{z}_\tau^n - \hat{z}_\tau\|_{\mathcal{B}} \mathrm{d}\tau \\ \le\ & \int_0^s |L_\gamma|_{C([0,s],R^+)} M_1 \|z_\tau^n - z_\tau\|_{\Omega_0} \mathrm{d}\tau \\ \le\ & |L_\gamma|_{C([0,b],R^+)} M_1 b \|z_s^n - z_s\|_{\Omega_0} \to 0, \quad n \to \infty. \end{aligned}$$

From the assumption (H_4), we obtain

$$f(s, \hat{z}_s^n + y_s, \int_0^s \gamma(s,\tau,\hat{z}_\tau^n + y_\tau)\mathrm{d}\tau) \to f(s, \hat{z}_s + y_s, \int_0^s \gamma(s,\tau,\hat{z}_\tau + y_\tau)\mathrm{d}\tau), \quad n \to \infty.$$

By the Lebegue dominated convergence theorem, for any $\varepsilon > 0$ be given, there exists $N > 0$ such that

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,\hat{z}_s^n+y_s,\int_0^s\gamma(s,\tau,\hat{z}_\tau^n+y_\tau)\mathrm{d}\tau)\\
&-f(s,\hat{z}_s+y_s,\int_0^s\gamma(s,\tau,\hat{z}_\tau+y_\tau)\mathrm{d}\tau)\|_{D(A)}\mathrm{ds}\\
\leq\ & \varepsilon,
\end{aligned}$$

for $t \in [0, b], n \geq N$. Then for $t \in [0, b], n \geq N$, we see that

$$\begin{aligned}
& \|(P_2z^n)(t)-(P_2z)(t)\|\\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,\hat{z}_s^n+y_s,\int_0^s\gamma(s,\tau,\hat{z}_\tau^n+y_\tau)\mathrm{d}\tau)\\
&-f(s,\hat{z}_s+y_s,\int_0^s\gamma(s,\tau,\hat{z}_\tau+y_\tau)\mathrm{d}\tau)\|_{D(A)}\mathrm{ds}\\
&+\int_0^t\|S'(t-s)\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}[f(\tau,\hat{z}_\tau^n+y_\tau,\int_0^\tau\gamma(\tau,\xi,\hat{z}_\xi^n+y_\xi)\mathrm{d}\xi)\\
&-f(\tau,\hat{z}_\tau+y_\tau,\int_0^\tau\gamma(\tau,\xi,\hat{z}_\xi+y_\xi)\mathrm{d}\xi)]\mathrm{d}\tau\|\mathrm{ds}\\
\leq\ & \varepsilon+\varepsilon\int_0^t\varphi_A(t-s)\mathrm{ds}\\
\leq\ & (1+b\|\varphi_A\|_{L^1([0,b],R^+)})\varepsilon,
\end{aligned}$$

which implies that $P_2z^n \to P^2z$ as $n \to \infty$. So P_2 is continuous on $[0, b]$.

Next, we will show that $\{P_2z, z \in \mathbb{B}_{k_0}\}$ is equicontinuous on $[0, b]$.

Let $\hat{f}_z(s) = f(s, \hat{z}_s + y_s, \int_0^s \gamma(s, \tau, \hat{z}_\tau + y_\tau)\mathrm{d}\tau)$. By the assumptions $(H_3) \sim (H_4)$, we have

$$\begin{aligned}
& \|\hat{f}_z(s)\|_{D(A)}\\
=\ & \|f(s,\hat{z}_s+y_s,\int_0^s\gamma(s,\tau,\hat{z}_\tau+y_\tau)\mathrm{d}\tau)\|_{D(A)}\\
\leq\ & K_f[\|\hat{z}_s+y_s\|_{\mathcal{B}}+K_\gamma b(\|\hat{z}_s+y_s\|_{\mathcal{B}}+1)+1]\\
\leq\ & K_f(K_\gamma b+1)(C_0+1)=C_3.
\end{aligned}$$

For $t \in [0, b]$ and $h > 0$ such that $t + h \in [0, b]$, we get

$$\|F_z(t+h)-F_z(t)\|_{D(A)}$$

$$
\begin{aligned}
&= \|\frac{1}{\Gamma(\alpha)}\int_0^{t+h}(t+h-s)^{\alpha-1}\hat{f}_z(s)\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_0^{t}(t-s)^{\alpha-1}\hat{f}_z(s)\mathrm{d}s\|_{D(A)}\\
&\le \frac{1}{\Gamma(\alpha)}\int_0^{t}[(t-s)^{\alpha-1}-(t+h-s)^{\alpha-1}]\|\hat{f}_z(s)\|_{D(A)}\mathrm{d}s\\
&\quad +\frac{1}{\Gamma(\alpha)}\int_t^{t+h}(t+h-s)^{\alpha-1}\|\hat{f}_z(s)\|_{D(A)}\mathrm{d}s\\
&\le \frac{C_3}{\alpha\Gamma(\alpha)}[t^\alpha-(t+h)^\alpha+h^\alpha]+\frac{C_3}{\alpha\Gamma(\alpha)}h^\alpha \le \frac{2C_3}{\alpha\Gamma(\alpha)}h^\alpha.
\end{aligned}
$$

We infer that there exists $0<\delta<b$ such that $\|F_z(t+h)-F_z(t)\|_{D(A)} \le \varepsilon$ and $h\|\varphi_A\|_{L^1([t,t+h],R^+)} \le \varepsilon$ for all $z \in \mathbb{B}_{k_0}$ and every $0<h<\delta$. Then we have

$$
\begin{aligned}
&\|(P_2z)(t+h)-(P_2z)(t)\|\\
\le\ & \|F_z(t+h)-F_z(t)\|_{D(A)}\\
&+\|\int_0^{t+h}S'(t+h-s)F_z(s)\mathrm{d}s-\int_0^{t}S'(t-s)F_z(s)\mathrm{d}s\|\\
\le\ & \varepsilon+\int_0^{h}\|S'(t+h-s)F_z(s)\|\mathrm{d}s+\int_0^{t}\|S'(t-s)[F_z(s+h)-F_z(s)]\|\mathrm{d}s\\
\le\ & \varepsilon+C_2h\|\varphi_A\|_{L^1([t,t+h],R^+)}+\varepsilon b\|\varphi_A\|_{L^1([0,b],R^+)}\\
\le\ & (b\|\varphi_A\|_{L^1([0,b],R^+)}+C_2+1)\varepsilon.
\end{aligned}
$$

which implies $\{P_2z, z\in\mathbb{B}_{k_0}\}$ is right equicontinuous at $t\in(0,b)$. Using the same argument, we can get $\{P_2z, z\in\mathbb{B}_{k_0}\}$ is left equicontinuous at $t\in(0,b]$ and the right equicontinuous at zero. Thus $\{P_2z, z\in\mathbb{B}_{k_0}\}$ is equicontinuous on $[0,b]$.

Let $t\in(0,b]$ and $0<\varepsilon<min\{t,1\}$. From Bochner integral (see [23, Lemma 2.1.3]), for $x\in\mathbb{B}_{k_0}$, we have

$$
\begin{aligned}
F_z(t) &= \frac{1}{\Gamma(\alpha)}\int_0^{\varepsilon}(t-\varepsilon)^{\alpha-1}\hat{f}_z(s)\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_\varepsilon^{t}(t-s)^{\alpha-1}\hat{f}_z(s)\mathrm{d}s\\
&\in B_{\frac{C_3\varepsilon^\alpha}{\alpha\Gamma(\alpha)}}(0,E)+\frac{1}{\Gamma(\alpha)}(t-\varepsilon)\overline{co((t-s)^{\alpha-1}\hat{f}_z(s): s\in[\varepsilon,t])},
\end{aligned}
$$

where $co(S)$ denote the convex hull of a set S.

On the other hand, since $S'(.)\in C([0,b],E)$, for the ε, there exist numbers $0=s_0<s_1<......<s_{n+1}=t$ such that $|s_i-s_{i+1}|\le\varepsilon$ for all $i=0,1,...n$ and $\|S'(s)-S'(s_j)\|_{C([0,b],E)}\le\frac{\varepsilon}{1+b}, s\in[s_j,s_{j+1}], j=1,...n$.

Then for $t \in (0, b]$, $z \in \mathbb{B}_{k_0}$, we get

$$\int_0^t S'(t-s)F_z(s)\mathrm{ds} = \int_0^{s_1} S'(s)F_z(t-s)\mathrm{ds} + \sum_{i=1}^{n} \int_{s_i}^{s_{i+1}} [S'(s) - S'(s_i)]F_z(t-s)\mathrm{ds} + \sum_{i=1}^{n} S'(s_i) \int_{s_i}^{s_{i+1}} F_z(t-s)\mathrm{ds}.$$

Noting that

$$\| \int_0^{s_1} S'(s)F_z(t-s)\mathrm{ds}\| \leq C_2 \|\varphi_A\|_{L^1([0,\varepsilon],R^+)},$$

$$\| \sum_{i=1}^{n} \int_{s_i}^{s_{i+1}} [S'(s) - S'(s_i)]F_z(t-s)\mathrm{ds}\| \leq \frac{bC_2\varepsilon}{b+1},$$

we obtain that

$$\{(P_2 z)(t), z \in B_{k_0}\} \subset C_\varepsilon + K_\varepsilon,$$

where $K_\varepsilon \subset E$ is a compact set and $\mathrm{Diam}(C_\varepsilon) = \frac{C_3\varepsilon^\alpha}{\alpha\Gamma(\alpha)} + C_2\|\varphi_A\|_{L^1([0,\varepsilon],R^+)} + \frac{bC_2\varepsilon}{b+1} \to 0$ as $\varepsilon \to 0$. This proves that the set $\{(P_2 z)(t), z \in \mathbb{B}_{k_0}\}$ is relatively compact on $[0, b]$.

Hence P_2 is a completely continuous operator.

Using Lemma (2.3), we obtain that $P_1 + P_2$ has a fixed point on $\mathbb{B}_{k_0}$, which means that the Cauchy problem (1.2) has a mild solution. □

3.2. Uniqueness of solutions

We now discuss the uniqueness of mild solutions by using the Banach contraction principle.

Theorem 3.2. *Assume the condition (i) in Theorem 3.1 and* $(H_2) \sim (H_5)$ *hold. Also* $L_g(0)M_1 < 1$. *Then the Cauchy problem* (1.2) *has a unique mild solution on* $(-\infty, b]$ *for some* $0 < b \leq a$.

Proof. Since $L_g(0)M_1 < 1$, $|L_g|_{C([0,c],R^+)}M_1 \to L_g(0)M_1$ and $\|\varphi_A\|_{L^1([0,c],R^+)} \to 0$ as $c \to 0$, there exists $0 < b \leq a$ such that

$$[|L_g|_{C([0,b],R^+)} + \frac{b^\alpha |L_f|_{C([0,b],R^+)}}{\alpha\Gamma(\alpha)}(K_\gamma b + 1)]M_1(1 + b\|\varphi_A\|_{L^1([0,b],R^+)}) < 1. \quad (3.4)$$

For $k>0, x\in\Omega$ and $\|x\|_{C(J,E)}\le k$, we define an operator $\Phi:\Omega\to\Omega$ by

$$(\Phi x)(t)=\begin{cases} G(t)+F(t)+\int_0^t S'(t-s)[G(s)+F(s)]\mathrm{ds}, & t\in J,\\ \phi(t) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad , & t\in(-\infty,0],\end{cases}$$

where $G(t)$ and $F(t)$ are as Definition 2.3. From Theorem 3.1, we see that Φ is well defined.

Similar as Theorem 3.1, we also define the operator $P:\Omega_0\to\Omega_0$ by

$$(Pz)(t)=G_z(t)+F_z(t)+\int_0^t S'(t-s)[G_z(s)+F_z(s)]\mathrm{ds},\quad t\in[0,b],$$

where $\Omega_0, G_z(t)$ and $F_z(t)$ are as Theorem 3.1.

It is clear that the operator Φ is a contraction on Ω is equivalent to P is a contraction on Ω_0. In fact, for $z_1,z_2\in\Omega_0$ and $t\in[0,b]$,

$$\begin{aligned}\|(Pz_1)(t)-(Pz_2)(t)\| \ \le\ & \|G_{z_1}(t)-G_{z_2}(t)\|_{D(A)}+\|F_{z_1}(t)-F_{z_2}(t)\|_{D(A)}\\ &+\int_0^t\|S'(t-s)[G_{z_1}(s)-G_{z_2}(s)]\|\mathrm{ds}\\ &+\int_0^t\|S'(t-s)[F_{z_1}(s)-F_{z_2}(s)]\|\mathrm{ds}.\end{aligned}$$

From the assumption (H_2) and (A1), we obtain

$$\begin{aligned}\|G_{z_1}(t)-G_{z_2}(t)\|_{D(A)} &= \|g(t,\hat{z}_{1t}+y_t)-g(t,\hat{z}_{2t}+y_t)\|_{D(A)}\\ &\le |L(g)|_{C([0,b],R^+)}\|\hat{z}_{1t}-\hat{z}_{2t}\|_{\mathcal{B}}\\ &\le |L(g)|_{C([0,b],R^+)}M_1\|z_1-z_2\|_{\Omega_0}.\end{aligned}$$

By (H_3) and (H_5), we see that

$$\begin{aligned}&\|F_{z_1}(t)-F_{z_2}(t)\|_{D(A)}\\ \le\ & \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|f(s,\hat{z}_{1s}+y_s,\int_0^s\gamma(s,\tau,\hat{z}_{1\tau}+y_\tau)\mathrm{d}\tau)\\ &-f(s,\hat{z}_{2s}+y_s,\int_0^s\gamma(s,\tau,\hat{z}_{2\tau}+y_\tau)\mathrm{d}\tau)\|_{D(A)}\mathrm{ds}\\ \le\ & \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|L_f|_{C([0,s],R^+)}[\|\hat{z}_{1s}-\hat{z}_{2s}\|_{\mathcal{B}}\\ &+\int_0^s\|\gamma(s,\tau,\hat{z}_{1\tau}+y_\tau)-\gamma(s,\tau,\hat{z}_{2\tau}+y_\tau)\|_{\mathcal{B}}\mathrm{d}\tau]\mathrm{ds}\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|L_f|_{C([0,t],R^+)}[M_1\|z_1-z_2\|_{\Omega_0}\\
&\quad +\int_0^s K_\gamma\|\hat{z}_{1s}-\hat{z}_{2s}\|_{\mathcal{B}}\mathrm{d}\tau]\mathrm{ds}\\
&\leq \frac{b^\alpha|L_f|_{C([0,b],R^+)}}{\alpha\Gamma(\alpha)}[M_1\|z_1-z_2\|_{\Omega_0}+K_\gamma bM_1\|z_1-z_2\|_{\Omega_0}]\\
&\leq \frac{b^\alpha|L_f|_{C([0,b],R^+)}}{\alpha\Gamma(\alpha)}M_1(K_\gamma b+1)\|z_1-z_2\|_{\Omega_0}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\quad \|(Pz_1)(t)-(Pz_2)(t)\|\\
&\leq [|L_g|_{C([0,b],R^+)}M_1+\frac{b^\alpha|L_f|_{C([0,b],R^+)}}{\alpha\Gamma(\alpha)}M_1(K_\gamma b+1)]\|z_1-z_2\|_{\Omega_0}\\
&\quad +[|L_g|_{C([0,b],R^+)}M_1+\frac{b^\alpha|L_f|_{C([0,b],R^+)}}{\alpha\Gamma(\alpha)}M_1(K_\gamma b+1)]b\|\varphi_A\|_{L^1([0,b],R^+)}\|z_1-z_2\|_{\Omega_0}\\
&\leq [|L_g|_{C([0,b],R^+)}+\frac{b^\alpha|L_f|_{C([0,b],R^+)}}{\alpha\Gamma(\alpha)}(K_\gamma b+1)]M_1(b\|\varphi_A\|_{L^1([0,b],R^+)}+1)\|z_1-z_2\|_{\Omega_0}.
\end{aligned}
$$

According to (3.4), we see that P is a contraction on Ω_0. By the Banach contraction principle, we know the Cauchy problem (1.2) has a unique mild solution. □

3.3 Continuous dependence of solutions

In the next result we investigate the continuous dependence of the mild solutions on the initial condition.

Theorem 3.3. *For each $\phi_1,\phi_2\in\mathcal{B}$, suppose that there exists a constant K_* such that $\|\phi_1(0)-\phi_2(0)\|_{D(A)}\leq K_*\|\phi_1-\phi_2\|_{\mathcal{B}}$ and suppose the conditions in Theorem 3.2 are satisfied. Then for the corresponding mild solutions $x_1(t),x_2(t)$ of the problem*

$$
\begin{cases} D^\alpha[x(t)-g(t,x_t)]=Ax(t)+f(t,x_t,\int_0^t\gamma(t,s,x_s)\mathrm{ds}), & t\in[0,a],\\ x(t)=\phi_i(t) & ,\quad t\in(-\infty,0], \end{cases} \tag{3.5}
$$

we have the inequality

$$
\|x_{1t}-x_{2t}\|_{\mathcal{B}}\leq(\frac{M_1C_5}{1-C_6}+C_4)\|\phi_1-\phi_2\|_{\mathcal{B}} \tag{3.6}
$$

where $t \in [0,b]$, $C_4 = M_1 K_*(1 + b\|\varphi_A\|_{L^1(C([0,b],R^+))}) + M_2$,

$$\begin{aligned} C_5 &= (1 + b\|\varphi_A\|_{L^1(C([0,b],R^+))})[|L_g|_{C([0,b],R^+)}(C_4 + 1) \\ &\quad + \frac{b^\alpha |L_f|_{C([0,b],R^+)}(|L_\gamma|_{C([0,b],R^+)}b + 1)}{\alpha\Gamma(\alpha)} C_4], \end{aligned}$$

$$C_6 = (1+b\|\varphi_A\|_{L^1(C([0,b],R^+))})[|L_g|_{C([0,b],R^+)} + \frac{b^\alpha |L_f|_{C([0,b],R^+)}(|L_\gamma|_{C([0,b],R^+)}b + 1)}{\alpha\Gamma(\alpha)}]M_1.$$

From Theorem 3.2, we know that $C_6 < 1$.

Proof. From Theorem 3.2, each of the solutions of the Cauchy problem (3.5) exists and is unique on $(-\infty, b]$ for some $0 < b \le a$.

We rewrite $x_i(.)$ as $x_i(t) = \hat{z}_i(t) + y_i(t) (i = 1, 2), 0 \le t \le b$, where $\hat{z}_i(t), y_i(t)$ are defined as Theorem 3.1. Furthermore, $x_{it} = \hat{z}_{it} + y_{it}, 0 \le t \le b$ and $z_i(.)$ satisfies

$$z_i(t) = G_{z_i}(t) + F_{z_i}(t) + \int_0^t S'(t-s)[G_{z_i}(s) + F_{z_i}(s)]\mathrm{ds},$$

where $G_{z_i}(t) = g(t, \hat{z}_{it} + y_{it}) - g(0, \phi_i)$,
$F_{z_i}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \hat{z}_{is} + y_{is}, \int_0^s \gamma(s, \tau, \hat{z}_{i\tau} + y_{i\tau})\mathrm{d}\tau)\mathrm{ds}$.

Consequently, for $t \in [0,b]$

$$\begin{aligned} z_1(t) - z_2(t) &= [G_{z_1}(t) - G_{z_2}(t)] + [F_{z_1}(t) - F_{z_2}(t)] \\ &\quad + \int_0^t S'(t-s)[G_{z_1}(s) - G_{z_2}(s)]\mathrm{ds} \\ &\quad + \int_0^t S'(t-s)[F_{z_1}(s) - F_{z_2}(s)]\mathrm{ds}. \end{aligned}$$

Using the assumptions, we obtain

$$\begin{aligned} &\|G_{z_1}(t) - G_{z_2}(t)\|_{D(A)} \\ \le\ &\|g(t, \hat{z}_{1t} + y_{1t}) - g(t, \hat{z}_{2t} + y_{2t})\|_{D(A)} + \|g(0,\phi_1) - g(0,\phi_2)\|_{D(A)} \\ \le\ &|L_g|_{C([0,b],R^+)}[\|\hat{z}_{1t} - \hat{z}_{2t}\|_{\mathcal{B}} + \|y_{1t} - y_{2t}\|_{\mathcal{B}} + \|\phi_1 - \phi_2\|_{\mathcal{B}}] \\ \le\ &|L_g|_{C([0,b],R^+)}[M_1\|z_1 - z_2\|_{\Omega_0} + M_1 \sup_{0\le s\le t} \|y_1(s) - y_2(s)\| \\ &+ M_2\|\phi_1 - \phi_2\|_{\mathcal{B}} + \|\phi_1 - \phi_2\|_{\mathcal{B}}] \\ \le\ &|L_g|_{C([0,b],R^+)}[M_1\|z_1 - z_2\|_{\Omega_0} + M_1(\|\phi_1(0) - \phi_2(0)\|_{D(A)} \\ &+ \int_0^t \|S'(t-s)(\phi_1(0) - \phi_2(0))\|_{D(A)}\mathrm{ds}) + M_2\|\phi_1 - \phi_2\|_{\mathcal{B}} + \|\phi_1 - \phi_2\|_{\mathcal{B}}] \\ \le\ &|L_g|_{C([0,b],R^+)}[M_1\|z_1 - z_2\|_{\Omega_0} + (C_4 + 1)\|\phi_1 - \phi_2\|_{\mathcal{B}}], \end{aligned}$$

and

$$
\begin{aligned}
& \|F_{z_1}(t)-F_{z_2}(t)\|_{D(A)} \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,\hat{z}_{1s}+y_{1s},\int_0^s \gamma(s,\tau,\hat{z}_{1\tau}+y_{1\tau})\mathrm{d}\tau) \\
& -f(s,\hat{z}_{2s}+y_{2s},\int_0^s \gamma(s,\tau,\hat{z}_{2\tau}+y_{2\tau})\mathrm{d}\tau)\|_{D(A)} \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|L_f|_{C([0,s],R^+)}[\|\hat{z}_{1s}+y_{1s}-\hat{z}_{2s}-y_{2s}\|_{\mathcal{B}} \\
& +\int_0^s |L_\gamma|_{C([0,\tau],R^+)}\|\hat{z}_{1\tau}+y_{1\tau}-\hat{z}_{2\tau}-y_{2\tau}\|_{\mathcal{B}}\mathrm{d}\tau]\mathrm{ds} \\
\leq\ & \frac{b^\alpha|L_f|_{C([0,b],R^+)}(|L_\gamma|_{C([0,b],R^+)}b+1)}{\alpha\Gamma(\alpha)}\|\hat{z}_{1t}+y_{1t}-\hat{z}_{2t}-y_{2t}\|_{\mathcal{B}} \\
\leq\ & \frac{b^\alpha|L_f|_{C([0,b],R^+)}(|L_\gamma|_{C([0,b],R^+)}b+1)}{\alpha\Gamma(\alpha)}(M_1\|z_1-z_2\|_{\Omega_0}+C_4\|\phi_1-\phi_2\|_{\mathcal{B}}).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \|z_1(t)-z_2(t)\| \\
\leq\ & \|G_{z_1}(t)-G_{z_2}(t)\|_{D(A)}+\|F_{z_1}(t)-F_{z_2}(t)\|_{D(A)} \\
& +\|\int_0^t S'(t-s)[G_{z_1}(s)-G_{z_2}(s)]\mathrm{ds}\|+\|\int_0^t S'(t-s)[F_{z_1}(s)-F_{z_2}(s)]\mathrm{ds}\| \\
\leq\ & C_6\|z_1-z_2\|_{\Omega_0}+C_5\|\phi_1-\phi_2\|_{\mathcal{B}}.
\end{aligned}
$$

Thus

$$\|z_1-z_2\|_{\Omega_0}\leq\frac{C_5}{1-C_6}\|\phi_1-\phi_2\|_{\mathcal{B}}.$$

On the other hand, since $x_{it}=\hat{z}_{it}+y_{it}, 0\leq t\leq b$, we have

$$
\begin{aligned}
& \|x_{1t}-x_{2t}\|_{\mathcal{B}} \\
\leq\ & \|\hat{z}_{1t}-\hat{z}_{2t}\|_{\mathcal{B}}+\|y_{1t}-y_{2t}\|_{\mathcal{B}} \\
\leq\ & M_1\|z_1-z_2\|_{\Omega_0}+C_4\|\phi_1-\phi_2\|_{\mathcal{B}} \\
\leq\ & \frac{M_1C_5}{1-C_6}\|\phi_1-\phi_2\|_{\mathcal{B}}+C_4\|\phi_1-\phi_2\|_{\mathcal{B}} \\
\leq\ & (\frac{M_1C_5}{1-C_6}+C_4)\|\phi_1-\phi_2\|_{\mathcal{B}}.
\end{aligned}
$$

Therefore the inequality (3.6) is held. $\square$

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