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# A class of BVPs for nonlinear fractional differential equations with $p$-Laplacian operator*

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**Abstract** In this paper, we study a class of integral boundary value problems for nonlinear differential equations of fractional order with $p$-Laplacian operator. Under some suitable assumptions, a new result on the existence of solutions is obtained by using a standard fixed point theorem. An example is included to show the applicability of our result.

**Keywords** Fractional differential equations; Boundary valued problems; $p$-Laplacian operator; Schaefer's fixed point theorem.

**Mathematics Subject Classification:** 26A33, 34K37

## 1 Introduction

In this paper, we consider the following boundary value problems (BVPs for short) for nonlinear fractional differential equations with $p$-Laplacian operator:

$$\begin{cases} D^{\beta}_{0^+}\phi_p(D^{\alpha}_{0^+}u(t)) = f(t, u(t), D^{\alpha}_{0^+}u(t)), \quad t \in [0, 1], \\ u(0) = \mu \displaystyle\int_0^1 u(s)ds + \lambda u(\xi), \\ D^{\alpha}_{0^+}u(0) = kD^{\alpha}_{0^+}u(\eta), \end{cases} \tag{1}$$

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where $\phi_p(s) = |s|^{p-2}s,\ p > 1$, $0 < \alpha,\ \beta \leq 1,\ 1 < \alpha+\beta \leq 2,\ \mu,\ \lambda,\ k \in \mathbb{R},\ \xi,\ \eta \in [0,1]$, $D_{0+}^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha$ and $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function.

It is easy to know that if $\lambda = k = \pm 1,\ \xi = \eta = 1,\ \mu = 0$, then the Eq.(1) reduces to periodic and anti-periodic BVP, respectively. Furthermore, the nonlinear operator $D_{0+}^{\beta}\phi_p(D_{0+}^{\alpha})$ reduces to the linear operator $D_{0+}^{\beta}D_{0+}^{\alpha}$ when $p = 2$ and the additive index law $D_{0+}^{\beta}D_{0+}^{\alpha}u(t) = D_{0+}^{\beta+\alpha}u(t)$ holds under some reasonable constraints on the function $u(t)$ (see [10, 17, 19]).

Fractional calculus is a generalization of ordinary differentiation and integration on an arbitrary order that can be non-integer. More and more researchers have found that fractional differential equations play important roles in many research areas, such as physics, chemical technology, population dynamics, biotechnology and economics (see [10, 13, 15, 17, 19]). A fractional derivative arises from many physical processes, such as propagations of mechanical waves in viscoelastic media (see[14]), a non-Markovian diffusion process with memory (see[16]), charge transport in amorphous semiconductors (see[23]), etc. Moreover, phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are also described by fractional differential equations (see[7–9]). For instance, Pereira et al. (see[18]) considered the following fractional Van der Pol equation

$$D^{\alpha}x(t) + \lambda(x^2(t) - 1)x'(t) + x(t) = 0, \quad 1 < \alpha < 2, \tag{2}$$

where $D^{\alpha}$ is the fractional derivative of order $\alpha$ and $\lambda$ is a control parameter which reflects the degree of nonlinearity of the system. Eq.(2) is obtained by substituting a fractance for the capacitance in the nonlinear RLC circuit model.

The turbulent flow in a porous medium is a fundamental mechanics phenomenon. For studying this type of phenomena, Leibenson (see[12]) introduced the $p$-Laplacian equation as follows

$$(\phi_p(u'(t)))' = f(t, u(t), u'(t)), \tag{3}$$

where $\phi_p(s) = |s|^{p-2}s,\ p > 1$. Obviously, $\phi_p$ is invertible and $(\phi_p)^{-1} = \phi_q$, where $q > 1$ such that $1/p + 1/q = 1$.

In the past few decades, many important results with certain boundary value conditions related to Eq.(3) had been obtained (see [3, 11, 29] and the references therein). However, the research of BVPs for fractional $p$-Laplacian equations has just begun in recent years. For example, T. Chen et al. [5] investigated the existence of solutions of the boundary value problem for fractional $p$-Laplacian equation with the following form

$$\begin{cases} D_{0+}^{\beta}\phi_p(D_{0+}^{\alpha}u(t)) = f(t, u(t), D_{0+}^{\alpha}u(t)), \quad t \in [0,1], \\ D_{0+}^{\alpha}u(0) = D_{0+}^{\alpha}u(1) = 0, \end{cases}$$

where $0 < \alpha,\ \beta \leq 1,\ 1 < \alpha + \beta \leq 2,\ D_{0+}^{\alpha}$ is the Caputo fractional derivative, and $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

Later, in [6], T. Chen and W. Liu studied an anti-periodic boundary value problem for the fractional $p$-Laplacian equation:

$$\begin{cases} D_{0+}^{\beta}\phi_p(D_{0+}^{\alpha}u(t)) = f(t, u(t)), \qquad t \in [0,1], \\ u(0) = -u(1),\ D_{0+}^{\alpha}u(0) = -D_{0+}^{\alpha}u(1), \end{cases}$$

where $0 < \alpha,\ \beta \leq 1,\ 1 < \alpha + \beta \leq 2,\ D_{0+}^{\alpha}$ is a Caputo fractional derivative, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. Under certain nonlinear growth conditions of the nonlinearity, the existence result was obtained by using Schaefer's fixed point theorem.

In [28], J. Wang and H. Xiang have considered the following $p$-Laplacian fractional boundary value problem:

$$\begin{cases} D_{0+}^{\gamma}\phi_p(D_{0+}^{\alpha}u(t)) = f(t, u(t)), \qquad 0 < t < 1, \\ u(0) = 0,\ u(1) = au(\xi),\ D_{0+}^{\alpha}u(0) = 0,\ D_{0+}^{\alpha}u(1) = bD_{0+}^{\alpha}u(\eta)\ , \end{cases}$$

where $1 < \gamma,\ \alpha \leq 2,\ 0 \leq a,\ b \leq 1,\ 0 < \xi,\ \eta < 1$, and $D_{0+}^{\alpha}$ is the standard Riemann-Liouville fractional differential operator of order $\alpha$. Using upper and lower solutions method, they get some existence results on the existence of positive solution.

Since the $p$-Laplacian operator and fractional calculus arises from many applied fields, such as turbulent filtration in porous media, blood flow problems, rheology, modeling of viscoplasticity, material science, it is worth studying the fractional $p$-Laplacian equations. As far as we know, there are relatively few results on BVPs for fractional $p$-Laplacian equations, and no paper is concerned with the existence results for fractional $p$-Laplacian BVPs (1). In this context, we study the problem (1) with integral boundary condition.

Integral boundary conditions have various applications in applied fields such as underground water flow, blood flow problems, chemical engineering, thermo-elasticity, population dynamics, and so on. For a detailed description of the integral boundary conditions, we refer the reader to some recent papers [1, 2, 4, 21, 24–26] and the references therein.

The rest of this paper is organized as follows: In section 2, we present some material to prove our main results. In section 3, by applying a standard fixed point principle, we prove the existence of solutions for nonlinear fractional differential equations with $p$-Laplacian operator. Finally, an example is given to illustrate the main result in section 4.

## 2 Preliminaries and lemmas

Firstly, we recall the following known definitions, which can be found, for instance, in [10, 19].

**Definition 2.1.** The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right side integral is pointwise defined on $(0, +\infty)$.

**Definition 2.2.** The Riemann-Liouville derivative of order $\alpha > 0$ for a function $f$ : $[0,\ \infty) \to \mathbb{R}$ can be written as

$${}^{L}\mathbf{D}_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{f(s)}{(x-s)^{\alpha-n+1}} ds,$$

where $n$ is the smallest integer greater than $\alpha$.

**Definition 2.3.** The Caputo fractional derivative of order $\alpha > 0$ for a function $f : [0,\ \infty) \to \mathbb{R}$ can be written as

$$D_{0+}^{\alpha} f(x) = {}^{L}\mathbf{D}_{0+}^{\alpha} \left[ f(x) - \sum_{k=0}^{n-1} \frac{x^k}{k!} f^{(k)}(0) \right],$$

where $n$ is the smallest integer greater than $\alpha$.

**Remark 2.4.** If $f \in AC^n[0,\ \infty)$, then

$$D_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(s)}{(x-s)^{\alpha+1-n}} ds,$$

where $n$ is the smallest integer greater than $\alpha$. Furthermore, the Caputo derivative of a constant is equal to zero.

For $n \in \mathbb{N}^+ := \{1, 2, \cdots\}$, $AC^n[a,b]$ denotes the space of real-valued functions $f(x)$ which have continuous derivatives up to order $n-1$ on $[a,b]$ such that $f^{(n-1)}(x) \in AC[a,b]$:

$$AC^n[a,b] = \{f : [a,b] \to R \ \ and \ \ f^{(n-1)}(x) \in AC[a,b]\},$$

where $AC[a,b]$ be the space of functions $f$ which are absolutely continuous on $[a,b]$.

**Lemma 2.5** ([10], p96). Let $\alpha > 0$. Assume that $u \in AC^n[0,1]$, $D^{\alpha}_{0+}u \in L(0,1)$. Then the following equality holds:

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) + c_0 + c_1 t + ... + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}, i = 0, 1, \cdots, n-1$; here $n$ is the smallest integer greater than or equal to $\alpha$.

**Definition 2.6** ([20]). Let $\Omega \subset \mathbb{R}^n$ be a domain. A function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is said to satisfy the Carathéodory conditions if

(i) $f(x, u)$ is a continuous function of $u$ for almost all $x \in \Omega$;

(ii) $f(x, u)$ is a measurable function of $x$ for all $u \in \mathbb{R}^m$.

Given a function $f$ satisfying the Carathéodory conditions and a function $u : \Omega \to \mathbb{R}^m$, we can define another function by composition

$$N(u)(x) := f(x, u(x)).$$

The composition operator $N$ is called a *Nemytskii operator*.

**Lemma 2.7** ([22, 27], Schaefer's fixed point theorem). Let $X$ be a Banach space and $T : X \to X$ be a completely continuous operator. If the set $E = \{u \in X \mid u = \rho T u, \ 0 < \rho \leq 1\}$ is bounded, then $T$ has at least a fixed point in $X$.

# 3 Main results

In this section, we deal with the existence of solutions of the problem (1). Define $||u||_\infty = \max_{t\in[0,1]} |u(t)|$ and let $X = \{u : \ u \in C[0,1]$ and $D^{\alpha}_{0+}u \in C[0,1]\}$ with the norm $||u|| = \max\{||u||_\infty, \ ||D^{\alpha}_{0+}u||_\infty\}$. By means of the linear functional analysis theory, we can prove that $X$ is a Banach space.

**Lemma 3.1.** Let $h(t) \in C[0,1]$, $\alpha \in (0,1]$, $\mu, \ \lambda \in \mathbb{R}$. If $\mu + \lambda \neq 1$, then $u \in AC[0,1]$ is a solution of the following fractional differential equation:

$$\begin{cases} D_{0+}^{\alpha}u(t) = h(t), \quad t \in [0,1], \\ u(0) = \mu \displaystyle\int_0^1 u(s)ds + \lambda u(\xi), \quad \xi \in [0,1], \end{cases} \tag{4}$$

if and only if $u \in C[0,1]$ is a solution of the fractional integral equation

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_0^1 \frac{\mu(1-s)^{\alpha}}{(1-\mu-\lambda)\Gamma(\alpha+1)} h(s)ds \\ &\quad + \int_0^{\xi} \frac{\lambda(\xi-s)^{\alpha-1}}{(1-\mu-\lambda)\Gamma(\alpha)} h(s)ds. \end{aligned} \tag{5}$$

**Proof.** Assume that $u \in AC[0,1]$ satisfies (4). For any $t \in [0,1]$, by Lemma 2.5, the first equality of (4) can be written as

$$u(t) = I_{0+}^{\alpha}h(t) + c_0 = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + c_0,$$

substituting it into the second equality of (4), we have

$$\begin{aligned} c_0 &= \mu\left(\int_0^1 \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} h(\tau)d\tau + c_0\right)ds\right) + \lambda\left(\int_0^{\xi} \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + c_0\right) \\ &= \frac{\mu}{\Gamma(\alpha)} \int_0^1 h(\tau) \int_{\tau}^1 (s-\tau)^{\alpha-1} ds\ d\tau + \mu c_0 + \frac{\lambda}{\Gamma(\alpha)} \int_0^{\xi} (\xi-s)^{\alpha-1} h(s)ds + \lambda c_0 \\ &= (\mu+\lambda)c_0 + \frac{\mu}{\Gamma(\alpha+1)} \int_0^1 (1-\tau)^{\alpha} h(\tau)d\tau + \frac{\lambda}{\Gamma(\alpha)} \int_0^{\xi} (\xi-s)^{\alpha-1} h(s)ds, \end{aligned}$$

thus, we get

$$c_0 = \int_0^1 \frac{\mu(1-s)^{\alpha}}{(1-\mu-\lambda)\Gamma(\alpha+1)} h(s)ds + \int_0^{\xi} \frac{\lambda(\xi-s)^{\alpha-1}}{(1-\mu-\lambda)\Gamma(\alpha)} h(s)ds.$$

That is

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_0^1 \frac{\mu(1-s)^{\alpha}}{(1-\mu-\lambda)\Gamma(\alpha+1)} h(s)ds + \int_0^{\xi} \frac{\lambda(\xi-s)^{\alpha-1}}{(1-\mu-\lambda)\Gamma(\alpha)} h(s)ds.$$

On the other hand, assume that $u$ satisfies (5). Using the fact that $D_{0+}^{\alpha}$ is the left inverse of $I_{0+}^{\alpha}$, we get (4), which completes our proof.

**Lemma 3.2.** Let $\varphi(t) \in C[0,1]$, $\alpha,\ \beta \in (0,1]$, $\mu,\ \lambda,\ k \in \mathbb{R}$ such that $k \neq 1$, $\mu + \lambda \neq 1$, then a function $u \in \{u \mid u \in AC[0,1] \text{ and } D_{0+}^{\alpha}u \in AC[0,1]\}$ is a solution of the following fractional differential equation:

$$\begin{cases} D_{0+}^{\beta}\phi_p(D_{0+}^{\alpha}u(t)) = \varphi(t), \quad t \in [0,1], \\ u(0) = \mu \displaystyle\int_0^1 u(s)ds + \lambda u(\xi), \\ D_{0+}^{\alpha}u(0) = kD_{0+}^{\alpha}u(\eta), \ \xi, \ \eta \in [0,1], \end{cases} \tag{6}$$

if and only if $u \in C[0,1]$ is a solution of the fractional integral equation

$$\begin{aligned} u(t) &= I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}\varphi(t) + F_1\varphi(t)) + F_2\varphi(t) \\ &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\bigg(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\varphi(\tau)d\tau + F_1\varphi(s)\bigg)ds + F_2\varphi(t) \end{aligned} \tag{7}$$

where, for any $t \in [0,1]$

$$\begin{aligned} F_1\varphi(t) &\equiv \frac{\phi_p(k)}{1-\phi_p(k)}\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)ds, \\ F_2\varphi(t) &\equiv \int_0^1 \frac{\mu(1-s)^{\alpha}}{(1-\mu-\lambda)\Gamma(\alpha+1)}\phi_q\bigg(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\varphi(\tau)d\tau + F_1\varphi(s)\bigg)ds \\ &\quad + \int_0^{\xi} \frac{\lambda(\xi-s)^{\alpha-1}}{(1-\mu-\lambda)\Gamma(\alpha)}\phi_q\bigg(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\varphi(\tau)d\tau + F_1\varphi(s)\bigg)ds. \end{aligned}$$

**Proof.** At first, assume that $u$ is a solution of (6). For any $t \in [0,1]$, by Lemma 2.5, the first equality of (6) can be written as

$$\phi_p(D_{0+}^{\alpha}u(t)) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)ds + \bar{c}_0.$$

Then $\phi_p(D_{0+}^{\alpha}u(0)) = \bar{c}_0$. Combining the fact that $D_{0+}^{\alpha}u(0) = kD_{0+}^{\alpha}u(\eta)$, then we have

$$\bar{c}_0 = \frac{\phi_p(k)}{1-\phi_p(k)}\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)ds =: F_1\varphi(t).$$

Hence,

$$\phi_p(D_{0+}^{\alpha}u(t)) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)ds + F_1\varphi(t).$$

Then the equation (6) can be written as follows

$$\begin{cases} D_{0+}^{\alpha}u(t) = \phi_q\big(\displaystyle\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)ds + F_1\varphi(t)\big), \quad t \in [0,1], \\ u(0) = \mu \displaystyle\int_0^1 u(s)ds + \lambda u(\xi), \quad \xi \in [0,1]. \end{cases}$$

When we set $h(t)=\phi_q\big(\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)ds+F_1\varphi(t)\big)$, then by Lemma 3.1, we have (7).

Conversely, we can obtain that the solution of (7) is the solution of the BVP (6) by calculation. This completes the proof.

Now, we consider the fractional differential equation (1). Let us define an operator $T : X \to X$ as following by

$$
\begin{aligned}
Tu(t) &= I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}Nu(t)+F_1Nu(t))+F_2Nu(t)\\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\bigg(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}Nu(\tau)d\tau+\frac{\phi_p(k)}{1-\phi_p(k)}\int_0^\eta \frac{(\eta-\tau)^{\beta-1}}{\Gamma(\beta)}Nu(\tau)d\tau\bigg)ds\\
&\quad+\int_0^1 \frac{\mu(1-s)^{\alpha}}{(1-\mu-\lambda)\Gamma(\alpha+1)}\\
&\quad\times\phi_q\bigg(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}Nu(\tau)d\tau+\frac{\phi_p(k)}{1-\phi_p(k)}\int_0^\eta \frac{(\eta-\tau)^{\beta-1}}{\Gamma(\beta)}Nu(\tau)d\tau\bigg)ds\\
&\quad+\int_0^\xi \frac{\lambda(\xi-s)^{\alpha-1}}{(1-\mu-\lambda)\Gamma(\alpha)}\\
&\quad\times\phi_q\bigg(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}Nu(\tau)d\tau+\frac{\phi_p(k)}{1-\phi_p(k)}\int_0^\eta \frac{(\eta-\tau)^{\beta-1}}{\Gamma(\beta)}Nu(\tau)d\tau\bigg)ds,
\end{aligned}
$$

where $N$ is a Nemytskii operator defined by

$$Nu(t)=f(t,u(t),D_{0+}^{\alpha}u(t)),\quad t\in[0,1]. \tag{8}$$

Clearly, a fixed point of the operator $T$ is a solution of the problem (1).

**Lemma 3.3.** The operator $T : X \to X$ is completely continuous.

**Proof.** At first, we show that $T : X \to X$ is continuous.

Let $\{u_n\} \subseteq X$ be a sequence with $u_n \to u$ in $X$. We will show that $\|Tu_n - Tu\| \to 0$. Since $f : J \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, it is easy to see $\lim_{n\to\infty} Nu_n(t) = Nu(t)$ uniformly for $t \in [0,1]$. By the continuity of $\phi_q$, we have

$$
\begin{aligned}
\lim_{n\to\infty} Tu_n(t) &= \lim_{n\to\infty}\big(I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}Nu_n(t)+F_1Nu_n(t))+F_2Nu_n(t)\big)\\
&= I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}Nu(t)+F_1Nu(t))+F_2Nu(t)=Tu(t)\ \text{ uniformly for } t\in[0,1],
\end{aligned}
$$

thus, $\|Tu_n - Tu\|_\infty \to 0$ as $n\to\infty$ in $X$. Moreover,

$$\begin{aligned}\lim_{n\to\infty} D^\alpha_{0+}Tu_n(t) &= \lim_{n\to\infty} D^\alpha_{0+}\big(I^\alpha_{0+}\phi_q(I^\beta_{0+}Nu_n(t)+F_1Nu_n(t))+F_2Nu_n(t)\big)\\ &= \lim_{n\to\infty}\phi_q(I^\beta_{0+}Nu_n(t)+F_1Nu_n(t))\\ &= \phi_q(I^\beta_{0+}Nu(t)+F_1Nu(t)) = D^\alpha_{0+}Tu(t) \ \ \text{uniformly for } t\in[0,1],\end{aligned}$$

thus $\|D^\alpha_{0+}Tu_n - D^\alpha_{0+}Tu\|_\infty \to 0$ as $n\to\infty$ in $X$. And as a consequence, we have $\|Tu_n - Tu\|\to 0$ in $X$. This shows that $T: X\to X$ is continuous.

Next, we prove that $T(\overline{\Omega})$ and $D^\alpha_{0+}T(\overline{\Omega})$ are relatively compact in $C[0,1]$ respectively.

Let $\Omega\subset X$ be an open bounded subset, then for any $u\in\overline{\Omega}$, there exists $M_0>0$ such that $||u||\le M_0$. Since $f$ is a continuous function, there exists $M_1>0$ such that $|f(t,u(t),D^\alpha_{0+}u(t))|\le M_1$, $t\in[0,1]$, $u\in\overline{\Omega}$. Then

$$\begin{aligned}|I^\beta_{0+}Nu(t)+F_1Nu(t)| &\le \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}|f(s,u(s),D^\alpha_{0+}u(s))|ds\\ &\quad +\left|\frac{\phi_p(k)}{1-\phi_p(k)}\right|\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}|f(s,u(s),D^\alpha_{0+}u(s))|ds\\ &\le \frac{M_1}{\Gamma(\beta+1)}+\left|\frac{\phi_p(k)}{1-\phi_p(k)}\right|\frac{M_1\eta^\beta}{\Gamma(\beta+1)} := L,\quad t\in[0,1],\end{aligned}$$

$$\begin{aligned}|F_2Nu(t)| &= \bigg|\int_0^1 \frac{\mu(1-s)^\alpha}{(1-\mu-\lambda)\Gamma(\alpha+1)}\phi_q\Big(I^\beta_{0+}Nu(s)+F_1Nu(s)\Big)ds\\ &\quad +\int_0^\xi \frac{\lambda(\xi-s)^{\alpha-1}}{(1-\mu-\lambda)\Gamma(\alpha)}\phi_q\Big(I^\beta_{0+}Nu(s)+F_1Nu(s)\Big)ds\bigg|\\ &\le \int_0^1\frac{|\mu|(1-s)^\alpha}{|1-\mu-\lambda|\Gamma(\alpha+1)}L^{q-1}ds+\int_0^\xi\frac{|\lambda|(\xi-s)^{\alpha-1}}{|1-\mu-\lambda|\Gamma(\alpha)}L^{q-1}ds\\ &= \frac{|\mu|L^{q-1}}{|1-\mu-\lambda|\Gamma(\alpha+2)}+\frac{|\lambda|\xi^\alpha L^{q-1}}{|1-\mu-\lambda|\Gamma(\alpha+1)} := L_1,\quad t\in[0,1].\end{aligned}$$

For $t\in[0,1]$ and $u\in\overline{\Omega}$, we have

$$\begin{aligned}|Tu(t)| &= |I^\alpha_{0+}\phi_q(I^\beta_{0+}Nu(t)+F_1Nu(t))+F_2Nu(t)|\\ &\le \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q(|I^\beta_{0+}Nu(s)+F_1Nu(s)|)ds+|F_2Nu(t)|\\ &\le \frac{L^{q-1}}{\Gamma(\alpha+1)}+L_1,\end{aligned}$$

$$\begin{aligned}|D_{0^+}^{\alpha}Tu(t)| &= |D_{0^+}^{\alpha}\big(I_{0^+}^{\alpha}\phi_q(I_{0^+}^{\beta}Nu(t)+F_1Nu(t))+F_2Nu(t)\big)| \\ &= |\phi_q(I_{0^+}^{\beta}Nu(t)+F_1Nu(t))| \\ &\leq \phi_q(|I_{0^+}^{\beta}Nu(t)+F_1Nu(t)|)\leq L^{q-1}.\end{aligned}$$

Then, we get that $||Tu||_{\infty}\leq \frac{L^{q-1}}{\Gamma(\alpha+1)}+L_1$ and $||D_{0^+}^{\alpha}Tu||_{\infty}\leq L^{q-1}$. Thus, $T(\overline{\Omega})$ and $D_{0^+}^{\alpha}T(\overline{\Omega})$ are uniformly bounded in $C[0,1]$ respectively.

For $0\leq t_1<t_2\leq 1, u\in\overline{\Omega}$, we have

$$\begin{aligned}|Tu(t_2)-Tu(t_1)| &= \big|I_{0^+}^{\alpha}\phi_q(I_{0^+}^{\beta}Nu(t)+F_1Nu(t))|_{t=t_2}-I_{0^+}^{\alpha}\phi_q(I_{0^+}^{\beta}Nu(t)+F_1Nu(t))|_{t=t_1}\big| \\ &= \bigg|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\phi_q\big(I_{0^+}^{\beta}Nu(s)+F_1Nu(s)\big)ds \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\phi_q\big(I_{0^+}^{\beta}Nu(s)+F_1Nu(s)\big)ds\bigg| \\ &\leq \frac{L^{q-1}}{\Gamma(\alpha)}\bigg\{\int_0^{t_1}|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}|ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\bigg\} \\ &= \frac{L^{q-1}}{\Gamma(\alpha+1)}[t_1^{\alpha}-t_2^{\alpha}+2(t_2-t_1)^{\alpha}],\end{aligned}$$

Since $t^{\alpha}$ is uniformly continuous on $[0,1]$, we obtain that $Tu(t)$ is uniformly continuous on $[0,1]$. Thus, we get that $T(\overline{\Omega})\subset C[0,1]$ is equicontinuous. Besides, for $0\leq t_1<t_2\leq 1, u\in\overline{\Omega}$, we also have

$$\begin{aligned}&\big|(I_{0^+}^{\beta}Nu(t)+F_1Nu(t))|_{t=t_2}-(I_{0^+}^{\beta}Nu(t)+F_1Nu(t))|_{t=t_1}\big| \\ = &\ |\int_0^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}f(s,u(s),D_{0^+}^{\alpha}u(s))ds-\int_0^{t_1}\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}f(s,u(s),D_{0^+}^{\alpha}u(s))ds| \\ \leq &\ \frac{M_1}{\Gamma(\beta)}\int_0^{t_1}|(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}|ds+\int_{t_1}^{t_2}(t_2-s)^{\beta-1}ds \\ = &\ \frac{M_1}{\Gamma(\beta+1)}[t_1^{\beta}-t_2^{\beta}+2(t_2-t_1)^{\beta}].\end{aligned}$$

Thus $I_{0^+}^{\beta}Nu(t)+F_1Nu(t)$ is uniformly continuous on $[0,1]$. This, together with the uniformly continuity of $\phi_q(s)$ on $[-D,D],\ D>0$, yields that $D_{0^+}^{\alpha}Tu(t)=\phi_q(I_{0^+}^{\beta}Nu(t)+F_1Nu(t))$ is uniformly continuous. Thus, we get that $D_{0^+}^{\alpha}T(\overline{\Omega})\subset X$ is equicontinuous too.

Hence, $T(\overline{\Omega})$ and $D_{0^+}^{\alpha}T(\overline{\Omega})$ are relatively compact in $C[0,1]$ respectively.

Finally, we are going to show that $T$ is relatively compact in $X$.

Assume that $\{u_n\} \subset X$ is a bounded sequence. By using the Arzelá-Ascoli theorem, we can select a subsequence $\{Tu_{n_k}\}$ of $\{Tu_n\}$ which is convergent with respect to the norm $||u||_\infty$ in $C[0,1]$. That is

$$\lim_{k\to\infty} ||Tu_{n_k} - Tu||_\infty = 0.$$

Then, by using the Arzelá-Ascoli theorem again, we can select a subsequence $\{D_{0^+}^\alpha Tu_{n_i}\} \subseteq \{D_{0^+}^\alpha Tu_{n_k}\}$ which is convergent with respect to the norm $||D_{0^+}^\alpha u||_\infty$ in $C[0,1]$. Thus, we have $\lim\limits_{i\to\infty} ||D_{0^+}^\alpha Tu_{n_i} - D_{0^+}^\alpha Tu||_\infty = 0$.

Hence, $\lim\limits_{i\to\infty} ||Tu_{n_i} - Tu|| = 0$ in $X$, which shows that $T$ is relatively compact in $X$.

As a consequence of above discussion, the operator $T : X \to X$ is completely continuous. The proof is completed.

**Theorem 3.4.** Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Assume that

($\mathrm{H}_1$) there exist nonnegative functions $a,\ b,\ c \in C[0,1]$ such that

$$|f(t,u,v)| \le a(t) + b(t)|u|^{p-1} + c(t)|v|^{p-1}, \quad t \in [0,1],\ (u,v) \in \mathbb{R}^2;$$

($\mathrm{H}_2$) there exist constants $\mu,\ \lambda,\ k \in \mathbb{R}$, such that $k \neq 1,\ \mu + \lambda \neq 1$ and $A^{p-1}B(||b||_\infty + ||c||_\infty) < 1$, where

$$A = \frac{1}{\Gamma(\alpha+1)} + \frac{|\mu| + |\lambda|(1+\alpha)\xi^\alpha}{|1-\mu-\lambda|\Gamma(\alpha+2)}, \quad B = \frac{1}{\Gamma(\beta+1)}\big(1 + \eta^\beta\Big|\frac{\phi_p(k)}{1-\phi_p(k)}\Big|\big).$$

Then the problems (1) has at least one solution.

**Proof.** Set $\Omega = \{u \in X \mid u = \rho Tu,\ \rho \in (0,1]\}$. Now we are going to show that the set $\Omega$ is bounded. From ($\mathrm{H}_1$), we have

$$\begin{aligned}
|F_1 Nu(t)| &\le \Big|\frac{\phi_p(k)}{1-\phi_p(k)}\Big| \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} |f(s,u(s),D^\alpha u(s))| ds \\
&\le \Big|\frac{\phi_p(k)}{1-\phi_p(k)}\Big| \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \Big(a(s) + b(s)||u||_\infty^{p-1} + c(s)||D^\alpha u||_\infty^{p-1}\Big) ds \\
&\le \frac{\eta^\beta}{\Gamma(\beta+1)} \Big|\frac{\phi_p(k)}{1-\phi_p(k)}\Big| \Big(||a||_\infty + (||b||_\infty + ||c||_\infty)||u||^{p-1}\Big), \quad \forall t \in [0,1],
\end{aligned}$$

which, together with the monotonicity of $s^{q-1}$, yields that

$$
\begin{aligned}
|F_2 Nu(t)| &\leq \int_0^1 \frac{|\mu|(1-s)^{\alpha}}{|1-\mu-\lambda|\Gamma(\alpha+1)}\left|\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}Nu(\tau)d\tau + F_1 Nu(t)\right|^{q-1} ds \\
&\quad + \int_0^{\xi} \frac{|\lambda|(\xi-s)^{\alpha-1}}{|1-\mu-\lambda|\Gamma(\alpha)}\left|\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}Nu(\tau)d\tau + F_1 Nu(t)\right|^{q-1} ds \\
&\leq \frac{|\mu|+|\lambda|(1+\alpha)\xi^{\alpha}}{|1-\mu-\lambda|\Gamma(\alpha+2)}\left[\frac{1}{\Gamma(\beta+1)}(1+\eta^{\beta}\left|\frac{\phi_p(k)}{1-\phi_p(k)}\right|)\right]^{q-1} \\
&\quad \times\Big(||a||_\infty + (||b||_\infty + ||c||_\infty)||u||^{p-1}\Big)^{q-1}, \quad \forall\, t\in[0,1].
\end{aligned}
$$

For $u \in \Omega$, we get $u(t) = \rho T u(t)$. Thus, we obtain that

$$
\begin{aligned}
|u(t)| &\leq |I_{0+}^{\alpha}\big(\phi_q(I_{0+}^{\beta}Nu(t) + F_1 Nu(t))\big)| + |F_2 Nu(t)| \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\bigg(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}|Nu(t)| + |F_1 Nu(t)|\bigg)ds + |F_2 Nu(t)| \\
&\leq \bigg(\frac{1}{\Gamma(\alpha+1)} + \frac{|\mu|+|\lambda|(1+\alpha)\xi^{\alpha}}{|1-\mu-\lambda|\Gamma(\alpha+2)}\bigg)\left[\frac{1}{\Gamma(\beta+1)}(1+\eta^{\beta}\left|\frac{\phi_p(k)}{1-\phi_p(k)}\right|)\right]^{q-1} \\
&\quad \times\Big(||a||_\infty + (||b||_\infty + ||c||_\infty)||u||^{p-1}\Big)^{q-1} \\
&\leq AB^{q-1}\Big(||a||_\infty + (||b||_\infty + ||c||_\infty)||u||^{p-1}\Big)^{q-1}, \quad \forall t\in[0,1],
\end{aligned}
$$

$$
\begin{aligned}
|D_{0+}^{\alpha}u(t)| &= |\phi_q\big(I_{0+}^{\beta}Nu(t) + F_1 Nu(t)\big)| \\
&\leq \phi_q\bigg(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}|Nu(t)| + |F_1 Nu(t)|\bigg)ds \\
&\leq \left[\frac{1}{\Gamma(\beta+1)}(1+\eta^{\beta}\left|\frac{\phi_p(k)}{1-\phi_p(k)}\right|)\right]^{q-1}\Big(||a||_\infty + (||b||_\infty + ||c||_\infty)||u||^{p-1}\Big)^{q-1} \\
&\leq AB^{q-1}\Big(||a||_\infty + (||b||_\infty + ||c||_\infty)||u||^{p-1}\Big)^{q-1}, \quad \forall t\in[0,1],
\end{aligned}
$$

where for $\alpha \in (0,1]$, $0 < \Gamma(\alpha+1) \leq 1$, $A = \frac{1}{\Gamma(\alpha+1)} + \frac{|\mu|+|\lambda|(1+\alpha)\xi^{\alpha}}{|1-\mu-\lambda|\Gamma(\alpha+2)} \geq 1$. Thus, we have

$$
||u|| \leq AB^{q-1}\Big(||a||_\infty + (||b||_\infty + ||c||_\infty)||u||^{p-1}\Big)^{q-1}, \quad u\in\Omega.
$$

In view of condition ($H_2$), and from the above inequality, we can see that there exists a constant $M > 0$ such that

$$
||u||^{p-1} \leq \Big(1 - A^{p-1}B(||b||_\infty + ||c||_\infty)\Big)^{-1}||a||_\infty A^{p-1}B < M, \quad u\in\Omega.
$$

This shows that the set $\Omega$ is bounded. By Lemma 3.3, the operator $T : X \to X$ is completely continuous. As a consequence of Schaefer's fixed point theorem, $T$ has at least a fixed point which is a solution of the Eq.(1). The proof is completed.

# 4 Example

In this section, we give an example to illustrate the usefulness of our main results.

**Example 4.1.** Consider the following boundary value problem consisting of the equation

$$\begin{cases} D_{0+}^{\frac{2}{3}}\big(\phi_3(D_{0+}^{\frac{1}{2}})u(t)\big) = \dfrac{7t^2}{\cos t} + \dfrac{u^2(t)}{(9+e^t)} + \dfrac{\sin t}{5+|u(t)|}\phi_3(D_{0+}^{\frac{1}{2}}u(t)),\ t \in [0,1], \\ u(0) = \dfrac{1}{2}u(1) + \displaystyle\int_0^1 u(s)ds, \qquad D_{0+}^{\frac{1}{2}}u(0) = -5D_{0+}^{\frac{1}{2}}u(\frac{1}{2}). \end{cases} \tag{9}$$

It is not difficult to verify that problem (9) is of the form (1). For the particular case $p = 3,\ q = \frac{3}{2},\ \alpha = \frac{1}{2},\ \beta = \frac{2}{3},\ \lambda = \frac{1}{2},\ \xi = \mu = 1,\ k = -5,\ \eta = \frac{1}{2}$ and

$$f(t,u,v) = \frac{7t^2}{\cos t} + \frac{1}{(9+e^t)}u^2 + \frac{\sin t}{5+|u|}|v|v, \quad t \in [0,1].$$

Moreover, there exist nonnegative functions $a(t) = \dfrac{7t^2}{\cos t},\ b(t) = \dfrac{1}{(9+e^t)},\ c(t) = \dfrac{1}{4}\sin t,\ t \in [0,1]$ such that the condition $(H_1)$ holds. Through some calculation, we can get that $||a||_\infty \approx 12.9557,\ ||b||_\infty = \frac{1}{10},\ ||c||_\infty = \frac{1}{5}\sin 1 \approx 0.1683$ and

$$\begin{aligned} A &= \frac{1}{\Gamma(\frac{1}{2}+1)} + \frac{1+\frac{1}{2}\times(1+\frac{1}{2})\times 1^{1/2}}{|1-1-\frac{1}{2}|\ \Gamma(\frac{1}{2}+2)} = \frac{1}{\Gamma(\frac{3}{2})} + \frac{7}{2\Gamma(\frac{5}{2})}, \\ B &= \frac{1}{\Gamma(\frac{2}{3}+1)}(1+(\frac{1}{2})^{\frac{2}{3}}\left|\frac{\phi_3(-5)}{1-\phi_3(-5)}\right|) = \frac{1}{\Gamma(\frac{5}{3})}(1+2^{-\frac{2}{3}}\times\frac{25}{26}), \\ A^{p-1}&B(||b||_\infty + ||c||_\infty) = A^2B\big(\frac{1}{100}+\frac{1}{25}\sin^2 1\big) \approx 0.7748 < 1. \end{aligned}$$

Obviously, the problem (9) satisfies all assumptions of Theorem 3.4. Hence, it has at least one solution.

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