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Existence results for Caputo type fractional differential equations with four-point nonlocal fractional integral boundary conditions

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Abstract

This paper presents some existence and uniqueness results for a boundary value problem of fractional differential equations of order $\alpha \in (1, 2]$ with four-point nonlocal fractional integral boundary conditions. Our results are based on some standard tools of fixed point theory and nonlinear alternative of Leray-Schauder type. Some illustrative examples are also discussed.

Key words and phrases: Fractional differential equations; nonlocal integral boundary conditions; existence; fixed point theorems.
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1 Introduction

In recent years, a variety of problems involving differential equations of fractional order have been investigated by several researchers with the sphere of study ranging from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. Fractional differential equations appear naturally in a number of fields such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. It is also found that the differential equations of arbitrary order provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. With these features, the fractional order models become more realistic and practical than the classical integer-order models. For details and examples, see ([1]-[5]). The recent development of the subject can be found in a series of papers ([8]-[21]). Recently, Ahmad and Nieto [22] studied a problem involving Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions.

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In this paper, motivated by [22], we discuss the existence of solutions for a four-point nonlocal boundary value problem for Caputo type fractional differential equations of order $\alpha \in (1, 2]$ with fractional integral boundary conditions given by

$$\begin{cases} {}^cD^\alpha x(t) = f(t, x(t)), \quad 1 < \alpha \le 2, \quad t \in [0, 1], \\ x(0) = aI^{\alpha-1}x(\eta) = a\displaystyle\int_0^\eta \frac{(\eta - s)^{\alpha-2}}{\Gamma(\alpha - 1)} x(s)ds, \\ x(1) = bI^{\alpha-1}x(\sigma) = b\displaystyle\int_0^\sigma \frac{(\sigma - s)^{\alpha-2}}{\Gamma(\alpha - 1)} x(s)ds, \quad 0 < \eta < \sigma < 1, \end{cases} \tag{1}$$

where a and b are arbitrary real constants, ${}^cD^\alpha$ denotes the Caputo fractional derivative of order α and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

2 Preliminaries

Let us recall some basic definitions on fractional calculus ([1]-[3]).

Definition 2.1 *The Riemann-Liouville fractional integral of order q for a continuous function $g : [0, \infty) \to \mathbb{R}$, is defined as*

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t - s)^{1-q}} ds, \quad q > 0,$$

provided the integral exists.

Definition 2.2 *For an at least $n-$times continuously differentiable function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as*

$${}^cD^q g(t) = \frac{1}{\Gamma(n - q)} \int_0^t (t - s)^{n-q-1} g^{(n)}(s)ds, \quad n - 1 < q < n, n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q.

Lemma 2.1 *For any $y \in C[0, 1]$, the unique solution of the linear fractional boundary value problem*

$$\begin{cases} {}^cD^\alpha x(t) = y(t), \quad 1 < \alpha \le 2, \quad t \in [0, 1], \\ x(0) = aI^{\alpha-1}x(\eta) = a\displaystyle\int_0^\eta \frac{(\eta - s)^{\alpha-2}}{\Gamma(\alpha - 1)} x(s)ds, \\ x(1) = bI^{\alpha-1}x(\sigma) = b\displaystyle\int_0^\sigma \frac{(\sigma - s)^{\alpha-2}}{\Gamma(\alpha - 1)} x(s)ds, \quad 0 < \eta < \sigma < 1, \end{cases} \tag{2}$$

is

$$x(t) = I^\alpha y(t) + (\Delta_1 - \Delta_4 t)I^{2\alpha-1}y(\eta) + (\Delta_2 + \Delta_3 t)[bI^{2\alpha-1}y(\sigma) - I^\alpha y(1)], \tag{3}$$

where

$$\begin{aligned}
&\Delta_1 = \frac{a}{\Delta}\Big(1 - \frac{b\sigma^{\alpha}}{\Gamma(\alpha+1)}\Big), \quad \Delta_2 = \frac{a\eta^{\alpha}}{\Delta\Gamma(\alpha+1)},\\
&\Delta_3 = \frac{1}{\Delta}\Big(1 - \frac{a\eta^{\alpha-1}}{\Gamma(\alpha)}\Big), \quad \Delta_4 = \frac{a}{\Delta}\Big(1 - \frac{b\sigma^{\alpha-1}}{\Gamma(\alpha)}\Big),\\
&\Delta = \Big(1 - \frac{b\sigma^{\alpha}}{\Gamma(\alpha+1)}\Big)\Big(1 - \frac{a\eta^{\alpha-1}}{\Gamma(\alpha)}\Big) + \frac{a\eta^{\alpha}}{\Gamma(\alpha+1)}\Big(1 - \frac{b\sigma^{\alpha-1}}{\Gamma(\alpha)}\Big).
\end{aligned} \tag{4}$$

Proof. For some constants $c_0, c_1 \in \mathbb{R}$ and $1 < \alpha \leq 2$, the general solution of the equation ${}^cD^{\alpha}x(t) = y(t)$ can be written as

$$x(t) = I^{\alpha}y(t) + c_0 + c_1 t, \tag{5}$$

Using the boundary conditions for the problem (2) in (5), we find that

$$c_0 = \frac{1}{\Delta}\Big\{a\Big(1 - \frac{b\sigma^{\alpha}}{\Gamma(\alpha+1)}\Big)I^{2\alpha-1}y(\eta) + \frac{a\eta^{\alpha}}{\Gamma(\alpha+1)}\Big(bI^{2\alpha-1}y(\sigma) - I^{\alpha}y(1)\Big)\Big\},$$

$$c_1 = \frac{1}{\Delta}\Big\{\Big(1 - \frac{a\eta^{\alpha-1}}{\Gamma(\alpha)}\Big)\Big(bI^{2\alpha-1}y(\sigma) - I^{\alpha}y(1)\Big) - a\Big(1 - \frac{b\sigma^{\alpha-1}}{\Gamma(\alpha)}\Big)I^{2\alpha-1}y(\eta)\Big\}.$$

Substituting the values of c_0 and c_1 in (5), we get (3). This completes the proof. □

3 Existence results

Let $\mathcal{C} = C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$.

To define a fixed point problem equivalent to (1), we make use of Lemma 2.1 to define an operator $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ as

$$\begin{aligned}
(\mathcal{T}x)(t) \quad &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds + (\Delta_1 - \Delta_4 t)\int_0^{\eta}\frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}f(s,x(s))ds\\
&+(\Delta_2 + \Delta_3 t)\Big(b\int_0^{\sigma}\frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}f(s,x(s))ds - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds\Big),
\end{aligned} \tag{6}$$

where $\Delta_1, \Delta_2, \Delta_3$ and Δ_4 are given by (4).

Observe that the problem (1) has solutions if and only if the operator $\mathcal{T}$ has fixed points.

For the forthcoming analysis, we set

$$\kappa = \frac{1}{\Gamma(\alpha+1)} + \frac{(|\Delta_1| + |\Delta_4|)\eta^{2\alpha-1}}{\Gamma(2\alpha)} + (|\Delta_2| + |\Delta_3|)\Big(\frac{\sigma^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{1}{\Gamma(\alpha+1)}\Big). \tag{7}$$

Theorem 3.1 *Assume that $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying the assumption*

(H_1) $|f(t,x)-f(t,y)|\le L|x-y|,\quad \forall t\in[0,1]\quad ,x,y\in\mathbb{R},\ L>0.$

Then the problem (1) has a unique solution if $L<1/\kappa$, where κ is given by (7).

Proof. Let us set $\sup_{t\in[0,1]}|f(t,0)|=M$ and define $B_r=\{x\in\mathcal{C}:|x|\le r\}$, where

$$r\ge M\kappa/(1-L\kappa).$$

As a first step, we show that $\mathcal{T}B_r\subset B_r$. For $x\in B_r$, we have

$$\begin{aligned}
\|\mathcal{T}x\| &\le \sup_{t\in[0,1]}\Big\{\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(|f(s,x(s))-f(s,0)|+|f(s,0)|)ds\\
&+|\Delta_1-\Delta_4 t|\int_0^\eta\frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}(|f(s,x(s))-f(s,0)|+|f(s,0)|)ds\\
&+|\Delta_2+\Delta_3 t|(b\int_0^\sigma\frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)})(|f(s,x(s))-f(s,0)|+|f(s,0)|)ds\Big\}\\
&\le (Lr+M)\Big\{\frac{1}{\Gamma(\alpha+1)}+\frac{(|\Delta_1|+|\Delta_4|)\eta^{2\alpha-1}}{\Gamma(2\alpha)}+(|\Delta_2|+|\Delta_3|)\\
&\times\Big(\frac{\sigma^{2\alpha-1}}{\Gamma(2\alpha)}+\frac{1}{\Gamma(\alpha+1)}\Big)\Big\}\\
&=(Lr+M)\kappa\le r,
\end{aligned}$$

where we have used (7). Thus, $\mathcal{T}B_r\subset B_r$. Now for $x,y\in\mathcal{C}$, we obtain

$$\begin{aligned}
\|\mathcal{T}x-\mathcal{T}y\| &\le \sup_{t\in[0,1]}\Big\{\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s))-f(s,y(s))|ds\\
&+|\Delta_1-\Delta_4 t|\int_0^\eta\frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}|f(s,x(s))-f(s,y(s))|ds\\
&+|\Delta_2+\Delta_3 t|(b\int_0^\sigma\frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)})|f(s,x(s))-f(s,y(s))|ds\Big\}\\
&\le L\Big\{\frac{1}{\Gamma(\alpha+1)}+\frac{(|\Delta_1|+|\Delta_4|)\eta^{2\alpha-1}}{\Gamma(2\alpha)}+(|\Delta_2|+|\Delta_3|)\\
&\times\Big(\frac{\sigma^{2\alpha-1}}{\Gamma(2\alpha)}+\frac{1}{\Gamma(\alpha+1)}\Big)\Big\}\|x-y\|\\
&=L\kappa\|x-y\|.
\end{aligned}$$

Since $L<1/\kappa$, therefore, the operator $\mathcal{T}$ is a contraction. Hence, by Banach's contraction principle, the problem (1) has a unique solution. This completes the proof. □

Our next existence result is based on Krasnoselskii's fixed point theorem [23].

Theorem 3.2 *Let N be a closed convex and nonempty subset of a Banach space M. Let A, B be the operators such that (i) $Ax + By \in N$ whenever $x, y \in N$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z \in N$ such that $z = Az + Bz$.*

Theorem 3.3 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumptions (H_1) and*

(H_2) $|f(t,x)| \leq \mu(t)$ $,\forall (t,x) \in [0,1] \times \mathbb{R}$ *where* $\mu \in C([0,1], \mathbb{R}^+)$.

If

$$L\Big\{\frac{(|\Delta_1| + |\Delta_4|)\eta^{2\alpha-1}}{\Gamma(2\alpha)} + (|\Delta_2| + |\Delta_3|)\Big(\frac{\sigma^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{1}{\Gamma(\alpha+1)}\Big)\Big\} < 1, \tag{8}$$

then the problem (1) has at least one solution on $[0,1]$.

Proof. In view of (H_2), we define $\sup_{t\in[0,1]} |\mu(t)| = \|\mu\|$, and consider the set $B_{\bar{r}} = \{x \in \mathrm{C} : \|x\| \leq \bar{r}\}$, where $\bar{r} \geq \|\mu\|\kappa$, where κ is given by (7). Introduce the operators Φ and Ψ on $B_{\bar{r}}$ as

$$\begin{aligned}
(\Phi x)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds, \\
(\Psi x)(t) &= (\Delta_1 - \Delta_4 t)\int_0^\eta \frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} f(s,x(s))ds \\
&+ (\Delta_2 + \Delta_3 t)\Big(b\int_0^\sigma \frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} f(s,x(s))ds - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds\Big).
\end{aligned}$$

For $x,\ y \in B_{\bar{r}}$, we find that

$$||\Phi x + \Psi y\| \leq \|\mu\|\kappa \leq \bar{r}.$$

Thus, $\Phi x + \Psi y \in B_{\bar{r}}$.
Notice that Ψ is a contraction mapping by the condition (8). Continuity of f implies that the operator Φ is continuous. Also, Φ is uniformly bounded on $B_{\bar{r}}$ as

$$\|\Phi x\| \leq \frac{\|\mu\|}{\Gamma(\alpha+1)}.$$

Now,we prove the compactness of the operator Φ. By the condition (H_1), let

$$\sup_{(t,x)\in[0,1]\times B_{\bar{r}}} \|f(t,x)\| = f_1.$$

Then, for $0 < t_1 < t_2 < 1$, we get

$$|(\Phi x)(t_2) - (\Phi y)(t_1)| \leq \frac{f_1}{\Gamma(\alpha)}\Big(\Big|\int_{t_1}^{t_2} (t_2-s)^{\alpha-1}ds\Big| + \Big|\int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]ds\Big|\Big),$$

which is independent of x and tends to zero as $t_2 \to t_1$. So Φ is relatively compact on $B_{\bar{r}}$. Hence, By the Arzela Ascoli theorem, Φ is compact on $B_{\bar{r}}$. Thus all the assumptions of Theorem 3.2 are satisfied. Therefore, the problem (1) has at least one solution on $[0,1]$. This completes the proof. $\square$

To prove the next existence result for the problem (1), we recall the following fixed point theorem [23].

Theorem 3.4 *Let X be a Banach space. Assume that $T : X \to X$ is a completely continuous operator and the set $V = \{u \in X \mid u = \mu T u, 0 < \mu < 1\}$ is bounded. Then T has a fixed point in X.*

Theorem 3.5 *Assume that there exists a positive constant L_1 such that $|f(t,x)| \leq L_1$ for $t \in [0,1]$, $x \in \mathbb{R}$. Then the problem (1) has at least one solution.*

Proof. As a first step, we show that the operator $\mathcal{T}$ is completely continuous. Observe that continuity of $\mathcal{T}$ follows from the continuity of f. Let $\Omega \subset \mathcal{C}$ be a bounded set. Then $\forall x \in \Omega$, we get

$$\begin{aligned}
\|(\mathcal{T}x)\| &\leq L_1 \sup_{t\in[0,1]} \Big\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + (|\Delta_1| + |\Delta_4|) \Big| \int_0^\eta \frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} ds \\
&+ (|\Delta_2| + |\Delta_3|) \Big(b \int_0^\sigma \frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} ds + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \Big) \Big\} \\
&\leq L_1 \Big\{ \frac{1}{\Gamma(\alpha+1)} + \frac{(|\Delta_1| + |\Delta_4|)\eta^{2\alpha-1}}{\Gamma(2\alpha)} + (|\Delta_2| + |\Delta_3|) \Big(\frac{\sigma^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{1}{\Gamma(\alpha+1)} \Big) \Big\} \\
&\leq \kappa L_1 = L_2,
\end{aligned}$$

where we have used (7). Furthermore,

$$\begin{aligned}
\|(\mathcal{T}x)'\| &\leq \sup_{t\in[0,1]} \Big\{ \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,x(s))| ds + |\Delta_4| \int_0^\eta \frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} |f(s,x(s))| ds \\
&+ |\Delta_3| \Big(b \int_0^\sigma \frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} |f(s,x(s))| ds + \int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} |f(s,x(s))| ds \Big) \Big\} \\
&\leq L_1 \{ \frac{1}{\Gamma(\alpha-1)} + \frac{1}{\Gamma(2\alpha)} (|\Delta_4|\eta^{2\alpha-1} + b|\Delta_3|\sigma^{2\alpha-1} + |\Delta_3)| \} = L_3.
\end{aligned}$$

Hence for $t_1, t_2 \in [0,1]$, we have

$$|(\mathcal{T}x)(t_1) - (\mathcal{T}x)(t_2)| \leq \int_{t_1}^{t_2} |(\mathcal{T}x)'(s)| ds \leq L_3(t_2 - t_1).$$

Thus, the operator $\mathcal{T}$ is equicontinuous. Hence, by Arzela-Ascoli theorem, $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

Next, we consider the set $V = \{x \in \mathcal{C} : x = \mu \mathcal{T} x,\ 0 < \mu < 1\}$. In order to show that V is bounded, let $x \in V,\ t \in [0,1]$. Then

$$\begin{aligned}(\mathcal{T}x)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds + (\Delta_1 - \Delta_4 t)\int_0^\eta \frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} f(s,x(s))ds \\ &+ (\Delta_2 + \Delta_3 t)(b\int_0^\sigma \frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} f(s,x(s))ds - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds)\end{aligned}$$

As before, it can be shown that

$$\|x\| = \sup_{t\in[0,1]} |\mu(\mathcal{T}x)(t)| \le L_1 \kappa = M_1.$$

This implies that the set V is bounded. Hence, by Theorem 3.4, it follows that the problem (1) has at least one solution on $[0,1]$. $\square$

Our final result is based on Leray-Schauder Nonlinear Alternative [24].

Lemma 3.1 *(Nonlinear alternative for single valued maps) Let E be a Banach space, M a closed, convex subset of E, U is an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is continuous, compact (that is, $F(U)$ is a relatively compact subset of C) map. Then either (i) F has a fixed point in $\overline{U}$, or (ii) there is a $u \in \partial U$, and $\lambda \in (0,1)$ with $u = \lambda F(U)$.*

Theorem 3.6 *Suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and the following conditions hold:*

(H_3) *there exist a function $p \in C([0,1], \mathbb{R}^+)$, and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t,x)| \le p(t)\psi(\|x\|)$, $\forall (t,x) \in [0,1] \times \mathbb{R}$;*

(H_4) *there exists a constant $M > 0$ such that*

$$\frac{M}{\psi(M)\|p\|\left\{\frac{1}{\Gamma(\alpha+1)} + \frac{(|\Delta_1|+|\Delta_4|)\eta^{2\alpha-1}}{\Gamma(2\alpha)} + (|\Delta_2| + |\Delta_3|)\left(\frac{\sigma^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{1}{\Gamma(\alpha+1)}\right)\right\}} > 1.$$

Then the problem (1) has at least one solution on $[0,1]$.

Proof. Consider the operator $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ defined by (6). The proof consists of several steps. As a first step, it will be shown that $\mathcal{T}$ maps bounded sets into bounded sets in

$\mathcal{C}$. For a positive number δ, let $B_\delta = \{x \in \mathcal{C} : \|x\| \le \delta\}$ be bounded set in $\mathcal{C}$, then for $x \in B_\delta$, we have

$$\begin{aligned}
\|\mathcal{T}x\| &\le \sup_{t\in[0,1]} \Big\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)\psi(\|x\|)ds \\
&+ (|\Delta_1| + |\Delta_4|)| \int_0^\eta \frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} p(s)\psi(\|x\|)ds \\
&+ (|\Delta_2| + |\Delta_3|)\Big(b\int_0^\sigma \frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} ds + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)\psi(\|x\|)ds \Big) \Big\} \\
&\le \psi(\delta)\|p\| \Big\{ \frac{1}{\Gamma(\alpha+1)} + \frac{(|\Delta_1| + |\Delta_4|)\eta^{2\alpha-1}}{\Gamma(2\alpha)} \\
&+ (|\Delta_2| + |\Delta_3|)\Big(\frac{\sigma^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{1}{\Gamma(\alpha+1)} \Big) \Big\}.
\end{aligned}$$

Next, we show that $\mathcal{T}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_\delta$, where B_δ is a bounded set of $\mathcal{C}$. Then we obtain

$$\begin{aligned}
&|(\mathcal{T}x)(t_2) - (\mathcal{T}x)(t_1)| \\
&\le \Big| \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)\psi(r)ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)\psi(r)ds \\
&-\Delta_4(t_2-t_1) \int_0^\eta \frac{(\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} p(s)\psi(r)ds \\
&+\Delta_3(t_2-t_1)\Big\{ b\int_0^\sigma \frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} p(s)\psi(r)ds - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)\psi(r)ds \Big\} \Big| \\
&\le \psi(r)\|p\| \Big(\Big| \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} ds \Big| + \Big| \int_0^{t_1} \frac{(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}}{\Gamma(\alpha)} ds \Big| \Big) \\
&+\Big| \Delta_4(t_2-t_1) \int_0^\eta \frac{|\eta-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} ds \Big| \\
&+|\Delta_3(t_2-t_1)| \Big(\Big| b\int_0^\sigma \frac{(\sigma-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} ds \Big| + \Big| \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds \Big| \Big).
\end{aligned}$$

Clearly, the right hand side of the above inequality tends to zero independently of $x \in B_\delta$ as $t_2 \to t_1$. Thus, it follows by the Arzela-Ascoli theorem that $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

Let x be a solution of problem (1). Then, for $t \in [0,1]$, for $\lambda \in (0,1)$, as before we have

$$\|x\| = \sup_{t\in[0,1]} \|\lambda(\mathcal{T}x)(t)\|$$

$$\leq \quad \psi(\|x\|)\|p\|\Big\{\frac{1}{\Gamma(\alpha+1)}+\frac{(|\Delta_1|+|\Delta_4|)\eta^{2\alpha-1}}{\Gamma(2\alpha)}+(|\Delta_2|+|\Delta_3|)\Big(\frac{\sigma^{2\alpha-1}}{\Gamma(2\alpha)}+\frac{1}{\Gamma(\alpha+1)}\Big)\Big\}.$$

Consequently, we have

$$\frac{\|x\|}{\psi(\|x\|)\|p\|\Big\{\frac{1}{\Gamma(\alpha+1)}+\frac{(|\Delta_1|+|\Delta_4|)\eta^{2\alpha-1}}{\Gamma(2\alpha)}+(|\Delta_2|+|\Delta_3|)\Big(\frac{\sigma^{2\alpha-1}}{\Gamma(2\alpha)}+\frac{1}{\Gamma(\alpha+1)}\Big)\Big\}}\leq 1.$$

In view of (H_4) ,there exists M such that $\|x\|\neq M$. Let us set

$$U=\{x\in\mathcal{C}([0,1],\mathbb{R}):\|x\|<M+1\}.$$

Note that the operator $\mathcal{T}:\overline{U}\to\mathcal{C}([0,1],\mathbb{R})$ is continuous and completely continuous. From the choice of U , there is no $x\in\partial U$ such that $x=\lambda\mathcal{T}(x)$ for some $\lambda\in(0,1)$. In consequence, by the nonlinear alternative of Leray-Schauder type (Lemma 3.1), we deduce that $\mathcal{T}$ has a fixed point $x\in\overline{U}$ which is a solution of the problem (1). This completes the proof. □

Example 3.1 *Consider the problem*

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t)=\dfrac{1}{(t^2+2)}\dfrac{|x|}{1+|x|}+sin^2t, \quad t\in[0,1],\\ x(0)=I^{1/2}x(1/4), \quad x(1)=I^{1/2}x(2/3), \end{cases} \tag{9}$$

where $\alpha=3/2,\ a=b=1,\ \eta=1/4\ ,\ \sigma=2/3$. *Clearly* $L=1/2$ *as*

$$|f(t,x)-f(t,y)|\leq\frac{1}{t^2+2}|x-y|\leq\frac{1}{2}|x-y|,$$

and

$$\kappa=\frac{1}{\Gamma(\alpha+1)}+\frac{(|\Delta_1|+|\Delta_4|)\eta^{2\alpha-1}}{\Gamma(2\alpha)}+(|\Delta_2|+|\Delta_3|)\Big(\frac{\sigma^{2\alpha-1}}{\Gamma(2\alpha)}+\frac{1}{\Gamma(\alpha+1)}\Big)=1.4835. \tag{10}$$

As $\kappa<1/L$, *therefore, the conclusion of Theorem 3.1 applies to the problem (9).*

Example 3.2 *Consider the fractional boundary value problem*

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t)=\dfrac{3e^{-cos^2x}cos2t}{3+sinx}, \quad t\in[0,2],\\ x(0)=I^{1/2}x(1/4), \quad x(1)=I^{1/2}x(2/3). \end{cases} \tag{11}$$

Obviously

$$|f(t,x)|=\Big|\frac{3e^{-\cos^2x}\cos 2t}{3+\sin x}\Big|\leq 3/2=L_1. \tag{12}$$

Therefore, by the conclusion of Theorem 3.5, there exists at least one solution for problem (11).

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