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A Criterion for the Global Attractivity of Scalar Population Models with Delay

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Dedicated to Prof. L. Hatvani and Prof. I. Györi, on their 60th birthday

Abstract

We establish sufficient conditions for the global attractivity of the positive equilibrium for some scalar delayed population models. As an illustration of the criterion established, a model generalizing the well-known food-limited population model with delay is studied.

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1. Introduction

In the present work, we consider scalar functional differential equations (FDEs) in the phase space $C := C([-r, 0]; \mathbb{R})$ of continuous functions from $[-r, 0]$ to $\mathbb{R}$, $r > 0$, equipped with the sup norm $|\varphi|_C = \max_{-r\leq\theta\leq 0} |\varphi(\theta)|$. Let

$$\dot{N}(t) = N(t)f(t, N_t), \quad t \geq 0, \tag{1.1}$$

be a scalar FDE, where $f : [0, \infty) \times C \to \mathbb{R}$ is a continuous function, and, as usual, N_t denotes the function in C defined by $N_t(\theta) = N(t + \theta), -r \leq \theta \leq 0$. In population dynamics, Eq. (1.1) is often taken as a model for the growth of a single population species, where $N(t)$ is the density of the population at time t, r represents the maturation period of the species and $f(t, N_t)$ is the growth function. For most models, it is natural to assume that $f(t, \cdot)$ is a decreasing function for each $t \geq 0$, where the order in C is defined by

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$\varphi \le \psi$ if and only if $\varphi(\theta) \le \psi(\theta)$ for all $\theta \in [-r, 0]$. Due to the biological interpretation of the model, we are interested only in *positive* solutions of (1.1). Therefore, together with (1.1), we consider *admissible* initial conditions $N_0 = \psi$, with $\psi \in C$ such that $\psi(\theta) \ge 0$ for $\theta \in [-r, 0)$ and $\psi(0) > 0$.

Many models of the form (1.1) can be written as

$$\dot{x}(t) = (1 + x(t))F(t, x_t), \quad t \ge 0, \tag{1.2}$$

where $F : [0, \infty) \times C \to \mathbb{R}$ is a continuous function. In fact, suppose that (1.1) has a unique positive equilibrium N_*. From a biological point of view, it is of interest to investigate the global attractivity of N_*. In order to study the stability of N_*, one can translate the equilibrium to the origin, e.g. by the change of variables $x(t) = \frac{N(t)}{N_*} - 1$, which transforms (1.1) into (1.2), with $F : I \times C \to \mathbb{R}$ defined by $F(t, \varphi) = f(t, N_*(1+\varphi))$. (Here and in what follows, for a real constant $c \in \mathbb{R}$, we use c to denote also the constant function $\varphi(\theta) = c, -r \le \theta \le 0$.) This justifies the study of the scalar FDE (1.2). Clearly, *admissible* initial conditions for (1.2) read now as $x_0 = \varphi$, where $\varphi \in C_{-1}$ and $C_{-1} = \{\varphi \in C : \varphi(\theta) \ge -1 \text{ for } \theta \in [-r, 0),\ \varphi(0) > -1\}$. Throughout this paper, even if it is not mentioned, we only consider *admissible* solutions $N(t)$ of (1.1) (or $x(t)$ of (1.2)), i.e., solutions with admissible initial conditions. An equilibrium N_* of (1.1) is said to be *globally attractive* or *globally asymptotically stable* if all admissible solutions of (1.1) tend to N_* as $t \to \infty$. Analogously, we define the global attractivity of the zero solution of (1.2).

The paper is organized as follows. In Section 2, we give sufficient conditions for the global attractivity of the equilibrium zero of (1.2) following the work in [1]. In case the growth function $F(t, \cdot)$ is non-increasing, a useful corollary is presented. In Section 3, the results in Section 2 are applied to a delayed model that generalizes the so-called "food-limited" population model with delay.

2. A criterion for global attractivity

Although we have in mind the study of population models (1.1), in this section we consider the scalar FDE

$$\dot{x}(t) = (1 + x(t))F(t, x_t), \quad t \ge 0, \tag{2.1}$$

where $F : [0, \infty) \times C \to \mathbb{R}$ is continuous, and initial conditions

$$x_0 = \varphi, \quad \text{with} \quad \varphi \in C_{-1}. \tag{2.2}$$

Uniqueness of solutions for the initial value problems (2.1)-(2.2) is assumed.

For F as in (2.1), we impose the following assumptions:

(A1) if $w : [-r, \infty) \to \mathbb{R}$ is continuous with $\lim_{t\to\infty} w(t) = w^* > -1, w^* \neq 0$, then $\int_0^\infty F(s, w_s)\, ds$ diverges.

(A2) there exists a piecewise continuous function $b : [0, \infty) \to [0, \infty)$ such that

$$-b(t)M(\varphi) \leq F(t, \varphi) \leq b(t)M(-\varphi), \quad \text{for} \quad t \geq 0, \varphi \in C_{-1}, \tag{2.3}$$

where $M(\varphi)$ is the *Yorke functional* (cf. [3] and [7]) defined by

$$M(\varphi) = \max\{0, \sup_{\theta\in[-r,0]} \varphi(\theta)\};$$

(A3) for b as in (A2), there is $T \geq 0$ such that $\int_{t-r}^t b(s)ds \leq \frac{3}{2}$, for $t \geq T$.

The following criterion for the global attractivity of the zero solution of (2.1) follows from [1].

Theorem 2.1. *Assume (A1), (A2) and (A3). Then all solutions of (2.1) with admissible initial conditions are defined and bounded away from* -1 *on* $[0, \infty)$*, and satisfy* $x(t) \to 0$ *as* $t \to \infty$*.*

Proof. From (A2), we have $F(t, \varphi) \leq b(t)$ for $t \geq 0, \varphi \in C_{-1}$, thus $F(t, \varphi)$ is bounded from above on $[0, \alpha] \times C_{-1}$, for each $\alpha > 0$. From [1, Lemma 3.1], we deduce that all solutions $x(t)$ of (2.1) with admissible initial conditions are defined on $[0, \infty)$. Since (A2) and (A3) hold, Theorem 3.3 and Lemma 3.5 in [1] imply that the solutions $x(t)$ are bounded and bounded away from -1 on $[0, \infty)$, and that $x(t) \to 0$ as $t \to \infty$ if $x(t)$ is oscillatory.

Now consider the case of a solution $x(t)$ of (2.1) with admissible initial condition that is eventually non-negative. If $x(t) \geq 0$ for $t \geq t_0 - r$, from (2.3) we have $F(t, x_t) \leq 0, t \geq t_0$. Since $1 + x(t) > 0, t \geq 0$, from (2.1) we get $\dot{x}(t) \leq 0, t \geq t_0$, i.e., $x(t)$ is eventually non-increasing. Define $c = \lim_{t\to\infty} x(t) \geq 0$. If $c > 0$, from (A1) we obtain

$$1 + x(t) = (1 + x(t_0))e^{\int_{t_0}^\infty F(s,x_s)\, ds} \to 0 \quad \text{as} \quad t \to \infty,$$

a contradiction. Thus $c = 0$. If $x(t)$ is eventually non-positive, the proof is analogous. ∎

Remark 2.1. Hypothesis (A1) is imposed to force non-oscillatory solutions to zero, as t goes to infinity, whereas (A2) and (A3) are used to deal with oscillatory solutions. Instead of (A1), a hypothesis slightly stronger was assumed in [1]. We note that (A1) was first introduced in [6], not in the setting of Eq. (2.1), but for the study of a general scalar FDE $\dot{x}(t) = F(t, x_t)$.

We now analyse an FDE having the form (2.1), with F non-increasing on $\varphi \in C_{-1}$, and depending only on one discrete delay. To be more precise, let F have the form $F(t, \varphi) = b(t)g(t, \varphi(-r))$ where $g(t, \cdot)$ is non-increasing on $x \in [-1, \infty)$ for each $t \geq 0$.

Corollary 2.2. *Consider*

$$\dot{x}(t) = (1 + x(t))b(t)g(t, x(t - r)), \quad t \geq 0, \tag{2.4}$$

where $b : [0, \infty) \to [0, \infty), g : [0, \infty) \times [-1, \infty) \to \mathbb{R}$ *are continuous functions such that:*

(i) for each $c > -1, c \neq 0$, *the integral* $\int_0^\infty b(s)g(s, c)\, ds$ *diverges;*

(ii) $g(t, \cdot)$ *is non-increasing for* $x \geq -1$, *for each* $t \geq 0$;

(iii)

$$|g(t, x)| \leq |x|, \quad x \geq -1, t \geq 0;$$

(iv) there is $T \geq 0$ *such that* $\int_{t-r}^{t} b(t)dt \leq \frac{3}{2}$, *for* $t \geq T$.

Then all admissible solutions $x(t)$ *of (2.4) are defined on* $[0, \infty)$, *and satisfy* $x(t) \to 0$ *as* $t \to \infty$.

Proof. Define $F(t, \varphi) = b(t)g(t, \varphi(-r))$, so that (2.4) reads as (2.1). Since $[-1, \infty) \ni x \mapsto g(t, x)$ is non-increasing for each $t \geq 0$, condition (i) implies that (A1) holds; on the other hand, (A2) follows easily from (iii). ∎

As an immediate consequence of this corollary, consider the standard example of the logistic equation with delay and a non-constant coefficient,

$$\dot{N}(t) = b(t)N(t)\Big[1 - \frac{N(t - r)}{K}\Big], \tag{2.5}$$

where $r > 0$ is the maturation period of the species, $K > 0$ is the carrying capacity of the habitat and $b(t)$ is the growth rate of the species. Translating the positive equilibrium $N_* = K$ to the origin by the change $x(t) = N(t)/K - 1$, we get a generalization of the so-called Wright equation (for which the growth rate $b(t)$ is constant):

$$\dot{x}(t) = -b(t)(1 + x(t))x(t - r). \tag{2.6}$$

Eq. (2.6) has the form (2.4) with $g(x) = -x$. Applying Corollary 2.2, we immediately deduce the following criterion established in [5]: if

$$\int_{t-r}^{t} b(s)ds \leq \frac{3}{2} \quad \text{and} \quad \int_0^\infty b(s)ds = \infty,$$

then the positive equilibrium $N_* = K$ is a global attractor of all positive solutions.

3. An example

Consider the scalar FDE with one discrete delay,

$$\dot{N}(t) = r(t)N(t)\Big[\frac{K - aN(t - \tau)}{K + \lambda(t)N(t - \tau)}\Big]^\alpha, \quad t \geq 0, \tag{3.1}$$

where $r, \lambda : [0, \infty) \to [0, \infty)$ are continuous, $a, K, \tau > 0$ and $\alpha \geq 1$ is the ratio of two odd integers. For $\alpha = 1$, Eq. (3.1) has been extensively studied (see [1], [2] and [3] and references therein), since it has been proposed as an alternative model of the logistic equation (2.5) for a food limited single population model. Note that for $\lambda(t) \equiv 0$ and $\alpha = 1$, (3.1) reduces to the form (2.5). For $\alpha \geq 1$ a ratio of two odd integers, see [4]. Clearly $N_* = K/a$ is the unique positive equilibrium of (3.1).

Theorem 3.1. *Assume*

$$\int_0^\infty \frac{r(s)}{(1+\lambda(s))^\alpha}\, ds = \infty. \tag{3.2}$$

Assume also that one of the following conditions hold:

(i) $\lambda(t) \geq a$ *for all* $t \geq 0$ *and*

$$\int_{t-\tau}^t r(s)\, ds \leq 3/2 \quad \text{for large } t; \tag{3.3}$$

(ii) $0 < \lambda(t) \leq a$ *for all* $t \geq 0$, *and*

$$a^\alpha \int_{t-\tau}^t \frac{r(s)}{\lambda(s)^\alpha}\, ds \leq 3/2 \quad \text{for large } t; \tag{3.4}$$

(iii)

$$\alpha \int_{t-\tau}^t r(s)\, ds \leq 3/2 \quad \text{for large } t. \tag{3.5}$$

Then $N_* = K/a$ *is globally attractive (in the set of all positive solutions of (3.1)).*

Proof. Clearly, in (3.1) we obtain $a = 1$ by replacing K and $\lambda(t)$ by $K_0 = K/a$ and $\lambda_0(t) = \lambda(t)/a$, respectively. On the other hand, considering separately the cases $a \geq 1$ and $0 < a < 1$, one can see that (3.2) holds if and only if

$$\int_0^\infty \frac{r(s)}{(1+a^{-1}\lambda(s))^\alpha}\, ds = \infty.$$

Hence only the case $a = 1$ is considered.

Let $a = 1$. Through the change of variables $x(t) = \frac{N(t)}{K} - 1$, (3.1) becomes

$$\dot{x}(t) = -r(t)(1+x(t))\Big[\frac{x(t-\tau)}{1+\lambda(t)(1+x(t-\tau))}\Big]^\alpha, \quad t \geq 0. \tag{3.6}$$

Case 1. Assume (i). Eq. (3.6) has the form (2.4) for $b(t) = r(t)$ and

$$g(t,x) = -\Big(\frac{x}{1+\lambda(t)(1+x)}\Big)^\alpha, \quad t \geq 0, x \geq -1.$$

For each $t \geq 0$, note that $[-1, \infty) \ni x \mapsto g(t,x)$ is decreasing, and

$$|g(t,x)| \leq |x|^\alpha \leq |x|, \quad \text{if } -1 \leq x \leq 1.$$

Since $\lambda(t) \geq 1$ on $[0, \infty)$, for all $t \geq 0$ we have

$$|g(t,x)| \leq 1 \leq |x|, \quad \text{if } x \geq 1.$$

Thus conditions (ii) and (iii) of Corollary 2.2 are fulfilled. On the other hand, if (3.2) holds, a comparison criterion shows that

$$\int_0^\infty r(s) g(s,c) ds = -c^\alpha \int_0^\infty \frac{r(s)}{[1+\lambda(s)(1+c)]^\alpha} ds$$

diverges for each $c > -1$. Invoking Corollary 2.2, we conclude that in this case the equilibrium $N_* = K$ is globally attractive.

Case 2. Assume (ii). We write (3.6) in the form (2.4),

$$\dot{x}(t) = (1 + x(t)) b(t) g(t, x(t-r)), \quad t \geq 0,$$

with the functions $b(t)$ and $g(t,x)$ defined now by

$$b(t) = \frac{r(t)}{\lambda(t)^\alpha}, \quad g(t,x) = -\Big(\frac{\lambda(t)x}{1+\lambda(t)(1+x)}\Big)^\alpha, \quad t \geq 0, x \geq -1.$$

Clearly, $[-1, \infty) \ni x \mapsto g(t,x)$ is decreasing for each $t \geq 0$. Since $0 < \lambda(t) \leq 1$ for $t \geq 0$, we obtain

$$\begin{gathered} |g(t,x)| \leq \lambda(t)^\alpha |x|^\alpha \leq |x|, \quad \text{if } -1 \leq x \leq 1, \\ |g(t,x)| \leq 1 \leq |x|, \quad \text{if } x \geq 1. \end{gathered}$$

Thus conditions (ii) and (iii) of Corollary 2.2 are satisfied. Assumptions (3.2) and (3.4) imply that conditions (i) and (iv) of the same corollary hold, hence the conclusion follows.

Case 3. Assume (3.5). In this case we effect the change of variables

$$x(t) = \Big(\frac{N(t)}{K}\Big)^\alpha - 1,$$

so that (3.1) is transformed into

$$\dot{x}(t) = \alpha r(t)(1+x(t)) \Big[\frac{1-(1+x(t-\tau))^{1/\alpha}}{1+\lambda(t)(1+x(t-\tau))^{1/\alpha}}\Big]^\alpha, \quad t \geq 0, \tag{3.7}$$

which has the form (2.4) for $b(t) = \alpha r(t)$ and $g(t,x) = \big(g_0(t,x)\big)^\alpha$, where

$$g_0(t,x) = \frac{1-(1+x)^{1/\alpha}}{1+\lambda(t)(1+x)^{1/\alpha}}, \quad t \geq 0, x \geq -1.$$

The function $x \mapsto g_0(t,x)$ is decreasing on $[-1,\infty)$, for each $t \geq 0$. Now, we want to prove that $|g(t,x)| \leq |x|$ for $t \geq 0, x \geq -1$, which is equivalent to show that

$$|g_0(t,x)| \leq |x|^{1/\alpha}, \quad t \geq 0, x \geq -1. \tag{3.8}$$

Since $\alpha \geq 1$, we have $|g_0(t,x)| \leq |1-(1+x)^{1/\alpha}| \leq |x|^{1/\alpha}$ for $t \geq 0, x \geq -1$, thus (3.8) follows. Hence, conditions (ii) and (iii) of Corollary 2.2 are fulfilled. Clearly, (3.2) and (3.5) imply conditions (i) and (iv) in Corollary 2.2. The proof is complete. ■

Remark 3.1. The specific model (3.1) with $K = a = 1$ was studied by Liu [4]. For both situations, $\lambda(t) \geq 1$ for $t \geq 0$, and $0 < \lambda(t) \leq 1$ for $t \geq 0$, the author proved the global attractivity of the positive equilibrium N_* under (3.2) and weaker assumptions than (3.3), and (3.4), respectively. However, case (iii) in Theorem 3.1 was not addressed in [4]. On the other hand, a significant advantage of the criterion established in Corollary 2.2 is that it applies to many models, avoiding the study of each specific model by itself.

References

1. T. Faria, Global attractivity in scalar delayed differential equations with applications to population models, to appear in *J. Math. Anal. Appl.*

2. K. Gopalsamy, *Stability and Oscillation in Delay Differential Equations of Population Dynamics*, Kluwer Academic Publishers, Dordrecht, 1992.

3. Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, London, 1993.

4. Y. Liu, Global attractivity for a differential-difference population model, *Appl. Math. E-Notes* **1** (2001), 56–64.

5. J. W.-H. So and J. S. Yu, Global attractivity for a population model with time delay, *Proc. Amer. Math. Soc.* **123** (1995), 2687–2694.

6. J. W.-H. So, J. S. Yu and M.-P. Chen, Asymptotic stability for scalar delay differential equations, *Funkcial. Ekvac.* **39** (1996), 1–17.

7. J. A. Yorke, Asymptotic stability for one dimensional delay-differential equations, *J. Differential Equations* **7** (1970), 189–202.

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