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Global Attractivity of Solutions for Nonlinear Fractional Order Riemann-Liouville Volterra-Stieltjes Partial Integral Equations

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Abstract

This paper deals with the existence and the attractivity of solutions of a class of fractional order functional Riemann-Liouville Volterra-Stieltjes partial integral equations. Our results are obtained by using the Schauder fixed point theorem.

Key words and phrases: Volterra-Stieltjes quadratic integral equation, left-sided mixed Riemann-Liouville integral of fractional order, attractivity, solution, fixed point.
AMS (MOS) Subject Classifications: 26A33, 45D05, 45G05, 45M10.

1 Introduction

Integral equations are one of the most useful mathematical tools in both pure and applied analysis. This is particularly true of problems in mechanical vibrations and the related fields of engineering and mathematical physics. We can find numerous applications of differential and integral equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc., [10, 16, 25]. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Abbas *et al.* [5], Kilbas *et al.* [19], Miller and Ross [20], Podlubny [22], Samko *et al.* [24], and the papers of Abbas

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et al. [1, 2, 3, 4, 6, 7], Ahmad *et al.* [8, 9], Banaś and Zając [12], Chen *et al.* [13], Darwish *et al.* [15], Diethelm and Ford [17] and the references therein.

In [1], Abbas *et al.* used the techniques of some fixed point theorems, for the study of the existence and the stability of solutions for some classes of nonlinear quadratic integral equations of fractional order.

Motivated by that paper, this work deals with the existence and the attractivity of solutions to the following nonlinear fractional order Riemann-Liouville Volterra-Stieltjes quadratic partial integral equations of the form,

$$\begin{aligned} u(t,x) &= f(t,x,u(t,x),u(\alpha(t),x)) + \tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\ &\quad \times h(t,x,s,y,u(s,y),u(\gamma(s),y))dy d_s g(t,s);\ (t,x)\in J := \mathbb{R}_+\times[0,b], \end{aligned} \tag{1}$$

where $b>0,\ r_1,r_2\in(0,\infty),\ \mathbb{R}_+=[0,\infty),\ \alpha,\ \beta,\ \gamma:\mathbb{R}_+\to\mathbb{R}_+,\ f:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R},\ g:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R},\ h:J'\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions, $\lim_{t\to\infty}\alpha(t)=\infty$, $J'=\{(t,x,s,y)\in J^2: s\le t,\ y\le x\}$ and $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi)=\int_0^\infty t^{\xi-1}e^{-t}dt;\ \xi>0.$$

We use the Schauder fixed point theorem for the existence of solutions of the equation (1), and we prove that all solutions are uniformly globally attractive. Finally, we present an example illustrating the applicability of the imposed conditions.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $L^1([0,p]\times[0,q])$, for $p,q>0$, we denote the space of Lebesgue-integrable functions $u:[0,p]\times[0,q]\to\mathbb{R}$ with the norm

$$\|u\|_1=\int_0^p\int_0^q|u(t,x)|dxdt.$$

By $BC:=BC(J)$ we denote the Banach space of all bounded and continuous functions from J into $\mathbb{R}$ equipped with the standard norm

$$\|u\|_{BC}=\sup_{(t,x)\in J}|u(t,x)|.$$

For $u_0\in BC$ and $\eta\in(0,\infty)$, we denote by $B(u_0,\eta)$, the closed ball in BC centered at u_0 with radius η.

Definition 2.1 *[26] Let* $r=(r_1,r_2)\in(0,\infty)\times(0,\infty),\ \theta=(0,0)$ *and* $u\in L^1([0,p]\times[0,q])$. *The left-sided mixed Riemann-Liouville integral of order* r *of* u *is defined by*

$$(I_\theta^r u)(t,x)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1}u(s,t)dsd\tau.$$

In particular,

$$(I_\theta^\theta u)(t,x)=u(t,x),\ (I_\theta^\sigma u)(t,x)=\int_0^t\int_0^x u(\tau,s)dsd\tau;$$

for almost all $(t,x)\in[0,p]\times[0,q]$, where $\sigma=(1,1)$.
For instance, $I_\theta^r u$ exists for all $r_1,r_2\in(0,\infty)$, when $u\in L^1([0,p]\times[0,q])$. Note also that when $u\in C([0,p]\times[0,q])$, then $(I_\theta^r u)\in C([0,p]\times[0,q])$, moreover

$$(I_\theta^r u)(t,0)=(I_\theta^r u)(0,x)=0;\ t\in[0,p],\ x\in[0,q].$$

Example 2.2 *Let* $\lambda,\omega\in(-1,\infty)$ *and* $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, *then*

$$I_\theta^r t^\lambda x^\omega=\frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)}t^{\lambda+r_1}x^{\omega+r_2},\ \textit{for almost all } (t,x)\in[0,p]\times[0,q].$$

If u is a real function defined on the interval $[a,b]$, then the symbol $\bigvee_a^b u$ denotes the variation of u on $[a,b]$. We say that u is of bounded variation on the interval $[a,b]$ whenever $\bigvee_a^b u$ is finite. If $w:[a,b]\times[c,b]\to\mathbb{R}$, then the symbol $\bigvee_{t=p}^q w(t,s)$ indicates the variation of the function $t\to w(t,s)$ on the interval $[p,q]\subset[a,b]$, where s is arbitrarily fixed in $[c,d]$. In the same way we define $\bigvee_{s=p}^q w(t,s)$. For the properties of functions of bounded variation we refer to [21].

If u and φ are two real functions defined on the interval $[a,b]$, then under some conditions (see [21]) we can define the Stieltjes integral (in the Riemann-Stieltjes sense)

$$\int_a^b u(t)d\varphi(t)$$

of the function u with respect to φ. In this case we say that u is Stieltjes integrable on $[a,b]$ with respect to φ. Several conditions are known guaranteeing Stieltjes integrability [21]. One of the most frequently used requires that u is continuous and φ is of bounded variation on $[a,b]$.

In what follows we use the following properties of the Stieltjes integral ([23], section 8.13).
If u is Stieltjes integrable on the interval $[a,b]$ with respect to a function φ of bounded variation, then

$$\left|\int_a^b u(t)d\varphi(t)\right|\leq\int_a^b|u(t)|d\left(\bigvee_a^t\varphi\right).$$

If u and v are Stieltjes integrable functions on the interval $[a, b]$ with respect to a nondecreasing function φ such that $u(t) \le v(t)$ for $t \in [a, b]$, then

$$\int_a^b u(t)d\varphi(t) \le \int_a^b v(t)d\varphi(t).$$

In the sequel we also consider Stieltjes integrals of the form

$$\int_a^b u(t)d_s g(t, s),$$

and Riemann-Liouville Stieltjes integrals of fractional order of the form

$$\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}u(s)d_s g(t, s),$$

where $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R},\ r \in (0, \infty)$ and the symbol d_s indicates the integration with respect to s.

Let $\emptyset \neq \Omega \subset BC$, and let $G : \Omega \to \Omega$, and consider the solutions of equation

$$(Gu)(t, x) = u(t, x). \tag{2}$$

Inspired by the definition of the attractivity of solutions of integral equations (for instance [11]), we introduce the following concept of attractivity of solutions for equation (2).

Definition 2.3 *Solutions of equation (2) are locally attractive if there exists a ball $B(u_0, \eta)$ in the space BC such that, for arbitrary solutions $v = v(t, x)$ and $w = w(t, x)$ of equations (2) belonging to $B(u_0, \eta) \cap \Omega$, we have that, for each $x \in [0, b]$,*

$$\lim_{t\to\infty}(v(t, x) - w(t, x)) = 0. \tag{3}$$

When the limit (3) is uniform with respect to $B(u_0, \eta) \cap \Omega$, solutions of equation (2) are said to be uniformly locally attractive (or equivalently that solutions of (2) are locally asymptotically stable).

Definition 2.4 *[11] The solution $v = v(t, x)$ of equation (2) is said to be globally attractive if (3) hold for each solution $w = w(t, x)$ of (2). If condition (3) is satisfied uniformly with respect to the set Ω, solutions of equation (2) are said to be globally asymptotically stable (or uniformly globally attractive).*

Definition 2.5 *[13] The zero solution $u(t, x)$ of equation (2) is globally attractive if every solution of (2) tends to zero as $t \to \infty$.*

Lemma 2.6 *[14] Let $D \subset BC$. Then D is relatively compact in BC if the following conditions hold:*
(a) D is uniformly bounded in BC.
(b) The functions belonging to D are almost equicontinuous on $\mathbb{R}_+ \times [0,b]$,
i.e., *equicontinuous on every compact subset of $\mathbb{R}_+ \times [0,b]$.*
(c) The functions from D are equiconvergent, that is, given $\epsilon > 0$, $x \in [0,b]$ there corresponds $T(\epsilon, x) > 0$ such that $|u(t,x) - \lim_{t\to\infty} u(t,x)| < \epsilon$ for any $t \geq T(\epsilon, x)$ and $u \in D$.

3 Main Results

In this section, we are concerned with the existence and the uniform global attractivity of solutions for the equation (1). Let us start by defining what we mean by a solution of the equation (1).

Definition 3.1 *By a solution of equation (1) we mean a function $u \in BC$ such that u satisfies equation (1) on J.*

The following hypotheses will be used in the sequel.

(H_1) There exist positive constants M and L such that

$$|f(t,x,u_1,u_2) - f(t,x,v_1,v_2)| \leq \frac{M|u_1 - v_1| + L|u_2 - v_2|}{1 + \alpha(t)},$$

for $(t,x) \in J$ and $u_1, u_2, v_1, v_2 \in \mathbb{R}$.

(H_2) The function $t \to f(t,x,0,0)$ is bounded on J with

$$f^* = \sup_{(t,x)\in\mathbb{R}_+\times[0,b]} f(t,x,0,0) \ \ \textit{and} \ \ \lim_{t\to\infty} |f(t,x,0,0)| = 0, \ x \in [0,b].$$

(H_3) For all $t_1, t_2 \in \mathbb{R}_+$ such that $t_1 < t_2$ the function $s \mapsto g(t_2, s) - g(t_1, s)$ is nondecreasing on $\mathbb{R}_+$.

(H_4) The function $s \mapsto g(0, s)$ is nondecreasing on $\mathbb{R}_+$.

(H_5) The functions $s \mapsto g(t,s)$ and $t \mapsto g(t,s)$ are continuous on $\mathbb{R}_+$ for each fixed $t \in \mathbb{R}_+$ or $s \in \mathbb{R}_+$, respectively.

(H_6) There exist continuous functions $p_1, p_2 : J' \to \mathbb{R}_+$ such that

$$|h(t,x,s,y,u,v)| \leq \frac{p_1(t,x,s,y)|u| + p_2(t,x,s,y)|v|}{1 + |u| + |v|};$$

for $(t,x,s,y)\in J'$, $u,v\in\mathbb{R}$. Moreover, assume that

$$\lim_{t\to\infty}\int_0^{\beta(t)}(\beta(t)-s)^{r-1}p_i(t,x,s,y)d_sg(t,s)=0;\ i=1,2.$$

Remark 3.2 *Set*

$$p_i^*:=\sup_{(t,x)\in J}\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x(\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}p_i(t,x,s,y)dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right);$$

for $i=1,2$. From hypothesis (H_6), we infer that p_i^* is finite, for $i=1,2$.

Theorem 3.3 *Assume that hypotheses $(H_1)-(H_6)$ hold. If*

$$M+L+p_1^*+p_2^*<1, \tag{4}$$

then the equation (1) has at least one solution in the space BC. Moreover, solutions of equation (1) are uniformly globally attractive.

Proof: Let us define the operator N such that, for any $u\in BC$,

$$\begin{aligned}(Nu)(t,x) &= f(t,x,u(t,x),u(\alpha(t),x))+\tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x(\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}\\ &\quad\times h(t,x,s,y,u(s,x),u(\gamma(s),x))dyd_sg(t,s);\ (t,x)\in J.\end{aligned} \tag{5}$$

From the assumptions of this theorem, we infer that $N(u)$ is continuous on J.

Now we prove that $N(u)\in BC$ for any $u\in BC$. For arbitrarily fixed $(t,x)\in J$, we have

$$\begin{aligned}|(Nu)(t,x)| &= \Big|f(t,x,u(t,x),u(\alpha(t),x))\\ &\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x(\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}\\ &\quad\times h(t,x,s,y,u(s,y),u(\gamma(s),y))dyd_sg(t,s)\Big|\\ &\le \Big|f(t,x,u(t,x),u(\alpha(t),x))-f(t,x,0,0)+f(t,x,0,0)\Big|\\ &\quad+\Big|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x(\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}\\ &\quad\times h(t,x,s,y,u(s,y),u(\gamma(s),y))dyd_sg(t,s)\Big|\\ &\le \frac{M}{1+\alpha(t)}|u(t,x)|+\frac{L}{1+\alpha(t)}|u(\alpha(t),x)|+\Big|f(t,x,0,0)\Big|\\ &\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x(\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}\end{aligned}$$

$$
\begin{aligned}
&\times \frac{p_1(t,x,s,y)|u(s,y)| + p_2(t,x,s,y)|u(\gamma(s),y)|}{1+|u(s,y)|+|u(\gamma(s),y)|} dyd_s \left(\bigvee_{k=0}^{s} g(t,k)\right) \\
&\le \Big| f(t,x,0,0)\Big| + M|u(t,x)| + L|u(\alpha(t),x)| \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\
&\times (p_1(t,x,s,y)+p_2(t,x,s,y)) dyd_s \left(\bigvee_{k=0}^{s} g(t,k)\right) \\
&\le f^* + (M+L+p_1^*+p_2^*)\|u\|_{BC}.
\end{aligned}
$$

Thus

$$
\|N(u)\| \le f^* + (M+L+p_1^*+p_2^*)\|u\|_{BC}. \tag{6}
$$

Hence $N(u) \in BC$. From (4) and (6), we infer that N transforms the ball $B_\eta := B(0,\eta)$ into itself, where

$$
\eta \ge \frac{f^*}{1-M-L-p_1^*-p_2^*}.
$$

We shall show that $N : B_\eta \to B_\eta$ satisfies the assumptions of Schauder's fixed point theorem [18]. The proof will be given in several steps and cases.

Step 1: *N is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_η. Then, for each $(t,x) \in J$, we have

$$
\begin{aligned}
& |(Nu_n)(t,x) - (Nu)(t,x)| \\
\le\ & |f(t,x,u_n(t,x),u_n(\alpha(t),x)) - f(t,x,u(t,x),u(\alpha(t),x))| \\
& + \tfrac{1}{\Gamma(r_1)\Gamma(r_2)} \textstyle\int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\
\times\ & |h(t,x,s,y,u_n(s,y),u_n(\gamma(s),y)) - h(t,x,s,y,u(s,y),u(\gamma(s),y))| dyd_s g(t,s) \\
\le\ & (M+L)\|u_n-u\|_{BC} + \tfrac{1}{\Gamma(r_1)\Gamma(r_2)} \textstyle\int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\
\times\ & |h(t,x,s,y,u_n(s,y),u_n(\gamma(s),y)) \\
& -h(t,x,s,y,u(s,y),u(\gamma(s),y))| dyd_s \left(\textstyle\bigvee_{k=0}^{s} g(t,k)\right).
\end{aligned} \tag{7}
$$

Case 1. If $(t,x) \in [0,a] \times [0,b];\ a > 0$, then, since $u_n \to u$ as $n \to \infty$ and g, h are continuous, (7) gives

$$
\|N(u_n) - N(u)\|_{BC} \to 0 \quad \text{as } n \to \infty.
$$

Case 2. If $(t,x)\in(a,\infty)\times[0,b];\ a>0$, then from (H_3) and (7), for each $(t,x)\in J$, we get

$$\begin{array}{rl} & |(Nu_n)(t,x)-(Nu)(t,x)| \\ \leq & (M+L)\|u_n-u\|+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x(\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\ \times & \Big(p_1(t,x,s,y)(|u_n(s,x)|+|u(s,x)|) \\ + & p_2(t,x,s,y)(|u_n(\gamma(s),x)|+|u(\gamma(s),x)|)\Big)dyd_s\left(\bigvee_{k=0}^s g(t,k)\right) \\ \leq & (M+L)\|u_n-u\|+\frac{2\eta}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x(\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\ \times & (p_1(t,x,s,y)+p_2(t,x,s,y))dyd_s\left(\bigvee_{k=0}^s g(t,k)\right) \end{array} \tag{8}$$

From (H_6) and since $u_n\to u$ as $n\to\infty$ and $t\to\infty$, then (8) gives

$$\|N(u_n)-N(u)\|_{BC}\to 0 \quad \text{as } n\to\infty.$$

Step 2: $N(B_\eta)$ *is uniformly bounded.*
This is clear since $N(B_\eta)\subset B_\eta$ and B_η is bounded.

Step 3: $N(B_\eta)$ *is equicontinuous on every compact subset* $[0,a]\times[0,b]$ *of* J, $a>0$. Let $(t_1,x_1),(t_2,x_2)\in[0,a]\times[0,b]$, $t_1<t_2,\ x_1<x_2$ and let $u\in B_\eta$. Also without loss of generality, suppose that $\beta(t_1)\leq\beta(t_2)$. Thus we have

$$\begin{aligned} & |(Nu)(t_2,x_2)-(Nu)(t_1,x_1)| \\ & \leq |f(t_2,x_2,u(t_2,x_2),u(\alpha(t_2),x_2))-f(t_2,x_2,u(t_1,x_1),u(\alpha(t_1),x_1))| \\ & +|f(t_2,x_2,u(t_1,x_1),u(\alpha(t_1),x_1))-f(t_1,x_1,u(t_1,x_1),u(\alpha(t_1),x_1))| \\ & +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big|\int_0^{\beta(t_2)}\int_0^{x_2}(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1} \\ & \times[h(t_2,x_2,s,y,u(s,y),u(\gamma(s),y))-h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))]dyd_sg(t,s)\Big| \\ & +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big|\int_0^{\beta(t_2)}\int_0^{x_2}(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1} \\ & \times h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))dyd_sg(t,s) \\ & -\int_0^{\beta(t_1)}\int_0^{x_2}(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1}h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))dyd_sg(t,s)\Big| \\ & +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big|\int_0^{\beta(t_1)}\int_0^{x_2}(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1} \\ & \times h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))dyd_sg(t,s) \\ & -\int_0^{\beta(t_1)}\int_0^{x_1}(\beta(t_1)-s)^{r_1-1}(x_1-y)^{r_2-1}h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))dyd_sg(t,s)\Big| \\ & \leq M|u(\alpha(t_2),x_2)-u(\alpha(t_1),x_1)|+L|u(t_2,x_2)-u(t_1,x_1)| \\ & +|f(t_2,x_2,u(t_1,x_1),u(\alpha(t_1),x_1))-f(t_1,x_1,u(t_1,x_1),u(\alpha(t_1),x_1))| \end{aligned}$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t_2)}\int_0^{x_2}(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1}$$

$$\times\Big|h(t_2,x_2,s,y,u(s,y),u(\gamma(s),y))-h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))\Big|dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right)$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\beta(t_1)}^{\beta(t_2)}\int_0^{x_2}(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1}$$

$$\times\Big|h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))\Big|dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right)$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t_1)}\int_0^{x_1}\Big|(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1}-(\beta(t_1)-s)^{r_1-1}(x_1-y)^{r_2-1}\Big|$$

$$\times\Big|h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))\Big|dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right)$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t_1)}\int_{x_1}^{x_2}|(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1}|$$

$$\times\Big|h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))\Big|dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right)$$

$$\leq M|u(\alpha(t_2),x_2)-u(\alpha(t_1),x_1)|+L|u(t_2,x_2)-u(t_1,x_1)|$$

$$+|f(t_2,x_2,u(t_1,x_1),u(\alpha(t_1),x_1))-f(t_1,x_1,u(t_1,x_1),u(\alpha(t_1),x_1))|$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t_2)}\int_0^{x_2}(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1}$$

$$\times\Big|h(t_2,x_2,s,y,u(s,y),u(\gamma(s),y))-h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))\Big|dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right)$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\beta(t_1)}^{\beta(t_2)}\int_0^{x_2}(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1}$$

$$\times\Big(p_1(t_1,x_1,s,y)+p_2(t_1,x_1,s,y)\Big)dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right)$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t_1)}\int_0^{x_1}\Big|(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1}-(\beta(t_1)-s)^{r_1-1}(x_1-y)^{r_2-1}\Big|$$

$$\times\Big(p_1(t_1,x_1,s,y)+p_2(t_1,x_1,s,y)\Big)dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right)$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t_1)}\int_{x_1}^{x_2}|(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1}|$$

$$\times\Big(p_1(t_1,x_1,s,y)+p_2(t_1,x_1,s,y)\Big)dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right).$$

From continuity of $\alpha, \beta, f, g, h, p_1, p_2, u$ and as $t_1 \to t_2$ and $x_1 \to x_2$, the right-hand side of the above inequality tends to zero.

Step 4: *$N(B_\eta)$ is equiconvergent.*
Let $(t,x) \in J$ and $u \in B_\eta$, then we have

$$\begin{aligned}
|(Nu)(t,x)| &\le \Big|f(t,x,u(t,x),u(\alpha(t),x)) - f(t,x,0,0) + f(t,x,0,0)\Big| \\
&\quad + \Big|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\
&\times h(t,x,s,y,u(s,y),u(\gamma(s),y))dyd_s g(t,s)\Big| \\
&\le |f(t,x,0,0)| + \frac{M}{1+\alpha(t)}|u(t,x)| + \frac{L}{1+\alpha(t)}|u(\alpha(t),x)| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\
&\quad \times (p_1(t,x,s,y)+p_2(t,x,s,y))d_s\left(\bigvee_{k=0}^{s}g(t,k)\right) \\
&\le |f(t,x,0,0)| + \frac{\eta(M+L)}{1+\alpha(t)} \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\
&\quad \times (p_1(t,x,s,y)+p_2(t,x,s,y))d_s\left(\bigvee_{k=0}^{s}g(t,k)\right).
\end{aligned}$$

Then, since $\alpha(t) \to \infty$ as $t \to \infty$, we deduce that, for each $x \in [0,b]$, we get

$$|(Nu)(t,x)| \to 0, \ \textit{as } t \to +\infty.$$

Hence,

$$|(Nu)(t,x) - (Nu)(+\infty,x)| \to 0, \ \textit{as } t \to +\infty.$$

As a consequence of Steps 1 to 4 together with the Lemma 2.6, we can conclude that $N : B_\eta \to B_\eta$ is continuous and compact. From an application of Schauder's theorem [18], we deduce that N has a fixed point u which is a solution of the equation (1).

Step 5: *The uniform global attractivity of solutions.*
Now we investigate the stability of solutions of equation (1). Let us assume that u and

v are two solutions of the equation (1) with the conditions of this theorem. Then, for each $(t,x) \in J$, we have

$$
\begin{aligned}
& |u(t,x) - v(t,x)| = |(Nu)(t,x) - (Nv)(t,x)| \\
\leq\ & |f(t,x,u(t,x),u(\alpha(t),x)) - f(t,x,v(t,x),v(\alpha(t),x))| \\
& + \tfrac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\
& \times |h(t,x,s,y,u(s,y),u(\gamma(s),y)) - h(t,x,s,y,v(s,y),v(\gamma(s),y))| dyd_s g(t,s) \\
\leq\ & M\|u(t,x) - v(t,x)\| + L\|u(\alpha(t),x) - v(\alpha(t),x)\| \\
& + \tfrac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\
& \times |h(t,x,s,y,u(s,y),u(\gamma(s),y)) - h(t,x,s,y,v(s,y),v(\gamma(s,y)))| dyd_s \left(\textstyle\bigvee_{k=0}^{s} g(t,k)\right) \\
\leq\ & M\|u(t,x) - v(t,x)\| + L\|u(\alpha(t),x) - v(\alpha(t),x)\| \\
& + \tfrac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\
& \times (p_1(t,x,s,y) + p_2(t,x,s,y)) dyd_s \left(\textstyle\bigvee_{k=0}^{s} g(t,k)\right).
\end{aligned}
\tag{9}
$$

By using (4), (9) and the fact that $\alpha(t) \to \infty$ as $t \to \infty$, we deduce that

$$
\begin{aligned}
& \lim_{t\to\infty} |u(t,x) - v(t,x)| \leq \lim_{t\to\infty} \frac{1}{(1-M-L)\Gamma(r_1)\Gamma(r_2)} \\
\times\ & \int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\
& \times (p_1(t,x,s,y) + p_2(t,x,s,y)) dyd_s \left(\bigvee_{k=0}^{s} g(t,k)\right) \\
=\ & 0.
\end{aligned}
$$

Hence,

$$\lim_{t\to\infty} |u(t,x) - v(t,x)| = 0.$$

Consequently, all solutions of equation (1) are uniformly globally attractive.

As a consequence of Theorem 3.3, we prove the following result.

Theorem 3.4 *Assume that hypotheses $(H_1) - (H_6)$ and the inequality (4) hold. If*

$$f(t,x,0,0) + \int_0^{\beta(t)} \int_0^x \frac{(\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} h(t,x,s,y,0;0) dyd_s g(t,s) = 0; \quad (10)$$

for $(t,x) \in J$, then the zero solution of equation (1) is globally attractive.

Proof. From the equation (10), it is clear that $v(t,x) = 0$ is a solution of our equation (1). Let us assume that u is any solution of the equation (1). Then, for each $(t,x) \in J$, we have

$$\begin{aligned}
& |u(t,x)| = |u(t,x)-v(t,x)| = |(Nu)(t,x)-(Nv)(t,x)| \\
\leq\ & |f(t,x,u(t,x),u(\alpha(t),x)) - f(t,x,0,0)| \\
& +\tfrac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\
& \times |h(t,x,s,y,u(s,y),u(\gamma(s),y)) - h(t,x,s,y,0,0)| dy d_s g(t,s) \\
\leq\ & M\|u(t,x)\| + L\|u(\alpha(t),x)\| \\
& +\tfrac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\
& \times |h(t,x,s,y,u(s,y),u(\gamma(s),y)) - h(t,x,s,y,0,0)| dy d_s \left(\bigvee_{k=0}^s g(t,k)\right) \\
\leq\ & M\|u(t,x)\| + L\|u(\alpha(t),x)\| \\
& +\tfrac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} p_1(t,x,s,y) dy d_s \left(\bigvee_{k=0}^s g(t,k)\right).
\end{aligned} \tag{11}$$

By using (4), (11) and the fact that $\alpha(t) \to \infty$ as $t \to \infty$, we deduce that

$$\begin{aligned}
\lim_{t\to\infty} |u(t,x)| \ &\leq\ \lim_{t\to\infty} \frac{\int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}}{(1-M-L)\Gamma(r_1)\Gamma(r_2)} \\
&\times\ p_1(t,x,s,y) dy d_s \left(\bigvee_{k=0}^s g(t,k)\right) = 0.
\end{aligned}$$

Thus, $\lim_{t\to\infty} |u(t,x)| = 0$. Hence, the zero solution of equation (1) is globally attractive.

4 An Example

As an application of our results we consider the following nonlinear fractional order Riemann-Liouville Volterra-Stieltjes quadratic partial integral equation of the form

$$\begin{aligned}
u(t,x) \ &=\ f(t,x,u(t,x),u(\alpha(t),x)) + \tfrac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\
&\times h(t,x,s,y,u(s,y),u(\gamma(s),y)) dy d_s g(t,s), (t,x) \in J := \mathbb{R}_+ \times [0,1],
\end{aligned} \tag{12}$$

where $r_1 = \frac{1}{4},\ r_2 = \frac{1}{2},\ \alpha(t) = \beta(t) = \gamma(t) = t;\ t \in \mathbb{R}_+$,

$$f(t,x,u,v) = \frac{xe^{-t}|uv|}{8(1+t)(1+|u|+2|v|)},\ (t,x) \in J \ \text{ and } u,v \in \mathbb{R},$$

$$g(t,s) = s,\ (t,s) \in \mathbb{R}_+^2,$$

$$\begin{cases}
h(t,x,s,y,u,v) = \dfrac{cxs^{\frac{-3}{4}}(1+|u|)\sin\sqrt{t}\sin s}{(1+y^2+t^2)(1+|u|+|v|)}; \\
\quad if\ (t,x,s,y) \in J',\ s \neq 0,\ y \in [0,1]\ and\ u,v \in \mathbb{R}, \\
h(t,x,0,y,u,v) = 0;\ if\ (t,x) \in J,\ y \in [0,1]\ and\ u,v \in \mathbb{R},
\end{cases}$$

$$c=\frac{\pi}{8e\Gamma(\frac{1}{4})}\ \ and\ \ J'=\{(t,x,s,y)\in J^2: s\le t\ and\ x\le y\}.$$

First, we can see that $\lim_{t\to 0}\alpha(t)=0$. Next, the function f is a continuous, and

$$|f(t,x,u_1,u_2)-f(t,x,v_1,v_2)|\le\frac{1}{8(1+t)}(|u_1-v_1|+2|u_2-v_2|);\ (t,x)\in J,\ u,v\in\mathbb{R}.$$

Then, the assumption (H_1) is satisfied with $M=\frac{1}{8},\ L=\frac{1}{4}$, and (H_2) is satisfied with $f^*=0$. Also, we can easily see that the function g satisfies the hypotheses $(H_3)-(H_5)$. The function h satisfies the assumption (H_6). Indeed, h is continuous and

$$|h(t,x,s,y,u,v)|\le p_1(t,x,s,y)|u|+p_2(t,x,s,y)|v|;\ (t,x,s,y)\in J',\ u,v\in\mathbb{R},$$

and for $i=1,2$,

$$\begin{cases} p_i(t,x,s,y)=\dfrac{cxs^{\frac{-3}{4}}\sin\sqrt{t}\sin s}{1+y^2+t^2};\ (t,x,s,y)\in J',\ y\in[0,1],\ s\neq 0,\\ p_i(t,x,0,y)=0; \qquad (t,x)\in J,\ y\in[0,1]. \end{cases}$$

Then, for $i=1,2$, we have

$$\begin{aligned}\left|\int_0^t (t-s)^{r-1}p_i(t,x,s,y)d_sg(t,s)\right| &\le \int_0^t (t-s)^{\frac{-3}{4}}cxs^{\frac{-3}{4}}|\sin\sqrt{t}\sin s|d_s\left(\bigvee_{k=0}^{s}g(t,k)\right)\\ &\le cx|\sin\sqrt{t}|\int_0^t (t-s)^{\frac{-3}{4}}s^{\frac{-3}{4}}ds\\ &\le \frac{cx\Gamma^2(\frac{1}{4})}{\sqrt{\pi}}\left|\frac{\sin\sqrt{t}}{\sqrt{t}}\right|\\ &\le \frac{cx\Gamma^2(\frac{1}{4})}{\sqrt{\pi t}}\longrightarrow 0\ as\ t\to\infty,\end{aligned}$$

and

$$\begin{aligned} p_i^* &:= \sup_{(t,x)\in J}\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}p_i(t,x,s,y)dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right)\\ &\le \sup_{(t,x)\in J}\frac{cx\Gamma(\frac{1}{4})}{\sqrt{\pi}}\left|\frac{\sin\sqrt{t}}{\sqrt{t}}\right|\\ &= \frac{c\Gamma(\frac{1}{4})}{\pi}=\frac{1}{8e}.\end{aligned}$$

Finally, we can see that

$$M+L+p_1^*+p_2^*\le\frac{2+3e}{8e}<1.$$

Hence the condition (4) is satisfied. Consequently, by Theorem 3.3, the equation (12) has a solution defined on $\mathbb{R}_+ \times [0, 1]$ and solutions of this equation are uniformly globally attractive.

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