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SOLUTIONS OF TWO-TERM TIME FRACTIONAL ORDER DIFFERENTIAL EQUATIONS WITH NONLOCAL INITIAL CONDITIONS

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ABSTRACT. We study the existence of mild solutions for the two-term fractional order abstract differential equation

$$D_t^{\alpha+1}u(t)+\mu D_t^{\beta}u(t)-Au(t)=D_t^{\alpha-1}f(t,u(t)),\quad t\in[0,1],\quad 0<\alpha\leq\beta\leq 1,\mu\geq 0,$$

with nonlocal initial conditions and where A is a linear operator of sectorial type. To achieve our goal, we use a new mixed method, which combines a generalization of the theory of C_0-semigroups, Hausdorff measure of noncompactness and a fixed point argument.

1. INTRODUCTION

Let X be a Banach space. Our concern in this paper is the study of existence of mild solutions for fractional order differential equations of the form

$$D_t^{\gamma}u(t)+\mu D_t^{\beta}u(t)-Au(t)=F(s,u(s)),\ 0<\gamma\leq 2,\ 0<\beta\leq 1,\ \mu\geq 0,\ t\in[0,1], \tag{1.1}$$

with prescribed nonlocal initial conditions $u(0)=0$ and $u'(0)=g(u)$, where $A:D(A)\subset X\to X$ is a sectorial operator, F and g are vector-valued functions, and D_t^{γ} denotes the Caputo fractional derivative of order γ.

Evolution equations involving fractional derivatives in time have, in some cases, better effects in applications than traditional evolution equations of integer order in time (cf., e.g. [1, 4, 11, 12, 14, 17, 18], the survey paper [10] and the references therein). The class of fractional evolution equations can provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. What it need to emphasize is that this is the main advantage of fractional models in comparison with integer-order models, in which such effects are in fact neglected.

Equation (1.1) is a general model that include recent investigations in the subject. Indeed, in [16] Lizama obtained existence and uniqueness of solutions for the abstract equation (1.1) in the special case $\gamma=\beta+1$ and, in [19], Stojanović and Gorenflo studied the nonlinear two-term time fractional diffusion wave equation (1.1) with $0<\gamma<\beta$ and $A=\frac{d^2}{dx^2}$. In all the foregoing cases, the initial value problem was considered, but the study of existence of solutions for the equation (1.1) with *nonlocal* initial conditions was left open. Anticipating a wide interest in problems modeled by (1.1), this paper contributes in filling this important gap.

Stimulated by the observation that nonlocal initial conditions are more realistic than usual ones in treating physical problems, the study of fractional evolution equations with nonlocal

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initial conditions has been recently initiated (see [5], [6] and [7] for more detailed information about the importance of nonlocal initial conditions in applications). For example, in [20] Wang and Yang and, in [21] Zhang and Liu, obtained existence of mild solutions for the the nonlocal problem (1.1) in the border case $\mu = 0$ and $0 < \gamma < 1$. The main hypothesis in both cases is that A is the generator of a C_0-semigroup with some qualitative property (e.g. compactness, see [8] and references therein).

On the other hand, in [13] Keyantuo, Lizama and Warma proved that it is possible to give an abstract operator approach to equation (1.1) by a method of regularization, defining first an ad-hoc solution family of strongly continuous operators $S_{\gamma,\beta}(t)$ for (1.1) in case $F \equiv 0$. It turns out, that it is a particular case of an (a,k)-regularized family [15]. Then, the solution of the linear part in equation (1.1) can be written in terms of a kind of variation of constants formula. It give us the necessary framework to apply an operator theoretical approach in the analysis of mild solutions for the abstract fractional order differential equation (1.1).

In this paper, we will show that the use of the above described method of representation of the solution is successful to obtain existence of mild solutions for the following regularized version of the equation (1.1):

(1.2)
$$D_t^{\alpha+1}u(t) + \mu D_t^{\beta}u(t) - Au(t) = \int_0^t g_{1-\alpha}(t-s)f(s,u(s))ds,\ t \geq 0,\ 0 < \alpha < \beta \leq 1, \quad \mu > 0,$$

where $g_\gamma(t) := \frac{t^{\gamma-1}}{\Gamma(\gamma)}$ for $\gamma > 0$, $u(0) = 0$ and nonlocal initial condition $u'(0) = g(u)$. Indeed, in this case, and following [13], a mild solution of (1.2) can be defined as a fixed point of the equation

$$u(t) = (g_1 * S_{\alpha,\beta})(t)g(u) + (g_1 * S_{\alpha,\beta} * f)(t), \quad 0 < \alpha \leq \beta \leq 1,\ \mu > 0. \tag{1.3}$$

Then, we can use an strategy based of Hausdorff measure of noncompactness and a fixed point argument, recently used by Zhu, Song and Li [22] (see also the preprint by Zhang and Liu [21]), to obtain our main result (Theorem 3.1). However, in contrast with [22] it is remarkable that, in the use of this combination of methods, neither compactness or equicontinuity condition on the family $S_{\alpha,\beta}(t)$ to get a fixed point in (1.3) is needed (compare [22] and Remark 3.1), being this fact the main contribution of this paper. Finally, we conclude showing a concrete example to illustrate the feasibility of the abstract given hypothesis.

2. PRELIMINARIES

Let $\alpha > 0$, $m = \lceil \alpha \rceil$ and $u : [0,\infty) \to X$, where X is a complex Banach space. We denote by $\mathbb{R}_+$ the closed interval $[0,\infty)$. The Caputo fractional derivative of $u \in C(\mathbb{R}_+)$ of order α is defined by

$$D_t^{\alpha}u(t) := \frac{d^m}{dt^m}\int_0^t g_{m-\alpha}(t-s)u(s)ds, \quad t > 0,$$

where $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}$, $t > 0$, $\beta > 0$, and in case $\beta = 0$ we set $g_0(t) := \delta_0$, the Dirac measure concentrated at the origin. When $\alpha = n$ is integer, we define $D_t^n := \frac{d^n}{dt^n}, n \in \mathbb{N}$. On the other

hand, the convolution

$$D_t^{-\alpha}u(t) := \int_0^t g_\alpha(t-s)u(s)ds, \quad t>0,$$

defines the Riemann-Liouville integral of order α and stands for the fractional integral of order α of u.

Definition 2.1. *(*[13]*) Let $\mu \geq 0$ and $0 \leq \alpha, \beta \leq 1$ be given. Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. We call A the generator of an $(\alpha,\beta)_\mu$-regularized family if there exist $\omega \geq 0$ and a strongly continuous function $S_{\alpha,\beta} : \mathbb{R}_+ \to \mathscr{B}(X)$ such that $\{\lambda^{\alpha+1} + \mu\lambda^\beta : Re\lambda > \omega\} \subset \rho(A)$ and*

$$H(\lambda)x := \lambda^\alpha(\lambda^{\alpha+1} + \mu\lambda^\beta - A)^{-1}x = \int_0^\infty e^{-\lambda t}S_{\alpha,\beta}(t)xdt, \quad Re\lambda > \omega, \quad x \in X.$$

Because of the uniqueness theorem for the Laplace transform, if $\mu = 0$ and $\alpha = 0$, this corresponds to the case of a C_0-semigroup whereas the case $\mu = 0,\ \alpha = 1$ corresponds to the concept of cosine family. For more details on the Laplace transform approach to semigroups and cosine functions, we refer to the monograph [2].

Let us recall that a closed and densely defined operator A is said to be ω-sectorial of angle θ if there exists $\theta \in (0,\pi/2)$, $M > 0$ and $\omega \in \mathbb{R}$ such that its resolvent exists in the sector $\omega + S_\theta := \{\omega + \lambda : \lambda \in \mathbb{C}, |arg(\lambda)| < \frac{\pi}{2} + \theta\} \setminus \{\omega\}$, and

$$||(\lambda - A)^{-1}|| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \in \omega + S_\theta.$$

These are generators of holomorphic semigroups. In case $\omega = 0$ we merely say that A is sectorial of angle $\theta + \pi/2$. Sufficient conditions to obtain generators of an $(\alpha,\beta)_\mu$-regularized family are given in the following result.

Theorem 2.2. *(*[13]*) Let $0 < \alpha \leq \beta \leq 1$, $\mu > 0$ and A be a ω sectorial operator of angle $\beta\pi/2$. Then A generates a bounded $(\alpha,\beta)_\mu$-regularized family.*

We next consider the linear fractional differential equation

$$D_t^{\alpha+1}u(t) + \mu D_t^\beta u(t) - Au(t) = h(t), \quad t \geq 0, \quad 0 < \alpha \leq \beta \leq 1, \mu \geq 0, \tag{2.1}$$

with initial conditions $u(0) = x$, $u'(0) = y$ and A is a ω-sectorial operator of angle $\beta\pi/2$.

Recall that a function $u \in C^1(\mathbb{R}_+; X)$ is called a strong solution of (2.1) on $\mathbb{R}_+$ if $u(t) \in D(A)$ and (2.1) holds on $\mathbb{R}_+$. If merely $u(t) \in X$ instead of the domain of A, we say that u is a *mild solution* of the linear equation (2.1). We note that, by [13, Cor.3.4] and Theorem 2.2, a mild solution for (2.1) always exists and is given by:

$$u(t) = S_{\alpha,\beta}(t)x + (g_1 * S_{\alpha,\beta})(t)y + \mu(g_{1+\alpha-\beta} * S_{\alpha,\beta}(t))x + (S_{\alpha,\beta} * g_\alpha * h)(t), \tag{2.2}$$

where $0 < \alpha \leq \beta \leq 1$, $\mu > 0$, $x, y \in X$ and $S_{\alpha,\beta}(t)$ is the $(\alpha,\beta)_\mu$-regularized family generated by A.

3. Main Result

In this section, we use the Hausdorff measure of noncompactness and a fixed point argument to prove the existence of a mild solution for a special case of equation (1.1) with a nonlocal initial condition. More precisely, we consider

$$D_t^{\alpha+1}u(t)+\mu D_t^{\beta}u(t)-Au(t)=D^{\alpha-1}f(s,u(s)) \quad t\geq 0, \quad 0<\alpha\leq\beta\leq 1, \quad \mu\geq 0, \tag{3.1}$$

where $u(0)=0$ and $u'(0)=g(u)$, and $f:I\times X\to X$, $g:C([0,1];X)\to X$ are suitable functions. In this case, it follows from (2.2) that a mild solution corresponds, by definition, to a fixed point of the equation:

$$u(t)=(g_1*S_{\alpha,\beta})(t)g(u)+\int_0^t(g_1*S_{\alpha,\beta})(t-s)f(s,u(s))ds, \quad 0<\alpha\leq\beta\leq 1,\ \mu\geq 0.$$

As an example, note that in the particular border case of $\alpha=\beta=1$ and $\mu=0$ the equation to be considered is given by

$$u''(t)=Au(t)+f(t,u(t)),$$

with nonlocal initial conditions $u(0)=0$ and $u'(0)=g(u)$. In consequence, if A is the generator of a bounded cosine family $C(t)$, then $S_{1,1}(t)\equiv C(t)$ and the family $S(t):=g_1*S_{1,1}(t)$ corresponds to the sine family generated by A. Note the important fact that $S(t)$ is always norm continuous for $t>0$ whenever $C(t)$ is bounded. Moreover, according to the choice of initial values for the problem, the mild solution should satisfy

$$u(t)=S(t)g(u)+\int_0^t S(t-s)f(s,u(s))ds.$$

In order to give our main result, we consider the following hypothesis.

- (H1) $g:C([0,1];X)\to X$ is continuous, compact and there exists positive constants c and d such that $\|g(u)\|\leqslant c\|u\|+d$, $\forall u\in C([0,1];X)$.
- (H2) $f:[0,1]\times X\to X$ satisfies the Carathéodory type conditions, that is, $f(\cdot,x)$ is measurable for all $x\in X$ and $f(t,\cdot)$ is continuous for almost all $t\in[0,1]$.
- (H3) There exists a function $m\in L^1(0,1;\mathbb{R}^+)$ (here $L^1(0,1;\mathbb{R}^+)$ is the space of $\mathbb{R}^+$-valued Bochner functions on $[0,1]$ with the norm $\|x\|=\int_0^1\|x(s)\|ds$) and a nondecreasing continuous function $\Phi:\mathbb{R}^+\to\mathbb{R}^+$ such that

  $$\|f(t,x)\|\leqslant m(t)\Phi(\|x\|)$$

  for all $x\in X$ and almost all $t\in[0,1]$.
- (H4) There exists a function $H\in L^1(0,1;\mathbb{R}^+)$ such that for any bounded $B\subseteq X$

  $$\gamma(f(t,B))\leqslant H(t)\gamma(B)$$

  for almost all $t\in[0,1]$.

In $(H4)$ γ denote the Hausdorff measure of noncompactness which is defined by

$$\gamma(B)=\inf\{\varepsilon>0: B \text{ has a finite cover by balls of radius } \varepsilon\}.$$

We note that this measure of noncompactness satisfies interesting regularity properties (for more information, we refer to [3]).

(i) If $A \subseteq B$ then $\gamma(A) \leqslant \gamma(B)$.
(ii) $\gamma(A) = \gamma(\overline{A})$, where $\overline{A}$ denotes the closure of A.
(iii) $\gamma(A) = 0$ if and only if A is totally bounded.
(iv) $\gamma(\lambda A) = |\lambda|\gamma(A)$ with $\lambda \in \mathbb{R}$.
(v) $\gamma(A \cup B) = \max\{\gamma(A), \gamma(B)\}$
(vi) $\gamma(A+B) \leqslant \gamma(A) + \gamma(B)$, where $A+B = \{a+b : a \in A,\ b \in B\}$.
(vii) $\gamma(A) = \gamma(\overline{co}(A))$ where $\overline{co}(A)$ is the closed convex hull of A.

Remark 3.1. *It is notable that, thanks to the boundedness of $S_{\alpha,\beta}(t)$ granted by Theorem 2.2, the function $t \to g_1 * S_{\alpha,\beta}(t)$ is norm continuous for $t > 0$. Indeed, we have for $0 < t < s$*

$$\| \int_0^t S_{\alpha,\beta}(\tau)d\tau - \int_0^s S_{\alpha,\beta}(t)\| \leq \int_t^s \|S_{\alpha,\beta}(\tau)\|d\tau \leq \sup_{\tau \geq 0} \|S_{\alpha,\beta}(\tau)\||s-t|.$$

We denote $M := \sup\{||g_1 * S_{\alpha,\beta}(t)|| : t \in [0,1]\}$. We are now in position to establish the main result of this paper.

Theorem 3.2. *If the hypothesis (H1)-(H4) are satisfied and there exists a constant $R > 0$ such that*

$$M(cR+d) + M\Phi(R)\int_0^1 m(s)ds \leqslant R$$

then the problem (3.1) *has at least one mild solution.*

Proof. Define $F : C([0,1];X) \to C([0,1];X)$ by

$$(Fx)(t) = \int_0^t S_{\alpha,\beta}(\tau)g(x)d\tau + \int_0^t (g_1 * S_{\alpha,\beta})(t-s)f(s,x(s))ds,\ \ t \in [0,1]$$

for all $x \in C([0,1];X)$. First, we show that F is a continuous map. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq C([0,1];X)$ be a sequence such that $x_n \to x$ (in the norm of $C([0,1];X)$). Note that

$$\|F(x_n) - F(x)\| \leqslant M\|g(x_n) - g(x)\| + M\int_0^1 \|f(s,x_n(s)) - f(s,x(s))\|ds,$$

by (H1) and (H2) and dominated convergence theorem we conclude that $\|F(x_n) - F(x)\| \to 0$ when $n \to \infty$.
Let $B_R := \{x \in C([0,1];X) : \|x(t)\| \leqslant R \text{ for all } t \in [0,1]\}$. Clearly B_R is bounded and convex. For any $x \in B_R$ we have by hipothesis

$$\begin{aligned}\|(Fx)(t)\| &\leqslant \|g_1 * S_{\alpha,\beta}(t)g(x)\| + \left\|\int_0^t g_1 * S_{\alpha,\beta}(t-s)f(s,x(s))ds\right\| \\ &\leqslant M(cR+d) + M\Phi(R)\int_0^1 m(s)ds \leqslant R.\end{aligned}$$

Therefore $F : B_R \to B_R$ is a bounded operator and $F(B_R)$ is a bounded set. Moreover, by continuity of the function $t \to g_1 * S_{\alpha,\beta}(t)$ on $(0,+\infty)$, we conclude that $F(B_R)$ is an equicontinuous

set of functions. Define $B := \overline{co}(F(B_R))$. Then B is an equicontinuous set of functions and $F : B \to B$ is a continuous operator.

Let $\varepsilon > 0$. By [22, Lemma 2.4] there exists $\{y_n\}_{n\in\mathbb{N}} \subset F(B)$ such that

$$\begin{aligned} \gamma(FB(t)) &\leqslant 2\gamma(\{y_n(t)\}_{n\in\mathbb{N}}) + \varepsilon \leqslant 2\gamma\left(\int_0^t g_1 * S_{\alpha,\beta}(t-s) f(s,\{y_n(s)\}_{n\in\mathbb{N}})ds\right) + \varepsilon \\ &\leqslant 4M\int_0^t \gamma(f(s,\{y_n(s))\}_{n\in\mathbb{N}})ds + \varepsilon \leqslant 4M\int_0^t H(s)\gamma(\{y_n(s)\}_{n\in\mathbb{N}})ds + \varepsilon \\ &\leqslant 4M\gamma(\{y_n\})\int_0^t H(s)ds + \varepsilon \leqslant 4M\gamma(B)\int_0^t H(s)ds + \varepsilon. \end{aligned} \tag{3.2}$$

Since $H \in L^1(0,1;X)$ there exists $\varphi \in C([0,1];\mathbb{R}_+)$ such that $\int_0^1 |H(s)-\varphi(s)|ds < \alpha$ (where $\alpha < \dfrac{1}{4M}$). Let $N := \max\{\varphi(t) : t \in [0,1]\}$. Then

$$\gamma(FB(t)) \leqslant 4M\gamma(B)\left[\int_0^t |H(s)-\varphi(s)|ds + \int_0^t \varphi(s)ds\right] + \varepsilon \leqslant 4M\gamma(B)\left[\alpha + Nt\right] + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we obtain that

$$\gamma(FB(t)) \leqslant (a+bt)\gamma(B), \tag{3.3}$$

where $a = 4\alpha M$ and $b = 4MN$. Let $\varepsilon > 0$ be given. By [22, Lemma 2.4] there exists $\{y_n\}_{n\in\mathbb{N}} \subseteq \overline{co}(F(B))$ such that

$$\begin{aligned} \gamma(F^2(B(t))) &\leqslant 2\gamma\left(\int_0^t g_1 * S_{\alpha,\beta}(t-s) f(s,\{y_n(s)\}_{n\in\mathbb{N}})ds\right) + \varepsilon \\ &\leqslant 4M\int_0^t \gamma(f(s,\{y_n(s)\}_{n\in\mathbb{N}}))ds + \varepsilon \\ &\leqslant 4M\int_0^t H(s)\gamma(\overline{co}(FB(s)))ds + \varepsilon \\ &\leqslant 4M\int_0^t H(s)\gamma(FB(s))ds + \varepsilon \\ &\leqslant 4M\int_0^t \left[|H(s)-\varphi(s)| + \varphi(s)\right](a+bs)\gamma(B)ds + \varepsilon \\ &\leqslant 4M(a+bt)\int_0^t |H(s)-\varphi(s)|ds + 4MN\left(at + \frac{bt^2}{2}\right) + \varepsilon \\ &\leqslant a(a+bt) + b\left(at + \frac{bt^2}{2}\right) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary then

$$\gamma(F^2(B(t))) \leqslant \left(a^2 + 2abt + \frac{(bt)^2}{2}\right)\gamma(B),$$

where $0 < a < 1$ and $b > 0$. By an iterative process we obtain

$$\gamma(F^n(B(t))) \leqslant \left(a^n + C_n^1 a^{n-1} bt + C_n^2 a^{n-2} \frac{(bt)^2}{2!} + \cdots + \frac{(bt)^n}{n!}\right) \gamma(B).$$

By [22, Lemma 2.1] we obtain that

$$\gamma(F^n(B)) \leqslant \left(a^n + C_n^1 a^{n-1} b + C_n^2 a^{n-2} \frac{b^2}{2!} + \cdots + \frac{b^n}{n!}\right) \gamma(B).$$

From [22, Lemma 2.5] we know that there exists $n_0 \in \mathbb{N}$ such that

$$\left(a^{n_0} + C_{n_0}^1 a^{n_0-1} b + C_{n_0}^2 a^{n_0-2} \frac{b^2}{2!} + \cdots + \frac{b^{n_0}}{n_0!}\right) = r < 1.$$

We conclude that

$$\gamma(F^{n_0} B) \leqslant r\gamma(B).$$

By [22, Lemma 2.6] , F has a fixed point in B, and this fixed point is a mild solution of equation (3.1). □

4. Application

To finish, we present one example which do not aim at generality but indicate how our theorem can be applied to concrete problems. Let $X = L^2(\mathbb{R}^n), \mu > 0$ and $0 < \alpha \leq \beta \leq 1$. Consider the following integro-differential equation

$$\text{(4.1)} \qquad \begin{cases} \partial_t^{\alpha+1} u(t,x) + \mu \partial_t^{\beta} u(t,x) &= Au(t,x) + \partial_t^{\alpha-1}[t^{-1/3}\sin(u(t))] \quad t \in [0,1]; \\ u(0,x) &= 0; \\ u_t(0,x) &= \sum_{i=1}^{N} \int_{\mathbb{R}^n} \varepsilon k(x,y) u(t_i,y) dy, \quad x \in \mathbb{R}^n, \end{cases}$$

where N is a positive integer, $0 < t_1 < t_2 < \cdots < t_m < 1$; $k(x,y) \in L^2(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^+)$, the constants $\alpha, \beta\mu$ satisfy $0 < \alpha \leq \beta \leq 1$, $\mu > 0$, and the operator A is defined by

$$(Au)(t,x) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u(t,x)}{\partial z_i \partial z_j} + \sum_{i=1}^{n} b_i(x) \frac{u(t,x)}{\partial z_i} + \overline{c}(x) u(t,x),$$

with given coefficients a_{ij} , b_i , $\overline{c}$, $(i, j = 1, 2, \ldots, n)$ satisfying the usual uniformly ellipticity conditions, and $D(A) = \{v \in X : v \in H^2(\mathbb{R}^n)\}$. We will prove that there exists $\varepsilon > 0$ sufficiently small such that equation (4.1) has a mild solution on X. Indeed, note that system (4.1) takes the abstract form

$$\text{(4.2)} \qquad \begin{cases} D_t^{\alpha+1} u(t) + \mu D_t^{\beta} u(t) &= Au(t) + D_t^{\alpha-1} f(t, u(t)), \quad t \in [0,1]; \\ u(0) &= 0; \\ u'(0) &= g_\varepsilon(u). \end{cases}$$

where the function $g_\varepsilon : C([0,1],X) \to X$ is given by $g_\varepsilon(u)x = \varepsilon \sum_{i=1}^{m} k_g u(t_i)(x)$ with $(k_g v)(x) = \int_{\mathbb{R}^n} k(x,y)v(y)dy$, for $v \in X, x \in \mathbb{R}^n$, and the function $f : [0,1] \times X \to X$ is defined by $f(t,u(t)) = t^{-1/3}\sin(u(t))$. Observe that $\|f(t,u(t)) - f(t,v(t))\| \leqslant t^{-1/3}\|u-v\|$, and hence f satisfies $(H2)$. Note that $\|g_\varepsilon(v)\| \leqslant N\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varepsilon k^2(z,y)dydz\Big)^{1/2}\|v\|$, and the function k_g is completely continuous. It proves $(H1)$. In addition $\|f(t,u(t))\| \leqslant t^{-1/3}\Phi(\|u\|)$, with $\Phi(\|u\|) \equiv 1$, proving $(H3)$. Finally, given a bounded subset B of X, and from the properties of γ, we obtain $\gamma(f(t,B)) \leq t^{-1/2}\gamma(\sin(B)) \leq Ct^{-1/2}\gamma(B)$ for some constant $C > 0$ and therefore $(H4)$ is also satisfied.

On the other hand, it follows from the theory of C_0-semigroups (see e.g. [9]) that A generates an analytic, non compact semigroup $\{T(t)\}_{t\geqslant 0}$ on $L^2(\mathbb{R}^n)$. In particular, A is $\pi/2$-sectorial. Furthermore, there exists a constant $M > 0$ such that $M = \sup\{\|T(t)\| : t \geqslant 0\} < +\infty$. By Theorem 2.2, the operator A in equation (4.2) generates a bounded $(\alpha,\beta)_\mu$-regularized family $\{S_{\alpha,\beta}(t)\}_{t\geqslant 0}$. Let $K = \sup\{\|g_1 * S_{\alpha,\beta}(t)\| : t \in [0,1]\}$. Observe that there exists $\varepsilon > 0$ such that $Kc < 1$ where $c = \varepsilon N\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k^2(z,y)dydz\Big)^{1/2}$. Therefore, there exists $R > 0$ such that $KcR + \dfrac{3K}{2} < R$. It follows that equation (4.2) has at least a mild solution for all $\varepsilon > 0$ sufficiently small.

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